

Article

New Results about Fuzzy Differential Subordinations Associated with Pascal Distribution

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Abstract: Based upon the Pascal distribution series $\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r a_j \zeta^j$, we can obtain a set of fuzzy differential subordinations in this investigation. We also newly obtain class $P_{q,\lambda}^{F,r,m}(\eta)$ of univalent analytic functions defined by the operator $\mathcal{N}_{q,\lambda}^{r,m}$, give certain properties for the class $P_{q,\lambda}^{F,r,m}(\eta)$ and also obtain some applications connected with a special case for the operator. New research directions can be taken on fuzzy differential subordinations associated with symmetry operators.

Keywords: convolution; fuzzy differential subordination; Pascal distribution

MSC: 30C45; 30C50; 30C80



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1. Introduction

Let $\mathcal{H}_m(\omega)$ represent the class of holomorphic and univalent functions on ω such that $\omega \subset \mathbb{C}$ and let $\mathcal{H}(\omega)$ denote the class of holomorphic functions on ω . The class of holomorphic functions in the open unit disk of the complex plane $\Lambda = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ is denoted in this study by a note $\mathcal{H}(\Lambda)$, with $B_\Lambda = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ standing as the unit disk's boundary. For $m \in \mathbb{N} = \{1, 2, \dots\}$, we define

$$\mathcal{H}_m[\gamma] = \left\{ Y \in \mathcal{H}(\Lambda) : Y(\zeta) = \gamma + \sum_{j=m+1}^{\infty} a_j \zeta^j, \zeta \in \Lambda \right\},$$

$$\mathcal{A}_m = \left\{ Y \in \mathcal{H}(\Lambda) : Y(\zeta) = \zeta + \sum_{j=m+1}^{\infty} a_j \zeta^j, \zeta \in \Lambda \right\} \quad \text{with } \mathcal{A}_1 = \mathcal{A}, \quad (1)$$

and

$$\mathcal{S} = \{Y \in \mathcal{A}_m : Y \text{ is a univalent function in } \Lambda\}.$$

We denote by

$$\mathcal{C} = \left\{ Y \in \mathcal{A}_m : \Re \left(1 + \frac{\zeta Y''(\zeta)}{Y'(\zeta)} \right) > 0, \zeta \in \Lambda \right\},$$

which is the set of convex functions on Λ .

Let Y_1 and Y_2 be analytic in Λ . Then Y_1 is subordinate to Y_2 written as $Y_1 \prec Y_2$ if there exists a Schwarz function ϕ , which is analytic in Λ with $\phi(0) = 0$ and $|\phi(\zeta)| < 1$

for all $\zeta \in \Lambda$ such that $Y_1(\zeta) = Y_2(\phi(\zeta))$. Furthermore, if the function Y_2 is univalent in Λ , then we have the following equivalence (see [1,2]):

$$Y_1(\zeta) \prec Y_2(\zeta) \Leftrightarrow Y_1(0) = Y_2(0) \text{ and } Y_1(\Lambda) \subset Y_2(\Lambda).$$

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

Definition 1 ([3]). Assume that $\mathcal{T} \neq \emptyset$ is a Fuzzy subset and $\mathcal{F} : \mathcal{T} \rightarrow [0, 1]$ is an application. A pair of $(\Lambda, \mathcal{F}_\Lambda)$, where $\mathcal{F}_\Lambda : \mathcal{T} \rightarrow [0, 1]$, and

$$\mathcal{R} = \{x \in \mathcal{T} : 0 < \mathcal{F}_\Lambda(x) \leq 1\} = \sup(\Lambda, \mathcal{F}_\Lambda),$$

a fuzzy subset. The fuzzy set $(\Lambda, \mathcal{F}_\Lambda)$ is called a function $\mathcal{F}_\Lambda$.

Let $Y, g \in \mathcal{H}(\omega)$ be denoted by

$$Y(\omega) = \{Y(\zeta) : 0 < \mathcal{F}_{Y(\omega)} Y(\zeta) \leq 1, \zeta \in \omega\} = \sup(Y(\omega), \mathcal{F}_{Y(\omega)}), \quad (2)$$

and

$$g(\omega) = \{g(\zeta) : 0 < \mathcal{F}_{g(\omega)} g(\zeta) \leq 1, \zeta \in \omega\} = \sup(g(\omega), \mathcal{F}_{g(\omega)}). \quad (3)$$

Proposition 1 ([4]). (i) If $(\mathcal{B}, \mathcal{F}_\mathcal{B}) = (\mathcal{U}, \mathcal{F}_\mathcal{U})$, then we have $\mathcal{B} = \mathcal{U}$, where $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_\mathcal{B})$ and $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U})$; (ii) if $(\mathcal{B}, \mathcal{F}_\mathcal{B}) \subseteq (\mathcal{U}, \mathcal{F}_\mathcal{U})$, then we have $\mathcal{B} \subseteq \mathcal{U}$, where $\mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_\mathcal{B})$ and $\mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U})$.

Definition 2 ([4]). Let $\zeta_0 \in \omega$ be a fixed point and let the functions $Y, g \in \mathcal{H}(\omega)$. The function Y is said to be fuzzy subordinate to g , and we write $Y \prec_{\mathcal{F}} g$ or $Y(\zeta) \prec_{\mathcal{F}} g(\zeta)$, which satisfies the following conditions:

- (i) $Y(\zeta_0) = g(\zeta_0)$;
- (ii) $\mathcal{F}_{Y(\omega)} Y(\zeta) \leq \mathcal{F}_{g(\omega)} g(\zeta)$, $\zeta \in \omega$.

Proposition 2 ([4]). Assume that $\zeta_0 \in \omega$ is a fixed point and the functions $Y, g \in \mathcal{H}(\omega)$. If $Y(\zeta) \prec_{\mathcal{F}} g(\zeta)$, $\zeta \in \omega$, then

- (i) $Y(\zeta_0) = g(\zeta_0)$
- (ii) $Y(\omega) \subseteq g(\omega)$, $\mathcal{F}_{Y(\omega)} Y(\zeta) \leq \mathcal{F}_{g(\omega)} g(\zeta)$, $\zeta \in \omega$,

where $Y(\omega)$ and $g(\omega)$ are defined by (2) and (3), respectively.

Definition 3 ([5]). Assume that $h \in \mathcal{S}$ and $\Phi : \mathbb{C}^3 \times \Lambda \rightarrow \mathbb{C}$, $\Phi(\alpha, 0, 0; 0) = h(0) = \alpha$. If p satisfies the requirements of the second-order fuzzy differential subordination and is analytic in Λ , with $p(0) = \alpha$,

$$\mathcal{F}_{\Phi(\mathbb{C}^3 \times \Lambda)} \Phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta) \leq \mathcal{F}_{h(\Lambda)} h(\zeta). \quad (4)$$

If q is a fuzzy dominant of the fuzzy differential subordination solutions, then p is said to be a fuzzy solution of the fuzzy differential subordination and satisfies

$$\mathcal{F}_{p(\Lambda)} p(\zeta) \leq \mathcal{F}_{q(\Lambda)} q(\zeta), \text{ i.e., } p(\zeta) \prec_{\mathcal{F}} q(\zeta), \zeta \in \Lambda,$$

for each and every p satisfying (4).

Definition 4. A fuzzy dominant $\tilde{q}$ that satisfies

$$\mathcal{F}_{\tilde{q}(\Lambda)} \tilde{q}(\zeta) \leq \mathcal{F}_{q(\Lambda)} q(\zeta),$$

then

$$\tilde{q}(\zeta) \prec_{\mathcal{F}} q(\zeta), \quad \zeta \in \Lambda.$$

The fuzzy best dominant of (4) is referred to for all fuzzy dominants.

Assume the function $\Omega \in \mathcal{A}_m$ is given by

$$\Omega(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \psi_j \zeta^j, \quad \zeta \in \Lambda.$$

The Hadamard (or convolution) product of Y and Ω is defined as

$$(Y * \Omega)(\zeta) := \zeta + \sum_{j=m+1}^{\infty} a_j \psi_j \zeta^j, \quad \zeta \in \Lambda.$$

A variable x is said to have the *Pascal distribution* if it takes the values $0, 1, 2, 3, \dots$ with the probabilities $(1-q)^r, \frac{qr(1-q)^r}{1!}, \frac{q^2r(r+1)(1-q)^r}{2!}, \frac{q^3r(r+1)(r+2)(1-q)^r}{3!}, \dots$, respectively, where q and r are called the parameters, and thus we have the probability formula

$$P(X = k) = \binom{k+r-1}{r-1} q^k (1-q)^r, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now we present a power series whose coefficients are Pascal distribution probabilities, i.e.,

$$\begin{aligned} Q_{q,m}^r(\zeta) &:= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} q^{j-1} (1-q)^r \zeta^j, \quad \zeta \in \Lambda, \\ (m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1). \end{aligned}$$

We easily determine from the ratio test that the radius of convergence of the above power series is at least $\frac{1}{q} \geq 1$; hence, $Q_{q,m}^r \in \mathcal{A}_m$.

We define the functions

$$\begin{aligned} \mathcal{M}_{q,\lambda}^{r,m}(\zeta) &:= (1-\lambda)Q_{q,m}^r(\zeta) + \lambda z \left(Q_{q,m}^r(\zeta) \right)' \\ &= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r \zeta^j, \quad \zeta \in \Lambda, \\ (m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1, \lambda \geq 0). \end{aligned}$$

El-Deeb and Bulboacă [6] introduced the linear operator $\mathcal{N}_{q,\lambda}^{r,m} : \mathcal{A}_m \rightarrow \mathcal{A}_m$ defined by

$$\begin{aligned} \mathcal{N}_{q,\lambda}^{r,m} Y(\zeta) &:= \mathcal{M}_{q,\lambda}^{r,m}(\zeta) * Y(\zeta) \\ &= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r a_j \zeta^j, \quad \zeta \in \Lambda, \\ (m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1, \lambda \geq 0), \end{aligned}$$

where Y is given by (1), and the symbol “ $*$ ” stands for the *Hadamard (or convolution) product*.

Remark 1. (i) For $m = 1$, the operator $\mathcal{N}_{q,\lambda}^{r,m}$ reduces to $\mathcal{I}_{q,\lambda}^r := \mathcal{N}_{q,\lambda}^{r,1}$, introduced and studied by El-Deeb et al. [7]; (ii) for $m = 1$ and $\lambda = 0$, the operator Q_q^r reduces to $Q_q^r := \mathcal{N}_{q,0}^{r,1}$, introduced and studied by El-Deeb et al. [7].

Using the operator $\mathcal{N}_{q,\lambda}^{r,m}$, we create a class of analytical functions and derive several fuzzy differential subordinations for this class.

Definition 5. If the function $Y \in \mathcal{A}$ belongs to the class $P_{q,\lambda}^{F,r,m}(\eta)$ for all $\eta \in [0, 1)$ and satisfies the inequality

$$F_{(\mathcal{N}_{q,\lambda}^{r,m}Y)'}(\Lambda) \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' > \eta, \quad (\zeta \in \Lambda).$$

2. Preliminary

The following lemmas are needed to show our results.

Lemma 1 ([2]). Assume that $F \in \mathcal{A}$ and $\mathcal{G}(\zeta) = \frac{1}{\zeta} \int_0^\zeta F(t) dt$, $\zeta \in \Lambda$. If $\Re \left\{ 1 + \frac{\zeta F''(\zeta)}{F'(\zeta)} \right\} > \frac{-1}{2}$, $\zeta \in \Lambda$, then $\mathcal{G} \in \mathcal{C}$.

Lemma 2 (Theorem 2.6 in [8]). If F is a convex function such that $F(0) = \gamma$, $v \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with $\Re(v) \geq 0$. If $p \in \mathcal{H}_m[\gamma]$ such that $p(0) = \gamma$, $\Phi : \mathbb{C}^2 \times \Lambda \rightarrow \mathbb{C}$, $\Phi(p(\zeta), \zeta p'(\zeta); \zeta) = p(\zeta) + \frac{1}{v} \zeta p'(\zeta)$ is an analytic function in Λ and

$$\mathcal{F}_{\Phi(\mathbb{C}^2 \times \Lambda)} \left(p(\zeta) + \frac{1}{v} \zeta p'(\zeta) \right) \leq \mathcal{F}_{h(\Lambda)} h(\zeta) \rightarrow p(\zeta) + \frac{1}{v} \zeta p'(\zeta) \prec_{\mathcal{F}} h(\zeta), \quad \zeta \in \Lambda,$$

then

$$\mathcal{F}_{p(\Lambda)} p(\zeta) \leq \mathcal{F}_{q(\Lambda)} q(\zeta) \leq \mathcal{F}_{h(\Lambda)} h(\zeta) \rightarrow p(\zeta) \prec_{\mathcal{F}} q(\zeta), \quad \zeta \in \Lambda,$$

where

$$q(\zeta) = \frac{v}{m \zeta^{\frac{v}{m}}} \int_0^\zeta \psi(t) t^{\frac{v}{m}-1} dt, \quad \zeta \in \Lambda.$$

The function q is convex, and it is the fuzzy best dominant.

Lemma 3 (Theorem 2.7 in [8]). Let g be a convex function in Λ and $F(\zeta) = g(\zeta) + m \gamma \zeta g'(\zeta)$, where $\zeta \in \Lambda$, $m \in \mathbb{N}$ and $\gamma > 0$, if

$$p(\zeta) = g(0) + p_m \zeta^m + p_{m+1} \zeta^{m+1} + \dots \in \mathcal{H}(\Lambda),$$

and

$$\mathcal{F}_{p(\Lambda)} \left(p(\zeta) + \gamma \zeta p'(\zeta) \right) \leq \mathcal{F}_{\psi(\Lambda)} \psi(\zeta) \rightarrow p(\zeta) + \gamma \zeta p'(\zeta) \prec_{\mathcal{F}} \psi(\zeta), \quad \zeta \in \Lambda.$$

Then

$$\mathcal{F}_{p(\Lambda)} (p(\zeta)) \leq \mathcal{F}_{g(\Lambda)} g(\zeta) \rightarrow p(\zeta) \prec_{\mathcal{F}} g(\zeta), \quad \zeta \in \Lambda.$$

This result is sharp.

We define the fuzzy differential subordination general theory and its applications (see [9–13]). The method of fuzzy differential subordination is applied in the next section to obtain a set of fuzzy differential subordinations related to the operator $\mathcal{N}_{q,\lambda}^{r,m}$.

3. Main Results

Assume that $\eta \in [0, 1)$, $m \in \mathbb{N}$, $r \geq 1$, $0 \leq q \leq 1$, $\lambda \geq 0$ and $\zeta \in \Lambda$ are mentioned throughout this paper.

Theorem 1. Let k belong to $\mathcal{C}$ in Λ , and $h(\zeta) = k(\zeta) + \frac{1}{\rho+2}\zeta k'(\zeta)$. If $Y \in P_{q,\lambda}^{F,r,m}(\eta)$ and

$$\mathcal{G}(\zeta) = \mathcal{I}^\rho Y(\zeta) = \frac{\rho+2}{\zeta^{\rho+1}} \int_0^\zeta t^\rho Y(t) dt, \quad (5)$$

then

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} Y)'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m} Y(\zeta) \right)' \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(\mathcal{N}_{q,\lambda}^{r,m} Y(\zeta) \right)' \prec_{\mathcal{F}} h(\zeta), \quad (6)$$

implies

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G})'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \leq F_{k(\Lambda)} k(\zeta) \rightarrow \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \prec_{\mathcal{F}} k(\zeta).$$

Proof. Since

$$\zeta^{\rho+1} \mathcal{G}(\zeta) = (\rho+2) \int_0^\zeta t^\rho Y(t) dt,$$

by differentiating, we obtain

$$(\rho+1)\mathcal{G}(\zeta) + \zeta \mathcal{G}'(\zeta) = (\rho+2)Y(\zeta),$$

and

$$(\rho+1)\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) + \zeta \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' = (\rho+2)\mathcal{N}_{q,\lambda}^{r,m} Y(\zeta), \quad (7)$$

and also, by differentiating (7), we obtain

$$\left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' + \frac{1}{(\rho+2)} \zeta \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)'' = \left(\mathcal{N}_{q,\lambda}^{r,m} Y(\zeta) \right)'. \quad (8)$$

The fuzzy differential subordination (6) technique is used

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} Y)'(\Lambda)} \left(\left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' + \frac{1}{(\rho+2)} \zeta \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)'' \right) \leq F_{h(\Lambda)} \left(k(\zeta) + \frac{1}{(\rho+2)} \zeta k'(\zeta) \right). \quad (9)$$

We denote

$$q(\zeta) = \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)', \text{ so } q \in \mathcal{H}_1[n]. \quad (10)$$

Putting (10) in (9), we have

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} Y)'(\Lambda)} \left(q(\zeta) + \frac{1}{(\rho+2)} \zeta q'(\zeta) \right) \leq F_{h(\Lambda)} \left(k(\zeta) + \frac{1}{(\rho+2)} \zeta k'(\zeta) \right). \quad (11)$$

Using Lemma (3), we obtain

$$F_{q(\Lambda)} q(\zeta) \leq F_{k(\Lambda)} k(\zeta), \text{ i.e. } F_{(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta))'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \leq F_{k(\Lambda)} k(\zeta),$$

and therefore, $\left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \prec_{\mathcal{F}} k(\zeta)$, where k is the fuzzy best dominant. $\square$

Putting $m = 1$ and $\lambda = 0$ in Theorem 1, we obtain the following example since the operator Q_q^r reduces to $Q_q^r := \mathcal{N}_{q,0}^{r,1}$.

Example 1. Let k be an element of $\mathcal{C}$ in Λ and $h(\zeta) = k(\zeta) + \frac{1}{\rho+2}\zeta k'(\zeta)$. If $Y \in P_{q,\lambda}^{F,r,m}(\eta)$ and $\mathcal{G}$ is given by (5), then

$$F_{(Q_q^r Y)'(\Lambda)} \left(Q_q^r Y(\zeta) \right)' \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(Q_q^r Y(\zeta) \right)' \prec_{\mathcal{F}} h(\zeta),$$

implies

$$F_{(Q_q^r \mathcal{G})'(\Lambda)} \left(Q_q^r \mathcal{G}(\zeta) \right)' \leq F_{k(\Lambda)} k(\zeta) \rightarrow \left(Q_q^r \mathcal{G}(\zeta) \right)' \prec_{\mathcal{F}} k(\zeta).$$

Theorem 2. Assume that $h(\zeta) = \frac{1+(2\eta-1)\zeta}{1+\zeta}$, $\eta \in [0, 1)$, $\lambda > 0$ and $\mathcal{I}^\rho$ is given by (5), then

$$\mathcal{I}^\rho \left[P_{q,\lambda}^{F,r,m}(\eta) \right] \subset P_{q,\lambda}^{F,r,m}(\eta^*), \quad (12)$$

where

$$\eta^* = 2\eta - 1 + (\rho + 2)(2 - 2\eta) \int_0^1 \frac{t^{\rho+2}}{t+1} dt. \quad (13)$$

Proof. A function h belongs to $\mathcal{C}$, and we obtain from the hypothesis of Theorem 2 using the same technique as that in the proof of Theorem 1 that

$$F_{q(\Lambda)} \left(q(\zeta) + \frac{1}{(\rho+2)} \zeta q'(\zeta) \right) \leq F_{h(\Lambda)} h(\zeta),$$

where $q(\zeta)$ is defined in (10). By using Lemma 2, we obtain

$$F_{q(\Lambda)} q(\zeta) \leq F_{k(\Lambda)} k(\zeta) \leq F_{h(\Lambda)} h(\zeta),$$

which implies

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G})'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \leq F_{k(\Lambda)} k(\zeta) \leq F_{h(\Lambda)} h(\zeta),$$

where

$$\begin{aligned} k(\zeta) &= \frac{\rho+2}{\zeta^{\rho+2}} \int_0^\zeta t^{\rho+1} \frac{1+(2\eta-1)t}{1+t} dt \\ &= (2\eta-1) + \frac{(\rho+2)(2-2\eta)}{\zeta^{\rho+2}} \int_0^\zeta \frac{t^{\rho+1}}{1+t} dt \in \mathcal{C}, \end{aligned}$$

where $k(\Lambda)$ is symmetric with respect to the real axis, so we have

$$F_{(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G})'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m} \mathcal{G}(\zeta) \right)' \geq \min_{|\zeta|=1} F_{k(\Lambda)} k(\zeta) = F_{k(\Lambda)} k(1), \quad (14)$$

and $\eta^* = k(1) = 2\eta - 1 + (\rho + 2)(2 - 2\eta) \int_0^1 \frac{t^{\rho+2}}{t+1} dt. \quad \square$

Theorem 3. Assume that k belongs to $\mathcal{C}$ in Λ , that $k(0) = 1$, and that $h(\zeta) = k(\zeta) + \zeta k'(\zeta)$. When $Y \in \mathcal{A}$ and the fuzzy differential subordination is satisfied,

$$F_{(\mathcal{N}_{q,\lambda}^{r,m}Y)'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' \prec_{\mathcal{F}} h(\zeta), \quad (15)$$

holds, then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}Y(\Lambda)} \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta) \rightarrow \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta). \quad (16)$$

Proof. Let

$$\begin{aligned} q(\zeta) &= \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} = \frac{\zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r a_j \zeta^j}{\zeta} \\ &= 1 + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)] q^{j-1} (1-q)^r a_j \zeta^{j-1}, \end{aligned}$$

and we obtain that

$$q(\zeta) + \zeta q'(\zeta) = \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)',$$

so

$$F_{(\mathcal{N}_{q,\lambda}^{r,m}Y)'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' \leq F_{h(\Lambda)} h(\zeta)$$

implies

$$F_{q(\Lambda)} \left(q(\zeta) + \zeta q'(\zeta) \right) \leq F_{h(\Lambda)} h(\zeta) = F_{k(\Lambda)} \left(k(\zeta) + \zeta k'(\zeta) \right).$$

Using the Lemma 3, we obtain

$$F_{q(\Lambda)} q(\zeta) \leq F_{k(\Lambda)} k(\zeta) \rightarrow F_{\mathcal{N}_{q,\lambda}^{r,m}Y(\Lambda)} \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta),$$

and we obtain

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta).$$

□

Theorem 4. Consider $h \in \mathcal{H}(\Lambda)$, which satisfies $\Re \left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \right) > \frac{-1}{2}$ when $h(0) = 1$. If the fuzzy differential subordination

$$F_{(\mathcal{N}_{q,\lambda}^{r,m}Y)'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) \right)' \prec_{\mathcal{F}} h(\zeta), \quad (17)$$

then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}Y(\Lambda)} \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta) \quad \text{i.e.} \quad \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta), \quad (18)$$

where

$$k(\zeta) = \frac{1}{\zeta} \int_0^{\zeta} h(t) dt,$$

the function k is convex, and it is the fuzzy best dominant.

Proof. Let

$$q(\zeta) = \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} = 1 + \sum_{j=d+1}^{\infty} \frac{[j]_q!}{[\lambda+1]_{q,j-1}} a_j \psi_j \zeta^{j-1}, \quad q \in \mathcal{H}_1[1],$$

where $\Re\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) > \frac{-1}{2}$. From Lemma 1, we have

$$k(\zeta) = \frac{1}{\zeta} \int_0^{\zeta} h(t) dt \in$$

belongs to the class $\mathcal{C}$, which satisfies the fuzzy differential subordination (17). Since

$$k(\zeta) + \zeta k'(\zeta) = h(\zeta),$$

it is the fuzzy best dominant. We have

$$q(\zeta) + \zeta q'(\zeta) = \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)\right)',$$

then (17) becomes

$$F_{q(\Lambda)}\left(q(\zeta) + \zeta q'(\zeta)\right) \leq F_{h(\Lambda)}h(\zeta).$$

By using Lemma 3, we obtain

$$F_{q(\Lambda)}q(\zeta) \leq F_{k(\Lambda)}k(\zeta), \quad \text{i.e.} \quad F_{\mathcal{N}_{q,\lambda}^{r,m}Y(\Lambda)} \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)}k(\zeta),$$

then

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta).$$

□

Putting $h(\zeta) = \frac{1+(2v-1)\zeta}{1+\zeta}$ in Theorem 4. As a result, we have the following corollary:

Corollary 1. Let $h = \frac{1+(2v-1)\zeta}{1+\zeta}$ be a convex function in Λ , with $h(0) = 1$, $0 \leq \beta < 1$. If $Y \in \mathcal{A}$ and verifies the fuzzy differential subordination

$$F_{\left(\mathcal{N}_{q,\lambda}^{r,m}Y\right)'(\Lambda)} \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)\right)' \leq F_{h(\Lambda)}\left(\frac{1+(2v-1)\zeta}{1+\zeta}\right), \quad \text{i.e.} \quad \left(\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)\right)' \prec_{\mathcal{F}} \left(\frac{1+(2v-1)\zeta}{1+\zeta}\right),$$

then

$$F_{\mathcal{N}_{q,\lambda}^{r,m}Y(\Lambda)} \frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)}k(\zeta), \quad (19)$$

then

$$\frac{\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta),$$

where

$$k(\zeta) = 2v - 1 + \frac{2(1-v)}{\zeta} \ln(1 + \zeta),$$

the function k is convex and it is the fuzzy best dominant.

Putting $m = 1$ and $\lambda = 0$ in Corollary 1, we obtain the following example.

Example 2. Let $h = \frac{1+(2v-1)\zeta}{1+\zeta}$ be a convex function in Λ , with $h(0) = 1$, $0 \leq \beta < 1$. If $f \in \mathcal{A}$ and verifies the fuzzy differential subordination

$$F_{(Q_q^r Y)'(\Lambda)}(Q_q^r Y(\zeta))' \leq F_{h(\Lambda)}\left(\frac{1+(2v-1)\zeta}{1+\zeta}\right), \quad \text{i.e.} \quad (Q_q^r Y(\zeta))' \prec_{\mathcal{F}} \left(\frac{1+(2v-1)\zeta}{1+\zeta}\right)$$

then

$$F_{Q_q^r Y(\Lambda)} \frac{Q_q^r Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta) \quad \text{i.e.} \quad \frac{Q_q^r Y(\zeta)}{\zeta} \prec_{\mathcal{F}} k(\zeta), \quad (20)$$

where

$$k(\zeta) = 2v - 1 + \frac{2(1-v)}{\zeta} \ln(1 + \zeta).$$

4. Conclusions

All of the above results provide information about fuzzy differential subordinations for the operator $\mathcal{N}_{q,\lambda}^{r,m}$; we also provide certain properties for the class $P_{q,\lambda}^{F,r,m}(\eta)$ of univalent analytic functions. Using these classes and operators, we can create some simple applications.

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