


Article

The Hyperbolic Ptolemy's Theorem in the Poincaré Ball Model of Analytic Hyperbolic Geometry

Abraham A. Ungar 

Department of Mathematics, North Dakota State University, Fargo, ND 58105, USA; abraham.ungar@ndsu.edu

Abstract: Ptolemy's theorem in Euclidean geometry, named after the Greek astronomer and mathematician Claudius Ptolemy, is well known. We translate Ptolemy's theorem from analytic Euclidean geometry into the Poincaré ball model of analytic hyperbolic geometry, which is based on the Möbius addition and its associated symmetry gyrogroup. The translation of Ptolemy's theorem from Euclidean geometry into hyperbolic geometry is achieved by means of hyperbolic trigonometry, called *gyrotrigonometry*, to which the Poincaré ball model gives rise, and by means of the duality of trigonometry and gyrotrigonometry.

Keywords: Ptolemy's theorem; Poincaré ball model; hyperbolic geometry; Möbius addition; gyrogroups; gyrovector spaces; gyrotrigonometry

MSC: 51M10; 83A05



Citation: Ungar, A.A. The Hyperbolic Ptolemy's Theorem in the Poincaré Ball Model of Analytic Hyperbolic Geometry. *Symmetry* **2023**, *15*, 1487. <https://doi.org/10.3390/sym15081487>

Academic Editors: Stanisława Kanas and Sergei D. Odintsov

Received: 21 June 2023

Revised: 10 July 2023

Accepted: 25 July 2023

Published: 27 July 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Analytic hyperbolic geometry represents the study of hyperbolic geometry by Lobachevsky and Bolyai; it has been studied analytically since 1988 [1], in full analogy with the analytical study of Euclidean geometry [1–3]. In the analytical study of hyperbolic geometry, the Einstein addition and Möbius addition capture analogies with the vector addition, as follows:

- (1) The vector addition admits scalar multiplication, giving rise to vector spaces which, in turn, form the algebraic setting for analytic Euclidean geometry. In full analogy,
- (2) The Einstein addition, which pertains to relativistically admissible velocities, also permits scalar multiplication, giving rise to Einstein gyrovector spaces, which, in turn, form the algebraic setting for the Klein ball model of analytic hyperbolic geometry [1–5]. Accordingly, the Klein model of hyperbolic geometry is also known as the relativistic model of hyperbolic geometry [6,7]. Furthermore, in full analogy,
- (3) the Möbius addition [8] admits scalar multiplication, giving rise to Möbius gyrovector spaces, which, in turn, form the algebraic setting for the Poincaré ball model of analytic hyperbolic geometry [1–4].

In the same way that the Euclidean vector plane admits trigonometry, each Einstein gyrovector plane and Möbius gyrovector plane admits hyperbolic trigonometry, called *gyrotrigonometry*.

Ptolemy's theorem in Euclidean plane geometry, named after the Greek astronomer and mathematician Claudius Ptolemy, is well known.

- (1) In [6], we presented the well-known proof of Ptolemy's theorem in terms of the standard trigonometry of analytic Euclidean plane geometry. In particular, the associated *law of cosines* was employed.
- (2) In full analogy, in [6], we discovered the hyperbolic Ptolemy's theorem in the Klein (relativistic) model of analytic hyperbolic plane geometry. The proof of the resulting hyperbolic Ptolemy's theorem is obtained by means of the gyrotrigonometry that the

Klein model of analytic hyperbolic geometry admits. In particular, the associated *law of gyrocosines* was employed.

- (3) In full analogy, in this article, we discover the hyperbolic Ptolemy's theorem in the Poincaré ball model of analytic hyperbolic plane geometry. The proof of the resulting hyperbolic Ptolemy's theorem is obtained by means of the gyrotrigonometry that the Poincaré model of analytic hyperbolic geometry admits. In particular, the associated *law of gyrocosines* is employed.

Finally, we present an elegant application of the hyperbolic Ptolemy's theorem to *gyrodiametric* gyrotriangles, which results in a Pythagorean-like identity.

For a fruitful reading of this article, familiarity with Ref. [6] is recommended, where important background information about gyrogroups, gyrovectors spaces, and gyrotrigonometry is reviewed. In particular, the duality of trigonometry–gyrotrigonometry is reviewed and illustrated.

2. Möbius Addition and Scalar Multiplication

Let $s > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n -space, $n \in \mathbb{N}$, endowed with the common vector addition, $+$, and inner product, \cdot , and let \mathbb{R}_s^n be the s -ball given by

$$\mathbb{R}_s^n = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}. \quad (1)$$

Möbius addition is a binary operation, \oplus , in the s -ball \mathbb{R}_s^n , given by

$$\mathbf{u} \oplus \mathbf{v} = \frac{(1 + \frac{2}{s^2} \mathbf{u} \cdot \mathbf{v} + \frac{1}{s^2} \|\mathbf{v}\|^2) \mathbf{u} + (1 - \frac{1}{s^2} \|\mathbf{u}\|^2) \mathbf{v}}{1 + \frac{2}{s^2} \mathbf{u} \cdot \mathbf{v} + \frac{1}{s^4} \|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \quad (2)$$

where \cdot and $\|\cdot\|$ are the inner product and the norm that the ball \mathbb{R}_s^n inherits from its ambient space \mathbb{R}^n [8].

Note that in the Euclidean limit, $s \rightarrow \infty$, the s -ball \mathbb{R}_s^n expands to the whole of its ambient space \mathbb{R}^n , and Möbius addition tends to the common vector addition, $\mathbf{u} + \mathbf{v}$, in \mathbb{R}^n .

Möbius addition in \mathbb{R}_s^n , $n \geq 2$, is neither commutative nor associative. Hence, the pair (\mathbb{R}_s^n, \oplus) does not form a group. However, it forms a *gyrocommutative gyrogroup* as shown, for instance, in [2,3]. A gyrogroup is a group-like structure, a prominent concrete example of which is given by the Einstein addition of relativistically admissible velocities, studied, for instance, in [2,3,6]. The elegant road from Möbius to gyrogroups is revealed in [8].

The definition of gyrogroups, some of which are gyrocommutative, is presented, for instance, in [2,3,5,6]. Gyrogroups form a natural generalization of groups [9], giving rise to useful applications, such as in [10–28].

Möbius addition, \oplus , admits scalar multiplication, \otimes , turning Möbius gyrogroups into Möbius gyrovectors spaces $(\mathbb{R}_s^n, \oplus, \otimes)$. Möbius scalar multiplication is given by ([1] Section 3.6), ([2] Section 8.12), ([3], Section 6.14)

$$\begin{aligned} r \otimes \mathbf{v} &= s \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh\left(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \quad (3)$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{v} \neq \mathbf{0}$; and $r \otimes \mathbf{0} = \mathbf{0}$.

Gyrovectors spaces are naturally generalized vector spaces, the definition of which is presented, for instance, in [2,3,6].

The resulting (i) Möbius gyrovectors spaces form the algebraic setting for the Cartesian Poincaré ball model of analytic hyperbolic geometry, just as (ii) Einstein gyrovectors spaces

form the algebraic setting for the Cartesian Klein ball model of analytic hyperbolic geometry, just as (iii) vector spaces form the algebraic setting for the standard Cartesian model of analytic Euclidean geometry [1]. Interesting applications of gyrovector spaces are found, for instance, in [29–40].

Thus, the notions of gyrogroups and gyrovector spaces preserve the flavor of their classical counterparts, groups, and vector spaces [1]. In particular, in the same way that groups are linked to symmetry in geometry, gyrogroups are linked to symmetry in gyrogeometry. In this paper, the gyro-counterpart of Euclidean geometry is the Poincaré model of hyperbolic geometry.

Gyrovector planes admit hyperbolic trigonometry, called *gyrotrigonometry*, just as vector planes admit common trigonometry. The law of gyrocosines in the gyrotrigonometry of the Klein model of hyperbolic geometry is employed in [6] for the proof of Ptolemy's theorem in the Klein model. Similarly, the law of gyrocosines in the gyrotrigonometry of the Poincaré ball model of hyperbolic geometry is employed in this article for the proof of Ptolemy's theorem in the Poincaré model.

3. Gyrotrigonometry in Möbius Gyrovector Planes and Its Law of Gyrocosines

Möbius addition and scalar multiplication allow Möbius *gyrolines* to be expressed analytically in a way that is fully analogous to lines in analytic Euclidean geometry. The unique Euclidean line that passes through two distinct points, $A, B \in \mathbb{R}^n$, is given analytically by

$$A + (-A + B)t, \quad (4)$$

$t \in \mathbb{R}$. In full analogy, the unique Möbius gyroline that passes through two distinct points, $A, B \in \mathbb{R}_s^n$, in a Möbius gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, is given analytically by

$$A \oplus (\ominus A \oplus B) \otimes t, \quad (5)$$

$t \in \mathbb{R}$, where $\ominus A = -A$ is depicted in Figure 1 for $n = 2$.

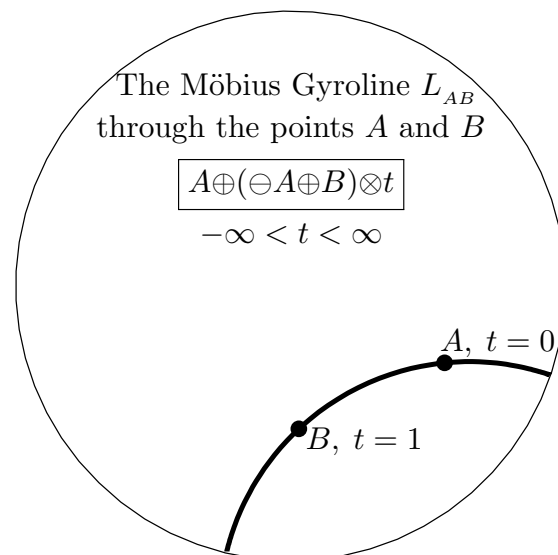


Figure 1. In the gyroformalism of analytic hyperbolic geometry, expressions that describe hyperbolic geometric objects take graceful forms analogous to their Euclidean counterparts. This is illustrated here, where the unique gyroline in a Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ through two given points $A, B \in \mathbb{R}_s^2$ is shown. When the gyroline parameter $t \in \mathbb{R}$ runs from $-\infty$ to ∞ , the point $P(t) = A \oplus (\ominus A \oplus B) \otimes t$ runs over the gyroline L_{AB} . In particular, at “time” $t = 0$, the point is at $P(0) = A$, and, owing to the left cancellation law of Möbius addition, at “time” $t = 1$, the point is at $P(1) = B$. The Möbius gyroline equation is shown in the box. The analogies it shares with the Euclidean straight line equation in the vector space approach and Euclidean geometry are obvious.

A Möbius gyroline (5) is a circular arc that approaches the boundary of the s -ball \mathbb{R}_s^n orthogonally. It passes at point A when $t = 0$, and at point B when $t = 1$, and it forms a geodesic in the Poincaré ball model of hyperbolic geometry [4].

The gyrosegment AB that links the two distinct points $A, B \in \mathbb{R}_s^n$ is given by (5) with $0 \leq t \leq 1$. The gyrolength of a gyrosegment AB is

$$|AB|_h := \|\ominus A \oplus B\|, \tag{6}$$

$A, B \in \mathbb{R}_s^n$, just as the length of a segment AB is

$$|AB|_e := \|\ominus A + B\|, \tag{7}$$

$A, B \in \mathbb{R}^n$.

A gyrotriangle ABC whose gyrosides are the gyrosegments AB, BC , and AC in \mathbb{R}_s^n , along with its gyroangles, is depicted in Figure 2. The gyrolengths of the gyrosides of gyrotriangle ABC are

$$\begin{aligned} a &:= |BC|_h = \|\ominus B \oplus C\| \\ b &:= |AC|_h = \|\ominus A \oplus C\| \\ c &:= |AB|_h = \|\ominus A \oplus B\|, \end{aligned} \tag{8}$$

as shown in Figure 2.

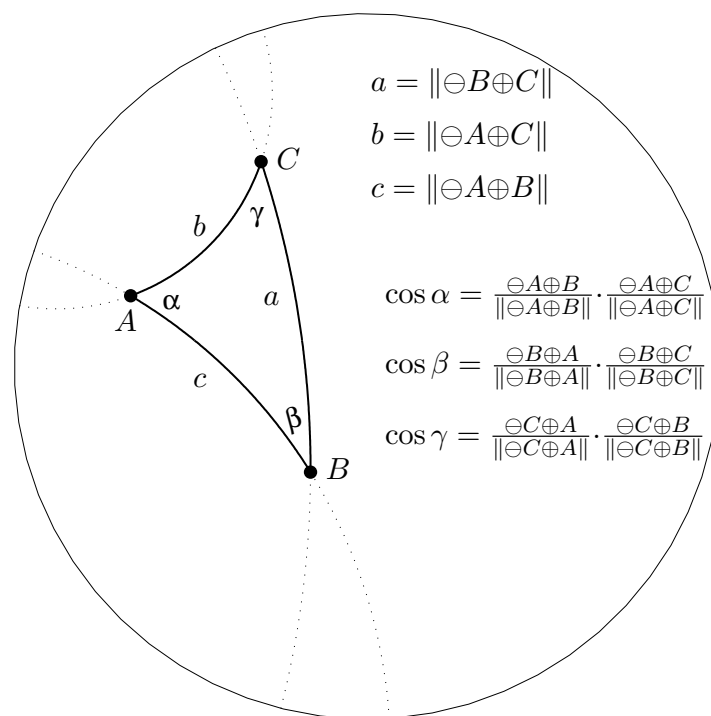


Figure 2. A gyrotriangle ABC in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$, along with its (i) gyrovertices A, B, C , (ii) gyroside gyrolengths a, b, c , and (iii) gyroangles α, β, γ and their gyrocosines.

Gyrotrigonometry in Möbius gyrovector spaces is studied, for instance, in ([3] Section 8.5). In gyrotrigonometry, the gyrocosine of a gyroangle $\alpha = \angle BAC$, shown in Figure 2, is given by

$$\cos \alpha = \frac{\|\ominus A \oplus B\| \cdot \|\ominus A \oplus C\|}{\|\ominus A \oplus B\| \cdot \|\ominus A \oplus C\|}, \tag{9}$$

just as in trigonometry, the cosine of an angle, α , in the Euclidean counterpart of Figure 2, is given by

$$\cos \alpha = \frac{-A + B}{\| -A + B \|} \cdot \frac{-A + C}{\| -A + C \|} \tag{10}$$

Accordingly, viewing $\cos \alpha$ trigonometrically yields (10), while viewing $\cos \alpha$ gyrotrigonometrically yields (9). The resulting *trigonometry–gyrotrigonometry duality* is illustrated in ([6] Section 11). The usefulness of the analogies that (5) and (4) share suggests that the analogies that (9) and (10) share will prove useful as well. We will see in this article that this is, indeed, the case.

An important trigonometric identity that we will view gyrotrigonometrically is

$$\sin \frac{\alpha}{2} \sin \frac{\gamma}{2} + \sin \frac{\beta}{2} \sin \frac{\delta}{2} = \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \tag{11}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$, where $\delta = 2\pi - (\alpha + \beta + \gamma)$. Identity (11) can be realized not only trigonometrically (i) in the Euclidean plane, but also gyrotrigonometrically (ii) in the Einstein gyrovector plane, and (iii) in the Möbius gyrovector plane. These three distinct realizations of (11) are consistent, owing to the duality of trigonometry and gyrotrigonometry, as explained in [6].

Indeed, (i) realizing (11) in trigonometry gives rise to Ptolemy’s theorem in Euclidean geometry, as shown in [6]; (ii) realizing (11) in Einstein gyrotrigonometry gives rise to Ptolemy’s theorem in the Klein ball (relativistic) model of hyperbolic geometry, as shown in [6]; and (iii) realizing (11) in Möbius gyrotrigonometry gives rise to Ptolemy’s theorem in the Poincaré ball model of hyperbolic geometry, as we will see in Section 4.

Having the gyrotriangle notation in Figure 2, we are now in the position to state the *law of gyrocosines* in a Möbius gyrovector plane, verified in ([2] Section 8.4) and ([3] Section 8.5).

Theorem 1 (The law of gyrocosines in Möbius gyrovector spaces). *Let ABC be a gyrotriangle in a Möbius gyrovector plane $(\mathbb{R}_s^n, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{R}_s^2$, side gyrolengths $0 < a, b, c < s$, and gyroangles α, β, γ , as shown in Figure 2. Then, the gyrosides and gyroangles of the gyrotriangle satisfy the law of gyrocosines*

$$c^2 = \frac{a^2 + b^2 - 2ab \cos \gamma}{1 + a_s^2 b_s^2 - 2a_s b_s \cos \gamma} \tag{12}$$

and, equivalently,

$$\cos \gamma = \frac{a^2 + b^2 - c^2 - a_s^2 b_s^2 c^2}{2ab(1 - c_s^2)} \tag{13}$$

where $a_s = a/s, b_s = b/s, c_s = c/s$, with similar relations involving the other gyrosides and gyroangles of the gyrotriangle.

Note that in the Euclidean limit, $s \rightarrow \infty$, the law of gyrocosines (12) descends to the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos \gamma \tag{14}$$

as expected. Also, note that in the context of Euclidean geometry, $\cos \gamma$ in (14) is viewed trigonometrically, as in (10), while in the context of hyperbolic geometry, $\cos \gamma$ in (12) and (13) is viewed gyrotrigonometrically, as in (9) and Figure 2.

4. Ptolemy’s theorem in the Poincaré Ball Model of Hyperbolic Geometry

Applying the law of gyrocosines (13) to the O -gyrovertex gyroangle $\alpha = \angle AOB$ of gyrotriangle ABO , shown in Figure 3, yields

$$\cos \alpha = \frac{2r^2 - |AB|_h^2 - \frac{r^4}{s^4} |AB|_h^2}{2r^2(1 - \frac{|AB|_h^2}{s^2})} \tag{15}$$

noting that a, b, c , and γ in (13) are, respectively, realized in (15) by $r, r, |AB|_h$, and α .

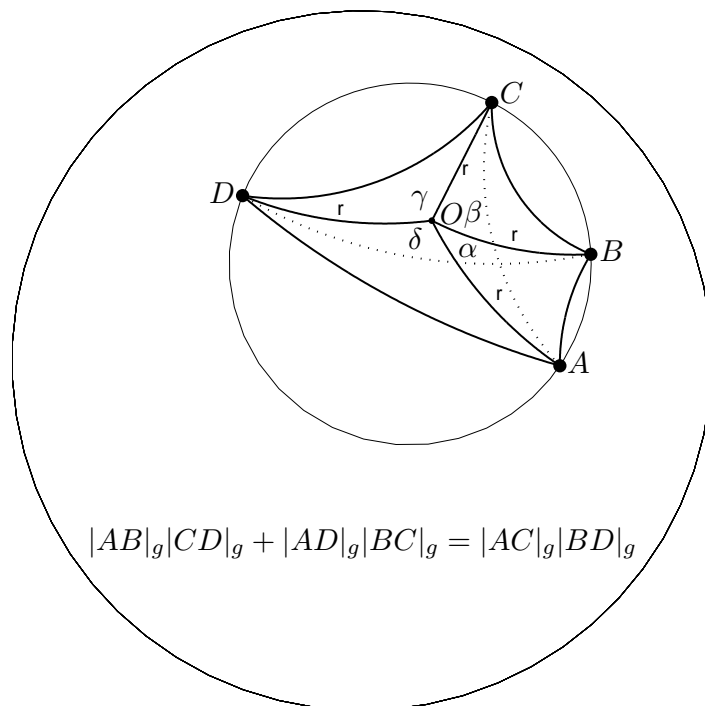


Figure 3. Ptolemy’s theorem in the hyperbolic plane regulated by the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. A gyrocyclic gyroquadrilateral $ABCD$ inscribed in its circumgyrocircle gyrocentered at its circumgyrocenter O , with gyroradius $r = \|\ominus O \oplus A\| = \|\ominus O \oplus B\| = \|\ominus O \oplus C\| = \|\ominus O \oplus D\|$ is shown. The O -gyrovertex gyroangles α, β, γ , and δ satisfy the following equation: $\alpha + \beta + \gamma + \delta = 2\pi$. The Hyperbolic Ptolemy’s theorem is fully analogous to its Euclidean counterpart, asserting that $|AB|_g|CD|_g + |AD|_g|BC|_g = |AC|_g|BD|_g$, where $|AB|_g$, etc., is defined by (20) along with (6).

We now employ the trigonometric (and, hence, gyrotrigonometric) identity $\sin^2(\alpha/2) = (1 - \cos \alpha)/2$, obtaining from (15) the equation

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} = \frac{(s^2 - r^2)^2}{4s^2r^2(s^2 - |AB|_h^2)} |AB|_h^2 \tag{16}$$

so that, finally,

$$\sin \frac{\alpha}{2} = \frac{s^2 - r^2}{2sr\sqrt{s^2 - |AB|_h^2}} |AB|_h = \frac{1 - r_s^2}{2r} \gamma_{|AB|_h} |AB|_h \tag{17}$$

where $r_s = r/s$, and where $\gamma_{|AB|_h}$ is the Lorentz gamma factor of $|AB|_h$, given by

$$\gamma_{|AB|_h} = \frac{1}{\sqrt{1 - \frac{|AB|_h^2}{s^2}}} \tag{18}$$

Repeating the result in (17) for the six O -gyrovertex gyrotriangle gyroangles $\alpha, \beta, \gamma, \delta, \alpha + \beta$, and $\beta + \gamma$ in Figure 3 yields the following six equations:

$$\begin{aligned}
\sin \frac{\alpha}{2} &= \frac{1-r_s^2}{2r} \gamma_{|AB|_h} |AB|_h =: \frac{1-r_s^2}{2r} |AB|_g \\
\sin \frac{\beta}{2} &= \frac{1-r_s^2}{2r} \gamma_{|BC|_h} |BC|_h =: \frac{1-r_s^2}{2r} |BC|_g \\
\sin \frac{\gamma}{2} &= \frac{1-r_s^2}{2r} \gamma_{|CD|_h} |CD|_h =: \frac{1-r_s^2}{2r} |CD|_g \\
\sin \frac{\delta}{2} &= \frac{1-r_s^2}{2r} \gamma_{|AD|_h} |AD|_h =: \frac{1-r_s^2}{2r} |AD|_g \\
\sin \frac{\alpha+\beta}{2} &= \frac{1-r_s^2}{2r} \gamma_{|AC|_h} |AC|_h =: \frac{1-r_s^2}{2r} |AC|_g \\
\sin \frac{\beta+\gamma}{2} &= \frac{1-r_s^2}{2r} \gamma_{|BD|_h} |BD|_h =: \frac{1-r_s^2}{2r} |BD|_g
\end{aligned} \tag{19}$$

where

$$|AB|_g = \gamma_{|AB|_h} |AB|_h \tag{20}$$

and so forth. We call $|AB|_g$ the g -modified gyrolength of gyroside AB , and so forth.

Substituting (19) into the gyrotrigonometric identity (11) yields the result (21) of Ptolemy's theorem in the Poincaré ball model of hyperbolic geometry.

Theorem 2 (Ptolemy's theorem in the Poincaré ball model of hyperbolic geometry). *Let $ABCD$ be a gyrocyclic gyroquadrilateral, shown in Figure 3. Then, the product of the g -modified gyrodiagonals equals the sum of the products of the g -modified opposite gyrosides; that is*

$$|AB|_g |CD|_g + |AD|_g |BC|_g = |AC|_g |BD|_g. \tag{21}$$

Clearly, in the Euclidean limit, $s \rightarrow \infty$, the hyperbolic Ptolemy's theorem (21) descends to its Euclidean counterpart, studied in [6].

5. Gyrodiametric Gyrotriangles

Definition 1. *A gyrotriangle is gyrodiametric if one of its gyrosides coincides with a gyrodiameter of its circumgyrocircle.*

In Euclidean geometry, a diametric triangle is right-angled, with the angle opposite to the diametric side being $\pi/2$. In contrast, non-Euclidean gyrodiametric gyrotriangles are not right-gyroangled; however, they obey a Pythagorean-like identity, as Theorem 3 asserts.

Definition 2. *A gyrocyclic gyroquadrilateral is gyrodiametric if its gyrodiagonals intersect at its circumgyrocenter, as shown in Figure 4.*

A gyrodiametric gyroquadrilateral $ABCD$, shown in Figure 4, gives rise to two gyrodiametric gyrotriangles, ABC and ADC . In the notation shown in Figure 4, we clearly have

$$\begin{aligned}
\alpha &= \gamma \\
\beta &= \delta \\
\alpha + \beta &= \beta + \gamma.
\end{aligned} \tag{22}$$

Hence, by (19), the equations in (22) yield, respectively, the equations

$$\begin{aligned}
|AB|_g &= |CD|_g \\
|BC|_g &= |AD|_g \\
|AC|_g &= |BD|_g.
\end{aligned} \tag{23}$$

Hence, it follows from (21) that

$$|AB|_g^2 + |BC|_g^2 = |AC|_g^2 \tag{24}$$

where $|AB|_g$, $|BC|_g$, and $|AC|_g$ are the g -modified gyrolengths of the gyrosides of the gyrodiametric gyrotriangle ABC , as shown in Figure 4.

Formalizing the result in (24), we proved the following theorem, where we use the notation in Figure 4.

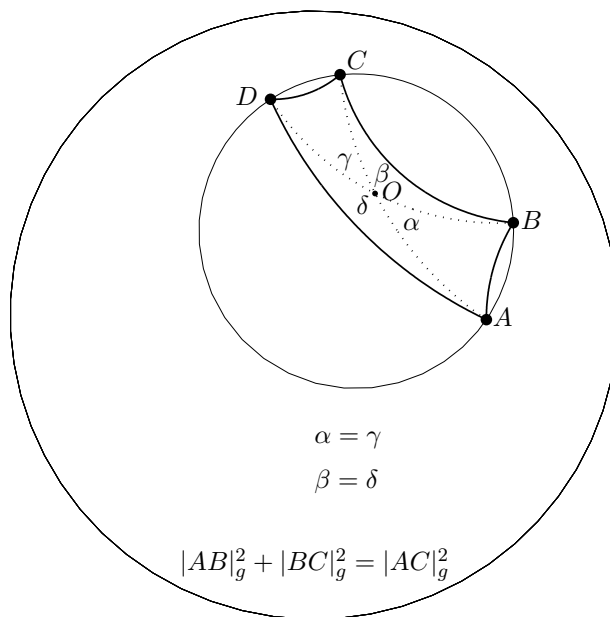


Figure 4. A gyrodiametric gyroquadrilateral in a Möbius gyrovector plane. The gyrocyclic gyroquadrilateral $ABCD$ of Figure 3 is depicted here in the special position where the two gyrodiameters, AC and BD , intersect at the gyroquadrilateral circumgyrocenter O , resulting in the two gyrodiametric gyrotriangles, ABC and ADC . The common gyroside, AC , of these gyrotriangles is a gyrodiameter of the gyroquadrilateral circumgyrocircle. The Pythagorean-like formula that the g -modified gyroside gyrolengths of a gyrodiametric gyrotriangle obey in the Poincaré ball model of hyperbolic geometry is shown.

Theorem 3 (Gyrodiametric Gyrotriangle Pythagorean-like Equation). *Let ABC be a gyrodiametric gyrotriangle in a Möbius gyrovector space $(\mathbb{R}_g^n, \oplus, \otimes)$, where AC is the gyrodiametric gyroside. Then, the gyrotriangle gyrosides obey the Pythagorean-like equation*

$$|AB|_g^2 + |BC|_g^2 = |AC|_g^2 \tag{25}$$

where

$$\begin{aligned} |AB|_g &= \gamma_{|AB|_h} |AB|_h \\ |AB|_h &= \|\ominus A \oplus B\|. \end{aligned} \tag{26}$$

Funding: This research received no external funding

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Ungar, A.A. *A Gyrovector Space Approach to Hyperbolic Geometry*; Morgan & Claypool Pub.: San Rafael, CA, USA, 2009.
2. Ungar, A.A. *Analytic Hyperbolic Geometry: Mathematical Foundations And Applications*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2005.

3. Ungar, A.A. *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2022.
4. Ungar, A.A. Gyrovector spaces and their differential geometry. *Nonlinear Funct. Anal. Appl.* **2005**, *10*, 791–834.
5. Ungar, A.A. *Hyperbolic Triangle Centers: The Special Relativistic Approach*; Springer: New York, NY, USA, 2010.
6. Ungar, A.A. Ptolemy's theorem in the relativistic model of analytic hyperbolic geometry. *Symmetry* **2023**, *15*, 649. [[CrossRef](#)]
7. Abraham A. Ungar. Hyperbolic trigonometry in the Einstein relativistic velocity model of hyperbolic geometry. *Comput. Math. Appl.* **2000**, *40*, 313–332.
8. Ungar, A.A. From Möbius to gyrogroups. *Amer. Math. Mon.* **2008**, *115*, 138–144. [[CrossRef](#)]
9. Suksumran, T. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem and isomorphism theorems. In *Essays in Mathematics and Its Applications: In Honor of Vladimir Arnold*; Rassias, T.M., Pardalos, P.M., Eds.; Springer: New York, NY, USA, 2016; pp. 369–437.
10. Ferreira, M. Factorizations of Möbius gyrogroups. *Adv. Appl. Clifford Algebr.* **2009**, *19*, 303–323. [[CrossRef](#)]
11. Ferreira, M.; Ren, G. Möbius gyrogroups: A Clifford algebra approach. *J. Algebra* **2011**, *328*, 230–253.
12. Ferreira, M. Gyrogroups in projective hyperbolic Clifford analysis. In *Hypercomplex Analysis and Applications*; Trends Math.; Birkhäuser/Springer: Basel, Switzerland, 2011; pp. 61–80.
13. Ferreira, M. Harmonic analysis on the Einstein gyrogroup. *J. Geom. Symmetry Phys.* **2014**, *35*, 21–60.
14. Ferreira, M. Harmonic Analysis on the Möbius Gyrogroup. *J. Fourier Anal. Appl.* **2015**, *21*, 281–317. [[CrossRef](#)]
15. Ferreira, M. Harmonic analysis on the proper velocity gyrogroup. *Banach J. Math. Anal.* **2017**, *11*, 21–49.
16. Ferreira, M.; Suksumran, T. Orthogonal gyrodecompositions of real inner product gyrogroups. *Symmetry* **2020**, *12*, 941 .
17. Ferreira, M. Gyroharmonic analysis on relativistic gyrogroups. *Math. Interdiscip. Res.* **2016**, *1*, 69–109.
18. Honma, T.; Hatori, O. A gyrogeometric mean in the einstein gyrogroup. *Symmetry* **2020**, *12*, 1333.
19. Wattanapan, J.; Atiponrat, W.; Suksumran, T. Embedding of strongly topological gyrogroups in path-connected and locally path-connected gyrogroups. *Symmetry* **2020**, *12*, 1817.
20. Suksumran, T. On metric structures of normed gyrogroups. In *Mathematical Analysis and Applications*; Springer: Cham, Switzerland, 2019; Volume 154, pp. 529–542.
21. Suksumran, T. Two theorems on direct products of gyrogroups. *AIMS Math.* **2023**, *8*, 6278–6287.
22. Suksumran, T. On Schreier graphs of gyrogroup actions. *J. Pure Appl. Algebra* **2022**, *226*, 107134.
23. Mahdavi, S.; Ashrafi, A.R.; Salahshour, M.A. Normal subgyrogroups of certain gyrogroups. *Quasigroups Relat. Syst.* **2022**, *30*, 115–122.
24. Li, P.; Shen, R. On topological gyrogroups. *Filomat* **2023**, *37*, 5087–5093.
25. Zhang, J.; Liang, Q. Extensions of some topological properties in topological gyrogroups. *Topol. Appl.* **2022**, *322*, 108328.
26. Lal, R.; Kakkar, V. Gyrogroups associated with groups. *Comm. Algebra* **2022**, *50*, 524–537.
27. Bao, M.; Lin, F. Feathered gyrogroups and gyrogroups with countable pseudocharacter. *Filomat* **2019**, *33*, 5113–5124.
28. Bao, M.; Lin, F. Submetrizability of strongly topological gyrogroups. *Houst. J. Math.* **2021**, *47*, 427–443.
29. Kim, S. Ordered gyrovector spaces. *Symmetry* **2020**, *12*, 1041. [[CrossRef](#)]
30. Kim, S. Distributivity on the gyrovector spaces. *Kyungpook Math. J.* **2015**, *55*, 13–20. [[CrossRef](#)]
31. Kim, S. Gyrovector spaces on the open convex cone of positive definite matrices. *Math. Interdisc. Res.* **2016**, *1*, 173–185.
32. Watanabe, K. A confirmation by hand calculation that the Möbius ball is a gyrovector space. *Nihonkai Math. J.* **2016**, *27*, 99–115.
33. Watanabe, K. Orthogonal gyroexpansion in Möbius gyrovector spaces. *J. Funct. Spaces* **2017**, *13*, 1518254. [[CrossRef](#)]
34. Watanabe, K. Continuous quasi gyrolinear functionals on möbius gyrovector spaces. *J. Funct. Spaces* **2020**, *2020*, 1950727. [[CrossRef](#)]
35. Watanabe, K. On quasi gyrolinear maps between möbius gyrovector spaces induced from finite matrices. *Symmetry* **2021**, *13*, 76. [[CrossRef](#)]
36. Abe, T.; Watanabe, K. Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space. *J. Math. Anal. Appl.* **2017**, *449*, 77–90. [[CrossRef](#)]
37. Abe, T.; Hatori, O. Generalized gyrovector spaces and a Mazur-Ulam theorem. *Publ. Math. Debr.* **2015**, *87*, 393–413. [[CrossRef](#)]
38. Hatori, O. Extension of isometries in generalized gyrovector spaces of the positive cones. In *Problems and Recent Methods in Operator Theory*; American Mathematical Society: Providence, RI, USA 2017; Volume 687, pp. 145–156.
39. Hatori, O. Examples and applications of generalized gyrovector spaces. *Results Math.* **2017**, *71*, 295–317.
40. Demirel, O. The first sharp gyrotriangle inequality in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$. *Forum Geom.* **2017**, *17*, 439–447.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.