Super-Connectivity of the Folded Locally Twisted Cube

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Abstract: The hypercube $Q_n$ is one of the most popular interconnection networks with high symmetry. To reduce the diameter of $Q_n$, many variants of $Q_n$ have been proposed, such as the $n$-dimensional locally twisted cube $LTQ_n$. To further optimize the diameter of $LTQ_n$, the $n$-dimensional folded locally twisted cube $FLTQ_n$ is proposed, which is built based on $LTQ_n$ by adding $2^{n-1}$ complementary edges. Connectivity is an important indicator to measure the fault tolerance and reliability of a network. However, the connectivity has an obvious shortcoming, in that it assumes all the adjacent vertices of a vertex will fail at the same time. Super-connectivity is a more refined index to judge the fault tolerance of a network, which ensures that each vertex has at least one neighbor. In this paper, we show that the super-connectivity $\kappa^{(1)}(FLTQ_n) = 2n$ for any integer $n \geq 6$, which is about twice $\kappa(LTQ_n)$.

Keywords: super-connectivity; folded locally twisted cube; fault tolerance; interconnection network; reliability

1. Introduction

High-performance computers can be widely used in many fields thanks to the development of high performance computing technology. The topological properties of interconnection networks are very important for high-performance computers. One typically uses an undirected graph $G = (V(G), E(G))$ to model the topology of a multiprocessor system $H$, where the processor set of $H$ is represented by $V(G)$ and the link set of $H$ is represented by $E(G)$.

Interconnection networks have many important properties, one of which is the connectivity denoted by $\kappa(G)$. A graph’s connectivity is the minimum number of vertices whose removal makes the graph disconnected or trivial [1]. Connectivity is an important indicator to measure the fault tolerance and reliability of a network. In a large interconnection network, each vertex has a large number of neighbors. This property has an obvious deficiency, in that it assumes that all the adjacent vertices of a vertex will fail at the same time. However, this situation does not happen frequently in real networks. To address this deficiency, Esfahanian et al. [2] introduced the concept of restricted connectivity by imposing additionally restricted conditions on a network. Super-connectivity is a special case of restricted connectivity. When determining the super-connectivity of a network, one needs to ensure that each vertex has at least one neighbor. Hence, super-connectivity is a more refined index to judge the fault tolerance of a network.

Let $K$ be a subset of $V(G)$. $G \setminus K$ (or $G - K$) denotes a graph obtained by removing all the vertices in $K$ and edges incident to at least one vertex in $K$ from $G$. If $G \setminus K$ is disconnected and each component of $G \setminus K$ has at least two vertices, then $K$ is called a super vertex cut. Let $S$ be a subset of $E(G)$. If $G \setminus S$ is disconnected and each component...
of $G \setminus S$ has at least two vertices, then $S$ is called a super edge cut. The super-connectivity of $G$ (or, respectively, the super edge connectivity), denoted by $\kappa^{(1)}(G)$ (or $\lambda^{(1)}(G)$), is the minimum cardinality of all super vertex cuts (or super edge cuts) in $G$, if any exist. Many relevant results have been obtained regarding super-connectivity and super edge connectivity [3–16].

The hypercube $Q_n$ has become one of the most popular interconnection networks, because of its many attractive properties, such as its regularity and symmetry. $Q_n$ is a Cayley graph and hence vertex-transitive and edge-transitive. However, the diameter of $Q_n$ is not optimal. In order to enhance the hypercube, researchers have proposed many variants, such as crossed cubes [17], locally twisted cubes [18], and spined cubes [19]. The $n$-dimensional locally twisted cube $LTQ_n$ was proposed by Yang et al. [18], whose diameter was only about half that of $Q_n$. Many research results have been published on the properties of $LTQ_n$ [20–25]. $LTQ_n$ is vertex-transitive if and only if $n \leq 3$, and it is edge-transitive if and only if $n = 2$ [25]. To further enhance the hypercube, inspired by the folded cube [26], Peng et al. [27] proposed a new network topology called the folded locally twisted cube $FLTQ_n$. So far there, no work has been reported on the super-connectivity of $FLTQ_n$. In this work, we studied the super-connectivity of $FLTQ_n$ and obtained the result that the super-connectivity $\kappa^{(1)}(FLTQ_n)$ is $2n$ for $n \geq 6$, which is about twice $\kappa(FLTQ_n)$.

2. Preliminaries

In this paper, we use the terms vertex and node interchangeably. We also use $(x, y)$ to denote an edge between vertices $x$ and $y$. For any vertex $x \in V(G)$, the neighboring set of $x$ is denoted by $N_G(x) = \{y | (x, y) \in E(G)\}$ (or $N(x)$ for short). Let $S \subset V(G)$. The neighboring set of $S$ is defined as $N_G(S) = (\cup_{x \in S} N(x)) \setminus S$ (or $N(S)$ for short). We define $N_G[S] = \cup_{x \in S} N(x)$ and $N_G[x] = N_G(x) \cup \{x\}$. We use $x_0x_1x_2 \cdots x_{n-1}$ to represent a binary string $\mu$ of length $n$, where $x_i \in \{0, 1\}$ for $1 \leq i \leq n$ is a part of $\mu$. $x_1$ is the first part of $\mu$, and $x_n$ is the $n$th part of $\mu$. The symbol $x_i$ is used to represent the complement of $x_i$. As a variant of $Q_n$, $LTQ_n$ has the same number of vertices as $Q_n$. Each vertex of $LTQ_n$ is denoted by a unique binary string of length $n$. The definition of $LTQ_n$ is given below.

**Definition 1** ([18]). For $n \geq 2$, an $n$-dimensional locally twisted cube, $LTQ_n$, is defined recursively as follows:

1. $LTQ_2$ is a graph consisting of four nodes labeled with $00$, $01$, $10$, and $11$, which are connected by four edges, $(00, 01)$, $(00, 10)$, $(01, 11)$, and $(10, 11)$.
2. For $n \geq 3$, $LTQ_n$ is built from two disjointed copies of $LTQ_{n-1}$ named $LTQ^0_{n-1}$ and $LTQ^1_{n-1}$. Let $LTQ^0_{n-1}$ (or, respectively, $LTQ^1_{n-1}$) be the graph obtained by prefixing the label of each node of one copy of $LTQ_{n-1}$ with $0$ (or with $1$); each node $x = 0x_nx_{n-2} \cdots x_2x_1$ of $LTQ^0_{n-1}$ is connected to the node $1(x_n-1 + x_1)x_{n-2} \cdots x_2x_1$ of $LTQ^1_{n-1}$ with an edge, where $+ replace$ represents modulo 2 addition.

$LTQ_3$ and $LTQ_4$ are demonstrated in Figure 1. Each node in $LTQ^0_{n-1}$ has only one adjacent node in $LTQ^1_{n-1}$. The set of edges between $LTQ^0_{n-1}$ and $LTQ^1_{n-1}$ is called a perfect matching $M$ of $LTQ_n$. Hence, we can write $LTQ_n = G(LTQ^0_{n-1}, LTQ^1_{n-1}, M)$.

In [18], Yang et al. also provided a non-recursive definition of $LTQ_n$.

**Definition 2** ([18]). Let $\mu = x_nx_{n-1} \cdots x_1$ and $\nu = y_ny_{n-1} \cdots y_1$ be any two distinct vertices of $LTQ_n$ for $n \geq 2$. $\mu$ and $\nu$ are connected if and only if one of the following conditions is satisfied:

1. There is an integer $3 \leq k \leq n$ such that
   (a) $x_k = y_k$;
   (b) $x_{k-1} = y_{k-1} + x_1$ (‘+’ represents modulo 2 addition);
   (c) all the remaining bits of $\mu$ and $\nu$ are the same.
2. There is an integer $1 \leq k \leq 2$ such that $\mu$ and $\nu$ only differ in the $k$th bit.
Let $\mu = x_nx_{n-1}x_{n-2} \ldots x_3x_2x_1$ be any vertex of $LTQ_n$. By Definition 2, all the $n$ neighbors of $\mu$ are listed as follows:

$\mu_1 = x_nx_{n-1}x_{n-2} \ldots x_3x_2x_1$;

$\mu_2 = x_nx_{n-1}x_{n-2} \ldots x_3x_2x_1$;

$\mu_3 = x_nx_{n-1}x_{n-2} \ldots x_3(x_2 + x_1)x_1$;

$\ldots$

$\mu_{n-1} = x_nx_{n-1}(x_{n-2} + x_1)x_{n-3} \ldots x_2x_1$;

$\mu_n = x_n(x_{n-1} + x_1)x_{n-2} \ldots x_3x_2x_1$.

We call $\mu_i$ the $i$th dimensional neighbor of $\mu$ for $1 \leq i \leq n$.

![Diagram of LTQ_3 and LTQ_4](image-url)

**Figure 1.** (a) The three-dimensional locally twisted cube $LTQ_3$; (b) the four-dimensional locally twisted cube $LTQ_4$.

**Definition 3** ([27]). For any integer $n \geq 2$, an $n$-dimensional folded locally twisted cube, denoted by $FLTQ_n$, is a graph constructed based on $LTQ_n$ by adding all complementary edges. Each vertex $x = x_nx_{n-1} \ldots x_1$ in $LTQ_n$ is incident to another vertex $\overline{x} = \overline{x}_nx_{n-1} \ldots \overline{x}_1$ through a complementary edge, where $\overline{x}_i = 1 - x_i$.

We call the added complementary edges $c$-links. $FLTQ_n$ has $2^{n-1}$ $c$-links, and each vertex $\mu = x_nx_{n-1} \ldots x_1$ is connected to a complementary vertex $\mu_c = \overline{x}_n\overline{x}_{n-1} \ldots \overline{x}_1$ by a $c$-link. The set of complementary edges between $LTQ_n^{0}_{n-1}$ and $LTQ_n^{1}_{n-1}$ is a perfect matching $C$ of $FLTQ_n$. Hence, we can write $FLTQ_n = G(LTQ_n^{0}_{n-1}, LTQ_n^{1}_{n-1}, M, C)$ or $G(LTQ_n, C)$. Each node $\mu \in V(FLTQ_n)$ in $LTQ_n^{0}_{n-1}$ (or, respectively, $LTQ_n^{1}_{n-1}$) has two neighbors, $\mu_n$ and $\mu_c$, in $LTQ_n^{0}_{n-1}$ (or $LTQ_n^{1}_{n-1}$) for $n \geq 3$. Compared with $LTQ_n$, each vertex in $FLTQ_n$ has one more neighbor. Then, the node degree of $FLTQ_n$ is $n + 1$ and $\kappa(FLTQ_n) = n + 1$ [27]. Figure 2 demonstrates $FLTQ_3$ and $FLTQ_4$, respectively, and Figure 3 demonstrates $FLTQ_5$. 
Figure 2. (a) The three-dimensional folded locally twisted cube \( \text{FLTQ}_3 \); (b) the four-dimensional folded locally twisted cube \( \text{FLTQ}_4 \).

Figure 3. The five-dimensional folded locally twisted cube \( \text{FLTQ}_5 \).

3. Super Connectivity of \( \text{FLTQ}_n \)

In this section, we study the super connectivity of \( \text{FLTQ}_n \) for any integer \( n \geq 6 \). Since \( \text{FLTQ}_n \) is composed of \( \text{LTQ}_n \) and the complementary edge set \( C \), we can use some properties of \( \text{LTQ}_n \) to prove the super-connectivity property of \( \text{FLTQ}_n \).
Lemma 1 ([18]). For $n \geq 2$, $\kappa(LTQ_n) = \lambda(LTQ_n) = n$.

Lemma 2 ([28]). For any two vertices $\mu, \nu \in V(LTQ_n)(n \geq 2)$, we have $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$.

Lemma 3 ([28]). Let $\mu$ and $\nu$ be any two distinct vertices in $LTQ_n(n \geq 4)$ such that $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| = 2$.

1. If $\mu \in V(LTQ^0_{n-1})$ and $\nu \in V(LTQ^1_{n-1})$, then the two common neighbors in $LTQ^0_{n-1}$ and the other one is in $LTQ^1_{n-1}$.
2. If $\mu, \nu \in V(LTQ^0_{n-1})$ or $V(LTQ^1_{n-1})$, then the two common neighbors are in $LTQ^0_{n-1}$ or $LTQ^1_{n-1}$.

Lemma 4. Let $\mu$ and $\nu$ be any two distinct vertices in the same $LTQ^i_{n-1}$ for $0 \leq i \leq 1$ and $n \geq 6$. If $\mu_n = \nu_c$ or $\mu_c = \nu_n$, then $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 1$.

Proof. Without loss of generality, we suppose that $\mu, \nu \in V(LTQ^0_{n-1})$, and $\mu_n = \nu_c$. Then, $\mu_n$ is the common neighbor for $\mu$ and $\nu$. Let $\mu = x_nx_{n-1}x_{n-2} \cdots x_3x_2x_1$ and $X = LTQ_n \setminus \{\mu_n\}$. Next, we consider the neighbors of $\mu$ and $\nu$ in $X$ according to different values of the first part $x_1$ of $\mu$.

Case 1. $x_1 = 0$.

$\mu_n = x_nx_{n-1}x_{n-2} \cdots x_3x_20 = \nu_c$ and $\nu = x_nx_{n-1}x_{n-2} \cdots x_3x_21$. We list $N_X(\mu)$ and $N_X(\nu)$ separately in Table 1.

<table>
<thead>
<tr>
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<tr>
<td>$\mu_2 = x_nx_{n-1}x_{n-2} \cdots x_3\bar{x}_20$</td>
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<tr>
<td>$\mu_3 = x_nx_{n-1}x_{n-2} \cdots \bar{x}_3x_20$</td>
<td>$\nu_3 = x_n\bar{x}<em>{n-1}\bar{x}</em>{n-2} \cdots \bar{x}_3x_21$</td>
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<tr>
<td>$\mu_{n-1} = x_n\bar{x}<em>{n-1}x</em>{n-2} \cdots x_3x_20$</td>
<td>$\nu_{n-1} = x_nx_{n-1}x_{n-2} \cdots \bar{x}_3\bar{x}_21$</td>
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<tr>
<td>$\mu_c = x_n\bar{x}<em>{n-1}x</em>{n-2} \cdots x_3x_21$</td>
<td>$\nu_c = x_nx_{n-1}x_{n-2} \cdots \bar{x}_3\bar{x}_21$</td>
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It is obvious that $|N_X(\mu) \cap N_X(\nu)| = 0$.

Case 2. $x_1 = 1$.

$\mu_n = x_nx_{n-1}x_{n-2} \cdots x_3x_21 = \nu_c$ and $\nu = x_nx_{n-1}\bar{x}_{n-2} \cdots \bar{x}_3x_20$. We list $N_X(\mu)$ and $N_X(\nu)$ separately in Table 2.

<table>
<thead>
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<td>$\mu_3 = x_nx_{n-1}x_{n-2} \cdots \bar{x}_3\bar{x}_21$</td>
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<td>$\mu_c = x_n\bar{x}<em>{n-1}x</em>{n-2} \cdots x_3\bar{x}_21$</td>
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It is obvious that $|N_X(\mu) \cap N_X(\nu)| = 0$.

Hence, $\mu$ and $\nu$ have only one common neighbor in $FLTQ_n$ and $|N_{FLTQ_n}(\mu) \cap N_{FLTQ_n}(\nu)| = 1$. □

Lemma 5. Let $\mu$ be any node in $FLTQ_n$, where $n \geq 6$ and $X = FLTQ_n \setminus \{\mu\}$. Then, $|N_X(\mu_n) \cap N_X(\mu_c)| = 0$. 

Proof. Let $\mu = x_nx_{n-1}x_{n-2} \ldots x_3x_2x_1$. We consider the different values of the first part $x_1$ of $\mu$.

Case 1. $x_1 = 0$.

Let $\alpha = \mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 0$ and $\beta = \mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots x_3 \bar{x}_2 1$. All the neighbors of $\alpha$ and $\beta$ in $X$ are listed separately in Table 3.

<table>
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It is obvious that $N_X(\alpha) \cap N_X(\beta) = \emptyset$.

Case 2. $x_1 = 1$.

Let $\alpha = \mu_n = \bar{x}_n x_{n-1} x_{n-2} \ldots x_3 x_2 1$ and $\beta = \mu_c = \bar{x}_n \bar{x}_{n-1} \bar{x}_{n-2} \ldots x_3 \bar{x}_2 0$. All the neighbors of $\alpha$ and $\beta$ in $X$ are listed separately in Table 4.

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It is obvious that $N_X(\alpha) \cap N_X(\beta) = \emptyset$.

Hence, $|N_X(\mu_n) \cap N_X(\mu_c)| = 0$. \(\square\)

Lemma 6. Let $\mu, \nu \in V(LTQ_n)$ where $n \geq 6$. Then $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$.

Proof. Since $LTQ_n$ is constructed from $LQ_n$ by adding the complementary edge set $C$, we can study this lemma based on $LQ_n$.

Case 1. $\mu, \nu$ are in the same $LTQ_{n-1}$ for $0 \leq i \leq 1$.

Without loss of generality, we suppose that $\mu, \nu \in V(LTQ_{n-1})$. According to Lemmas 2 and 3, $|N_{LQ_{n-1}}(\mu) \cap N_{LQ_{n-1}}(\nu)| \leq 2$ for $n \geq 6$, and the two common neighbors are in $LQ_{n-1}$. According to the definition of $LTQ_n$, we have $N_{LTQ_{n-1}}(\mu) = \{\mu_n, \mu_c\}$, $N_{LTQ_{n-1}}(\nu) = \{\nu_n, \nu_c\}$, where $\mu_n \neq \nu_n$ and $\mu_c \neq \nu_c$. If $\mu_c \neq \nu_c$, then $\mu$ and $\nu$ do not have the same neighbors in $LQ_{n-1}$. Hence, $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| \leq 2$. According to Lemma 4, if $\mu_c = \nu_c$ or $\mu_n = \nu_n$, then $\mu$ and $\nu$ have only one common neighbor in $LQ_n$ and $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| = 1 \leq 2$.

Case 2. $\mu$ and $\nu$ are in a different $LTQ_{n-1}$ for $0 \leq i \leq 1$.

Without loss of generality, we suppose that $\mu \in V(LTQ_{n-1})$ and $\nu \in V(LTQ_{n-1})$. According to Lemma 2, $|N_{LTQ_{n-1}}(\mu) \cap N_{LTQ_{n-1}}(\nu)| \leq 2$. Based on the definition of $LTQ_n$, we have $N_{LTQ_{n-1}}(\mu) = \{\mu_n, \mu_c\}$ and $N_{LTQ_{n-1}}(\nu) = \{\nu_n, \nu_c\}$. According to Lemma 5, $|N_{LTQ_n}(\mu) \cap N_{LTQ_n}(\nu)| = 0$. Hence, we cannot find a vertex $\mu' \in V(LTQ_{n-1})$, where $\mu'$ and $\mu \in V(LTQ_{n-1})$ have two common neighbors, nor can we find a vertex $\nu' \in V(LTQ_{n-1})$, where $\nu'$ and $\nu \in V(LTQ_{n-1})$ have
two common neighbors. Then, \( u \) and \( v \) cannot have three or four common neighbors in \( \text{FLTQ}_n \). Hence, \( |N_{\text{FLTQ}_n}(\mu) \cap N_{\text{FLTQ}_n}(v)| \leq 2 \). \( \square \)

**Lemma 7 ([28]).** If \( \mu \) and \( v \) are two vertices of \( \text{LTQ}_n \) and \( (\mu, v) \in E(\text{LTQ}_n) \), where \( n \geq 2 \), then \( |N_{\text{LTQ}_n}(\mu) \cap N_{\text{LTQ}_n}(v)| = 0 \).

**Lemma 8.** If \( \mu \) and \( v \) are two vertices of \( \text{FLTQ}_n \) and \( (\mu, v) \in E(\text{FLTQ}_n) \), where \( n \geq 3 \), then \( |N_{\text{FLTQ}_n}(\mu) \cap N_{\text{FLTQ}_n}(v)| = 0 \).

**Proof.** According to the position of \( \mu \) and \( v \), we consider two cases.

Case 1. \( \mu \) and \( v \) are in the same \( \text{LTQ}^i_{n-1} \) for \( 0 \leq i \leq 1 \).

Without loss of generality, we assume that \( \mu, v \in V(\text{LTQ}^0_{n-1}) \). According to Lemma 7, \( |N_{\text{LTQ}^0_{n-1}}(\mu) \cap N_{\text{LTQ}^0_{n-1}}(v)| = 0 \). We have \( N_{\text{LTQ}^0_{n-1}}(\mu) = \{\mu_n, \mu_0\} \) and \( N_{\text{LTQ}^0_{n-1}}(v) = \{v_0, v_c\} \). If \( N_{\text{LTQ}^0_{n-1}}(\mu) \cap N_{\text{LTQ}^0_{n-1}}(v) = \emptyset \), then \( |N_{\text{LTQ}^0_{n-1}}(\mu) \cap N_{\text{LTQ}^0_{n-1}}(v)| = 0 \). Otherwise, if \( \mu_0 = v_c \) or \( \mu_c = v_n \), then we let \( \mu = x_nx_{n-1}x_{n-2} \ldots x_2x_1 \). All the possible values of \( \mu \) and \( v \) are listed in Table 5.

| Table 5. The possible values of \( \mu \) and \( v \). |
|-----------------|-----------------|-----------------|
| \( \mu_n = v_c \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( x_1 = 0 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( v = x_nx_{n-2} \ldots x_3x_21 \) |
| \( \mu_n = v_c \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( x_1 = 1 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( v = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) |
| \( \mu_c = v_n \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( x_1 = 0 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( v = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) |
| \( \mu_c = v_n \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( x_1 = 1 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( v = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) |

It is obvious that \( (\mu, v) \notin E(\text{FLTQ}_n) \); then, we reach a contradiction, and all these values of \( \mu \) and \( v \) are impossible. Hence, \( |N_{\text{FLTQ}_n}(\mu) \cap N_{\text{FLTQ}_n}(v)| = 0 \).

Case 2. \( \mu \) and \( v \) are in a different \( \text{LTQ}^i_{n-1} \) for \( 0 \leq i \leq 1 \).

Without loss of generality, we assume that \( \mu \in V(\text{LTQ}^0_{n-1}) \) and \( v \in V(\text{LTQ}^1_{n-1}) \). Since \( (\mu, v) \in E(\text{LTQ}_n) \), \( v \) should be \( \mu_0 \) or \( \mu_c \). If \( \mu_0 = v \), let \( K = \{\mu, v, \mu_n, v_n\} \). Otherwise, If \( \mu_c = v \), let \( K = \{\mu, v, \mu_c, v_c\} \). Let \( \mu = x_nx_{n-1}x_{n-2} \ldots x_2x_1 \). All the possible values of \( K \) are listed in Table 6.

| Table 6. The possible values of \( \mu \) and \( v \). |
|-----------------|-----------------|-----------------|
| \( \mu_n = v_c \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( x_1 = 0 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( v = x_nx_{n-2} \ldots x_3x_21 \) |
| \( \mu_n = v_c \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( x_1 = 1 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( v = x_nx_{n-2} \ldots x_3x_21 \) |
| \( \mu_c = v_n \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) | \( x_1 = 0 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( v = x_nx_{n-2} \ldots x_3x_21 \) |
| \( \mu_c = v_n \) | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( x_1 = 1 \) |
| | \( \mu = x_nx_{n-1}x_{n-2} \ldots x_3x_21 \) | \( v = x_nx_{n-1}x_{n-2} \ldots x_3x_20 \) |

Because \( (\mu, v), (\mu_n, v), (\mu_n, v_n) \notin E(\text{FLTQ}_n) \), when \( \mu_c = v \), \( \mu \) and \( v \) do not have common neighbors. Hence, \( |N_{\text{FLTQ}_n}(\mu) \cap N_{\text{FLTQ}_n}(v)| = 0 \).

**Lemma 9.** Let \( \mu \) be any node in \( \text{LTQ}_n \) for any integer \( n \geq 3 \). Then, \( \text{LTQ}_n \setminus N_{\text{LTQ}_n}[\mu] \) is connected.

**Proof.** We use mathematical induction on \( n \) to prove this lemma. According to Lemma 1, we know that this lemma obviously holds when \( n = 3 \). Suppose that this lemma holds for \( n \leq k (k \geq 3) \). Let \( \mu \) be any node in \( \text{LTQ}_{k+1} \). Without loss of generality, we suppose that \( \mu \in V(\text{LTQ}^k_k) \). Then, by the induction hypothesis, \( \text{LTQ}^k_k \setminus N_{\text{LTQ}^k_k}[\mu] \) is connected. Since \( N_{\text{LTQ}^k_k}(\mu) = \{\mu_{k+1}\} \), according to Lemma 1, \( \text{LTQ}^k_k \setminus \{\mu_{k+1}\} \) is connected. Since each node in \( \text{LTQ}^0_k \) is connected to a node in \( \text{LTQ}^1_k \), \( \text{LTQ}^0_k \setminus N_{\text{LTQ}^0_k}[\mu] \) is connected to \( \text{LTQ}^1_k \setminus \{\mu_{k+1}\} \). Then, \( \text{LTQ}_{k+1} \setminus N_{\text{LTQ}_{k+1}}[\mu] \) is connected. Hence, this lemma holds. \( \square \)
prove that \( \kappa \)
vertices, then we have the lower bound
\[
\{ \text{FLTQ} \}
\]
and each vertex in \( \text{FLTQ} \)
Therefore, we have
\[
\kappa
(\text{FLTQ})
\]
Suppose that \( \kappa \)
is disconnected, and the edge \( \kappa \)
Consider an edge \( \kappa \)
\[
\{ \text{FLTQ} \}
\]
Then, we just need to find a super vertex cut
\[
\text{FLTQ}
\]
According to Lemma 8, \( |N_{\text{FLTQ}}(a) \cap N_{\text{FLTQ}}(x)| \leq 2 \) and \( |N_{\text{FLTQ}}(a) \cap N_{\text{FLTQ}}(y)| \leq 2 \). Since \( \kappa(\text{FLTQ}) = n + 1 \) and \( n + 1 - 2 - 2 \geq 1 \) for \( n \geq 6 \), \( a \) has at least one neighbor in \( K \). Hence, \( N_{\text{FLTQ}}(F) \) is a super vertex cut and \( \kappa(\text{FLTQ}) \leq 2n \) for \( n \geq 6 \).

**Lemma 10.** \( \kappa(\text{FLTQ}) \leq 2n \) for any integer \( n \geq 6 \).

**Proof.** Consider an edge \( (x, y) \in E(\text{FLTQ}) \). Let \( F = \{x, y\} \). Then, \( \text{FLTQ} \setminus N_{\text{FLTQ}}(F) \) is disconnected, and the edge \( (x, y) \) is one component of \( \text{FLTQ} \setminus N_{\text{FLTQ}}(F) \). According to Lemma 8, \( |N_{\text{FLTQ}}(F)| = (n + 1) + (n + 1) - 2 = 2n \). Let \( K = \text{FLTQ} \setminus N_{\text{FLTQ}}[F] \). To prove that \( K \) is a super vertex cut, we need to show that each vertex \( a \in V(K) \) has at least one neighbor. According to Lemma 6, \( |N_{\text{FLTQ}}(a) \cap N_{\text{FLTQ}}(x)| \leq 2 \) and \( |N_{\text{FLTQ}}(a) \cap N_{\text{FLTQ}}(y)| \leq 2 \). Since \( \kappa(\text{FLTQ}) = n + 1 \) and \( n + 1 - 2 - 2 \geq 1 \) for \( n \geq 6 \), \( a \) has at least one neighbor in \( K \). Hence, \( N_{\text{FLTQ}}(F) \) is a super vertex cut and \( \kappa(\text{FLTQ}) \leq 2n \) for \( n \geq 6 \).

**Lemma 11.** \( \kappa(\text{FLTQ}) \geq 2n \) for \( n \geq 6 \).

**Proof.** Suppose that \( F \) is a super vertex cut of \( \text{FLTQ} \). Then, \( \text{FLTQ} \setminus F \) is disconnected, and each vertex in \( \text{FLTQ} \setminus F \) has at least one neighbor. To prove \( \kappa(\text{FLTQ}) \geq 2n \), we will show that \( \text{FLTQ} \setminus F \) is connected when \( |F| \leq 2n - 1 \). Let \( F_i = F \cap LTQ^i_{n-1} \) for \( 0 \leq i \leq 1 \), \( K_0 = LTQ^0_{n-1} \setminus F_0 \), and \( K_1 = LTQ^1_{n-1} \setminus F_1 \). Without loss of generality, we suppose that \( |F_0| \geq |F_1| \). Then, \( |F_1| \leq n - 1 \).

Case 1. \( K_1 \) is connected.

Let \( a \) be any node in \( K_0 \). We have \( N_{LTQ^1_{n-1}}(a) = \{a, a\} \). If \( |N_{LTQ^1_{n-1}}(a) \cap F_1| \leq 1 \), then \( a \) is connected to \( K_1 \). Since \( K_1 \) is connected, then \( K_0 \cup K_1 \) is connected, which means that \( \text{FLTQ} \setminus F \) is connected. Otherwise, since each vertex in \( \text{FLTQ} \setminus F \) has at least one neighbor, there must be a vertex \( \beta \in K_0 \) such that \( (a, \beta) \in E(K_0) \). We have \( N_{LTQ^1_{n-1}}(\beta) = \{\beta, \beta\} \). If \( |N_{LTQ^1_{n-1}}(\beta) \cap F_1| \leq 1 \), then \( a \) can be connected to \( K_1 \) through vertex \( \beta \), and \( \text{FLTQ} \setminus F \) is connected. Otherwise, we have \( \{a, \alpha, \beta, \beta\} \in F_i \), \( |F_i| \geq 4 \), and \( |F_0| \leq 2n - 5 \). Let \( Y = N_{LTQ^0_{n-1}}(a) \cup N_{LTQ^1_{n-1}}(\beta) \setminus \{a, \alpha\} \). According to Lemma 8, \( |Y| = (n - 1) + (n - 1) - \)
2 = 2n − 4. Since |F₀| ≤ 2n − 5, we can find at least one vertex γ ∈ Y such that α and β are connected to K₁ through γ. Hence, FLTQₙ \ F is connected.

Case 2. K₁ is disconnected.

According to Lemma 1, we have κ(LTQₙ₋₁) = n − 1. Since K₁ is disconnected, then |F₁| = n − 1 and |F₀| = n. There should be an isolated vertex ω in K₁ and F₁ = NLTQₙ−₁[ω].

According to Lemma 9, LTQ₁ₙ₋₁ \ NLTQₙ−₁[ω] is connected. For any vertex α in K₀ where (α, ω) ∈ EFLTQₙ, based on Lemma 8, α and ω do not have common neighbors. Then, there exists a neighbor α' of α in LTQₙ such that α' \ NLTQₙ−₁[ω]. Hence, α is connected to LTQ₁ₙ₋₁ \ NLTQₙ−₁[ω] through α'. For any vertex α in K₀ where (α, ω) \ EFLTQₙ, there must exist a neighbor β in K₀. Let Y = NLTQₙ−₁(α) \ NLTQₙ−₁(β). According to Lemma 8, |Y| = (n − 1) + (n − 1) = 2n − 2. Since |F₀| = n, we can find at least n − 2 vertices in Y connected to LTQ₁ₙ₋₁. Since there exist two neighbors in LTQ₁ₙ₋₁ for each vertex in Y and 2n − 4 > n − 1 when n ≥ 6, we can find a vertex γ in Y such that α and β are connected to LTQ₁ₙ₋₁ \ NLTQₙ−₁[ω] through γ. Hence, FLTQₙ − F is connected.

Thus, FLTQₙ \ F is connected when |F| ≤ 2n − 1 and κ(FLTQₙ) ≥ 2n for any integer n ≥ 6. □

According to Lemmas 10 and 11, we obtain the following result:

**Theorem 1.** κ(FLTQₙ) = 2n for n ≥ 6.

4. Conclusions

The folded locally twisted cube FLTQₙ was introduced based on the locally twisted cube LTQₙ and the folded hypercube FQₙ. In this paper, we studied the super-connectivity of folded locally twisted cubes, which is an important indicator to measure the fault tolerance and reliability of a network. The main contribution of this work was that we addressed the super-connectivity of FLTQₙ. We proved that κ(FLTQₙ) = 2n for any integer n ≥ 6. Independent spanning trees and mesh embedding could be considered as future research directions. Independent spanning trees could be applied to reliable communication protocols, reliable broadcasting, and so on [29]. Meshes are fundamental guest graphs on which many algorithms, such as linear algebra algorithms and combinatorial algorithms, can be efficiently performed [30]. The results of independent spanning trees and mesh embedding for FLTQₙ could be compared with the results of LTQₙ [31,32].

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