New Results about Fuzzy Differential Subordinations Associated with Pascal Distribution

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Abstract: Based upon the Pascal distribution series \( N_{q,A}^{p,r,m} \), we can obtain a set of fuzzy differential subordinations in this investigation. We also newly obtain class \( p_{q,A}^{F,r,m}(\eta) \) of univalent analytic functions defined by the operator \( N_{q,A}^{p,r,m} \), give certain properties for the class \( p_{q,A}^{F,r,m}(\eta) \) and also obtain some applications connected with a special case for the operator. New research directions can be taken on fuzzy differential subordinations associated with symmetry operators.

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1. Introduction

Let \( \mathcal{H}(\omega) \) represent the class of holomorphic and univalent functions on \( \omega \) such that \( \omega \subset \mathbb{C} \) and let \( \mathcal{H}(\omega) \) denote the class of holomorphic functions on \( \omega \). The class of holomorphic functions in the open unit disk of the complex plane \( \Lambda = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \) is denoted in this study by a note \( \mathcal{H}(\Lambda) \) with \( B_{\Lambda} = \{ \xi \in \mathbb{C} : |\xi| = 1 \} \) standing as the unit disk’s boundary. For \( m \in \mathbb{N} = \{1, 2, \ldots\} \), we define

\[
\mathcal{H}_m[\gamma] = \left\{ Y \in \mathcal{H}(\Lambda) : Y(\zeta) = \gamma + \sum_{j=m+1}^{\infty} a_j \zeta^j, \zeta \in \Lambda \right\},
\]

\[
\mathcal{A}_m = \left\{ Y \in \mathcal{H}(\Lambda) : Y(\zeta) = \zeta + \sum_{j=m+1}^{\infty} a_j \zeta^j, \zeta \in \Lambda \right\}
\]

with \( \mathcal{A}_1 = \mathcal{A} \) \hspace{1cm} (1)

and

\[
\mathcal{S} = \{ Y \in \mathcal{A}_m : Y \text{ is a univalent function in } \Lambda \}.
\]

We denote by

\[
\mathcal{C} = \left\{ Y \in \mathcal{A}_m : \Re \left( 1 + \frac{\zeta Y''(\zeta)}{Y'(\zeta)} \right) > 0, \zeta \in \Lambda \right\},
\]

which is the set of convex functions on \( \Lambda \).

Let \( Y_1 \) and \( Y_2 \) be analytic in \( \Lambda \). Then \( Y_1 \) is subordinate to \( Y_2 \) written as \( Y_1 \prec Y_2 \) if there exists a Schwarz function \( \phi \), which is analytic in \( \Lambda \) with \( \phi(0) = 0 \) and \( |\phi(\zeta)| < 1 \).
for all \( \zeta \in \Lambda \) such that \( Y_1(\zeta) = Y_2(\phi(\zeta)) \). Furthermore, if the function \( Y_2 \) is univalent in \( \Lambda \), then we have the following equivalence (see [1,2]):

\[
Y_1(\zeta) \prec Y_2(\zeta) \iff Y_1(0) = Y_2(0) \text{ and } Y_1(\Lambda) \subset Y_2(\Lambda).
\]

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and propositions:

**Definition 1** ([3]). Assume that \( \mathcal{T} \neq \emptyset \) is a fuzzy subset and \( \mathcal{F} : \mathcal{T} \to [0,1] \) is an application. A pair of \((\Lambda, \mathcal{F}_\Lambda)\), where \( \mathcal{F}_\Lambda : \mathcal{T} \to [0,1] \), and

\[
\mathcal{R} = \{ x \in \mathcal{T} : 0 < \mathcal{F}_\Lambda(x) \leq 1 \} = \sup(\Lambda, \mathcal{F}_\Lambda),
\]

is called a fuzzy subset. The fuzzy set \((\Lambda, \mathcal{F}_\Lambda)\) is called a function \( \mathcal{F}_\Lambda \).

Let \( Y, g \in \mathcal{H}(\omega) \) be denoted by

\[
Y(\omega) = \left\{ Y(\zeta) : 0 < \mathcal{F}_{Y(\omega)}Y(\zeta) \leq 1, \zeta \in \omega \right\} = \sup\left(Y(\omega), \mathcal{F}_{Y(\omega)}\right),
\]

and

\[
g(\omega) = \left\{ g(\zeta) : 0 < \mathcal{F}_{g(\omega)}g(\zeta) \leq 1, \zeta \in \omega \right\} = \sup\left(g(\omega), \mathcal{F}_{g(\omega)}\right).
\]

**Proposition 1** ([4]). (i) If \((B, \mathcal{F}_B) = (\mathcal{U}, \mathcal{F}_\mathcal{U})\), then we have \( \mathcal{B} = \mathcal{U} \), where \( \mathcal{B} = \sup(\mathcal{B}, \mathcal{F}_B) \) and \( \mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U}) \); (ii) if \((B, \mathcal{F}_B) \subseteq (\mathcal{U}, \mathcal{F}_\mathcal{U})\), then we have \( B \subseteq \mathcal{U} \), where \( B = \sup(\mathcal{B}, \mathcal{F}_B) \) and \( \mathcal{U} = \sup(\mathcal{U}, \mathcal{F}_\mathcal{U}) \).

**Definition 2** ([4]). Let \( \zeta_0 \in \omega \) be a fixed point and let the functions \( Y, g \in \mathcal{H}(\omega) \). The function \( Y \) is said to be fuzzy subordinate to \( g \), and we write \( Y \prec_X g \) or \( Y(\zeta) \prec_X g(\zeta) \), which satisfies the following conditions:

(i) \( Y(\zeta_0) = g(\zeta_0) \);

(ii) \( \mathcal{F}_{Y(\omega)}Y(\zeta) \leq \mathcal{F}_{g(\omega)}g(\zeta), \zeta \in \omega \).

**Proposition 2** ([4]). Assume that \( \zeta_0 \in \omega \) is a fixed point and the functions \( Y, g \in \mathcal{H}(\omega) \). If \( Y(\zeta) \prec_X g(\zeta), \zeta \in \omega \), then

(i) \( Y(\zeta_0) = g(\zeta_0) \);

(ii) \( \mathcal{F}_{Y(\omega)}Y(\zeta) \leq \mathcal{F}_{g(\omega)}g(\zeta), \zeta \in \omega \), where \( Y(\omega) \) and \( g(\omega) \) are defined by (2) and (3), respectively.

**Definition 3** ([5]). Assume that \( h \in \mathcal{S} \) and \( \Phi : \mathcal{C} \times \Lambda \to \mathcal{C}, \Phi(\alpha, 0, 0; 0) = h(0) = \alpha \). If \( p \) satisfies the requirements of the second-order fuzzy differential subordination and is analytic in \( \Lambda \), with \( p(0) = \alpha \),

\[
\mathcal{F}_{\Phi(\zeta, \zeta; \Lambda)}\Phi\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta)\right) \leq \mathcal{F}_{h(\Lambda)}h(\zeta). \tag{4}
\]

If \( q \) is a fuzzy dominant of the fuzzy differential subordination solutions, then \( p \) is said to be a fuzzy solution of the fuzzy differential subordination and satisfies

\[
\mathcal{F}_{p(\Lambda)}p(\zeta) \leq \mathcal{F}_{q(\Lambda)}q(\zeta), \text{ i.e., } p(\zeta) \prec_X q(\zeta), \zeta \in \Lambda,
\]

for each and every \( p \) satisfying (4).

**Definition 4.** A fuzzy dominant \( \tilde{q} \) that satisfies

\[
\mathcal{F}_{\tilde{q}(\Lambda)}\tilde{q}(\zeta) \leq \mathcal{F}_{q(\Lambda)}q(\zeta),
\]

is called a fuzzy dominant.
where

\( q(\zeta) \sim_f q(\zeta), \ z \in \Lambda. \)

The fuzzy best dominant of (4) is referred to for all fuzzy dominants.

Assume the function \( \Omega \in \mathcal{A}_m \) is given by

\[
\Omega(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \psi_j \zeta^j, \ z \in \Lambda.
\]

The Hadamard (or convolution) product of \( Y \) and \( \Omega \) is defined as

\[
(Y \ast \Omega)(\zeta) := \zeta + \sum_{j=m+1}^{\infty} a_j \psi_j \zeta^j, \ z \in \Lambda.
\]

A variable \( x \) is said to have the Pascal distribution if it takes the values \( 0, 1, 2, 3, \ldots \) with the probabilities \( (1 - q)^r \frac{q^r(1-q)^r}{r!}, \frac{q^r(1-q)^r}{2!}, \frac{q^r(1-q)^r}{3!}, \ldots \), respectively, where \( q \) and \( r \) are called the parameters, and thus we have the probability formula

\[
P(X = k) = \binom{k+r-1}{r-1} q^k (1-q)^r, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]

Now we present a power series whose coefficients are Pascal distribution probabilities, i.e.,

\[
Q_{q,m}(\zeta) := \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} q^j (1-q)^r \zeta^j, \ z \in \Lambda,
\]

\[
(m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1).
\]

We easily determine from the ratio test that the radius of convergence of the above power series is at least \( \frac{1}{q} \geq 1 \); hence, \( Q_{q,m} \in \mathcal{A}_m \).

We define the functions

\[
\mathcal{M}_{q,\lambda}^{r,m}(\zeta) := (1 - \lambda)Q_{q,m}(\zeta) + \lambda z \left( Q_{q,m}(\zeta) \right)^r
\]

\[
= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)]q^j (1-q)^r \zeta^j, \ z \in \Lambda,
\]

\[
(m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1, \lambda \geq 0).
\]

El-Deeb and Bulboacă [6] introduced the linear operator \( \mathcal{N}_{q,\lambda}^{r,m} : \mathcal{A}_m \rightarrow \mathcal{A}_m \) defined by

\[
\mathcal{N}_{q,\lambda}^{r,m}Y(\zeta) := \mathcal{M}_{q,\lambda}^{r,m}(\zeta) \ast Y(\zeta)
\]

\[
= \zeta + \sum_{j=m+1}^{\infty} \binom{j+r-2}{r-1} [1 + \lambda(j-1)]q^j (1-q)^r a_j \zeta^j, \ z \in \Lambda,
\]

\[
(m \in \mathbb{N}, r \geq 1, 0 \leq q \leq 1, \lambda \geq 0),
\]

where \( Y \) is given by (1), and the symbol \( \ast \) stands for the Hadamard (or convolution) product.

**Remark 1.** (i) For \( m = 1 \), the operator \( \mathcal{N}_{q,\lambda}^{r,m} \) reduces to \( \mathcal{T}_{q,\lambda}^r := \mathcal{N}_{q,\lambda}^{r,1} \), introduced and studied by El-Deeb et al. [7]; (ii) for \( m = 1 \) and \( \lambda = 0 \), the operator \( Q_{q,0}^r \) reduces to \( Q_{q}^r := \mathcal{N}_{q,0}^{r,1} \), introduced and studied by El-Deeb et al. [7].
Using the operator $N_{q,\lambda}^{r,m}$, we create a class of analytical functions and derive several fuzzy differential subordinations for this class.

**Definition 5.** If the function $Y \in A$ belongs to the class $P_{q,\lambda}^{r,m}(\eta)$ for all $\eta \in [0,1)$ and satisfies the inequality

$$F_{(A_{r,m}^{r,m}(Y))}^\prime(\lambda) \left( A_{q,\lambda}^{r,m}(\zeta) \right)^\prime > \eta, \quad (\zeta \in \Lambda).$$

2. Preliminary

The following lemmas are needed to show our results.

**Lemma 1** ([2]). Assume that $F \in A$ and $G(\zeta) = \frac{1}{2} \int_0^\zeta F(t)dt$, $\zeta \in \Lambda$. If $\Re\left\{1 + \frac{\nu}{p^{\prime}(\zeta)}\right\} > \frac{1}{r^2}$, $\zeta \in \Lambda$, then $G \in C$.

**Lemma 2** (Theorem 2.6 in [8]). If $F$ is a convex function such that $F(0) = \gamma$, $v \in C^* = C \setminus \{0\}$ with $\Re(v) \geq 0$. If $p \in H_m[\gamma]$ such that $p(0) = \gamma$, $\Phi : C^2 \times \Lambda \to C$, $\Phi(p(\xi), \xi p^\prime(\zeta); \xi) = p(\xi) + \frac{1}{v} \xi p^\prime(\zeta)$ is an analytic function in $\Lambda$ and

$${\mathcal{F}}_{\Phi(C^2 \times \Lambda)}\left( p(\zeta) + \frac{1}{v} \xi p^\prime(\zeta) \right) \leq {\mathcal{F}}_{h(\Lambda)}h(\zeta) \rightarrow p(\zeta) + \frac{1}{v} \xi p^\prime(\zeta) <_{\mathcal{F}} h(\zeta), \quad \zeta \in \Lambda,$$

then

$${\mathcal{F}}_{p(\Lambda)}p(\zeta) \leq {\mathcal{F}}_{q(\Lambda)}q(\zeta) \leq {\mathcal{F}}_{h(\Lambda)}h(\zeta) \rightarrow p(\zeta) <_{\mathcal{F}} q(\zeta), \quad \zeta \in \Lambda,$$

where

$$q(\zeta) = \frac{v}{m \zeta} \int_0^\zeta \psi(t)t^\pi dt, \quad \zeta \in \Lambda.$$

The function $q$ is convex, and it is the fuzzy best dominant.

**Lemma 3** (Theorem 2.7 in [8]). Let $g$ be a convex function in $\Lambda$ and $F(\zeta) = g(\zeta) + \gamma \xi g^\prime(\zeta)$, where $\zeta \in \Lambda$, $m \in \mathbb{N}$ and $\gamma > 0$, if

$$p(\zeta) = g(0) + pm \xi^m + pm+1 \xi^{m+1} + ... \in H(\Lambda),$$

and

$${\mathcal{F}}_{p(\Lambda)}\left( p(\zeta) + \gamma \xi p^\prime(\zeta) \right) \leq {\mathcal{F}}_{q(\Lambda)}q(\zeta) \rightarrow p(\zeta) + \gamma \xi p^\prime(\zeta) <_{\mathcal{F}} q(\zeta), \quad \zeta \in \Lambda.$$

Then

$${\mathcal{F}}_{p(\Lambda)}p(\zeta) \leq {\mathcal{F}}_{g(\Lambda)}g(\zeta) \rightarrow p(\zeta) <_{\mathcal{F}} g(\zeta), \quad \zeta \in \Lambda.$$

This result is sharp.

We define the fuzzy differential subordination general theory and its applications (see [9–13]). The method of fuzzy differential subordination is applied in the next section to obtain a set of fuzzy differential subordinations related to the operator $N_{q,\lambda}^{r,m}$.

3. Main Results

Assume that $\eta \in [0,1)$, $m \in \mathbb{N}$, $r \geq 1$, $0 \leq q \leq 1$, $\lambda \geq 0$ and $\zeta \in \Lambda$ are mentioned throughout this paper.
Theorem 1. Let \( k \) belong to \( \mathcal{C} \) in \( \Lambda \), and \( h(\xi) = k(\xi) + \frac{1}{\rho + 2}\xi k'(\xi) \). If \( Y \in P_{q,\lambda}^{r,m}(\eta) \) and

\[
G(\xi) = \mathcal{T}^\rho Y(\xi) = \frac{\rho + 2}{\xi^{\rho + 1}} \int_0^\xi t^\rho Y(t) dt,
\]

then

\[
F \left( \Lambda^{q,\nu}_\eta \right) \left( \Lambda^{r,m}_\lambda Y(\xi) \right) \leq F[h(\lambda)] h(\xi) \rightarrow \left( \Lambda^{r,m}_\lambda Y(\xi) \right)' <_F h(\xi),
\]

implies

\[
F \left( \Lambda^{q,\nu}_\eta \right) \left( \Lambda^{r,m}_\lambda G(\xi) \right) \leq F[k(\lambda)] k(\xi) \rightarrow \left( \Lambda^{r,m}_\lambda G(\xi) \right)' <_F k(\xi).
\]

Proof. Since

\[
\xi^{\rho + 1} G(\xi) = (\rho + 2) \int_0^\xi t^\rho Y(t) dt,
\]

by differentiating, we obtain

\[
(\rho + 1) G(\xi) + \xi G'(\xi) = (\rho + 2) Y(\xi),
\]

and

\[
(\rho + 1) \Lambda^{r,m}_\lambda G(\xi) + \xi \left( \Lambda^{r,m}_\lambda G(\xi) \right)' = (\rho + 2) \Lambda^{r,m}_\lambda Y(\xi),
\]

and also, by differentiating (7), we obtain

\[
\left( \Lambda^{r,m}_\lambda G(\xi) \right)' + \frac{1}{(\rho + 2)} \xi \left( \Lambda^{r,m}_\lambda G(\xi) \right)' = \left( \Lambda^{r,m}_\lambda Y(\xi) \right)'.
\]

The fuzzy differential subordination (6) technique is used

\[
F \left( \Lambda^{q,\nu}_\eta \right) \left( \Lambda^{r,m}_\lambda G(\xi) \right) \leq F[h(\lambda)] h(\xi) \rightarrow \left( \Lambda^{r,m}_\lambda G(\xi) \right)' \leq F[h(\lambda)] \left( \Lambda^{r,m}_\lambda Y(\xi) \right)'.
\]

We denote

\[
q(\xi) = \left( \Lambda^{r,m}_\lambda G(\xi) \right)', \text{ so } q \in \mathcal{H}_1[\eta].
\]

Putting (10) in (9), we have

\[
F \left( \Lambda^{q,\nu}_\eta \right) \left( q(\xi) + \frac{1}{(\rho + 2)} \xi q'(\xi) \right) \leq F[h(\lambda)] \left( k(\xi) + \frac{1}{(\rho + 2)} \xi k'(\xi) \right).
\]

Using Lemma (3), we obtain

\[
F[q(\lambda)] q(\xi) \leq F[k(\lambda)] k(\xi), \text{ i.e. } F \left( \Lambda^{r,m}_\lambda G(\xi) \right)' \left( \Lambda^{r,m}_\lambda G(\xi) \right)' \leq F[k(\lambda)] k(\xi),
\]

and therefore, \( \left( \Lambda^{r,m}_\lambda G(\xi) \right)' <_F k(\xi) \), where \( k \) is the fuzzy best dominant.
Putting \( m = 1 \) and \( \lambda = 0 \) in Theorem 1, we obtain the following example since the operator \( Q'_{q} \) reduces to \( Q'_{q} := N_{q,0}^{1} \).

**Example 1.** Let \( k \) be an element of \( C \) in \( \Lambda \) and \( h(\zeta) = k(\zeta) + \frac{1}{\rho+2} \zeta \xi'(\zeta) \). If \( Y \in P_{q,\Lambda}^{\rho,m}(\eta) \) and \( G \) is given by (5), then

\[
F_{(Q'_{q}Y)}(\Lambda) \left( Q'_{q}Y(\zeta) \right) \leq F_{h(\Lambda)}h(\zeta) \quad \Rightarrow \quad \left( Q'_{q}Y(\zeta) \right) \preceq_{F} h(\zeta),
\]

implies

\[
F_{(Q'_{q}G)}(\Lambda) \left( Q'_{q}G(\zeta) \right) \leq F_{k(\Lambda)}k(\zeta) \quad \Rightarrow \quad \left( Q'_{q}G(\zeta) \right) \preceq_{F} k(\zeta).
\]

**Theorem 2.** Assume that \( h(\zeta) = \frac{1+(2q-1)\zeta}{1+\xi} \), \( \eta \in [0,1) \), \( \lambda > 0 \) and \( T^{\rho} \) is given by (5), then

\[
T^{\rho} \left[ P_{q,\Lambda}^{\rho,m}(\eta) \right] \subset P_{q,\Lambda}^{\rho,m}(\eta^{*}),
\]

where

\[
\eta^{*} = 2\eta - 1 + (\rho + 2)(2 - 2\eta) \int_{0}^{1} \frac{t^{\rho+2}}{1+t} dt.
\]

**Proof.** A function \( h \) belongs to \( C \), and we obtain from the hypothesis of Theorem 2 using the same technique as that in the proof of Theorem 1 that

\[
F_{q(\Lambda)} \left( q(\zeta) + \frac{1}{(\rho + 2)} \zeta q'(\zeta) \right) \leq F_{h(\Lambda)}h(\zeta),
\]

where \( q(\zeta) \) is defined in (10). By using Lemma 2, we obtain

\[
F_{q(\Lambda)}q(\zeta) \leq F_{k(\Lambda)}k(\zeta) \leq F_{h(\Lambda)}h(\zeta),
\]

which implies

\[
F_{\left( N_{q,\Lambda}^{\rho,m} \right)}(\Lambda) \left( N_{q,\Lambda}^{\rho,m}G(\zeta) \right) \leq F_{k(\Lambda)}k(\zeta) \leq F_{h(\Lambda)}h(\zeta),
\]

where

\[
k(\zeta) = \frac{\rho + 2}{\zeta^{\rho+2}} \int_{0}^{\zeta} \frac{t^{\rho+1} \left( 2\eta - 1 \right)t}{1+t} dt
\]

\[
= (2\eta - 1) + (\rho + 2)(2 - 2\eta) \int_{0}^{\zeta} \frac{t^{\rho+1}}{1+t} dt \in C,
\]

where \( k(\Lambda) \) is symmetric with respect to the real axis, so we have

\[
F_{\left( N_{q,\Lambda}^{\rho,m} \right)}(\Lambda) \left( N_{q,\Lambda}^{\rho,m}G(\zeta) \right) \geq \min_{|s|=1} F_{k(\Lambda)}k(\zeta) = F_{k(\Lambda)}k(1),
\]

and \( \eta^{*} = k(1) = 2\eta - 1 + (\rho + 2)(2 - 2\eta) \int_{0}^{1} \frac{t^{\rho+2}}{1+t} dt. \) □
Theorem 3. Assume that \( k \) belongs to \( C \) in \( \Lambda \), that \( k(0) = 1 \), and that \( h(\zeta) = k(\zeta) + \zeta k'(\zeta) \). When \( Y \in \mathcal{A} \) and the fuzzy differential subordination is satisfied,

\[
F\left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)'(\Lambda) \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)' <_{\mathcal{F}} h(\zeta),
\]
holds, then

\[
F_{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\Lambda)} \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta) \rightarrow \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} <_{\mathcal{F}} k(\zeta).
\]

Proof. Let

\[
q(\zeta) = \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} = \frac{\zeta + \sum_{j=m+1}^{\infty} (j+r-2)(1+\lambda(j-1))q^{-1}(1-q)aq_j\zeta^j}{\zeta} = 1 + \sum_{j=m+1}^{\infty} \left(\frac{j+r-2}{r-1}\right)(1+\lambda(j-1))q^{-1}(1-q)aq_j\zeta^{j-1},
\]

and we obtain that

\[
q(\zeta) + \zeta q'(\zeta) = \left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)',
\]

so

\[
F\left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)'(\Lambda) \leq F_{h(\Lambda)} h(\zeta)
\]

implies

\[
F_{q(\Lambda)} \left(q(\zeta) + \zeta q'(\zeta)\right) \leq F_{h(\Lambda)} h(\zeta) = F_{k(\Lambda)} \left(k(\zeta) + \zeta k'(\zeta)\right).
\]

Using the Lemma 3, we obtain

\[
F_{q(\Lambda)} q(\zeta) \leq F_{k(\Lambda)} k(\zeta) \rightarrow F_{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\Lambda)} \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta),
\]

and we obtain

\[
\frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} <_{\mathcal{F}} k(\zeta).
\]

\( \square \)

Theorem 4. Consider \( h \in \mathcal{H}(\Lambda) \), which satisfies \( \Re\left(1 + \frac{h''(\zeta)}{h'(\zeta)}\right) > \frac{1}{2} \) when \( h(0) = 1 \). If the fuzzy differential subordination

\[
F\left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)'(\Lambda) \leq F_{h(\Lambda)} h(\zeta) \rightarrow \left(\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)\right)' <_{\mathcal{F}} h(\zeta),
\]

then

\[
F_{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\Lambda)} \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta) \quad i.e. \quad \frac{\mathcal{N}^{\tau,m}_{q,\Lambda} Y(\zeta)}{\zeta} <_{\mathcal{F}} k(\zeta),
\]

where

\[
k(\zeta) = \frac{1}{\zeta} \int_{0}^{\zeta} h(t) dt,
\]

the function \( k \) is convex, and it is the fuzzy best dominant.
Proof. Let
\[ q(\zeta) = \frac{N_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} = 1 + \sum_{j=d+1}^{\infty} \frac{[j]_q!}{[\lambda + 1]_{qj-1}} a_{j\psi} \zeta^{j-1}, \quad q \in \mathcal{H}_1[1], \]
where \( \Re \left( 1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \right) > -\frac{1}{2} \). From Lemma 1, we have
\[ k(\zeta) = \frac{1}{\zeta} \int_{0}^{\zeta} h(t) dt \in \]
belongs to the class \( C \), which satisfies the fuzzy differential subordination (17). Since
\[ k(\zeta) + \zeta k'(\zeta) = h(\zeta), \]
it is the fuzzy best dominant. We have
\[ q(\zeta) + \zeta q'(\zeta) = \left( N_{q,\lambda}^{r,m}Y(\zeta) \right)', \]
then (17) becomes
\[ F_{q(\Lambda)}(q(\zeta) + \zeta q'(\zeta)) \leq F_{h(\Lambda)}h(\zeta). \]
By using Lemma 3, we obtain
\[ F_{q(\Lambda)}q(\zeta) \leq F_{k(\Lambda)}k(\zeta), \quad i.e. \quad F_{N_{q,\lambda}^{r,m}Y(\Lambda)} N_{q,\lambda}^{r,m}Y(\zeta) \leq F_{k(\Lambda)}k(\zeta), \]
then
\[ \frac{N_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{F} k(\zeta). \]

Putting \( h(\zeta) = \frac{1+(2v-1)\zeta}{1+\zeta} \) in Theorem 4. As a result, we have the following corollary:

**Corollary 1.** Let \( h = \frac{1+(2v-1)\zeta}{1+\zeta} \) be a convex function in \( \Lambda \), with \( h(0) = 1, 0 \leq \beta < 1 \). If \( Y \in \mathcal{A} \) and verifies the fuzzy differential subordination
\[ F\left( N_{q,\lambda}^{r,m}Y(\zeta) \right)'(\Lambda) \left( N_{q,\lambda}^{r,m}Y(\zeta) \right) \leq F_{h(\Lambda)} \left( \frac{1+(2v-1)\zeta}{1+\zeta} \right), \quad i.e. \quad \left( N_{q,\lambda}^{r,m}Y(\zeta) \right)' \prec_{F} \left( \frac{1+(2v-1)\zeta}{1+\zeta} \right), \]
then
\[ F_{N_{q,\lambda}^{r,m}Y(\Lambda)} N_{q,\lambda}^{r,m}Y(\zeta) \leq F_{k(\Lambda)}k(\zeta), \]
then
\[ \frac{N_{q,\lambda}^{r,m}Y(\zeta)}{\zeta} \prec_{F} k(\zeta), \]
where
\[ k(\zeta) = 2v - 1 + \frac{2(1-v)}{\zeta} \ln(1 + \zeta), \]
the function \( k \) is convex and it is the fuzzy best dominant.

Putting \( m = 1 \) and \( \lambda = 0 \) in Corollary 1, we obtain the following example.
Example 2. Let \( h = \frac{1+(2\nu-1)\zeta}{1+\zeta} \) be a convex function in \( \Lambda \), with \( h(0) = 1, 0 \leq \beta < 1 \). If \( f \in A \) and verifies the fuzzy differential subordination

\[
F \left( Q_{r}^{\nu}Y(\zeta) \right)'(\Lambda) \leq F_{h(\Lambda)} \left( \frac{1+(2\nu-1)\zeta}{1+\zeta} \right), \quad \text{i.e.} \quad Q_{r}^{\nu}Y(\zeta)' \preceq F \left( \frac{1+(2\nu-1)\zeta}{1+\zeta} \right)
\]

then

\[
F_{Q_{r}^{\nu}Y(\Lambda)} \frac{Q_{r}^{\nu}Y(\zeta)}{\zeta} \leq F_{k(\Lambda)} k(\zeta), \quad \text{i.e.} \quad \frac{Q_{r}^{\nu}Y(\zeta)}{\zeta} \preceq F k(\zeta), \quad (20)
\]

where

\[
k(\zeta) = 2\nu - 1 + \frac{2(1-\nu)}{\zeta} \ln(1+\zeta).
\]

4. Conclusions

All of the above results provide information about fuzzy differential subordinations for the operator \( A^{(r,m),\nu}_{\eta,\Lambda} \); we also provide certain properties for the class \( P^{F,r,m}_{\eta,\Lambda}(\eta) \) of univalent analytic functions. Using these classes and operators, we can create some simple applications.

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References


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