

Article

Optimizing Emergency Plane Selection in Civil Aviation Using Extended Dombi Hybrid Operators

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Abstract: Airports located in densely populated areas often face challenges due to asymmetrical traffic patterns. Efficient management and careful planning are required to handle the disproportionate flow of passengers, aircraft, and ground services. The significance of symmetry and asymmetry in civil aviation extends to international regulations and agreements. By harmonizing standards and practices among different nations, it is possible to achieve symmetry in safety measures and operational procedures, thereby promoting a unified and secure global aviation system. Conversely, asymmetry in regulations, infrastructure development, or technological advancements among countries can create obstacles in establishing a cohesive and equitable international aviation framework. This article discusses the weaknesses of the existing score function in handling the MADM problem in an Interval-Valued Pythagorean Fuzzy (IVPF) environment. To tackle this issue, an enhanced score function is developed as a solution. The article proposes the IVPF Dombi hybrid arithmetic and IVPF Dombi hybrid geometric operators based on IVPF information. Furthermore, the article proves some fundamental properties of these operators. In the context of recently introduced techniques using IVPF settings, an effective method is developed for selecting the best airline. Additionally, a comparative investigation is carried out to demonstrate the legitimacy and viability of this unique strategy in comparison to earlier approaches.

Keywords: IVPF Dombi hybrid arithmetic operator; IVPF Dombi hybrid geometric operator; multi criteria decision making problems; civil aviation

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1. Introduction

In order to determine which option is best, the framework of multi-criteria decision-making considers a wide range of factors. Most individuals assume that clear numerical data represents all the information necessary to make a decision about an option based on its attributes and the relative relevance of those traits. However, in many instances in actual life when the boundaries and objectives are not clearly defined, there are additional considerations to be made. Zadeh supported the use of fuzzy sets (FSs), an extension of classical sets, for tackling this problem [1].

A fuzzy set uses a membership function to evaluate criteria. The authors in [2] introduce the theory of decision-making in a fuzzy environment. Furthermore, a theory addressing the challenges of decision-making in a “fuzzy” reality was proposed in [3,4].

Dombi [5] discussed a wide range of fuzzy operators in 1982. Decision making, engineering, IT, pattern recognition, medical diagnosis, among other fields, extensively employ and are significantly influenced by the concept of intuitionistic fuzzy sets. In 1986, Atanassov [6] introduced the concept of intuitionistic fuzzy sets, which expands the notion of fuzzy sets in novel directions. According to this theory, the sum of an element's membership and non-membership degrees cannot exceed one.

In 1986, Turksen [7] proposed the concept of interval-valued fuzzy set (IVFS) as an extension of fuzzy sets. In the real world, decision-makers often struggle to provide precise numerical estimates due to the prevalence of imprecise and partial information. To address this issue, Atanassov [8] introduced the idea of interval-valued intuitionistic fuzzy sets (IVIFS) in 1989. Instead of using a single number, IVIFS utilizes a range between 0 and 1 to indicate levels of membership or non-membership. De et al. [9] identified concentration, dilation, and normalization as the three fundamental procedures necessary for an IFS.

Since its inception, intuitionistic fuzzy set theory has garnered significant attention from the scientific community, prompting mathematicians to explore higher-order fuzzy sets as a direct consequence. In order to effectively address MCDM (Multi-Criteria Decision Making) issues, several researchers have developed a wide variety of aggregation operators and information measures based on IFSs and IVIFSs [10–15]. While the strategies mentioned above offer numerous benefits, they may not address certain real-world scenarios. In order to address this limitation, Yager proposed the notion of Pythagorean fuzzy sets (PFSs), which provide an expanded interpretation of both membership and non-membership [16]. The authors in [17] discussed the use of PFSs in decision making in 2014. More refined approaches for the IVIF settings can be found in [18–20]. The interval valued Pythagorean fuzzy set (IVPFS) was initially introduced by Zhang [21] in 2016. In addition, more recent developments on IVPFSs can be viewed in [22–33]. Peng and Yang [34] developed the fundamentals of IVPFS aggregation operators in 2016. Moreover, Dombi operators on various sets were introduced in [35–41].

Dombi operators can create a tool with a wide range of applications through the use of multi-purpose aggregation, decision-making skills, and operational features. In this case, the data are consolidated into a single value by using averaging procedures. Workers in the Dombi culture are known for their adaptability and quick decision making. Dombi operators are highly versatile in the face of varying operational conditions. The significance of these operators motivates us to solve MCDM problems within the framework of these operators. In the present study, we develop a mechanism for the selection of the best airline to travel with by means of hybrid Dombi aggregation operators based on IVPF settings.

Our method excels over other methods because it takes into account the connection between the arguments. Therefore, our strategy is more adaptable. The existing methodologies do not have the capability to dynamically alter the parameters in line with the risk aversion of the decision-makers, which makes the MCDM solution challenging to implement in a practical setting. Aviation connects continents, cultures, and economies. Teams from throughout the industry always promote aviation's benefits. It increases trade, tourism, and jobs. All relevant stakeholders must work together. To increase air travel's frequency and reach to fully reap its benefits and sustain aviation's growth. According to the inter-industry Air Transport Action Group, the global aviation industry had a USD2.7 trillion economic impact in 2014, or 3.5% of global GDP (ATAG). Air travel is one of the fastest-growing industries, doubling every 15 years. Some of the motivations for the current study are described as follows:

1. **Passion for Aviation:** A lot of people are just really interested in planes, flying, and the aviation business. They really like flying, and studying civil aviation gives them the chance to do what they love and be a part of this exciting area.
2. **Technological advancements:** The airline business is always changing and adapting to new technologies. People who study civil flight can stay on the cutting edge of

these changes and help come up with new ways to solve problems. A strong motivator can be the chance to work with cutting-edge technology and help the progress of flight.

3. Impact and contribution: Civil flight is a key part of connecting people, making trade easier, and making the economy grow. By learning about civil aviation, people can help this business run in a safe and efficient way, making it easier for people and goods to move around the world. Many people find it very important to feel like they are making a difference and adding to society.

IVPFSs are capable of handling ambiguity, uncertainty, and vagueness in a precise and effective manner. This greatly contributes to the effectiveness of flight selection approaches. Data on the different factors that go into choosing an airline is often unclear, and IVPFSs can solve this problem more correctly.

Our research is conducted with the following major goals in mind:

1. Create a revised score function that fixes the issues with the IVPF environment's current scoring functions. To produce a scoring system that is more reliable and accurate, this will require the incorporation of cutting-edge mathematical and statistical methodologies.
2. Create the underlying Dombi procedures for IVPFSs. To enable more precise analysis and outcome prediction, this will entail the creation of mathematical models that define the connections between various IVPFSs components.
3. Start looking into IVPFD aggregation operators. In order to develop more efficient aggregation techniques for IVPFSs, this will entail investigating how several IVPFD operators might be integrated.
4. Support the numerous fundamental characteristics of the newly created operators. To prove the legitimacy and efficacy of the suggested operators thorough mathematical analysis and explicit proofs will be required.
5. Outline a method for resolving Multiple Attribute Decision-Making (MADM) issues that makes use of IVPFD aggregation operators. This will entail creating a step-by-step procedure for applying the new operators to examine and assess challenging decision-making issues.
6. Apply the recently recommended method to choose the best airline. This will entail applying the suggested algorithm to actual situations.
7. Outline a comparative comparison of the suggested technique and currently used tactics to demonstrate its viability. In order to do this, real-world data sets and scenarios will be used to compare the effectiveness of the proposed algorithm to that of existing techniques.

After providing a concise overview of IVPFs at the beginning of this paper, the remaining sections are organized as follows: In Section 2, a number of fundamental definitions are reviewed for the purpose of understanding the novel nature of the work presented in this work. In Section 3, the limitations of the existing score function to resolve the MCDM problem under the IVPF environment are discussed, and an enhanced score function is formulated as a solution to this problem. In the fourth portion, certain Dombi aggregation operators based on IVPF knowledge are proposed. In Section 5, an efficient technique is designed to choose the best airline within the framework of the newly introduced strategies under IVPF settings. The validity and viability of this novel strategy in comparison to previous methods are also shown by the comparative analysis that is offered. Section 6 concludes the current study with a summary of its specific findings.

2. Preliminaries

In this section, some basic fundamental concepts related to the work presented in this article are reviewed.

Definition 1. [7] An IVFS A defined on X is given by $A = \{ \langle y, [\mu_A^L(y), \mu_A^U(y)] \rangle : y \in X \}$, where $0 \leq \mu_A^L(y) \leq \mu_A^U(y) \leq 1$.

Definition 2. [14] Let X represent the scope of discourse. Formally, a PFS A on X is defined as follows: $A = \{ \langle y, \mu_A(y), \nu_A(y) \rangle : y \in X \}$, where $\mu_A: X \rightarrow [0,1]$ and $\nu_A: X \rightarrow [0,1]$. These functions satisfy the condition $0 \leq \mu_A^2(y) + \nu_A^2(y) \leq 1$, the membership and non-membership functions, respectively, that satisfy the condition ensuring the validity of the PFS. Moreover $\pi_A(y) = \sqrt{1 - \mu_A^2(y) - \nu_A^2(y)}$ describes the hesitancy margin of the PFS A .

Definition 3. [18] An IVPFS A on X is defined as: $A = \{ \langle y, [\mu_A^L(y), \mu_A^U(y)], [\nu_A^L(y), \nu_A^U(y)] \rangle : y \in X \}$, where $[\mu_A^L(y), \mu_A^U(y)]$ and $[\nu_A^L(y), \nu_A^U(y)]$, respectively, represent the membership degree and non-membership degree of an element y that admit the conditions $1 \geq \mu_A^U(y) \geq \mu_A^L(y)$ and $0 \leq \nu_A^L(y) \leq \nu_A^U(y) \leq 1$. Moreover, $0 \leq (\mu_A^L(y))^2 + (\nu_A^L(y))^2 \leq 1$ and $0 \leq (\mu_A^U(y))^2 + (\nu_A^U(y))^2 \leq 1$. The hesitancy degree of the IVPFS A is defined as:

$$\pi_A(y) = [\pi_A^L(y), \pi_A^U(y)] = \left[\sqrt{1 - (\mu_A^L(y))^2 - (\nu_A^L(y))^2}, \sqrt{1 - (\mu_A^U(y))^2 - (\nu_A^U(y))^2} \right]$$

Definition 4. [35] Consider the three IVPF numbers $g = \langle [\mu_g^L, \mu_g^U], [\nu_g^L, \nu_g^U] \rangle$, $g_1 = \langle [\mu_{g_1}^L, \mu_{g_1}^U], [\nu_{g_1}^L, \nu_{g_1}^U] \rangle$ and $g_2 = \langle [\mu_{g_2}^L, \mu_{g_2}^U], [\nu_{g_2}^L, \nu_{g_2}^U] \rangle$. The basic operations on these numbers are defined in subsequent way;

- i. $g_1 \cup g_2 = \langle [\max\{\mu_{g_1}^L, \mu_{g_2}^L\}, \max\{\mu_{g_1}^U, \mu_{g_2}^U\}], [\min\{\nu_{g_1}^L, \nu_{g_2}^L\}, \min\{\nu_{g_1}^U, \nu_{g_2}^U\}] \rangle$
- ii. $g_1 \cap g_2 = \langle [\min\{\mu_{g_1}^L, \mu_{g_2}^L\}, \min\{\mu_{g_1}^U, \mu_{g_2}^U\}], [\max\{\nu_{g_1}^L, \nu_{g_2}^L\}, \max\{\nu_{g_1}^U, \nu_{g_2}^U\}] \rangle$
- iii. $g_1 \oplus g_2 = \left\langle \left[\frac{\sqrt{(\mu_{g_1}^L)^2 + (\mu_{g_2}^L)^2 - (\mu_{g_1}^L)^2 (\mu_{g_2}^L)^2}}{\sqrt{(\mu_{g_1}^U)^2 + (\mu_{g_2}^U)^2 - (\mu_{g_1}^U)^2 (\mu_{g_2}^U)^2}} \right], [\nu_{g_1}^L, \nu_{g_2}^L, \nu_{g_1}^U, \nu_{g_2}^U] \right\rangle$
- iv. $g_1 \otimes g_2 = \left\langle [\mu_{g_1}^L, \mu_{g_2}^L, \mu_{g_1}^U, \mu_{g_2}^U], \left[\frac{\sqrt{(\nu_{g_1}^L)^2 + (\nu_{g_2}^L)^2 - (\nu_{g_1}^L)^2 (\nu_{g_2}^L)^2}}{\sqrt{(\nu_{g_1}^U)^2 + (\nu_{g_2}^U)^2 - (\nu_{g_1}^U)^2 (\nu_{g_2}^U)^2}} \right] \right\rangle$
- v. ${}^X g = \left\langle \left[\sqrt{1 - (1 - (\mu_g^L)^2)^X}, \sqrt{1 - (1 - (\mu_g^U)^2)^X} \right], [(\nu_g^L)^X, (\nu_g^U)^X] \right\rangle, \forall X > 0$
- vi. ${}^g X = \left\langle [(\mu_g^L)^X, (\mu_g^U)^X], \left[\sqrt{1 - (1 - (\nu_g^L)^2)^X}, \sqrt{1 - (1 - (\nu_g^U)^2)^X} \right] \right\rangle, \forall X > 0$
- vii. $g^C = \langle [\nu_g^L, \nu_g^U], [\mu_g^L, \mu_g^U] \rangle$

Some specified sorts of triangle norms and conorms are covered in the upcoming definition.

Definition 5. [5] Let $\tilde{c}$ and $\tilde{d}$ be any two numbers in the real range. We characterize the Dombi t -norms and Dombi t -conorms as follows:

- i. $\text{Dom}(\tilde{c}, \tilde{d}) = \frac{1}{1 + \left\{ \left(\frac{1-\tilde{c}}{\tilde{c}} \right)^\alpha + \left(\frac{1-\tilde{d}}{\tilde{d}} \right)^\alpha \right\}^{\frac{1}{\alpha}}}$
- ii. $\text{Dom}^C(\tilde{c}, \tilde{d}) = \frac{1}{1 + \left\{ \left(\frac{\tilde{c}}{1-\tilde{c}} \right)^\alpha + \left(\frac{\tilde{d}}{1-\tilde{d}} \right)^\alpha \right\}^{\frac{1}{\alpha}}}$

where $\alpha \geq 1$ and $(\tilde{c}, \tilde{d}) \in [0, 1] \times [0, 1]$. Both (i) and (ii) above stand for the Dombi product and the Dombi sum, respectively.

Moreover, one can observe the behavior of the Domi operations at boundary points in the following way:

$$\begin{aligned}\text{Dom}(0,0) &= \infty, \text{Dom}^c(0,0) = 1, \\ \text{Dom}(0,1) &= 0, \text{Dom}^c(0,1) = 0, \\ \text{Dom}(1,0) &= 0, \text{Dom}^c(1,0) = 0, \\ \text{Dom}(1,1) &= 1, \text{Dom}^c(1,1) = \infty.\end{aligned}$$

Definition 6. [34] Any two IVPFNs $g_1 = \langle [\mu_{g_1}^L, \mu_{g_1}^U], [v_{g_1}^L, v_{g_1}^U] \rangle$ and $g_2 = \langle [\mu_{g_2}^L, \mu_{g_2}^U], [v_{g_2}^L, v_{g_2}^U] \rangle$ satisfy the following properties:

- i. $g_1 = g_2$ iff $\mu_{g_1}^L = \mu_{g_2}^L, \mu_{g_1}^U = \mu_{g_2}^U, v_{g_1}^L = v_{g_2}^L$ and $v_{g_1}^U = v_{g_2}^U$;
- ii. $g_1 < g_2$ iff $\mu_{g_1}^L < \mu_{g_2}^L, \mu_{g_1}^U < \mu_{g_2}^U, v_{g_1}^L > v_{g_2}^L$ and $v_{g_1}^U > v_{g_2}^U$.

Definition 7. [34] A PFD hybrid weighted average operator of dimension q is characterized by a function PFDHA: $PFE^q \rightarrow PFE$, with corresponding weight vector $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_q)^T$ such that

$$\begin{aligned}PFDHA(d_1, d_2, \dots, d_q)_{\hat{\omega}, \hat{\omega}} &= \bigoplus_{k=1}^q (\hat{\omega}_k \hat{d}_{n(k)}) \\ &= \left(\sqrt[q]{\frac{1}{1 + \left\{ \sum_{k=1}^q \hat{\omega}_k \left(\frac{\tilde{\mu}_k^2}{1 - \tilde{\mu}_k^2} \right)^{\hat{\omega}_k} \right\}^{\frac{1}{\hat{\omega}_k}}}}, \sqrt[q]{\frac{1}{1 + \left\{ \sum_{k=1}^q \hat{\omega}_k \left(\frac{1 - \tilde{v}_k^2}{\tilde{v}_k^2} \right)^{\hat{\omega}_k} \right\}^{\frac{1}{\hat{\omega}_k}}}} \right),\end{aligned}$$

where $\hat{\omega}_k > 0$ and $\sum_{k=1}^q \hat{\omega}_k = 1$. Note that, $\hat{d}_{n(k)}$ is the k th largest value of the weighted PFNs $\hat{d}_k (\hat{d}_k = q \tilde{Y}_k \hat{d}_k, k = 1, 2, \dots, q)$, q is the number of PFNs and $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_q)^T$ is the standard weight vector of $\hat{d}_k (k = 1, 2, \dots, q)$.

Definition 8. [34] A PFD hybrid weighted average operator of dimension q is characterized by a function PFDHA: $PFE^q \rightarrow PFE$, with corresponding weight vector $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_q)^T$ such that

$$\begin{aligned}PFDHG_{\tilde{Y}, \hat{\omega}}(d_1, d_2, \dots, d_q) &= \bigotimes_{k=1}^q (\hat{d}_{n(k)})^{\hat{\omega}_k} \\ &= \left(\sqrt[q]{\frac{1}{1 + \left\{ \sum_{k=1}^q \hat{\omega}_k \left(\frac{1 - \tilde{\mu}_k^2}{\tilde{v}_k^2} \right)^{\hat{\omega}_k} \right\}^{\frac{1}{\hat{\omega}_k}}}}, \sqrt[q]{1 - \frac{1}{1 + \left\{ \sum_{k=1}^q \hat{\omega}_k \left(\frac{\tilde{v}_k^2}{1 - \tilde{v}_k^2} \right)^{\hat{\omega}_k} \right\}^{\frac{1}{\hat{\omega}_k}}}} \right)\end{aligned}$$

where $\hat{\omega}_k > 0$ and $\sum_{k=1}^q \hat{\omega}_k = 1$. Note that $\hat{d}_{n(k)}$ is the k th largest value of the weighted PFE $\hat{d}_k (\hat{d}_k = (\hat{d}_k)^{q \tilde{Y}_k}, k = 1, 2, \dots, q)$, q is the number of PFEs and $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_q)^T$ is the standard weight vector of $\hat{d}_k (k = 1, 2, \dots, q)$.

Definition 9. [27] The score function for IVPF number $\tilde{p} = ([a, b], [c, d])$ is defined as:

$$S(\tilde{p}) = \frac{a^2 + b^2 - c^2 - d^2}{2} + \frac{a^2 - a^2 d^2 - a^4}{2(c^2 - b^2 c^2 - c^4 + 1)}$$

where $S(\tilde{p}) \in [-1, 1]$. In particular, if $S(\tilde{p}) = 1$, then α is the largest IVPFN, i.e., $\langle [1, 1], [0, 0] \rangle$, if $S(\tilde{p}) = -1$, then α is the smallest IVPFN, i.e., $\langle [0, 0], [1, 1] \rangle$. Moreover, the score function satisfies the following properties for any two IVPFNs $\tilde{p}$ and $\tilde{q}$

- i. $S(\tilde{p}) < S(\tilde{q})$ implies $\tilde{p} < \tilde{q}$,
- ii. $S(\tilde{p}) > S(\tilde{q})$ implies $\tilde{p} > \tilde{q}$,
- iii. $S(\tilde{p}) = S(\tilde{q})$ implies $\tilde{p} \sim \tilde{q}$.

Definition 10. [39] Let $k_1 = \langle [\mu_{k_1}^L, \mu_{k_1}^U], [\nu_{k_1}^L, \nu_{k_1}^U] \rangle$ and $k_2 = \langle [\mu_{k_2}^L, \mu_{k_2}^U], [\nu_{k_2}^L, \nu_{k_2}^U] \rangle$ be any two IVPFNs, $1 \leq \vartheta$ and $0 \leq \chi$. The Dombi operations of t -norms and t -conorms of IVPFNs are discussed in the following way:

$$\begin{aligned}
 \text{i. } k_1 \oplus k_2 &= \left(\left[\sqrt[1 + \left\{ \left(\frac{(\mu_{k_1}^L)^2}{1 - (\mu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(\mu_{k_2}^L)^2}{1 - (\mu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\mu_{k_1}^L)^2}{1 - (\mu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(\mu_{k_2}^L)^2}{1 - (\mu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \left[\sqrt[1 + \left\{ \left(\frac{(\mu_{k_1}^U)^2}{1 - (\mu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(\mu_{k_2}^U)^2}{1 - (\mu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\mu_{k_1}^U)^2}{1 - (\mu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(\mu_{k_2}^U)^2}{1 - (\mu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \right. \\
 &\quad \left. \left[\sqrt[1 + \left\{ \left(\frac{(1 - (\nu_{k_1}^L)^2)}{(\nu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(1 - (\nu_{k_2}^L)^2)}{(\nu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 + \frac{1}{\left(1 + \left\{ \left(\frac{(1 - (\nu_{k_1}^L)^2)}{(\nu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(1 - (\nu_{k_2}^L)^2)}{(\nu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \left[\sqrt[1 + \left\{ \left(\frac{(1 - (\nu_{k_1}^U)^2)}{(\nu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(1 - (\nu_{k_2}^U)^2)}{(\nu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 + \frac{1}{\left(1 + \left\{ \left(\frac{(1 - (\nu_{k_1}^U)^2)}{(\nu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(1 - (\nu_{k_2}^U)^2)}{(\nu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}} \right]; \\
 \text{ii. } k_1 \otimes k_2 &= \left(\left[\sqrt[1 + \left\{ \left(\frac{(1 - (\mu_{k_1}^L)^2)}{(\mu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(1 - (\mu_{k_2}^L)^2)}{(\mu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 + \frac{1}{\left(1 + \left\{ \left(\frac{(1 - (\mu_{k_1}^L)^2)}{(\mu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(1 - (\mu_{k_2}^L)^2)}{(\mu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \left[\sqrt[1 + \left\{ \left(\frac{(1 - (\mu_{k_1}^U)^2)}{(\mu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(1 - (\mu_{k_2}^U)^2)}{(\mu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}}}{1 + \frac{1}{\left(1 + \left\{ \left(\frac{(1 - (\mu_{k_1}^U)^2)}{(\mu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(1 - (\mu_{k_2}^U)^2)}{(\mu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \right. \\
 &\quad \left. \left[\sqrt[1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\nu_{k_1}^L)^2}{1 - (\nu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(\nu_{k_2}^L)^2}{1 - (\nu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}}{1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\nu_{k_1}^L)^2}{1 - (\nu_{k_1}^L)^2} \right)^\vartheta + \left(\frac{(\nu_{k_2}^L)^2}{1 - (\nu_{k_2}^L)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}}, \left[\sqrt[1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\nu_{k_1}^U)^2}{1 - (\nu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(\nu_{k_2}^U)^2}{1 - (\nu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}}{1 - \frac{1}{\left(1 + \left\{ \left(\frac{(\nu_{k_1}^U)^2}{1 - (\nu_{k_1}^U)^2} \right)^\vartheta + \left(\frac{(\nu_{k_2}^U)^2}{1 - (\nu_{k_2}^U)^2} \right)^\vartheta \right\}^{\frac{1}{\vartheta}}} \right)^{\frac{1}{\vartheta}}}} \right]^{\frac{1}{\vartheta}} \right];
 \end{aligned}$$

$$\begin{aligned}
 \text{iii. } \mathcal{X}.k_1 &= \left(\left[\sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(\mu_{k_1}^L)^2}{1 - (\mu_{k_1}^L)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}}, \sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(\mu_{k_1}^U)^2}{1 - (\mu_{k_1}^U)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}} \right], \right. \\
 &\quad \left. \left[\sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(1 - (\nu_{k_1}^L)^2)}{(\nu_{k_1}^L)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}}, \sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(1 - (\nu_{k_1}^U)^2)}{(\nu_{k_1}^U)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}} \right] \right); \\
 \text{iv. } (k_1)^{\mathcal{X}} &= \left(\left[\sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(1 - (\mu_{k_1}^L)^2)}{(\mu_{k_1}^L)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}}, \sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(1 - (\mu_{k_1}^U)^2)}{(\mu_{k_1}^U)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}} \right], \right. \\
 &\quad \left. \left[\sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(\nu_{k_1}^L)^2}{1 - (\nu_{k_1}^L)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}}, \sqrt{\frac{1}{1 + \left\{ \mathcal{X} \left(\frac{(\nu_{k_1}^U)^2}{1 - (\nu_{k_1}^U)^2} \right)^\vartheta} \right\}^\frac{1}{\vartheta}}} \right] \right).
 \end{aligned}$$

3. Shortcomings of the Existing Score Function of IVPFS and Its Improvement

In the following section, we present a modification of the existing score function of IVPFS and a supportive illustration of this phenomenon for a better understanding.

Example 1. Let $\mathfrak{n}_1 = \langle [0, \sqrt{0.25}], [0, 0.5] \rangle$ and $\mathfrak{n}_2 = \langle [0, \sqrt{0.3}], [0, \sqrt{0.3}] \rangle$ be any two IVPFNs. The application of Definition 9 on $\mathfrak{n}_1$ and $\mathfrak{n}_2$ yields that

$$S(\mathfrak{n}_1) = S(\mathfrak{n}_2) = 0$$

The above fact leads us to note that $\mathfrak{n}_1 \sim \mathfrak{n}_2$ but $\mathfrak{n}_1 \neq \mathfrak{n}_2$.

This demonstrates the shortcoming of the investigated score function. The preceding discussion has prompted us to enhance this score function with the subsequent definition.

Definition 11. Consider the IVPFN $\tilde{p} = \langle [a, b], [c, d] \rangle$. The enhanced score function $S(\tilde{p})$ is defined as follows:

$$S(\tilde{p}) = \frac{a^2 + b^2 - c^2 - d^2}{2} + \frac{b^2 d^2}{3} \quad (1)$$

Here, $h(\tilde{p})$ represents the range of the score function such that $-1 \leq h(\tilde{p}) \leq 1$.

In addition, the previously described score function fulfills the subsequent comparison law for all pairs of IVPFNs $\tilde{p}, \tilde{q}$.

- i. $S(\tilde{p}) < S(\tilde{q})$ implies $\tilde{p} < \tilde{q}$;
- ii. $S(\tilde{p}) > S(\tilde{q})$ implies $\tilde{p} > \tilde{q}$;
- iii. $S(\tilde{p}) = S(\tilde{q})$ implies $\tilde{p} \sim \tilde{q}$.

To demonstrate the efficacy of the proposed score function for IVPFNs, we present the subsequent illustrative example.

Example 2. By employing the enhanced score function $S(\tilde{p})$ as described in Equation (1) of example 1, we determine that $S(n_1) = 0.02$ and $S(n_2) = 0.03$. Consequently, considering property (i) outlined in Definition 11, it follows that $n_1 < n_2$. This observation indicates that the alternative n_2 is superior to the alternative n_1 .

Based on the preceding discussion, we can draw the conclusion that the proposed score function offers greater suitability and serves as an effective algorithm for the decision analysis process. This is in contrast to the existing score function, as discussed in definition 9 which exhibits limitations and fails to provide satisfactory results.

4. Fundamental Characteristic of IVPF Dombi Hybrid Operator

In this section, we introduce the concepts of IVPF Dombi aggregation operators and establish their various fundamental properties.

While the IVPFDA operator assigns weights solely based on the PF values, the Pythagorean fuzzy Ordered Weighted Arithmetic (IVPFOWA) operator assigns weights based solely on the ordered positions of the IVPF values, rather than the weights of the IVPF values themselves. Consequently, the weights in both operators, IVPFDA and IVPFOWA, exhibit different behaviors. To overcome this drawback, we propose the IVPF Dombi hybrid weighted arithmetic (IVPFDHA) operator.

Definition 12. Let $\alpha_\xi (\xi = 1, 2, \dots, s)$ be an accumulation of IVPFEs. An IVPFDHA operator of dimension s is characterized by a function $IVPFDHA: IVPFE^s \rightarrow IVPFE$, such that

$$IVPFDHA_{\delta, \hat{\omega}}(\alpha_1, \alpha_2, \dots, \alpha_s) = \bigoplus_{\xi=1}^s (\hat{\omega}_\xi \alpha_{n(\xi)}) = \hat{\omega}_1 \alpha_{n(1)} \oplus \hat{\omega}_2 \alpha_{n(2)} \oplus \dots \oplus \hat{\omega}_s \alpha_{n(s)}$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_s)^T$ is the associated weighted vector of IVPFDHA operator satisfying $\hat{\omega}_s > 0$ and $\sum_{\xi=1}^s \hat{\omega}_\xi = 1$. Note that $\alpha_{n(\xi)}$ is the ξ th largest value of the weighted IVPFE $\alpha_\xi (\alpha_\xi = s \tilde{Y}_\xi \alpha_\xi, \xi = 1, 2, \dots, s)$, s is the number of IVPFEs and $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_s)^T$ is the standard weight vector of $\alpha_\xi (\xi = 1, 2, \dots, s)$.

From the above discussion, we observe that:

- Initially, it weights the IVPFEs α_ξ by the associated weights $\tilde{Y}_\xi$ and hence obtains the weighted IVPFEs $\alpha_\xi = s \tilde{Y}_\xi \alpha_\xi (\xi = 1, 2, \dots, s)$;
- Secondly, it reorders the weighted arguments in descending order $(\alpha_{n(1)}, \alpha_{n(2)}, \dots, \alpha_{n(s)})$, where $\alpha_{n(\xi)}$ is the ξ th largest of $\alpha_\xi (\xi = 1, 2, \dots, s)$;
- It weights these ordered weighted IVPFEs $\alpha_{n(\xi)}$ by the IVPFDWA weights $\hat{\omega}_\xi (\xi = 1, 2, \dots, s)$ and then aggregates all these values into a single valued quantity.

Based on the proposed operational rules of the IVPFEs, we can derive the following result.

Theorem 1. For any collection of IVPFEs $\alpha_\xi = ([\mu_\xi^l, \mu_\xi^u], [v_\xi^l, v_\xi^u]) (\xi = 1, 2, \dots, s)$. The structure of the IVPFDHA operator is defined using the Dombi operations with $\delta > 0$.

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\omega}}(\alpha_1, \alpha_2, \dots, \alpha_s) &= \bigoplus_{\xi=1}^s (\hat{\omega}_{\xi} \alpha_{n(\xi)}) \\
 &= \left(\sqrt[1]{\sqrt[1]{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^l)^2}{(1 - (\tilde{\mu}_{n(\xi)}^l)^2)} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}}, \sqrt[1]{\sqrt[1]{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^u)^2}{(1 - (\tilde{\mu}_{n(\xi)}^u)^2)} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}}}, \right. \\
 &\quad \left. \sqrt[1]{\sqrt[1]{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(1 - (\tilde{\nu}_{n(\xi)}^l)^2)}{(\tilde{\nu}_{n(\xi)}^l)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}, \sqrt[1]{\sqrt[1]{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(1 - (\tilde{\nu}_{n(\xi)}^u)^2)}{(\tilde{\nu}_{n(\xi)}^u)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}} \right) \quad (2)
 \end{aligned}$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_s)^T$ is the associated weighted vector of IVPFDHA operator satisfying the $\hat{\omega}_s > 0$ and $\sum_{\xi=1}^s \hat{\omega}_{\xi} = 1$, $\alpha_{n(\xi)}$ is the ξ th largest value of the weighted IVPFE $\alpha_{\xi}(\alpha_{\xi} = s \check{Y}_{\xi} \check{\alpha}_{\xi}, \xi = 1, 2, \dots, s)$, s is the number of IVPFEs and $\check{Y} = (\check{Y}_1, \check{Y}_2, \dots, \check{Y}_s)^T$ is the standard weight vector of $\check{\alpha}_{\xi}(\xi = 1, 2, \dots, s)$.

Proof. We prove the statement by applying induction on s . Therefore, for $s = 2$ we have

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\omega}}(\alpha_1, \alpha_2) &= \bigoplus_{\xi=1}^2 (\hat{\omega}_{\xi} \alpha_{n(\xi)}) = \hat{\omega}_1 \alpha_{n(1)} \oplus \hat{\omega}_2 \alpha_{n(2)} \\
 &= \left(\sqrt[1]{\sqrt[1]{1 - \frac{1}{1 + \left\{ \hat{\omega}_1 \left(\frac{(\tilde{\mu}_{n(1)}^l)^2}{(1 - (\tilde{\mu}_{n(1)}^l)^2)} \right)^{\rho}} + \left\{ \hat{\omega}_2 \left(\frac{(\tilde{\mu}_{n(2)}^l)^2}{(1 - (\tilde{\mu}_{n(2)}^l)^2)} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}}}, \sqrt[1]{\sqrt[1]{1 - \frac{1}{1 + \left\{ \hat{\omega}_1 \left(\frac{(\tilde{\mu}_{n(1)}^u)^2}{(1 - (\tilde{\mu}_{n(1)}^u)^2)} \right)^{\rho}} + \left\{ \hat{\omega}_2 \left(\frac{(\tilde{\mu}_{n(2)}^u)^2}{(1 - (\tilde{\mu}_{n(2)}^u)^2)} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}}}, \right. \\
 &\quad \left. \sqrt[1]{\sqrt[1]{1 + \left\{ \hat{\omega}_1 \left(\frac{(1 - (\tilde{\nu}_{n(1)}^l)^2)}{(\tilde{\nu}_{n(1)}^l)^2} \right)^{\rho}} + \left\{ \hat{\omega}_2 \left(\frac{(1 - (\tilde{\nu}_{n(2)}^l)^2)}{(\tilde{\nu}_{n(2)}^l)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}}}, \sqrt[1]{\sqrt[1]{1 + \left\{ \hat{\omega}_1 \left(\frac{(1 - (\tilde{\nu}_{n(1)}^u)^2)}{(\tilde{\nu}_{n(1)}^u)^2} \right)^{\rho}} + \left\{ \hat{\omega}_2 \left(\frac{(1 - (\tilde{\nu}_{n(2)}^u)^2)}{(\tilde{\nu}_{n(2)}^u)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}} \right)
 \end{aligned}$$

This means that:

$$IVPFDHA_{\delta, \hat{\omega}}(\alpha_1, \alpha_2) = \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^2 \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^l)^2}{1 - (\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^2 \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^u)^2}{1 - (\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\ \left. \left. \left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^2 \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\nu}_{n(\xi)}^l)^2}{(\tilde{\nu}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^2 \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\nu}_{n(\xi)}^u)^2}{(\tilde{\nu}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right) \right)$$

Thus, Equation (1) satisfies for $s = 2$.

Suppose that the Equation (2) is valid for $s = q$. So

$$IVPFDHA_{\delta, \hat{\omega}}(\alpha_1, \alpha_2, \dots, \alpha_q) = \oplus_{\xi=1}^q (\hat{\omega}_{\xi} \alpha_{n(\xi)}) \\ = \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^l)^2}{1 - (\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^u)^2}{1 - (\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\ \left. \left. \left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\nu}_{n(\xi)}^l)^2}{(\tilde{\nu}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\nu}_{n(\xi)}^u)^2}{(\tilde{\nu}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right) \right)$$

Moreover, for $s = q + 1$, we have:

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\omega}}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_q, \hat{\alpha}_{q+1}) &= \bigoplus_{\hat{\omega}}^{q+1} (\hat{\omega}_{\hat{\omega}} \alpha_{n(\hat{\omega})}) \\
 &= \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^q \hat{\omega}_{\hat{\omega}} \left(\frac{(\tilde{\mu}_{n(\hat{\omega})}^l)^2}{1 - (\tilde{\mu}_{n(\hat{\omega})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^q \hat{\omega}_{\hat{\omega}} \left(\frac{(\tilde{\mu}_{n(\hat{\omega})}^u)^2}{1 - (\tilde{\mu}_{n(\hat{\omega})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\
 &\quad \left. \left[\sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^q \hat{\omega}_{\hat{\omega}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\omega})}^l)^2}{(\tilde{\nu}_{n(\hat{\omega})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^q \hat{\omega}_{\hat{\omega}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\omega})}^u)^2}{(\tilde{\nu}_{n(\hat{\omega})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right), \\
 &\oplus \left(\left[\sqrt{\frac{1}{1 + \left\{ \hat{\omega}_{q+1} \left(\frac{(\tilde{\mu}_{n(q+1)}^l)^2}{1 - (\tilde{\mu}_{n(q+1)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \hat{\omega}_{q+1} \left(\frac{(\tilde{\mu}_{n(q+1)}^u)^2}{1 - (\tilde{\mu}_{n(q+1)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\
 &\quad \left. \left[\sqrt{\frac{1}{1 + \left\{ \hat{\omega}_{q+1} \left(\frac{1 - (\tilde{\nu}_{n(q+1)}^l)^2}{(\tilde{\nu}_{n(q+1)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \hat{\omega}_{q+1} \left(\frac{1 - (\tilde{\nu}_{n(q+1)}^u)^2}{(\tilde{\nu}_{n(q+1)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right),
 \end{aligned}$$

This means that:

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\omega}}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_q, \hat{\alpha}_{q+1}) &= \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^{q+1} \hat{\omega}_{\hat{\omega}} \left(\frac{(\tilde{\mu}_{n(\hat{\omega})}^l)^2}{1 - (\tilde{\mu}_{n(\hat{\omega})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^{q+1} \hat{\omega}_{\hat{\omega}} \left(\frac{(\tilde{\mu}_{n(\hat{\omega})}^u)^2}{1 - (\tilde{\mu}_{n(\hat{\omega})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\
 &\quad \left. \left[\sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^{q+1} \hat{\omega}_{\hat{\omega}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\omega})}^l)^2}{(\tilde{\nu}_{n(\hat{\omega})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\omega}=1}^{q+1} \hat{\omega}_{\hat{\omega}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\omega})}^u)^2}{(\tilde{\nu}_{n(\hat{\omega})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right)
 \end{aligned}$$

Thus, Equation (2) is true for $s = q + 1$. Hence, we conclude that Equation (2) is true for all $s \in N$. $\square$

Theorem 2. The IVPFD weighted arithmetic and IVPFDO weighted arithmetic operators can be regarded as special cases of the IVPFD hybrid operator.

Proof. Let $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_s) = \left(\frac{1}{s}, \frac{1}{s}, \dots, \frac{1}{s}\right)^T$. Then:

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) &= \hat{\phi}_1 \alpha_{n(1)} \oplus \hat{\phi}_2 \alpha_{n(2)} \oplus \dots \oplus \hat{\phi}_s \alpha_{n(s)} \\
 &= \frac{1}{s} (\alpha_{n(1)} \oplus \alpha_{n(2)} \oplus \dots \oplus \alpha_{n(s)}) \\
 &= \delta_1 \alpha_1 \oplus \delta_2 \alpha_2 \oplus \dots \oplus \delta_s \alpha_s \\
 &= IVPFDWA_{\delta}(\alpha_1, \alpha_2, \dots, \alpha_s)
 \end{aligned}$$

Assume that $\delta = \left(\frac{1}{s}, \frac{1}{s}, \dots, \frac{1}{s}\right)^T$, then $\alpha_s = \alpha_s$ and

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) &= \hat{\phi}_1 \alpha_{n(1)} \oplus \hat{\phi}_2 \alpha_{n(2)} \oplus \dots \oplus \hat{\phi}_s \alpha_{n(s)} \\
 &= \hat{\phi}_1 \alpha_{n(1)} \oplus \hat{\phi}_2 \alpha_{n(2)} \oplus \dots \oplus \hat{\phi}_s \alpha_{n(s)} \\
 &= IVPFDWA_{\delta}(\alpha_1, \alpha_2, \dots, \alpha_s). \square
 \end{aligned}$$

Theorem 3. (Idempotency property) Let $\alpha_{\xi} = ([\mu_{\xi}^l, \mu_{\xi}^u], [v_{\xi}^l, v_{\xi}^u]) (\xi = 1, 2, \dots, s)$ be a collection of equal IVPFNs, i.e., $\alpha_{\xi} = \alpha$ for all t , then

$$IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) = \alpha \quad (3)$$

Proof. Since, $\alpha_{\xi} = ([\mu_{\xi}^l, \mu_{\xi}^u], [v_{\xi}^l, v_{\xi}^u]) = \alpha, \forall \xi = 1, 2, \dots, s$. Then, we have by Equation (2)

$$\begin{aligned}
 IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) &= \bigoplus_{\xi=1}^s (\hat{\phi}_{\xi} \alpha_{n(\xi)}) \\
 &= \left(\left[\sqrt{\frac{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\phi}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^l)^2}{1 - (\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\phi}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^u)^2}{1 - (\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}}} \right] \right) \\
 &= \left(\left[\sqrt{\frac{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\phi}_{\xi} \left(\frac{(1 - (\tilde{v}_{n(\xi)}^l)^2)}{(\tilde{v}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\phi}_{\xi} \left(\frac{(1 - (\tilde{v}_{n(\xi)}^u)^2)}{(\tilde{v}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}} \right] \right) \\
 &= \left(\left[\sqrt{\frac{1 - \frac{1}{1 + \left(\frac{(\tilde{\mu}^l)^2}{1 - (\tilde{\mu}^l)^2} \right) (\sum_{\xi=1}^s \hat{\phi}_{\xi})^{\frac{1}{\rho}}}}}{1 - \frac{1}{1 + \left(\frac{(\tilde{\mu}^u)^2}{1 - (\tilde{\mu}^u)^2} \right) (\sum_{\xi=1}^s \hat{\phi}_{\xi})^{\frac{1}{\rho}}}}} \right] \right) \\
 &= \left(\left[\sqrt{\frac{1 - \frac{1}{1 + \left(\frac{(1 - (\tilde{v}^l)^2)}{(\tilde{v}^l)^2} \right) (\sum_{\xi=1}^s \hat{\phi}_{\xi})^{\frac{1}{\rho}}}}}{1 - \frac{1}{1 + \left(\frac{(1 - (\tilde{v}^u)^2)}{(\tilde{v}^u)^2} \right) (\sum_{\xi=1}^s \hat{\phi}_{\xi})^{\frac{1}{\rho}}}}} \right] \right) \\
 &= \left(\left[\sqrt{\frac{1 - \frac{1}{1 + \left(\frac{(\mu^l)^2}{1 - (\mu^l)^2} \right)}}}{1 - \frac{1}{1 + \left(\frac{(\mu^u)^2}{1 - (\mu^u)^2} \right)}}}, \sqrt{\frac{1 - \frac{1}{1 + \left(\frac{(1 - (v^l)^2)}{(\tilde{v}^l)^2} \right)}}}{1 - \frac{1}{1 + \left(\frac{(1 - (v^u)^2)}{(\tilde{v}^u)^2} \right)}}} \right] \right)
 \end{aligned}$$

This means that

$$IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) = ([\mu^l, \mu^u], [v^l, v^u]) = \alpha. \square$$

Theorem 4. (Boundedness property) Let $\alpha_{\xi} = ([\mu_{\xi}^l, \mu_{\xi}^u], [v_{\xi}^l, v_{\xi}^u]) (\xi = 1, 2, \dots, s)$ be a collection of IVPFNs. If $\alpha^* = \min_{\xi} \{\alpha_{\xi}\} = ([(\mu^l)^*, (\mu^u)^*], [(v^l)^*, (v^u)^*])$ and $\alpha^{\bullet} = \max_{\xi} \{\alpha_{\xi}\} = ([(\mu^l)^{\bullet}, (\mu^u)^{\bullet}], [(v^l)^{\bullet}, (v^u)^{\bullet}])$, where $(\mu^l)^* = \min_{\xi} \{\mu_{\xi}^l\}, (\mu^u)^* = \min_{\xi} \{\mu_{\xi}^u\}, (v^l)^* = \max_{\xi} \{v_{\xi}^l\},$

$(v^u)^* = \max_{\xi} \{v_{\xi}^u\}, (\mu^l)^* = \max_{\xi} \{\mu_{\xi}^l\}, (\mu^u)^* = \max_{\xi} \{\mu_{\xi}^u\}, (v^l)^* = \min_{\xi} \{v_{\xi}^l\}$ and $(v^u)^* = \min_{\xi} \{v_{\xi}^u\}$. Then, $\hat{\alpha}^* \leq IVPFDHA_{\delta, \hat{\phi}}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_s) \leq \hat{\alpha}^*$.

Proof. In view of the given conditions, we have

$$\begin{aligned}
 & \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\mu_{n(\xi)}^{*l})^2}{(1 - (\mu_{n(\xi)}^{*l})^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \leq \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^l)^2}{(1 - (\tilde{\mu}_{n(\xi)}^l)^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \\
 & \leq \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\mu_{n(\xi)}^{*l})^2}{(1 - (\mu_{n(\xi)}^{*l})^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \\
 & \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\mu_{n(\xi)}^{*u})^2}{(1 - (\mu_{n(\xi)}^{*u})^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \leq \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{\mu}_{n(\xi)}^u)^2}{(1 - (\tilde{\mu}_{n(\xi)}^u)^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \\
 & \leq \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\mu_{n(\xi)}^{*u})^2}{(1 - (\mu_{n(\xi)}^{*u})^2)} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \\
 & \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (v_{n(\xi)}^{*l})^2}{(v_{n(\xi)}^{*l})^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \geq \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{v}_{n(\xi)}^l)^2}{(\tilde{v}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \\
 & \geq \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (v_{n(\xi)}^{*l})^2}{(v_{n(\xi)}^{*l})^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \\
 & \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (v_{n(\xi)}^{*u})^2}{(v_{n(\xi)}^{*u})^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \geq \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{v}_{n(\xi)}^u)^2}{(\tilde{v}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \\
 & \geq \sqrt{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (v_{n(\xi)}^{*u})^2}{(v_{n(\xi)}^{*u})^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}
 \end{aligned}$$

The application of Definition 4 in the above relationship yields that

$$\hat{\alpha}^* \leq IVPFDHA_{\delta, \hat{\phi}}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_s) \leq \hat{\alpha}^*. \quad \square$$

Theorem 5. (Monotonicity property) Let $\epsilon_r = ([\mu_r^l, \mu_r^u], [\nu_r^l, \nu_r^u])$ and $\alpha_r = ([\tilde{\mu}_r^l, \tilde{\mu}_r^u], [\tilde{\nu}_r^l, \tilde{\nu}_r^u])$, $r = 1, 2, \dots, s$ be any two collections of IVPFES. If $\mu_r^l \leq \tilde{\mu}_r^l, \mu_r^u \leq \tilde{\mu}_r^u, \nu_r^l \geq \tilde{\nu}_r^l$ and $\nu_r^u \geq \tilde{\nu}_r^u \forall r$, then.

$$IVPFDHA_{\delta, \hat{\phi}}(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \leq IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_1, \dots, \alpha_s)$$

Proof. In view of Definition 12, we have

$$IVPFDHA_{\delta, \hat{\phi}}(\epsilon_1, \epsilon_2, \dots, \epsilon_s) = \hat{\phi}_1 k_{n(1)} \oplus \hat{\phi}_2 k_{n(2)} \oplus \dots \oplus \hat{\phi}_s k_{n(s)}$$

and

$$IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_1, \dots, \alpha_s) = \hat{\phi}_1 \alpha_{n(1)} \oplus \hat{\phi}_2 \alpha_{n(2)} \oplus \dots \oplus \hat{\phi}_s \alpha_{n(s)}$$

Since $\epsilon_r \leq \alpha_r$ for all r , therefore

$$IVPFDHA_{\delta, \hat{\phi}}(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \leq IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_1, \dots, \alpha_s). \square$$

In the following example, we present the formal implementation of Definition 12.

Example 3. Let $\alpha_1 = ([0.2, 0.3], [0.5, 0.6])$, $\alpha_2 = ([0.4, 0.5], [0.4, 0.6])$ and $\alpha_3 = ([0.3, 0.4], [0.6, 0.7])$ be three IVPFES, whose weight vector is $\delta = (0.15, 0.35, 0.5)^T$ and the aggregation associated vector is $\hat{\phi} = (0.3, 0.2, 0.5)^T$. For $\rho = 1$, we compute the values of $\alpha_{\hat{\epsilon}}$ as follows;

$$\alpha_{\hat{\epsilon}} = 3\delta_{\hat{\epsilon}}\alpha_{\hat{\epsilon}}, \hat{\epsilon} = 1, 2, 3.$$

$$\alpha_1 = ([0.136, 0.206], [0.625, 0.745]), \alpha_2 = ([0.408, 0.509], [0.392, 0.591]) \text{ and}$$

$$\alpha_3 = ([0.359, 0.471], [0.522, 0.625]).$$

Moreover, the permuted values of IVPFES in the frame of Definition 11 are determined in the following way:

$$S(\alpha_1) = -0.435, S(\alpha_2) = -0.009, S(\alpha_3) = -0.127$$

Thus, based on the score values of $\tilde{\alpha}_{\hat{\epsilon}}$, we have $\alpha_{n(1)} = ([0.408, 0.509], [0.392, 0.591])$, $\alpha_{n(2)} = ([0.359, 0.471], [0.522, 0.625])$ and $\alpha_{n(3)} = ([0.136, 0.206], [0.625, 0.745])$

Furthermore, by substituting the above computed values in Equation (2) of Theorem 1, we obtain

$$IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \tilde{\alpha}_3) = \bigoplus_{\hat{\epsilon}=1}^3 (\hat{\phi}_{\hat{\epsilon}} \alpha_{n(\hat{\epsilon})})$$

$$\left(\left[\sqrt[1]{1 - \frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\phi}_{\hat{\epsilon}} \left(\frac{(\tilde{\mu}_{n(\hat{\epsilon})}^l)^2}{1 - (\tilde{\mu}_{n(\hat{\epsilon})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}}, \sqrt[1]{1 - \frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\phi}_{\hat{\epsilon}} \left(\frac{(\tilde{\mu}_{n(\hat{\epsilon})}^u)^2}{1 - (\tilde{\mu}_{n(\hat{\epsilon})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}}, \right. \right. \\ \left. \left[\sqrt[1]{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\phi}_{\hat{\epsilon}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\epsilon})}^l)^2}{(\tilde{\nu}_{n(\hat{\epsilon})}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt[1]{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\phi}_{\hat{\epsilon}} \left(\frac{1 - (\tilde{\nu}_{n(\hat{\epsilon})}^u)^2}{(\tilde{\nu}_{n(\hat{\epsilon})}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}} \right] \right)$$

Consequently,

$$IVPFDHA_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \tilde{\alpha}_3) = ([0.3, 0.394], [0.502, 0.663])$$

In the following definition, we commence the study of the IVPF Dombi hybrid geometric (IVPFDHG) operator.

Definition 13. Let α_{ξ} ($\xi = 1, 2, \dots, s$) be an accumulation of IVPFNs. An IVPFDHG operator of dimension s is characterized by a function $IVPFDHG: IVPFE^s \rightarrow IVPFE$, such that

$$IVPFDHG_{\delta, \hat{\omega}}(\alpha_1, \alpha_2, \dots, \alpha_s) = \otimes_{\xi=1}^s (\alpha_{n(\xi)})^{\hat{\omega}_{\xi}} = (\alpha_{n(1)})^{\hat{\omega}_1} \otimes (\alpha_{n(2)})^{\hat{\omega}_2} \otimes \dots \otimes (\alpha_{n(s)})^{\hat{\omega}_s}$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_s)^T$ is the associated weighted vector of IVPFDHG operator satisfying the $\hat{\omega}_s > 0$ and $\sum_{\xi=1}^s \hat{\omega}_{\xi} = 1$. Note that, $\tilde{\alpha}_{n(\xi)}$ is the ξ th largest value of the weighted IVPFE $\alpha_{\xi}(\alpha_{\xi} = (\alpha_{\xi})^{s\tilde{Y}_{\xi}}, \xi = 1, 2, \dots, s)$, s is the number of IVPFEs and $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_s)^T$ is the standard weight vector of $\alpha_{\xi}(\xi = 1, 2, \dots, s)$.

Baes on the proposed operational rules of the IVPFEs, we can derive the following result.

Theorem 6. For any collection of IVPFEs $\alpha_{\xi} = ([\mu_{\xi}^L, \mu_{\xi}^U], [v_{\xi}^L, v_{\xi}^U])(\xi = 1, 2, \dots, s)$. The structure of IVPFDHG operator is defined using the Dombi operations with $\delta > 0$.

$$IVPFDHG_{\delta, \hat{\omega}}(\alpha_1, \alpha_2, \dots, \alpha_s) = \otimes_{\xi=1}^s (\alpha_{n(\xi)})^{\hat{\omega}_{\xi}} = \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^L)^2}{(\tilde{\mu}_{n(\xi)}^L)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^U)^2}{(\tilde{\mu}_{n(\xi)}^U)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}, \left[\sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{v}_{n(\xi)}^L)^2}{1 - (\tilde{v}_{n(\xi)}^L)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^s \hat{\omega}_{\xi} \left(\frac{(\tilde{v}_{n(\xi)}^U)^2}{1 - (\tilde{v}_{n(\xi)}^U)^2} \right)^{\rho}} \right\}^{\frac{1}{\rho}}}} \right] \right] \quad (4)$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_s)^T$ is the associated weighted vector of IVPFDHG operator satisfying $\hat{\omega}_s > 0$ and $\sum_{\xi=1}^s \hat{\omega}_{\xi} = 1$, $\tilde{\alpha}_{n(\xi)}$ is the ξ th largest value of the weighted IVPFE $\alpha_{\xi}(\alpha_{\xi} = (\alpha_{\xi})^{s\tilde{Y}_{\xi}}, \xi = 1, 2, \dots, s)$, s is the number of IVPFEs and $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_s)^T$ is the standard weight vector of $\alpha_{\xi}(\xi = 1, 2, \dots, s)$.

Proof. We prove the statement by applying induction on s . Therefore, for $s = 2$, we have

This means that

Thus, Equation (4) satisfies for $s = 2$.

Suppose that the Equation (4) is valid for $s = q$. So

$$IVPFDHG_{\delta, \hat{\phi}}(\dot{\alpha}_1, \dot{\alpha}_2, \dots, \dot{\alpha}_q) = \bigotimes_{\xi=1}^q (\alpha_{n(\xi)})^{\hat{\phi}_{\xi}}$$

$$\left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^l)^2}{(\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^u)^2}{(\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \right. \right. \\ \left. \left[\sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{(\tilde{\nu}_{n(\xi)}^l)^2}{1 - (\tilde{\nu}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{(\tilde{\nu}_{n(\xi)}^u)^2}{1 - (\tilde{\nu}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}} \right] \right)$$

Moreover, for $s = q + 1$, we have

$$IVPFDHG_{\delta, \hat{\phi}}(\dot{\alpha}_1, \dot{\alpha}_2, \dots, \dot{\alpha}_q \dot{\alpha}_{q+1}) = \bigotimes_{\xi=1}^q (\alpha_{n(\xi)})^{\hat{\phi}_{\xi}} \otimes (\alpha_{n(q+1)})^{\hat{\phi}_{q+1}}$$

$$= \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^l)^2}{(\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^u)^2}{(\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \right. \right. \\ \left. \left[\sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{(\tilde{\nu}_{n(\xi)}^l)^2}{1 - (\tilde{\nu}_{n(\xi)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^q \hat{\phi}_{\xi} \left(\frac{(\tilde{\nu}_{n(\xi)}^u)^2}{1 - (\tilde{\nu}_{n(\xi)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}} \right] \right) \\ \otimes \left(\left[\sqrt{\frac{1}{1 + \left\{ \hat{\phi}_{q+1} \left(\frac{1 - (\tilde{\mu}_{n(q+1)}^l)^2}{(\tilde{\mu}_{n(q+1)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left\{ \hat{\phi}_{q+1} \left(\frac{1 - (\tilde{\mu}_{n(q+1)}^u)^2}{(\tilde{\mu}_{n(q+1)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \right. \right. \\ \left. \left[\sqrt{1 - \frac{1}{1 + \left\{ \hat{\phi}_{q+1} \left(\frac{(\tilde{\nu}_{n(q+1)}^l)^2}{1 - (\tilde{\nu}_{n(q+1)}^l)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left\{ \hat{\phi}_{q+1} \left(\frac{(\tilde{\nu}_{n(q+1)}^u)^2}{1 - (\tilde{\nu}_{n(q+1)}^u)^2} \right)^{\rho}} \right)^{\frac{1}{\rho}}}} \right] \right)$$

This means that

$$\begin{aligned}
& IVPFDHG_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_q \alpha_{q+1}) \\
&= \left(\left[\sqrt[p]{\frac{1}{1 + \left\{ \sum_{\xi=1}^{q+1} \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^l)^2}{(\tilde{\mu}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt[p]{\frac{1}{1 + \left\{ \sum_{\xi=1}^{q+1} \hat{\phi}_{\xi} \left(\frac{1 - (\tilde{\mu}_{n(\xi)}^u)^2}{(\tilde{\mu}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \right. \right. \\
&\quad \left. \left. \sqrt[p]{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^{q+1} \hat{\phi}_{\xi} \left(\frac{(\tilde{v}_{n(\xi)}^l)^2}{1 - (\tilde{v}_{n(\xi)}^l)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}}, \sqrt[p]{1 - \frac{1}{1 + \left\{ \sum_{\xi=1}^{q+1} \hat{\phi}_{\xi} \left(\frac{(\tilde{v}_{n(\xi)}^u)^2}{1 - (\tilde{v}_{n(\xi)}^u)^2} \right)^{\rho} \right\}^{\frac{1}{\rho}}}} \right] \right)
\end{aligned}$$

Thus, Equation (4) is true for $s = q + 1$. Hence, we conclude that Equation (4) is true for all $s \in N$. $\square$

Theorem 7. The IVPFD weighted geometric and IVPFDO weighted geometric operators can be regarded as special cases of the IVPFD hybrid weighted operator.

Proof. Proof of this theorem is analogous to that of Theorem 2. $\square$

Theorem 8. (Idempotency property) Let $\alpha_{\xi} = ([\mu_{\xi}^l, \mu_{\xi}^u], [v_{\xi}^l, v_{\xi}^u]) (\xi = 1, 2, \dots, s)$ be a collection of equal IVPFNs, i.e., $\alpha_{\xi} = \alpha$ for all t , then

$$IVPFDHG_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) = \alpha$$

Proof. Proof of this theorem is straightforward so omitted here.

Theorem 9. (Boundedness property) Let $\alpha_{\xi} = ([\mu_{\xi}^l, \mu_{\xi}^u], [v_{\xi}^l, v_{\xi}^u]) (\xi = 1, 2, \dots, s)$ be a collection of IVPFNs. If $\alpha^* = \min_{\xi} \{\alpha_{\xi}\} = ([(\mu^l)^*, (\mu^u)^*], [(v^l)^*, (v^u)^*])$ and $\alpha^* = \max_{\xi} \{\alpha_{\xi}\} = ([(\mu^l)^*, (\mu^u)^*], [(v^l)^*, (v^u)^*])$, where $(\mu^l)^* = \min_{\xi} \{\mu_{\xi}^l\}$, $(\mu^u)^* = \min_{\xi} \{\mu_{\xi}^u\}$, $(v^l)^* = \max_{\xi} \{v_{\xi}^l\}$, $(v^u)^* = \max_{\xi} \{v_{\xi}^u\}$, $(\mu^l)^* = \max_{\xi} \{\mu_{\xi}^l\}$, $(\mu^u)^* = \max_{\xi} \{\mu_{\xi}^u\}$, $(v^l)^* = \min_{\xi} \{v_{\xi}^l\}$ and $(v^u)^* = \min_{\xi} \{v_{\xi}^u\}$. Then, $\alpha^* \leq IVPFDHG_{\delta, \hat{\phi}}(\alpha_1, \alpha_2, \dots, \alpha_s) \leq \alpha^*$. $\square$

Proof. Proof of this theorem is similar to Theorem 4.

Theorem 10. (Monotonicity property) Let $\epsilon_r = ([\mu_r^l, \mu_r^u], [v_r^l, v_r^u])$ and $\alpha_r = ([\tilde{\mu}_r^l, \tilde{\mu}_r^u], [\tilde{v}_r^l, \tilde{v}_r^u])$, $r = 1, 2, \dots, s$ be any two collections of IVPFEs. If $\mu_r^l \leq \tilde{\mu}_r^l$, $\mu_r^u \leq \tilde{\mu}_r^u$, $v_r^l \geq \tilde{v}_r^l$ and $v_r^u \geq \tilde{v}_r^u \forall r$, then.

$$IVPFDHG_{\delta, \hat{\phi}}(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \leq IVPFDHG_{\delta, \hat{\phi}}(\alpha_1, \alpha_1, \dots, \alpha_s)$$

Proof. Proof of this theorem is similar to that of Theorem 5. $\square$

Example 4. Let $\alpha_1 = ([0.3, 0.4], [0.6, 0.7])$, $\alpha_2 = ([0.5, 0.55], [0.5, 0.6])$ and $\alpha_3 = ([0.5, 0.7], [0.4, 0.5])$ be three IVPFEs, whose weight vector is $\delta = (0.25, 0.35, 0.4)^T$ and the aggregation associated vector is $\hat{\phi} = (0.2, 0.4, 0.4)^T$. For $\rho = 1$, we can calculate $\alpha_{\xi} = (\alpha_{\xi})^{s \delta_{\xi}}$, $\xi = 1, 2, 3$.

$$\begin{aligned}
\alpha_1 &= [0.341, 0.450], [0.545, 0.647], \alpha_2 = [0.491, 0.541], [0.509, 0.609] \text{ and} \\
\alpha_3 &= [0.466, 0.667], [0.431, 0.535].
\end{aligned}$$

By applying Definition 11, the permuted values of IVPFEs are calculated as follows:

$$S(\alpha_1) = -0.17, S(\alpha_2) = -0.012, S(\alpha_3) = -0.137$$

Thus, based on the score values of $\alpha_{\hat{\epsilon}}$, we have $\alpha_{n(1)} = ([0.466, 0.667], [0.431, 0.535])$, $\alpha_{n(2)} = ([0.491, 0.541], [0.509, 0.609])$ and $\alpha_{n(3)} = ([0.341, 0.450], [0.545, 0.647])$

$$IVPFDHG_{\delta, \hat{\omega}}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \otimes_{\hat{\epsilon}=1}^3 (\alpha_{n(\hat{\epsilon})})^{\hat{\omega}_{\hat{\epsilon}}}$$

$$= \left(\left[\sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\omega}_{\hat{\epsilon}} \left(\frac{1 - (\tilde{\mu}_{n(\hat{\epsilon})}^l)^2}{(\tilde{\mu}_{n(\hat{\epsilon})}^l)^2} \right)^{\frac{1}{\rho}}}} \right]^{\frac{1}{\rho}}}, \sqrt{\frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\omega}_{\hat{\epsilon}} \left(\frac{1 - (\tilde{\mu}_{n(\hat{\epsilon})}^u)^2}{(\tilde{\mu}_{n(\hat{\epsilon})}^u)^2} \right)^{\frac{1}{\rho}}}} \right]^{\frac{1}{\rho}}}, \right. \\ \left. \left[\sqrt{1 - \frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\omega}_{\hat{\epsilon}} \left(\frac{(\tilde{\nu}_{n(\hat{\epsilon})}^l)^2}{1 - (\tilde{\nu}_{n(\hat{\epsilon})}^l)^2} \right)^{\frac{1}{\rho}}}} \right]^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left\{ \sum_{\hat{\epsilon}=1}^3 \hat{\omega}_{\hat{\epsilon}} \left(\frac{(\tilde{\nu}_{n(\hat{\epsilon})}^u)^2}{1 - (\tilde{\nu}_{n(\hat{\epsilon})}^u)^2} \right)^{\frac{1}{\rho}}}} \right]^{\frac{1}{\rho}}} \right] \right)$$

Consequently,

$$IVPFDHG_{\delta, \hat{\omega}}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = ([0.408, 0.541], [0.439, 0.614])$$

5. An Approach to Multi-Criteria Decision Making on the Basis of Dombi Operators with IVPF Information

In this section, we employ the IVPFDHA (or IVPFDHG) operator to devise a method for interpreting MADM concerns by utilizing IVPF data. As it can assign a set of possible values for the membership degree and non-membership degree of an element to a given set, the IVPFS is pretty close to the cognitive process that humans go through when evaluating alternatives. The IVPFS has seen widespread application in a wide variety of settings over the course of the past few years. For instance, assume that an expert is assigned to make a choice from a given set of possibilities $H_{\hat{\epsilon}}(\hat{\epsilon} = 1, 2, \dots, \zeta)$ and the criteria $\hat{C}_{\delta}(\delta = 1, 2, \dots, r)$, whose weight vectors are $\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_r$ with $\hat{\omega}_i \in [0, 1]$ and $\sum_{i=1}^r \hat{\omega}_i = 1$. Additionally, he must give weights $\hat{\Gamma}_i$ to each criterion in order to take into consideration the fact that various options may have variable priorities and benefits. Let $F = (a_{\delta\hat{\epsilon}}) = ([\mu_{\delta\hat{\epsilon}}^L, \mu_{\delta\hat{\epsilon}}^U], [\nu_{\delta\hat{\epsilon}}^L, \nu_{\delta\hat{\epsilon}}^U])$, where $[\mu_{\delta\hat{\epsilon}}^L, \mu_{\delta\hat{\epsilon}}^U]$ represents the interval value that the alternative $H_{\hat{\epsilon}}(\hat{\epsilon} = 1, 2, \dots, \zeta)$ satisfies the criteria $\hat{C}_{\delta}(\delta = 1, 2, \dots, r)$, where as $[\nu_{\delta\hat{\epsilon}}^L, \nu_{\delta\hat{\epsilon}}^U]$ represents the interval value that the alternative $H_{\hat{\epsilon}}(\hat{\epsilon} = 1, 2, \dots, \zeta)$ does not satisfy the criteria $\hat{C}_{\delta}(\delta = 1, 2, \dots, r)$. Based on the newly defined IVPFDHA (or IVPFDHG) operator, the following procedure makes up the bulk of the technique described for evaluating MADM issues using IVPF data.

Step 1. Cost attributes and benefit attributes are the two main types of attributes in most MADM problems. If all the attributes in the set $\hat{C}_{\delta}(\delta = 1, 2, \dots, r)$, are of the same type then there is no need to normalize the attribute values. If there are two types of attributes in a MADM problem, Xu and Hu's approach can be used to convert values from the cost type of attribute to the benefit type of attribute (2010) by using the following formula:

$$X_{\delta\hat{\epsilon}} = \begin{cases} ([\mu_{\delta\hat{\epsilon}}^L, \mu_{\delta\hat{\epsilon}}^U], [\nu_{\delta\hat{\epsilon}}^L, \nu_{\delta\hat{\epsilon}}^U]); & \text{for benefit type criteria} \\ ([\nu_{\delta\hat{\epsilon}}^L, \nu_{\delta\hat{\epsilon}}^U], [\mu_{\delta\hat{\epsilon}}^L, \mu_{\delta\hat{\epsilon}}^U]); & \text{for loss type criteria} \end{cases}$$

In this case, the decision matrix is transformed into a normalized matrix.

Step 2. Calculate $\tilde{\alpha}_{\delta\hat{\epsilon}} = r\hat{\Gamma}_{\delta}\hat{\alpha}_{\delta\hat{\epsilon}}$ (or $\tilde{\alpha}_{\delta\hat{\epsilon}} = (\hat{\alpha}_{\delta\hat{\epsilon}})^{r\hat{\Gamma}_{\delta}}$).

Step 3. Compute the score values $S(\tilde{\alpha}_i)$ using Definition 11. Moreover, use these score values to reorder the items to obtain the highest score values at the beginning.

Step 4. IVPFDHA (or IVPFDHG) operator is used to aggregate these values.

Step 5. Using Definition 11, calculate the score values and choose the best option.

5.1. Numerical Application of Decision Making

One of the two main categories of aviation is “civil aviation”, which refers to both private and commercial flights that are not under the control of a governing body. The majority of countries are members of the International Civil Aviation Organization (ICAO), which is an organization where members collaborate to develop standards and guidelines for use in the field of civil aviation on a worldwide scale.

The following are the three primary categories of civil aviation:

1. Transportation services provided by chartered aircraft, include both regularly scheduled and ad hoc flights for passengers and cargo.
2. Aerial labor is when a plane is used to perform certain duties such as farming, taking pictures, measuring land, rescuing people, and other similar activities.
3. The term “general aviation” (GA) refers to any and all other categories of flights, including public and private.

The Saudi Arabian government’s decision to invest in high-speed rail has made it more difficult for domestic airlines to sell their services within the kingdom. There has been a rise in the number of airlines that are attempting to entice customers by providing lower tickets. They rapidly realized, however, that the only factor that mattered in this extremely competitive housing market is the quality of the service that is provided. The Saudi Arabian Civil Aviation Authority (CAASA) is in the process of conducting research to identify which regional airline offers the highest quality of service. The findings of this research will be used to promote the adoption of best practices by other airlines. Therefore, the CAASA establishes a committee in order to conduct an investigation into the five most important domestic airlines, which are Flyadeal (ϵ_1), Flynas (ϵ_2), SaudiGulf airline (ϵ_3), Nesma (ϵ_4), and Saudi Arabian Airline (ϵ_5), in light of the following three major attributes.

Bespeaking and ticketing service ($\mathcal{E}_1$): Bespeaking and ticketing service in the context of best airline selection refers to the process of reserving and purchasing flight tickets. Bespeaking implies making a reservation in advance, typically to secure a seat on a specific flight or to avail special services.

Cabin service and responsiveness ($\mathcal{E}_2$): Cabin service and responsiveness encompass the airline’s customer service and the quality of in-flight amenities and services. A good airline should be responsive and attentive to passengers’ needs.

Cost and time ($\mathcal{E}_3$): cost and time are two critical factors when selecting an airline. Cost refers to the price of the flight ticket, including any additional fees or charges. Time, on the other hand, refers to the duration of the flight, including layovers or stopovers, if applicable.

In the following Table 1, an expert assigns the corresponding weights to the attributes on the basis of the above discussion of their respective significances.

Table 1. Corresponding weights of the attributes.

Attributes	Weights
Bespeaking and ticketing service	0.3
Cabin service and responsiveness	0.35
Cost and time	0.35

The weighted vector of three attributes given by the decision as $\hat{\phi} = (0.3, 0.35, 0.35)^T$. The second weight vector $\hat{\Gamma} = (0.4, 0.3, 0.3)^T$ is provided for each criterion to indicate that the most prominent component of the alternative is given a larger weight and the rest a

lesser weight, reflecting that different alternatives may emphasize different areas. On the basis of the information obtained from the available online sources, like saudiscoop.com, the evaluation of these alternatives is carried out using IVPF information and is summarized in the following Table 2.

Table 2. Decision matrix.

	ϵ_1	ϵ_2	ϵ_3
ϵ_1	([0.3,0.4], [0.5,0.6])	([0.2,0.3], [0.4,0.5])	([0.0,0.3], [0.6,0.8])
ϵ_2	([0.5,0.6], [0.4,0.5])	([0.3,0.4], [0.6,0.7])	([0.4,0.5], [0.2,0.6])
ϵ_3	([0.4,0.5], [0.1,0.3])	([0.1,0.3], [0.4,0.6])	([0.1,0.3], [0.5,0.6])
ϵ_4	([0.2,0.5], [0.5,0.6])	([0.3,0.4], [0.6,0.7])	([0.3,0.4], [0.5,0.6])
ϵ_5	([0.3,0.4], [0.2,0.5])	([0.5,0.6], [0.3,0.5])	([0.3,0.4], [0.1,0.4])

Step 1: Transform the decision matrix into a normalized decision matrix. Table 3 describes the outcomes of this process in the following way:

Table 3. Normalized decision matrix.

	ϵ_1	ϵ_2	ϵ_3
ϵ_1	([0.3,0.4], [0.5,0.6])	([0.2,0.3], [0.4,0.5])	([0.6,0.8], [0.0,0.3])
ϵ_2	([0.5,0.6], [0.4,0.5])	([0.3,0.4], [0.6,0.7])	([0.2,0.6], [0.4,0.5])
ϵ_3	([0.4,0.5], [0.1,0.3])	([0.1,0.3], [0.4,0.6])	([0.5,0.6], [0.1,0.3])
ϵ_4	([0.2,0.5], [0.5,0.6])	([0.3,0.4], [0.6,0.7])	([0.5,0.6], [0.3,0.4])
ϵ_5	([0.3,0.4], [0.2,0.5])	([0.5,0.6], [0.3,0.5])	([0.1,0.4], [0.3,0.4])

Step 2: Calculate $\tilde{\alpha}_{\epsilon\delta} = r\tilde{f}_t\alpha_{\epsilon\delta}$ as

$$\begin{aligned}
 \tilde{\alpha}_{11} &= ([0.326, 0.431], [0.466, 0.565]), \\
 \tilde{\alpha}_{12} &= ([0.19, 0.286], [0.418, 0.52]), \\
 \tilde{\alpha}_{13} &= ([0.58, 0.784], [0.0, 0.315]), \\
 \tilde{\alpha}_{21} &= ([0.535, 0.635], [0.37, 0.466]), \\
 \tilde{\alpha}_{22} &= ([0.286, 0.383], [0.62, 0.719]), \\
 \tilde{\alpha}_{23} &= ([0.19, 0.58], [0.418, 0.52]), \\
 \tilde{\alpha}_{31} &= ([0.431, 0.535], [0.091, 0.276]), \\
 \tilde{\alpha}_{32} &= ([0.095, 0.286], [0.418, 0.62]), \\
 \tilde{\alpha}_{33} &= ([0.48, 0.58], [0.105, 0.315]), \\
 \tilde{\alpha}_{41} &= ([0.218, 0.535], [0.466, 0.565]), \\
 \tilde{\alpha}_{42} &= ([0.286, 0.383], [0.62, 0.719]), \\
 \tilde{\alpha}_{43} &= ([0.48, 0.58], [0.315, 0.418]), \\
 \tilde{\alpha}_{51} &= ([0.326, 0.431], [0.183, 0.466]), \\
 \tilde{\alpha}_{52} &= ([0.48, 0.58], [0.315, 0.520]), \\
 \tilde{\alpha}_{53} &= ([0.095, 0.383], [0.315, 0.418]).
 \end{aligned}$$

Step 3: Compute the score values by means of the formula presented in Definition 11.

$$\begin{aligned}
 S(\tilde{\alpha}_{11}) &= -0.102, & S(\tilde{\alpha}_{12}) &= -0.155, & S(\tilde{\alpha}_{13}) &= 0.446, \\
 S(\tilde{\alpha}_{21}) &= 0.197, & S(\tilde{\alpha}_{22}) &= -0.311, & S(\tilde{\alpha}_{23}) &= -0.006, \\
 S(\tilde{\alpha}_{31}) &= 0.201, & S(\tilde{\alpha}_{32}) &= -0.224, & S(\tilde{\alpha}_{33}) &= 0.239, \\
 S(\tilde{\alpha}_{41}) &= -0.071, & S(\tilde{\alpha}_{42}) &= -0.312, & S(\tilde{\alpha}_{43}) &= 0.166, \\
 S(\tilde{\alpha}_{51}) &= 0.034, & S(\tilde{\alpha}_{52}) &= 0.130, & S(\tilde{\alpha}_{53}) &= -0.051.
 \end{aligned}$$

Moreover, the hybrid decision-making matrix is obtained by applying the strategy described in step 3 of the algorithm and is presented in the following Table 4.

Table 4. Hybrid decision matrix.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$
ϵ_1	([0.58,0.784], [0,0.315])	([0.326,0.431], [0.466,0.565])	([0.19,0.286], [0.418,0.52])
ϵ_2	([0.535,0.635], [0.37,0.466])	([0.19,0.58], [0.418,0.52])	([0.286,0.383], [0.62,0.719])
ϵ_3	([0.48,0.58], [0.105,0.315])	([0.431,0.535], [0.091,0.276])	([0.095,0.286], [0.418,0.62])
ϵ_4	([0.48,0.58], [0.315,0.418])	([0.218,0.535], [0.466,0.565])	([0.286,0.383], [0.62,0.719])
ϵ_5	([0.48,0.58], [0.315,0.520])	([0.326,0.431], [0.183,0.466])	([0.095,0.383], [0.315,0.418])

Step 4: To obtain the overall preference of alternatives $\epsilon_i (i = 1, 2, \dots, 5)$ apply IVPFDH operator for the operational parameter $\rho = 1$ and the corresponding weight vector $\hat{\phi} = (0.3, 0.35, 0.35)^T$ on the above decision matrix. We have

$$\begin{aligned} \epsilon_1 &= ([0.414, 0.609], [0, 0.43]), \epsilon_2 = ([0.376, 0.553], [0.443, 0.546]), \\ \epsilon_3 &= ([0.384, 0.494], [0.118, 0.342]), \epsilon_4 = ([0.349, 0.511], [0.425, 0.535]) \text{ and} \\ \epsilon_5 &= ([0.344, 0.475], [0.243, 0.46]). \end{aligned}$$

Step 5: Calculate the score values by using Definition 11.

$$S(\epsilon_1) = 0.202, S(\epsilon_2) = 0.007, S(\epsilon_3) = 0.14, S(\epsilon_4) = -0.017 \text{ and } S(\epsilon_5) = 0.053$$

Consequently, we conclude that Flyadeal is the best choice for the purposeful decision-making problem.

5.2. The Impact of the Operational Parameter ρ in This Technique

In the following discussion, we investigate the behavior of the alternatives depending on the different values of the operational parameter ρ in the framework of the aforementioned technique.

It is important to note that as the parameter value of the ρ increases, the score values of the alternatives ϵ_j continue to rise in a progressive manner; yet the ranking of the alternatives remains unchanged i.e., $\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$. Table 4 displays the results of an IVPFDHA operator-based ranking of the possibilities $\epsilon_j (j = 1, 2, \dots, 5)$. This process is summarized in the following Table 5.

Table 5. Score values and Preference ranking by IVPFDWA operator for various ρ .

ρ	$S(\epsilon_1)$	$S(\epsilon_2)$	$S(\epsilon_3)$	$S(\epsilon_4)$	$S(\epsilon_5)$	Preference Ranking
1	0.202	0.007	0.14	−0.017	0.053	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$
2	0.292	0.057	0.175	0.031	0.086	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$
5	0.382	0.122	0.208	0.099	0.138	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$
10	0.414	0.154	0.224	0.130	0.166	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$

From the above discussion, one can observe that isotonicity is a key feature of the proposed approach, that facilitates a decision-maker's choice of an appropriate value for the operational parameter ρ .

5.3. Comparative Analysis

In the following discussion, we conduct a comparative analysis (see Table 6) to prove the efficacy and viability of the proposed approach. We evaluate our methods in comparison to the IVPF Einstein hybrid weighted geometric (IVPFEHWG) operator, the IVPF hybrid geometric (IVPFHG) operator, and the IVPF Einstein hybrid weighted arithmetic (IVPFEHWA) operator.

Table 6. Comparison table with existing methods.

Techniques	Preference Order	Best Alternative
IVPFDHA	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$	ϵ_1
IVPFDHG	$\epsilon_3 > \epsilon_5 > \epsilon_1 > \epsilon_4 > \epsilon_2$	ϵ_3
IVPFHG [26]	$\epsilon_3 > \epsilon_5 > \epsilon_1 > \epsilon_2 > \epsilon_4$	ϵ_3
IVPFEHA [38]	$\epsilon_1 > \epsilon_3 > \epsilon_5 > \epsilon_2 > \epsilon_4$	ϵ_1
IVPFEHG [38]	$\epsilon_3 > \epsilon_5 > \epsilon_1 > \epsilon_2 > \epsilon_4$	ϵ_3

In contrast to existing methods, our approach can handle cases where there is a close connection between the arguments, and hence the proposed technique is more universal. Moreover, these approaches use significantly less computing power than those that are currently being utilized. In addition, the induction of the operational parameter into the new strategy possesses the ability to vary the preference values corresponding to certain requirements of the decisions. This makes it possible for the individual making the decision to use the value of the parameter that is ideal for them in terms of their own risk tolerance and the requirements of the situation.

6. Conclusions

In this paper, an enhanced score function has been introduced to overcome the deficiencies of the existing score function in the IVPF environment. The concepts of IVPFDHA operator and IVPFDHG operator have been presented, and their various important properties have been investigated. In the context of the recently disclosed strategies, an effective approach for selecting the ideal airline through the application of IVPF parameters has been designed in this article. Finally, a comparative analysis has been established in order to show the validity and feasibility of the proposed techniques with existing methods. We shall apply the newly designed strategies to many real-world phenomena, specifically the construction industry, renewable energy selection, project portfolio management, and supplier selection, in our future research to exhibit the versatility of these newly defined methods. In addition, we shall also investigate the behavior and significance of the Dombi operator under IV q-rung and IV spherical fuzzy environments. This will allow for the efficient and cost-effective resolution of a wide range of critical MADM issues.

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