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Applications of the Symmetric Quantum-Difference Operator for New Subclasses of Meromorphic Functions

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Abstract: Our goal in this article is to use ideas from symmetric q-calculus operator theory in the study of meromorphic functions on the punctured unit disc and to propose a novel symmetric q-difference operator for these functions. A few additional classes of meromorphic functions are then defined in light of this new symmetric q-difference operator. We prove many useful conclusions regarding these newly constructed classes of meromorphic functions, such as convolution, subordination features, integral representations, and necessary conditions. The technique presented in this article may be used to produce a wide variety of new types of generalized symmetric q-difference operators, which can subsequently be used to investigate a wide variety of new classes of analytic and meromorphic functions related to symmetric quantum calculus.

Keywords: meromorphic functions; meromorphic q-starlike functions; meromorphic q-convex functions; subordination; symmetric q-calculus; symmetric q-derivative operator

MSC: 05A30; 30C45; 11B65; 47B38

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1. Introduction and Definitions

Let $\mathcal{H}(J)$ denote the class of all analytic functions on the open unit disk

$$J = \{\xi : \xi \in \mathbb{C} \quad \text{and} \quad |\xi| < 1\}.$$

Let $\mathcal{A}$ denote the class of all analytic functions $g(\xi)$which are normalized by the conditions

$$g(0) = 0 \;\; \text{and} \;\; g'(0) = 1$$

and have a Taylor–Maclaurin series representation of the form

$$g(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \qquad (\forall\, \xi \in J). \tag{1}$$

Also, $\mathcal{S} \subset \mathcal{A}$ represents all univalent functions in J.

Let $\mathcal{P}$ represent the class of Carathéodory functions p, which are analytic in the open unit disk J and satisfy the following conditions:

$$p(0) = 1 \text{ and } Re\{p(\xi)\} > 0, \quad (\forall\, \xi \in J).$$

Every $p \in \mathcal{P}$ has a series representation of the form

$$p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n. \tag{2}$$

Let $\mathcal{M}$ denote the class of all meromorphic functions g of the form

$$g(\xi) = \frac{1}{\xi} + \sum_{n=0}^{\infty} a_n \xi^n, \tag{3}$$

which are analytic in the punctured open unit disk

$$J^* = \{\xi : \xi \in \mathbb{C} \ \text{ and } \ 0 < |\xi| < 1\} = J \backslash \{0\}$$

with a simple pole at the origin with residue 1.

Let g and $h \in \mathcal{M}$, where g is given by (3) and h is given by

$$h(\xi) = \frac{1}{\xi} + \sum_{n=0}^{\infty} b_n \xi^n,$$

then the Hadamard product (or convolution) $g * h$ is defined by

$$(g * h)(\xi) = \frac{1}{\xi} + \sum_{n=0}^{\infty} a_n b_n \xi^n = (h * g)(\xi).$$

The subordination of two analytic functions g and h, in open unit disk J, can be defined as

$$g(\xi) \prec h(\xi), \ \ (\xi \in J),$$

if there exists a Schwarz function t which satisfies the conditions

$$t(0) = 0 \ \text{ and } \ |t(\xi)| < 1, \ (\xi \in J)$$

such that

$$g(\xi) = h(t(\xi)), \ (\xi \in J).$$

Equivalently, if the function h is univalent in J, then we have the following equivalence relation:

$$g(\xi) \prec h(\xi) \Leftrightarrow g(0) = h(0), \quad (\xi \in J)$$

and

$$g(J) \subset h(J), \ \ (\xi \in J).$$

The meromorphic function $g \in \mathcal{MS}^*(\alpha)$ if it satisfies the inequality

$$Re\left(\frac{\xi g'(\xi)}{g(\xi)}\right) < -\alpha, \ \ (0 \leq \alpha < 1, \ \xi \in J) \tag{4}$$

and meromorphic function $g \in \mathcal{MK}(\alpha)$ if it satisfies the inequality

$$Re\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) < -\alpha, \ \ (0 \leq \alpha < 1, \ \xi \in J). \tag{5}$$

The function classes $\mathcal{MS}^*(\alpha)$ and $\mathcal{MK}(\alpha)$ are known as meromorphic starlike and convex functions of order α. It can be observed from (4) and (5) that

$$g \in \mathcal{MK}(\alpha) \Leftrightarrow -\xi g' \in \mathcal{MS}^*(\alpha).$$

The class of meromorphic δ-convex functions of order γ defined by Nunokawa and Ahuja in [1] is as follows:

$$Re\left\{(1-\delta)\left(\frac{\xi g'(\xi)}{g(\xi)}\right)+\delta\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)\right\}<-\gamma,$$

where

$$(\xi \in J,\ \delta \geq 0, \gamma < 1,\ g \in \mathcal{M}).$$

A similar concept is related to the subclasses of meromorphic functions, see [2,3] for examples.

Quantum (or q-) calculus is the name given to the limitless calculus that is used in several branches of mathematics and physics. Jackson [4] defined the q-derivative (D_q) and q-integral operators using the concept of quantum (or q-) calculus. Geometric Function Theory (GFT) has examined the many uses of the q-derivative operator, which makes GFT extraordinarily important. Historically speaking, the concept of q-starlike functions was initially introduced by Ismail et al. [5]. Srivastava [6] employed the fundamental (or q-) hypergeometric functions for the first time in Geometric Function Theory while the q-Mittag-Leffler functions were examined by Srivastava and Bansal in [7]. Arif et al., in [8,9], constructed and analyzed several novel subclasses of multivalent functions by applying the principles of the q-derivative operator to practical situations. Mahmood et al. [10] recently conducted extensive research on the class of meromorphic q-starlike functions connected to Janowski functions. Following that, Srivastava [11] outlined a few ways fractional q-calculus can be used in GFT for complex analysis. Al-Shbeil et al. [12] studied some remarkable results for a subclass of bi-univalent functions employing the q-Chebyshev polynomials more recently as an application of the q-derivative operator. We refer to [12–18]. for a study of some general classes of q-starlike functions connected to the Janowski domain.

Among the many areas where symmetric q-calculus has been shown to be useful are fractional calculus and quantum physics [19,20]. Sun et al. [21] presented and analyzed various properties of fractional symmetric q-integrals and symmetric q-derivatives. Using symmetric q-fractional integrals and fractional difference operators, they looked at problems related to non-local boundary conditions. Kanas et al. [22] explored several applications of the symmetric q-derivative operator in the conic domain and constructed a new class of analytic functions. The idea was subsequently used by Khan et al. [23] in the creation of the symmetric q-conic domain. Taking this area into consideration, they looked into several previously unexplored classes of analytic functions. Applications of the symmetric q-operator, a generalization of the conic domain, and an examination of subclasses of q-convex and q-starlike functions may all be found in the work of Al-Shbeil et al. [24]. Recent work by Khan et al. [24] explored several novel applications of symmetric quantum calculus for a class of harmonic functions. In their paper [25], Khan et al. introduced the concept of a symmetric q-derivative operator for multivalent functions and described its numerous interesting applications. This article offers a three-pronged explanation of symmetry: geometry—the most common kind of symmetry studied in school—physics, biology, chemistry, and other branches.

Here, we provide some foundational concepts and ideas of the symmetric q-calculus that will be used in the subsequent creation of new subclasses.

For $n \in \mathbb{N}$, we have a definition for the symmetric q-number, which is

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{6}$$

and for $n = 0$, then

$$\widetilde{[n]}_q = 0.$$

It is important to keep in mind that the symmetric q-number cannot be written as a q-number. The factorial representation of the symmetric q number is defined as

$$\widetilde{[n]}_q! = \widetilde{[n]}_q \widetilde{[n-1]}_q \widetilde{[n-2]}_q ... \widetilde{[2]}_q \widetilde{[1]}_q, \qquad n \geq 1,$$

and for $n = 0$, the factorial of the symmetric q-number will be equal to 1, and for $q \to 1-$, then

$$\widetilde{[n]}_q! = n!.$$

Definition 1 ([26])**.** *The symmetric q-derivative (q-difference) operator $\widetilde{D}_q g(\xi)$ for the function $g \in \mathcal{A}$ is defined by*

$$\begin{aligned} \widetilde{D}_q g(\xi) &= \frac{1}{\xi}\left(\frac{g(q\xi) - g(q^{-1}\xi)}{q - q^{-1}}\right), \ \xi \in J, \\ &= 1 + \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n \xi^{n-1}, \ (\xi \neq 0, \ q \neq 1) \end{aligned} \tag{7}$$

and

$$\widetilde{D}_q \xi^n = \widetilde{[n]}_q \xi^{n-1}, \quad \widetilde{D}_q \left\{ \sum_{n=1}^{\infty} a_n \xi^n \right\} = \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n \xi^{n-1}.$$

From (7), we can observe that

$$\lim_{q \to 1-} \widetilde{D}_q g(\xi) = g'(\xi).$$

Inspired by the research presented in [26], we define the symmetric q-derivative operator for meromorphic functions as

Definition 2. *The symmetric q-derivative operator $\widetilde{D}_q g(\xi)$ for the meromorphic function $g \in \mathcal{M}$ is defined by*

$$\left(\widetilde{D}_q g\right)(\xi) = \frac{g(q\xi) - g(q^{-1}\xi)}{(q - q^{-1})\xi} = \frac{-1}{\xi^2} + \sum_{n=0}^{\infty} \widetilde{[n]}_q a_n \xi^{n-1}, \quad (\forall \xi \in J^*). \tag{8}$$

We observe that

$$\lim_{q \to 1-} \left(\widetilde{D}_q g\right)(\xi) = g'(\xi), \text{ for } g \in \mathcal{M}.$$

Now, by considering the operator defined in (8), we define the following new subclasses of meromorphic functions associated with symmetric q-calculus.

Definition 3. *A function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{MS}}^*(q)$ of meromorphic symmetric q-starlike functions if it satisfies the inequality*

$$\Re\left(\frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)}\right) < 0, \ \ \xi \in J. \tag{9}$$

Definition 4. *A function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{MK}}(q)$ of meromorphic symmetric q-convex functions if it satisfies the inequality*

$$\Re\left(1 + \frac{\xi \widetilde{D}_q^2 g(\xi)}{\widetilde{D}_q g(\xi)}\right) < 0, \ \ \xi \in J. \tag{10}$$

It can be observed from (9) and (10) that

$$g \in \widetilde{\mathcal{MK}}(q) \Leftrightarrow -\xi \widetilde{D}_q g \in \widetilde{\mathcal{MS}}^*(q).$$

Definition 5. *A function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{MS}}^*(\alpha, q)$ of meromorphic symmetric q-starlike functions of order α if it satisfies the condition*

$$\Re\left(\frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)}\right) < -\alpha, \ \ (0 \leq \alpha < 1, \xi \in J). \tag{11}$$

Definition 6. *A function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{MK}}(\alpha, q)$ of meromorphic symmetric q-convex functions of order α if it satisfies the condition*

$$\Re\left(1 + \frac{\xi \widetilde{D}_q^2 g(\xi)}{\widetilde{D}_q g(\xi)}\right) < -\alpha, \ \ (0 \leq \alpha < 1, \xi \in J). \tag{12}$$

We note that

$$g \in \widetilde{\mathcal{MK}}(\alpha, q) \Leftrightarrow -\xi \widetilde{D}_q g \in \widetilde{\mathcal{MS}}^*(\alpha, q),$$

For $\alpha = 0$, we have

$$\begin{aligned} \widetilde{\mathcal{MS}}^*(0, q) &= \widetilde{\mathcal{MS}}^*(q) \\ \widetilde{\mathcal{MK}}(0, q) &= \widetilde{\mathcal{MK}}(q). \end{aligned}$$

Definition 7. *A function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{M}(\beta, q)}$ if it satisfies the inequality*

$$\Re\left(\frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)}\right) > -\beta, \ \ (\ \xi \in J, \ \beta > 1). \tag{13}$$

Also, a function $g \in \mathcal{M}$ is in the class $\widetilde{\mathcal{N}(\beta, q)}$ if and only if

$$-\xi \widetilde{D}_q g \in \widetilde{\mathcal{M}(\beta, q)}. \tag{14}$$

Example 1. *Clearly, from (14), for $\beta > 1$, we know that if $g(\xi) = \xi^{-1} \in \widetilde{\mathcal{N}(\beta, q)}$, then $\xi \widetilde{D}_q g(\xi) = \xi^{-1} \in \widetilde{\mathcal{M}(\beta, q)}$.*

Here, we give another example of the functions $g \in \widetilde{\mathcal{M}(\beta, q)}$ and $g \in \widetilde{\mathcal{N}(\beta, q)}$, respectively.

Example 2.

$$\xi^{-1}(1 - \xi)^{2(1-\beta)} \in \widetilde{\mathcal{M}(\beta, q)}, \ \ (\xi \in J^*)$$

and

$$-\int_{\xi_0}^{\xi} u^{-2}(1 - u)^{2(1-\beta)} du \in \widetilde{\mathcal{N}(\beta, q)}, \ \ (\xi \in J^*, \ 0 < |\xi_0| < |\xi|).$$

Proof. From (13), we know that

$$\frac{\beta + \frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)}}{\beta - 1} = \frac{1 + \xi}{1 - \xi}, \ \ (\xi \in J). \tag{15}$$

From (15), we know that

$$\frac{\widetilde{D_q}g(\xi)}{g(\xi)}+\frac{1}{\xi}=\frac{2(\beta-1)}{1-\xi},\ (\xi\in J^*).$$

Integrating the above equation, we have

$$g(\xi)=\xi^{-1}(1-\xi)^{2(1-\beta)}\in\widetilde{\mathcal{M}(\beta,q)},\ \ (\xi\in J^*).$$

Moreover, we have

$$g\in\widetilde{\mathcal{N}(\beta,q)}\Leftrightarrow-\xi\widetilde{D_q}g\in\widetilde{\mathcal{M}(\beta,q)},$$

and we deduce that

$$-\int_{\xi_0}^{\xi}u^{-2}(1-u)^{2(1-\beta)}du\in\widetilde{\mathcal{N}(\beta,q)},\ (\xi\in J^*,\ 0<|\xi_0|<|\xi|).$$

□

In this section, we gave some new preliminary results, which will be used to prove our main results.

2. Preliminaries

Lemma 1. *A function g of the form (3) belongs to the class* $\widetilde{\mathcal{M}(\beta,q)}$ *if and only if*

$$-\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}\prec\frac{1-(2\beta-1)\xi}{1-\xi},\ \ (\xi\in J). \tag{16}$$

Proof. Let

$$v(\xi)=\frac{\beta+\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}}{\beta-1},\ \ \left(\xi\in J,\ g\in\widetilde{\mathcal{M}(\beta,q)}\right).$$

Since $s\in\mathcal{P}$, which implies that

$$\frac{\beta+\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}}{\beta-1}=\frac{1+s(\xi)}{1-s(\xi)},\ \ \left(\xi\in J,\ g\in\widetilde{\mathcal{M}(\beta,q)}\right), \tag{17}$$

where s is analytic in J along the conditions $s(0)=0$ and $|s(\xi)|<1$. Thus, from (17), we find that

$$-\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}=\frac{1-(2\beta-1)s(\xi)}{1-s(\xi)},\ \ (\xi\in J).$$

Then

$$-\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}\prec\frac{1-(2\beta-1)\xi}{1-\xi},\ \ (\xi\in J).$$

The relationship (16) is proved. For the converse part of Lemma 1, we follow the above deductive process. Hence, the proof of Lemma 1 is completed. □

Lemma 2. *A function g of the form (3) belongs to the class* $\widetilde{\mathcal{N}(\beta,q)}$ *if and only if*

$$-\left(1+\frac{\xi\widetilde{D_q^2}g(\xi)}{\widetilde{D_q}g(\xi)}\right)\prec\frac{1-(2\beta-1)\xi}{1-\xi},\ \ (\xi\in J).$$

Proof. Using the method for the proof of Lemma 1, we obtain the proof of Lemma 2. □

Lemma 3. *A function g of the form (3) satisfies the inequality*

$$\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q+\mu\right|+\left|\widetilde{[n]}_q-\mu\right|\right)|a_n|\leq 2\mu,\ \ 0<\mu\leq 1. \tag{18}$$

Then, $g\in\widetilde{\mathcal{MS}}^*(q)$.

Proof. To prove $g\in\widetilde{\mathcal{MS}}^*(q)$, it suffices to show that

$$\left|\frac{\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}+\mu}{\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}-\mu}\right|<1,\ \ (0<\mu\leq 1,\ \xi\in J).$$

We can write (18) as follows:

$$\begin{aligned}&(1+\mu)-\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q-\mu\right|\right)|a_n|\\ \geq\ &(1-\mu)+\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q+\mu\right|\right)|a_n|>0.\end{aligned}$$

Therefore,

$$0<\frac{(1-\mu)+\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q+\mu\right|\right)|a_n|}{(1+\mu)-\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q-\mu\right|\right)|a_n|}\leq 1. \tag{19}$$

We can deduce from (3) and (19) that

$$\begin{aligned}\left|\frac{\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}+\mu}{\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}-\mu}\right| &= \left|\frac{-1+\mu+\sum_{n=0}^{\infty}\left(\widetilde{[n]}_q+\mu\right)a_n\xi^{n-1}}{-1-\mu+\sum_{n=0}^{\infty}\left(\widetilde{[n]}_q-\mu\right)a_n\xi^{n-1}}\right|\\ &< \frac{1-\mu+\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q+\mu\right|\right)|a_n|}{1+\mu-\sum_{n=0}^{\infty}\left(\left|\widetilde{[n]}_q-\mu\right|\right)|a_n|}\leq 1.\end{aligned}$$

This completes the proof of the first part of Lemma 3. For the converse part of Lemma 3, we follow the above deductive process. □

Lemma 4. *A function $g\in\mathcal{M}$ satisfies the inequality*

$$\sum_{n=0}^{\infty}\left|\widetilde{[n]}_q\right|\left(\left|\widetilde{[n]}_q+\mu\right|+\left|\widetilde{[n]}_q-\mu\right|\right)|a_n|\leq 2\mu,\ \ 0<\mu\leq 1.$$

Then, $g\in\widetilde{\mathcal{MK}(q)}$.

Proof. By using a similar method to the proof of Lemma 3, we can prove Lemma 4. □

Lemma 5. *Let $\beta>1$. Suppose also that the sequence $\{A_{m+1}\}_{m=0}^{\infty}$ can be defined by*

$$A_1=(\beta-1),\qquad if\quad m=0,$$

$$A_{m+1} = \frac{2(\beta-1)}{2+\widetilde{[m]}_q}\left(1+\sum_{k=0}^{m-1} A_{k+1}\right), \quad if \quad m \in \mathbb{N}, \tag{20}$$

then

$$A_{m+1} = \frac{2(\beta-1)}{2\beta+\widetilde{[m]}_q}\prod_{k=0}^{m}\frac{2\beta+\widetilde{[k]}_q}{2+\widetilde{[k]}_q}, \ m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{21}$$

Proof. To prove the assertion (21) of Lemma 5, we use the principle of mathematical induction. Now, from (20), we have

$$\begin{aligned} A_2 &= \frac{2(\beta-1)}{\widetilde{[1]}_q+2}\left(1+\sum_{k=0}^{0} A_{1+k}\right) \\ &= \frac{2(\beta-1)}{\widetilde{[1]}_q+2}(1+A_1) \\ &= \frac{2\beta(\beta-1)}{\widetilde{[1]}_q+2}. \end{aligned}$$

From (21), we have

$$A_2 = \frac{2\beta(\beta-1)}{2+\widetilde{[1]}_q}.$$

Hence, (21) holds true for $m = 1$. Let (21) hold true for $m = i$, $(i \geq 2)$, then

$$A_{i+1} = \frac{2(\beta-1)}{2\beta+\widetilde{[i]}_q}\prod_{k=0}^{i}\frac{2\beta+\widetilde{[k]}_q}{2+\widetilde{[k]}_q}. \tag{22}$$

Combining (20) and (22), we find that

$$\begin{aligned} A_{i+2} &= \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}\left(1+\sum_{k=0}^{i} A_{1+k}\right) \\ &= \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}\left(1+\sum_{k=0}^{i-1} A_{k+1}\right) + \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}A_{i+1} \\ &= \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}\left(\frac{\widetilde{[i]}_q+2}{2(\beta-1)}\right)A_{i+1} + \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}A_{i+1} \\ &= \frac{2(\beta-1)}{2+\widetilde{[i+1]}_q}\left(\frac{\widetilde{[i]}_q+2\beta}{2(\beta-1)}\right)A_{i+1} \\ &= \left(\frac{2\beta+\widetilde{[i]}_q}{2+\widetilde{[i+1]}_q}\right)A_{i+1} \\ &= \frac{2(\beta-1)}{2\beta+\widetilde{[i+1]}_q}\prod_{k=0}^{i+1}\frac{2\beta+\widetilde{[k]}_q}{2+\widetilde{[k]}_q}. \end{aligned}$$

Hence, (21) holds true for $m = i+1$. This completes the proof of Lemma 5. □

In this section, we investigate our main results.

3. Main Results

Coefficient inequality for the function class $\widetilde{\mathcal{M}(\beta,q)}$.

Theorem 1. *A function $g \in \mathcal{M}$ satisfies the inequality*

$$\sum_{n=0}^{\infty}\left(\widetilde{[n]_q}+(2\beta-\eta)+\left|\widetilde{[n]_q}+\eta\right|\right)|a_n| \leq 2(\beta-1),\ 0\leq\eta\leq 1,\ \beta>1,$$

then, $g \in \widetilde{\mathcal{M}(\beta,q)}$.

Proof. To prove $g \in \widetilde{\mathcal{M}(\beta,q)}$, it suffices to show that

$$\left|\frac{\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}+\eta}{\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}+(2\beta-\eta)}\right|<1,\ \ (0\leq\eta\leq 1,\ \beta>1,\ \xi\in J).$$

By using a similar procedure to the proof of Lemma 3, we find that the assertion of Theorem 1 holds true. □

Integral representation of functions belonging to the class $\widetilde{\mathcal{M}(\beta,q)}$.

Theorem 2. *Let $g \in \widetilde{\mathcal{M}(\beta,q)}$, then*

$$g(\xi)=\xi^{-1}\exp\left(2(\beta-1)\int_0^{\xi}\frac{s(t)}{t(1-s(t))}dt\right),\ \ (\xi\in J^*), \tag{23}$$

where s is analytic in J and given in (17).

Proof. For $g \in \widetilde{\mathcal{M}(\beta,q)}$ and Lemma 1, we see that (16) holds true, then

$$\frac{\xi\widetilde{D_q}g(\xi)}{g(\xi)}=\frac{(2\beta-1)s(\xi)-1}{1-s(\xi)},\ (\xi\in J), \tag{24}$$

where s is analytic in J and given in (17).

We next find from (24) that

$$\frac{\widetilde{D_q}g(\xi)}{g(\xi)}+\frac{1}{\xi}=\frac{2(\beta-1)s(\xi)}{\xi(1-s(\xi))},\ \ (\xi\in J^*),$$

which, upon integration, yields

$$\log(\xi g(\xi))=2(\beta-1)\int_0^{\xi}\frac{s(t)}{t(1-s(t))}dt. \tag{25}$$

Now, we can easily derive assertion (23) of Theorem 2 from (25). □

Convolution property for the class $\widetilde{\mathcal{M}(\beta,q)}$.

Theorem 3. *Let* $g \in \widetilde{\mathcal{M}(\beta, q)}$*, then*

$$g(\xi) * \left\{ \begin{array}{c} \left(\frac{-\xi^{-1}+2}{(1-\xi)^2}\right)(1-e^{i\theta}) \\ +\frac{\xi^{-1}}{1-\xi}\{1-(2\beta-1)e^{i\theta}\} \end{array} \right\} \neq 0, \tag{26}$$

where $\xi \in J^*, 0 < \theta < 2\pi$.

Proof. Let $g \in \widetilde{\mathcal{M}(\beta, q)}$, and by using Lemma 1, we see that (16) holds true, which implies that

$$-\frac{\xi \widetilde{D_q} g(\xi)}{g(\xi)} \neq \frac{1-(2\beta-1)e^{i\theta}}{1-e^{i\theta}}, \quad (\xi \in J,\ 0 < \theta < 2\pi). \tag{27}$$

After some simple calculation, we have

$$\left(1-e^{i\theta}\right)\xi \widetilde{D_q} g(\xi) + \left\{1-(2\beta-1)e^{i\theta}\right\} g(\xi) \neq 0,\ (\xi \in J^*,\ 0 < \theta < 2\pi). \tag{28}$$

We note that

$$g(\xi) = g(\xi) * \left(\xi^{-1} + \ldots + \frac{1}{\xi} + 1 + \frac{\xi}{1-\xi}\right) = g(\xi) * \frac{\xi^{-1}}{1-\xi}, \tag{29}$$

and

$$\begin{aligned} \xi \widetilde{D_q} g(\xi) &= g(\xi) * \left(-\xi^{-1} + \ldots - \frac{1}{\xi} + \frac{\xi}{(1-\xi)^2}\right) \\ &= g(\xi) * \frac{-\xi^{-1}+2}{(1-\xi)^2}. \end{aligned} \tag{30}$$

Thus, by using (28)–(30), we can easily complete the assertion (26) of Theorem 3. □

Coefficient estimates of functions belonging to the class $\widetilde{\mathcal{M}(\beta, q)}$.

Theorem 4. *Let*

$$g(\xi) = \xi^{-1} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \widetilde{\mathcal{M}(\beta, q)},\ m \in \mathbb{N}_0. \tag{31}$$

Then,

$$|a_{m+1}| \leq \left\{ \begin{array}{c} \frac{2(\beta-1)}{2\beta+\widetilde{[m]_q}} \\ \times \prod_{k=0}^{m} \frac{2\beta+\widetilde{[k]_q}}{2+\widetilde{[k]_q}},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \end{array} \right\}. \tag{32}$$

Proof. Suppose that

$$v(\xi) = 1 + v_1\xi + v_2\xi^2 + \ldots \in \mathcal{P}$$

is defined by (2) and

$$\xi \widetilde{D_q} g(\xi) = (\beta-1)g(\xi)v(\xi) - \beta g(\xi). \tag{33}$$

We now find from (31) and (33) that

$$\begin{aligned} & -\xi^{-2} + \widetilde{[1]}_q a_1 \xi + 2a_2\xi^2 + \ldots + \widetilde{[1+m]}_q a_{m+1}\xi^{m+1} + \ldots \\ = \; & (\beta - 1)\Big(\xi^{-1} + a_1\xi + \ldots + a_{m+1}\xi^{m+1} + \ldots\Big) \\ & \times\Big(1 + v_1\xi + v_2\xi^2 + \ldots + v_m\xi^m + \ldots\Big) \\ & -\beta\Big(\xi^{-1} + a_1\xi^1 + \xi^2 + \ldots + a_{m+1}\xi^{m+1} + \ldots\Big). \end{aligned} \tag{34}$$

By evaluating the coefficients of ξ^{m+1} on both sides of (34), we obtain

$$\widetilde{[m+1]}_q a_{m+1} = (\beta - 1)(v_{2m+1} + a_1 v_m + \ldots + a_{m+1}) - \beta a_{m+1}. \tag{35}$$

On the other hand, it is well known that

$$|v_k| \leq 2, \;\; k \in \mathbb{N}. \tag{36}$$

Combining (35) and (36), we easily obtain

$$|a_1| \leq \beta - 1$$

and

$$|a_{m+1}| \leq \frac{2(\beta-1)}{2 + \widetilde{[m+1]}_q}\left(1 + \sum_{k=0}^{m-1} |a_{k+1}|\right), \quad m \in \mathbb{N}. \tag{37}$$

Suppose that $\beta > 1$. We define the sequence $\{A_{m+1}\}_{m=0}^{\infty}$ as follows:

$$\begin{cases} A_1 = \beta - 1, & m = 0 \\ A_{m+1} = \frac{2(\beta-1)}{2+\widetilde{[1+m]}_q}\left(1 + \sum\limits_{k=0}^{m-1} A_{k+1}\right), & m \in \mathbb{N}. \end{cases} \tag{38}$$

In order to prove that

$$|a_{m+1}| \leq A_{m+1}, \;\; m \in \mathbb{N}_0.$$

Using the principle of mathematical induction cannot verify that

$$|a_1| \leq A_1 = \beta - 1.$$

Thus, assuming that

$$|a_{i+1}| \leq A_{i+1}, \;\; i = 0, 1, \ldots m, \; m \in \mathbb{N}_0,$$

we find from (37) and (38) that

$$\begin{aligned} |a_{m+2}| \;\; & \leq \;\; \frac{2(\beta-1)}{2 + \widetilde{[m+1]}_q}\left(1 + \sum_{k=0}^{m} |a_{k+1}|\right) \\ & \leq \;\; \frac{2(\beta+\lambda)}{2 + \widetilde{[m+1]}_q}\left(1 + \sum_{k=0}^{m} A_{k+1}\right) \\ & = \;\; A_{m+2}, \;\; (m \in \mathbb{N}_0). \end{aligned}$$

Therefore, we have

$$|a_{m+1}| \leq A_{m+1}, \;\; (m \in \mathbb{N}_0). \tag{39}$$

From Lemma 5 and (38), for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we know that

$$A_{m+1} = \frac{2(\beta-1)}{2\beta + \widetilde{[m]}_q} \prod_{k=0}^{m} \frac{2\beta + \widetilde{[k]}_q}{2 + \widetilde{[k]}_q}. \tag{40}$$

From (39) and (40), we obtain the coefficient estimates (32) of Theorem 4. □

Sufficient condition for the meromorphic q-starlike functions for the class $\widetilde{\mathcal{M}(\beta,q)}$. Firstly, we find the bounds of β for the meromorphic q-starlikeness of

$$g(\xi) = \xi^{-1} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \widetilde{\mathcal{M}(\beta,q)},\ m \in \mathbb{N}.$$

For this, we consider the inequality

$$\sum_{n=2}^{\infty} \widetilde{[n]}_q |a_n| \le \sum_{n=2}^{\infty} \frac{\mu\left(\widetilde{[n]}_q + \beta\right)}{\beta - 1}|a_n| \le \mu,\ (0 < \mu \le 1).$$

The above inequality can be written as

$$\sum_{n=2}^{\infty} \left\{ \left(\mu - \widetilde{[n]}_q\right)\beta + (1+\mu)\widetilde{[n]}_q \right\} |a_n| \ge 0.$$

Now, we let

$$\mathcal{F}(n,\mu,\beta,q) = \left(\mu - \widetilde{[n]}_q\right)\beta + (1+\mu)\widetilde{[n]}_q.$$

Thus, by Lemma 3, we conclude that if $g \in \mathcal{M}$, satisfies the condition

$$\sum_{n=2}^{\infty} \mathcal{F}(n,\mu,\beta,q)|a_n| \ge 0, \quad \beta > 1, \tag{41}$$

then $g \in \widetilde{\mathcal{MS}}^*(q)$.

Theorem 5. *If*

$$g(\xi) = \xi^{-1} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \mathcal{M},\ m \in \mathbb{N}$$

satisfies the inequality

$$\sum_{n=2}^{\infty} \left(\widetilde{[n]}_q + \beta\right)|a_n| \le \beta - 1,$$

where

$$1 < \beta \le 1 + \frac{\left(\widetilde{[n]}_q + 1\right)\mu}{\widetilde{[n]}_q - \mu},\ \ n \in \mathbb{N},\ 0 < \mu \le 1, \tag{42}$$

then $g \in \widetilde{\mathcal{MS}}^*(q)$.

Proof. Suppose that (42) holds true. We can easily find that

$$\mathcal{F}(n,\mu,\beta,q) = \left(\mu - \widetilde{[n]}_q\right)\beta + (1+\mu)\widetilde{[n]}_q \ge 0.$$

Hence, (41) also holds true. Therefore, we conclude that $g \in \widetilde{\mathcal{MS}}^*(q)$. □

Properties of the Function Class $\widetilde{\mathcal{N}(\beta,q)}$

Theorem 6. *Let $0 \leq \eta \leq 1$ and $\beta > 1$, let a function $g \in \mathcal{M}$ satisfy the inequality*

$$\sum_{n=0}^{\infty} \left|\widetilde{[n]_q}\right| \left(\widetilde{[n]_q} + (2\beta - \eta) + \left|\widetilde{[n]_q} + \eta\right|\right) |a_n| \leq 2(\beta - 1),$$

where $0 \leq \eta \leq 1$ and $\beta > 1$, then, $g \in \widetilde{\mathcal{N}(\beta,q)}$.

Proof. By using a similar method to the proof of Theorem 1, we can prove Theorem 6. □

Theorem 7. *Let $g \in \widetilde{\mathcal{N}(\beta,q)}$, then*

$$g(\xi) = -\int_{\xi_0}^{\xi} u^{-2}.\exp\left(2(\beta - 1)\int_0^u \frac{s(t)}{t(1 - s(t))}dt\right)du,$$

where $\xi, u \in J^*$, $0 < |\xi_0| < |\xi|$, $g(\xi_0) = 0$, and s is analytic in J and given in (17).

Proof. By using a similar method to the proof of Theorem 2, we can prove Theorem 7. □

Theorem 8. *Let*

$$g(\xi) = \xi^{-1} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \widetilde{\mathcal{N}(\beta,q)},\ m \in \mathbb{N}_0,$$

then

$$\left\{|a_{m+1}| \leq \frac{2(\beta - 1)}{\left(\widetilde{[m]_q} + 1\right)\left(2\beta + \widetilde{[m]_q}\right)} \prod_{k=0}^{m} \frac{2\beta + \widetilde{[k]_q}}{2 + \widetilde{[k]_q}},\ \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right\}.$$

Proof. By using a similar method to the proof of Theorem 4, we can prove Theorem 8. □

Theorem 9. *If*

$$g(\xi) = \xi^{-1} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \mathcal{M},\ m \in \mathbb{N}$$

satisfies the inequality

$$\sum_{n=2}^{\infty} \widetilde{[n]_q}\left(\widetilde{[n]_q} + \beta\right)|a_n| \leq \beta - 1,$$

where β is given by (42), then $g \in \mathcal{MK}(q)$.

Proof. By using a similar method to the proof of Theorem 5, we can prove Theorem 9. □

Example 3. *Let $q \to 1-$ and $\beta = 1.5$, and let*

$$g(\xi) = \frac{1}{\xi} + \sum_{l=m}^{\infty} a_{l+1}\xi^{l+1} \in \mathcal{M},\ m \in \mathbb{N},$$

which satisfies the inequality

$$\sum_{n=2}^{\infty} n(n + 1.5)|a_n| \leq 0.5,$$

then $g \in \mathcal{MK}(q \to 1-)$.

4. Conclusions

Many researchers have recently established new subclasses of q-starlike and q-convex functions using quantum calculus in the field of Geometric Function Theory. In contrast, we defined some new subclasses of q-starlike and q-convex functions by using the idea of symmetric q-calculus operator theory to discover the unique symmetric q-difference operator for meromorphic functions. This paper contains three sections. In Section 1, we discussed some basic introductions to the previous literature, and the basic definitions of q-calculus, the symmetric q-difference operator, and other definitions. In Section 2, we proved some new preliminary results, which will be used to prove our main results. In Section 3, we proved our main results, such as convexity, compactness, the radii of q-starlike and q-convex functions, and necessary and sufficient conditions. Many of these articles make use of complete characteristics by establishing certain new subclasses of analytic functions connected with symmetric quantum calculus, which may be extended and explored using the approach presented in this article (see [27–34]).

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