Article

Tensor Products and Crossed Differential Graded Lie Algebras in the Category of Crossed Complexes

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Abstract: The study of algebraic structures endowed with the concept of symmetry is made possible by the link between Lie algebras and symmetric monoidal categories. This relationship between Lie algebras and symmetric monoidal categories is useful and has resulted in many areas, including algebraic topology, representation theory, and quantum physics. In this paper, we present analogous definitions for Lie algebras within the framework of whiskered structures, bimorphisms, crossed complexes, crossed differential graded algebras, and tensor products. These definitions, given for groupoids in existing literature, have been adapted to establish a direct correspondence between these algebraic structures and Lie algebras. We show that a 2-truncation of the crossed differential graded Lie algebra, obtained from our adapted definitions, gives rise to a braided crossed module of Lie algebras. We also construct a functor to simplicial Lie algebras, enabling a systematic mapping between different Lie algebraic categories, which supports the validity of our adapted definitions and establishes their compatibility with established categories.

Keywords: crossed module; Lie algebra; tensor product

1. Introduction

The theory of groups and algebras exhibits numerous valuable connections, which are effectively utilized in areas such as homological algebra. Within this context, various intriguing extensions of groups have been explored, including groupoids, crossed complexes, crossed modules, double groupoids, and $\omega$-groupoids. Similarly, as groupoids extend the notion of groups, algebroids arise as a generalization of algebras, a theory developed by Mitchell [1].

Brown and Higgins established a correspondence between the class of $\omega$-groupoids and complexes (over groupoids). In Mosa’s thesis [2], a similar framework is constructed within an algebraic setting, aiming to explore potential applications in non-abelian homological and homotopical algebra. Mosa successfully demonstrates an equivalence between the category of crossed modules (over algebroids) and the category of special double algebroids with connections.

The definition of braided crossed modules captures symmetries inherent in the underlying crossed modules. Braided crossed modules expand the study of Lie algebras and open up possibilities for applications across a wide range of subjects where symmetry is important. Brown and Gilbert in [3], when defining braided regular crossed modules over groupoids as algebraic models for homotopy-connected 3-types, first considered the monoidal closed category of crossed modules over groupoids. Then, they took into account dimensions one and two of a crossed complex $C$ of groupoids with a multiplication $\mu$ over $C$. The 2-truncated crossed complex with multiplication in this structure, $tr_2(C, \mu)$, results in a braided regular crossed module [3], where the object set $C_0$ representing the primary component of $C$ embodies a group structure, and the binary operation is supplied by $\mu_{00}$. Taken as a trivial group, $C_0$ yields a crossed complex $C$ with $\mu$ that is reduced, and; this structure gives rise to a crossed differential graded algebra over groups, as referenced...
in [4]). Consequently, a braided crossed module of groups results from the 1-truncation of a crossed differential graded algebra of groups. Ulualan established the notion of braiding for a crossed module of Lie algebras as a Lie algebraic version of that provided by Brown and Gilbert in [3]. Particularly, Lie algebras have been used extensively to examine the symmetries and transformations that preserve the basics of mathematical structures, including vector spaces, manifolds, and groups. Ulualan also demonstrated that the category of braided crossed modules of associative algebras is equivalent to that of braided categorical associative algebras.

In this work, we will give a definition of an appropriate tensor product “⊗” in the category of crossed complexes of Lie algebras and then we obtain a definition of crossed differential graded Lie algebra (dgla) \((C, \mu, \otimes)\), where \(C\) is a crossed complex of Lie algebras and \(\mu\) is a multiplication map from \(C \otimes C\) to \(C\), satisfying particular conditions given in Section 2. Using this structure, we can demonstrate that a braided crossed module of Lie algebras, as described in Ulualan, results from the 2-truncation of a crossed dgla, \(tr_2(C, \mu, \otimes)\). In Section 3, using the tensor product \(\otimes\) for the category of crossed complexes of Lie algebras, we define a multiplication map \(\mu : L \otimes L \to L\) as a bimorphism in the category of crossed complexes of Lie algebras to define a crossed differential graded algebra (dgla) of Lie algebras. Then, we show that a 2-truncated crossed dgla, together with a non-zero map \(\mu_{11} : L_1 \otimes L_1 \to L_2\) and zero maps \(\mu_{ij} : L_i \otimes L_j \to L_{i+j}\) for \(i + j \geq 3\), is a braided crossed module of Lie algebras. In [5], the construction of a functor from the category of simplicial Lie algebras to the category of braided crossed modules of Lie algebras is presented. However, it is mentioned that the inverse of this functor, which would allow for the transformation of objects from the category of braided crossed modules back to the category of simplicial Lie algebras, is left as an open problem for readers to investigate.

In Section 4, as an application of braided crossed modules of Lie algebras, we construct a functor from the category of braided crossed modules of Lie algebras to that of reduced simplicial Lie algebras.

In order to develop a theory of crossed complexes over algebras and the desire to define a tensor product for crossed complexes of algebroids, we reflect upon the open question proposed by Brown in [6]. Brown stated that crossed complexes over groupoids exhibit valuable connections and have applications in homological and homotopical algebra. However, this closed structure does not exist in the reduced case (i.e., one vertex case) or for groups, which limits the scope of the theory. To overcome this limitation, one would expect to develop a satisfactory theory of crossed complexes over algebroids. Keeping in mind what is done for groups and groupoids, our aim is to define the tensor product of crossed complexes of Lie algebras. We explore the definition of appropriate tensor products and crossed differential graded Lie algebra (dgla) structures.

2. Preliminaries

Let \(L\) be a fixed commutative ring. By a Lie algebra, we mean a unitary \(L\)-bimodule \(A\) endowed with a bilinear transformation \(L \times L \to L\) called a Lie bracket, satisfying antisymmetry and the Jacobi identity. That is, if \([X, Y] = -[Y, X]\) for all elements \(X\) and \(Y\) in the Lie algebra, the Lie bracket operation is antisymmetric, and the Jacobi identity expresses the compatibility of the Lie bracket operation with triple brackets, i.e., for any three elements \((X, Y,\) and \(Z)\) in the Lie algebra, the following equalities hold:

\[
[X, Y] = -[Y, X] \quad \text{(Anticommutativity)}
\]

and

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{(Jacobi identity)}
\]

A Lie algebra morphism, or a homomorphism of Lie algebras, is a mapping between two Lie algebras that preserve the Lie bracket operation. More formally, let \(L\) and \(L'\) be two Lie algebras. A Lie algebra morphism is a linear map \(f : L \to L'\) that satisfies the following condition:
\[ f([X,Y]) = [f(X), f(Y)] \]

for all elements \( X,Y \), in \( L \). The category of all Lie algebras will be denoted by \( \text{LieAlg} \).

**Definition 1** ([7]). Let \( L_1 \) and \( L_2 \) be two Lie \( k \)-algebras, then a Lie algebra action of \( L_1 \) on \( L_2 \) is a \( k \)-bilinear map

\[
\begin{array}{c}
L_1 \times L_2 \rightarrow L_2 \\
(l_1, l_2) \mapsto l_1 \triangleright l_2
\end{array}
\]

that satisfies the following axioms:

\[
\begin{align*}
k(l_1 \triangleright l_2) &= (kl_1) \triangleright l_2 = l_1 \triangleright (kl_2) \\
l_1 \triangleright (l_2 + l_2') &= l_1 \triangleright l_2 + l_1 \triangleright l_2' \\
(l_1 + l_1') \triangleright l_2 &= l_1 \triangleright l_2 + l_1' \triangleright l_2 \\
[l_1, l_1'] \triangleright l_2 &= l_1 ([l_1, l_2] - l_2 [l_1, l_2]) \\
l_1 \triangleright [l_2, l_2'] &= [l_1 \triangleright l_2, l_2'] + [l_2, l_1 \triangleright l_2']
\end{align*}
\]

for \( l_i, l_i' \in L_i \) for \( i = 1, 2 \) and \( k \in k \).

**Example 1.** Let \( A \) be a unitary and commutative ring and let \( L \) be an algebra over \( A \). Define the function

\[
[-,-] : L \times L \rightarrow L \\
(l_1, l_2) \mapsto [l_1, l_2] = l_1 l_2 - l_2 l_1
\]

then we have

\[
[l_1, l_2] = l_1 l_2 - l_2 l_1 = (-1) (l_2 l_1 - l_1 l_2) = - [l_2, l_1]
\]

and

\[
[l_1, [l_2, l_3]] + [l_2, [l_3, l_1]] + [l_3, [l_1, l_2]] = [l_1, l_2 l_3 - l_3 l_2] + [l_2, l_3 l_1 - l_1 l_3] + [l_3, l_1 l_2 - l_2 l_1] = l_1 (l_2 l_3 - l_3 l_2) - (l_2 l_3 - l_3 l_2) l_1 + l_2 (l_3 l_1 - l_1 l_3) = -(l_3 l_1 - l_1 l_3) l_2 + l_3 (l_1 l_2 - l_2 l_1) - (l_1 l_2 - l_2 l_1) l_3 = 0
\]

for \( l_1, l_2, l_3 \in L \). Thus, \( L \) is a Lie algebra with \([-,-]\).

**Definition 2** ([7,8]). A pre-crossed module of Lie algebras consists of a homomorphism of Lie algebras \( \partial : L_2 \rightarrow L_1 \) with the action of \( L_1 \) on \( L_2 \), denoted by \( (l_1, l_2) \mapsto l_1 \triangleright l_2 \), for \( l_1 \in L_1 \), \( l_2 \in L_2 \). One requires that the following identity hold:

\[
\partial(l_1 \triangleright l_2) = [l_1, \partial(l_2)]
\]

for all \( l_2 \in L_2 \) and \( l_1 \in L_1 \). If we also have

\[
\partial(l_2) \triangleright l_2' = [l_2, l_2']
\]

then \( \partial : L_2 \rightarrow L_1 \) is called a crossed module.
Definition 3 ([7,8]). A morphism of crossed modules of Lie algebras from \( L_2 \to L_1 \) to \( L'_2 \to L'_1 \) is given by a commutative diagram

\[
\begin{array}{ccc}
L_2 & \xrightarrow{\partial} & L_1 \\
\downarrow f_1 & & \downarrow f_0 \\
L'_2 & \xrightarrow{\delta'} & L'_1
\end{array}
\]

such that \( f_1(l_1 \triangleright l_2) = f_1(l_2) \triangleright (l_1) \).

We denote the category of braided crossed modules of Lie algebras by \( \text{XMod}_{\text{Lie}} \).

Example 2. Let \( L \) be any \( A \)-bimodule. \( L \) is a Lie algebra with multiplication \( L \times L \to L, (l_1, l_2) \mapsto [l_1, l_2] = 0 \). Define the action of \( A \) on \( L \) as

\[
A \times L \to L, \quad (a, l) \mapsto a \triangleright l = [a, l]
\]

for \( a \in A \) and \( l \in L \). Since for the zero morphism \( 0 : L \to A \), we have

\[
0(a \triangleright l) = 0[a, l] = 0 = [a, 0(l)]
\]

for \( a \in A \) and \( l \in L \) and

\[
0(l_1) \triangleright l_2 = [0(l_1), l_2] = 0 = [l_1l_2]
\]

for \( l_1, l_2 \in L \), the zero morphism \( 0 : L \to A \) is a crossed module of Lie algebras.

Definition 4 ([5]). A braided crossed module of Lie algebras is a crossed module of Lie algebras \( \delta_2 : L_2 \to L_1 \) together with a braiding map \( \{ \cdot \otimes \cdot \} : L_1 \otimes L_1 \to L_2 \), satisfying the following conditions

1. \( \delta_2\{l_1, l'_1\} + [l_1, l'_1] = [l_1, l'_1] \)
2. \( \{l_1, \delta_2(l_2)\} + l_2 \triangleright l_1 = l_1 \triangleright l_2 \)
3. \( \{\delta_2(l_2) \otimes l_1\} + l_1 \triangleright l_2 = l_2 \triangleright l_1 \)
4. \( \{\delta_2(l_2'), \delta_2(l'_2)\} + [l'_2, l'_2] = [l'_2, l'_2] \)
5. \( \{l_1, [l'_1, l''_1]\} = l'_1 \triangleright \{l_1, l''_1\} + \{l_1, l'_1\} \triangleright l''_1 \)
6. \( \{[l_1, l'_1], l''_1\} = l'_1 \triangleright \{l_1, l''_1\} + \{l_1, l'_1\} \triangleright l''_1 \)

for all \( l_i \in L_i \) for \( i = 1, 2 \).

Morphisms of braided crossed modules of Lie algebras are defined for crossed modules of Lie algebras. A morphism of braided crossed modules of Lie algebras from \( L_2 \to L_1 \) to \( L'_2 \to L'_1 \) is given by a commutative diagram:

\[
\begin{array}{ccc}
L_1 \times L_1 & \xrightarrow{\{\cdot, \cdot\}} & L_2 \\
\downarrow \phi_1 \times \phi_0 & & \downarrow \phi_0 \\
L'_1 \times L'_1 & \xrightarrow{\{\cdot, \cdot\}'} & L'_2 \\
\end{array}
\]

preserving the action of \( L_1 \) on \( L_2 \). We denote the category of braided crossed modules of Lie algebras by \( \text{BXMod}_{\text{Lie}} \).
A Lie algebroid $L$ is a category equipped with a $k$-module structure on each hom set, denoted as $L_1(x,y)$, for objects $x$ and $y$ in the set of objects $L_0$. The composition in $L$ is required to be $k$-bilinear.

To define a Lie algebroid $L$ on a set of objects $L_0$, we start with a directed graph $L = (L_1, L_0)$, where functions $s$ and $t$ are the source and target maps, respectively, and $e$ is the identity map on $L_0$. These maps satisfy $se = te = 1_{L_0}$.

We denote $L_1(x,y)$ as the set of morphisms in $L$ from object $x$ to object $y$. For an element $l$ in $L$, which is an element of $L_1(x,y)$, we write it as $l : x \rightarrow y$. The notation $1_x$ represents the identity morphism in $L_1$ for object $x$.

A Lie algebroid $L = (L_1, L_0, s, t, e, c)$ consists of the following:

(i) Each $L_1(x,y)$, for $x$ and $y$ in $L_0$, is equipped with a $k$-module structure.

(ii) There exists a $k$-bilinear function, denoted as $\circ : L_1 \times L_1 \rightarrow L_1$, which represents the composition of morphisms in $L$. For elements $a$ and $b$ in $L_1$, $a \circ b$ is their composition.

The axioms for Lie algebroid $L$ are as follows:

Associativity: Composition in $L$ is associative, meaning that for morphisms $a$, $b$, and $c$ in $L_1$, $(a \circ b) \circ c = a \circ (b \circ c)$ whenever defined.

Identity: The elements $1_x$ and $1_y$, for objects $x$ and $y$ in $L_0$, act as identities for composition. If $a : x \rightarrow y$, then $1_x \circ a = a \circ 1_y = a$.

The zero element of $L_1(x,y)$ is denoted as $0_{xy}$. The Lie algebroid $L = (L_1, L_0, s, t, e, c)$ can be represented by a diagram, as follows:

$$L : (L_1 \xrightarrow{s} L_0 \xleftarrow{t} L_0)$$

The notion of “whiskering” for an arbitrary category $C$ with a set of objects written as $C_0$ is defined by Brown [9]. Next, we adapt this definition for Lie algebroids.

**Definition 5.** A whiskering on the Lie algebroid $L$ consists of operations

$$\mu_{ij} : L_i \times L_j \rightarrow L_{i+j}, \quad i, j = 0, 1$$

satisfying the conditions W1–W5:

**W1.** $\mu_{00}$ gives a Lie algebra structure on $L_0$.

**W2.**

$$s(\mu_{01}(l_0, l_1)) = \mu_{00}(l_0, s(l_1)), \quad t(\mu_{01}(l_0, l_1)) = \mu_{00}(l_0, t(l_1)),$$

$$s(\mu_{10}(l_1, l_0)) = \mu_{00}(s(l_1), l_0), \quad t(\mu_{10}(l_1, l_0)) = \mu_{00}(t(l_1), l_0)$$

**W3.**

$$\mu_{01}(\mu_{00}(l_0, l'_0), l_1) = \mu_{01}(l_0, \mu_{01}(l'_0, l_1)), \quad \mu_{10}(l_1, \mu_{00}(l_0, l'_0)) = \mu_{10}(\mu_{10}(l_1, l_0), l'_0).$$

**W4.** If $t(l_1) = s(l'_1)$:

$$\mu_{01}(x, l_1 \circ l'_1) = \mu_{01}(x, l_1) \circ \mu_{01}(x, l'_1),$$

$$\mu_{10}(l_1 \circ l'_1, l_0) = \mu_{10}(l_1, l_0) \circ \mu_{10}(l'_1, l_0),$$

$$\mu_{01}(l_0, e(l'_0)) = e(\mu_{00}(l_0, l'_0)) = \mu_{10}(e(l_0), l'_0)$$

**W5.**

$$k(\mu_{01}(l_0, l_1)) = \mu_{01}(l_0, kl_1), \quad (\mu_{01}(l_0, l_1))k = \mu_{01}(l_0, l_1k),$$

$$k(\mu_{10}(l_0, l_1)) = \mu_{10}(kl_0, l_1), \quad (\mu_{10}(l_0, l_1))k = \mu_{10}(l_1k, l_0)$$
and if \( t(l_1) = t(l'_1), s(l_1) = s(l'_1) \)

\[
\begin{align*}
\mu_{01}(l_0, l_1 + l'_1) &= \mu_{01}(l_0, l_1) + \mu_{01}(l_0, l'_1), \\
\mu_{10}(l_1 + l'_1, l_0) &= \mu_{10}(l_1, l_0) + \mu_{10}(l'_1, l_0)
\end{align*}
\]

where \( k \in k, l_i, l'_i \in L_i \) for \( i = 0, 1 \).

If \( L \) is a whiskered Lie algebroid, then the two multiplications \( l, r \) on \( L_1 \) can be defined as

\[
l(l_1, l'_1) = \mu_{01}(s(l_1), l'_1) \circ \mu_{10}(l_1, t(l'_1)), \quad r(l_1, l'_1) = \mu_{10}(l_1, s(l'_1)) \circ \mu_{01}(t(l_1), l'_1)
\]

which implies

\[
l(l_1, l'_1) \circ r(l_1, l'_1) : \mu_{00}(s(l_1), s(l'_1)) \to \mu_{00}(t(l_1), t(l'_1)).
\]

and the Lie bracket of \( l_1, l'_1 \) is given by

\[
[l_1, l'_1] = -r(l_1, l'_1) + l(l_1, l'_1).
\]

In order to define the crossed module in the category of Lie algebroids, the notion of a Lie algebroid action must first be defined. We can explain the connection between Lie algebroids and their transformations using this notion, which is similar to the one that appeared in G.H. Mosa’s Ph.D. thesis [2].

Let \( L_0 \) be a set and \( L_1, L_2 \) be two Lie algebroids over \( L_0 \). Assume that \( L_1 \) acts on \( L_2 \) on the right and on the left as follows:

Let \( l_2 \in L_2(x, y) \) and \( l_1 \in L_1(w, x), l'_1 \in L_1(y, z) \), then we denote the right action \( l_2 \triangleright l'_1 \in L_2(x, z) \), and the left action of \( l_1 \) on \( l_2 \) by \( l_1 \triangleright l_2 \in L_2(w, y) \), such that these actions satisfy the following axioms:

\[
\begin{align*}
(i) \quad & l_1 \triangleright (l_2 \triangleright l'_1) = l_1 \triangleright (l_2 \triangleright l'_1) \quad & l_1 \triangleright (l_1 \triangleright l_2) = [l'_1, l_1] \triangleright l_2 \\
(ii) \quad & (l_2 \triangleright l_1) \triangleright l'_1 = l_2 \triangleright [l_1, l'_1] \quad & (l_1 \triangleright l_2) \triangleright l_1 = [l'_1, l_1] \triangleright l_2 \\
(iii) \quad & l_2 \triangleright (l_1 + l'_1) = l_2 \triangleright l_1 + l_2 \triangleright l'_1 \quad & (l_1 + l'_1) \triangleright l_2 = l_1 \triangleright l_2 + l'_1 \triangleright l_2 \\
(iv) \quad & (l_2 + l'_2) \triangleright l_1 = l_2 \triangleright l_1 + l'_2 \triangleright l_1 \quad & l_1 \triangleright (l_2 + l'_2) = l_1 \triangleright l_2 + l_1 \triangleright l'_2 \\
(v) \quad & (k l_2) \triangleright l'_1 = k (l_2 \triangleright l'_1) = l_2 \triangleright (k l'_1) \quad & l_1 \triangleright (k l_2) = k (l_1 \triangleright l_2) = (k l_1) \triangleright l_2 \\
(vi) \quad & l_3 \triangleright l_2 = l_2 = l_2 \triangleright 1_y
\end{align*}
\]

for all \( l_i, l'_i \in L_i \) for \( i = 1, 2, x, y \in L_0 \) and \( k \in k \).

**Definition 6.** Let \( L_1, L_2 \) be two Lie algebroids over the same set \( L_0 \). A morphism \( \partial : L_2 \to L_1 \) is called a crossed module of Lie algebroids if there exists left and right actions of \( L_1 \) on \( L_2 \), satisfying the following axioms:

**XMLA1.** \( \partial(l_2 \triangleright l'_2) = [\partial(l_2), l'_1], \quad \partial(l_1 \triangleright l_2) = [l_1, \partial(l_2)] \)

**XMLA2.** \( l_2 \triangleright \partial(l_2) = [l_2, l'_2] = \partial(l_2) \triangleright l'_2 \)

for all \( l_i, l'_i \in L_i \) for \( i = 1, 2 \) and both sides are defined. We can show a crossed module of Lie algebroids \( \partial \) from \( L_2 \) to \( L_1 \) by the following diagram:
Morphisms of crossed modules of Lie algebroids

\((\varphi, \phi) : (L_2, L_1, \partial) \longrightarrow (L'_2, L'_1, \partial')\)

are pairs of Lie algebra morphisms \(\varphi : L_2 \to L'_2, \phi : L_1 \to L'_1\), such that the following diagram

\[
\begin{array}{ccc}
L_2 & \xrightarrow{\varphi} & L'_2 \\
\downarrow{\partial} & & \downarrow{\partial'} \\
L_1 & \xrightarrow{\phi} & L'_1
\end{array}
\]

commutes, and \(\varphi(l_1 \triangleright l_2) = \phi(l_1) \triangleright \varphi(l_2)\) and \(\varphi(l_2 \triangleright l_1) = \varphi(l_2) \triangleright \phi(l_1)\), for all \(l_i \in L_i\) for \(i = 1, 2\). Thus, the category of crossed modules of Lie algebroids is obtained, and it is denoted by \(\text{XMod}_{\text{Algrd}}\).

### 3. Crossed Complexes over Lie Algebra(oid)s

Mosa, in [2], defined crossed complexes of algebroids and explained the relations between truncated cases of crossed complexes of algebroids and \(n\)-tuple algebroids, particularly for dimension 3. In this section, we define crossed complexes over Lie algebroids.

A crossed complex \(LC\) of Lie algebroids consists of Lie algebroid morphisms over \(L_0\)

\[
\begin{array}{ccc}
\cdots & L_3 & L_2 & L_1 \\
\downarrow{\sigma_3} & \downarrow{\sigma_2} & \downarrow{\sigma_1} \\
L_0 & L_0 & L_0
\end{array}
\]

satisfying the following conditions:

(i) each \(\sigma_n : L_n \to L_{n-1}, n \geq 2\), is the identity on \(L_0\);

(ii) \(L_1\) acts on each \(L_n\) for \(n \geq 2\) with the Lie algebroid action;

(iii) \(\sigma_2 : L_2 \to L_1\) is a crossed module of Lie algebroids and \(\sigma(L_2)\) induced action on \(L_n\) for \(n \geq 3\);

(iv) for \(n \geq 2\), \(\sigma_n : L_n \to L_{n-1}\) preserves the actions of \(L_1\) on \(L_n\), where \(L_1\) acts on itself by the composition;

(v) for \(n \geq 3\), \(\sigma_n \sigma_{n+1} = 0\).

A reduced crossed complex \(LC\) of Lie algebroids with \(L_0 = \{\ast\}\) is a crossed complex of Lie algebras. Next, we give the definition of the crossed complex of Lie algebras.

**Definition 7.** A crossed complex of Lie algebras is a chain complex of Lie algebras

\[
\begin{array}{ccc}
\cdots & L_3 & L_2 & L_1 & \{\ast\} \\
\downarrow{\sigma_3} & \downarrow{\sigma_2} & \downarrow{\sigma_1} \\
L_0 & L_0 & L_0 & \{\ast\}
\end{array}
\]

in which

(i) \(\sigma_2 : L_2 \to L_1\) is a crossed module of Lie algebras;

(ii) for \(n > 2\), \(L_n\) is a \(L_1\)-module and \(\sigma_2(C_2)\) acts trivially. That is, \(\sigma_n\) is an \(L_1\)-module homomorphism;

(iii) for \(n \geq 2\), \(\sigma_n \sigma_{n+1} = 0\).

We will denote crossed complexes of Lie algebras with \(\text{CrsC}_{\text{Lie}}\). In light of this, we consider a crossed complex to be a chain complex of Lie algebras formed up of a base crossed module of Lie algebras and a tail that is a complex of modules.
Next, our aim is to define an appropriate tensor product “ ⊗ ” for the category of crossed complexes over Lie algebras, with the notion of the coproduct for crossed modules of Lie algebras given in [10] to ensure the universality of tensor product in each dimension. The two prerequisites that Baues lists [11] when considering a tensor product are equivalent on the definition we provide, is also a significant influence. Thus, we start by defining the Higgins tensor product for crossed complexes [12] (see also [4,13], which has a major effect on the definition we provide, is also a significant influence. Thus, we start by defining the term “tensor product of crossed complexes of Lie algebras” in the manner that will be the most helpful to us.

**Definition 8.** Let $L$, $M$, and $N$ be reduced crossed complexes of Lie algebroids over the singleton \{*\}. A bimorphism $\theta : (L, M) \to N$ is a family of maps

$$\theta_{x,y} : L_x \times M_y \to N_{x+y}$$

for $x \geq 0, y \geq 0$, which satisfy the following conditions: where $l_x \in L_m$ and $m_x \in M_y$,

1. 
   $$\theta_{x,y}(l_x, m_1 \bullet m_y) = \theta_{0,1}(*, m_1) \bullet \theta_{x,y}(l_x, m_y) \quad \text{if } y \geq 2, x \geq 1$$
   $$\theta_{0,0}(*, m_1 \bullet m_y) = \theta_{0,1}(*, m_1) \bullet \theta_{0,y}(*, m_y)$$
   $$\theta_{x,y}(l_x \bullet m_1, m_y) = \theta_{x,y}(l_x, m_y) \bullet \theta_{0,1}(*, m_1)$$
   $$\theta_{0,y}(x, m_1 \bullet m_y) = \theta_{0,y}(m_1, m_y) \bullet \theta_{x,y}(l_x, *)$$
   $$\theta_{0,y}(l_x \bullet l_1, m_y) = \theta_{x,y}(l_x, m_y) \bullet \theta_{0,0}(l_1, *)$$

2. 
   $$\theta_{x,0}(l_x + l'_x, m_y) = \theta_{x,0}(l_x, *) + \theta_{x,0}(l'_x, *) \quad \text{if } x \geq 1, y = 0$$
   $$\theta_{x,y}(l_x + l'_x, m_y) = \theta_{x,y}(l_x, m_y) + \theta_{x,y}(l'_x, m_y) \quad \text{if } x \geq 1, y \geq 1$$
   $$\theta_{x,r}(k l_x + l'_x, m_y) = k \theta_{x,y}(l_x, m_y) + \theta_{x,y}(l'_x, m_y) \quad \text{for } k \in k$$

3. 
   $$\theta_{x,r}(l_x \circ l'_x, m_y) = \theta_{x,0}(l_x, *) \bullet \theta_{x,r}(l'_x, m_y) \circ \theta_{x,y}(l_x, m_y) \bullet \theta_{x,0}(l'_x, *)$$

4. 
   $$\theta_{x,y}(*, m_y + m'_y) = \theta_{0,0}(*, m_y) + \theta_{0,0}(*, m'_y) \quad \text{if } x = 0, y \geq 1$$
   $$\theta_{x,y}(l_x, m_y + m'_y) = \theta_{x,y}(l_x, m_y) + \theta_{x,y}(l_x, m'_y) \quad \text{if } x \geq 1, y \geq 1$$
   $$\theta_{y,m}(l_x, k m_y + m'_y) = k \theta_{y,m}(l_x, m_y) + \theta_{y,m}(l_x, m'_y) \quad \text{if } k \in k$$

5. 
   $$\sigma_{x}(\theta_{x,y}(l_x, *)) = \theta_{x,0}(\sigma_x(l_x), *) \quad \text{if } x \geq 2, y = 0$$
   $$\sigma_{x}(\theta_{0,1}(*, m_y)) = \theta_{0,1}(*, m_y) \quad \text{if } x \geq 2, y = 0$$
   $$\sigma_{x}(\theta_{1,1}(l_1, m_1)) = \theta_{1,1}(l_1, *) \circ \theta_{0,1}(*, m_1) - \theta_{0,1}(l_1, m_1) \circ \theta_{1,0}(l_1, *) \quad \text{if } x = y = 1$$
   $$\sigma_{x,y}(\theta_{x,y}(l_x, m_y)) = \theta_{x,y}(\sigma_x(l_x), m_y) + \theta_{x,y}(l_x, \sigma_y(m_y)) \quad \text{if } y \geq 1, x \geq 2$$
   $$\sigma_{x,y}(\theta_{x,y}(l_x, m_y)) = \sigma_{x,y}(\theta_{x,y}(l_x, m_y)) \quad \text{if } x = 1, y \geq 2$$
   $$\sigma_{x,y}(\theta_{x,y}(l_x, m_y)) = -\theta_{x,y}(l_x, \sigma_y(m_y)) - \theta_{x,y}(l_x, \sigma_y(m_y)) \quad \text{if } x = 1, y \geq 2$$

We will now define and construct a tensor product for Lie algebraic crossed complexes using the constructions (coproduct, free [10]) of Lie algebraic crossed modules. The Brown–Higgins tensor product for crossed complexes [12] (see also [4,13], which has a major effect on the definition we provide, is also a significant influence. Thus, we start by defining the term “tensor product of crossed complexes of Lie algebras” in the manner that will be the most helpful to us.
Definition 9. The tensor product $L \otimes M$ of the crossed complexes $L$ and $M$ is then given by the universal morphism $(L,L) \rightarrow L \otimes M$, defined as follows:

Dimension 1: $(L \otimes M)_1$ is the coproduct of algebras $L_1$ and $M_1$. It is a free product $L_1 \ast M_1$ with generators $\ast \otimes m_1$ and $l_1 \otimes \ast$, where $l_1 \otimes \ast$ is the image of $l_1$ by the canonical inclusion $L_1 \rightarrow (L \otimes M)_1$, and similarly, $\ast \otimes m_1$ is the image of $m_1$ by the canonical inclusion $M_1 \rightarrow (L \otimes M)_1$. Then, we have

$$ (l_1 \otimes \ast) + (l_1' \otimes \ast) = (l_1 + l_1') \otimes \ast \quad \text{and} \quad (\ast \otimes m_1) + (\ast \otimes m_1') = (\ast \otimes m_1 + m_1') $$

$$ [l_1 \otimes \ast], (l_1' \otimes \ast) = [l_1', l_1 \otimes \ast] \quad \text{and} \quad [(\ast \otimes m_1), (\ast \otimes m_1')] = (\ast \otimes [m_1, m_1']) $$

$$ k[l_1 \otimes \ast], (l_1' \otimes \ast)] = k[l_1', l_1 \otimes \ast] \quad \text{and} \quad [(\ast \otimes m_1), (\ast \otimes m_1') k] = (\ast \otimes [m_1, m_1']) k $$

for $l_1, l_1' \in L_1$, $m_1, m_1' \in M_1$ and $k \in k$.

Dimension 2: In dimension 2, the second term of the tensor crossed complex, $(L \otimes M)_2$, is the crossed $(L \otimes M)_1$-module of Lie algebras defined by the coproduct of $L_2'/P$, $(M_2'/P)$ and $(L_1 \times M_1)^*$, where $P$ is the Peiffer ideal and $L_2'$ and $M_2'$ are the pre-crossed $(L \otimes M)_1^*$-modules induced by the canonical maps $L_1 \rightarrow (L \otimes M)_1$ and $M_1 \rightarrow (L \otimes M)_1$, and $(L_1 \times M_1)^*$ is the crossed $(L \otimes M)_1$-module associated with the double algebroid of Lie algebras $(L_1 \times M_1, \ast \times M_1, L_1 \times \{\ast\}, \{\ast\} \times \{\ast\})$. Thus, $(L \otimes M)_2$ has generators $l_2 \otimes \ast, \ast \otimes m_2$ and $l_1 \otimes m_1$ with the relations:

$$ (l_2 \otimes \ast) + (l_2' \otimes \ast) = (l_2 + l_2') \otimes \ast \quad \text{and} \quad (l_1 \otimes m_1) + (l_1' \otimes m_1) = (l_1 + l_1') \otimes \ast $$

$$ (l_1 \otimes \ast) \triangleright (l_2 \otimes \ast) = (l_1 \otimes l_2) \otimes \ast \quad \text{and} \quad (l_2 \otimes m_2) \triangleright (l_1 \otimes \ast) = (l_2 \otimes l_1) \otimes \ast $$

and

$$ (l_1 \otimes m_1, l_1' \otimes m_1) = (l_1 \otimes \ast) \triangleright (l_1' \otimes m_1) \quad \text{and} \quad (l_1 \otimes \ast) \triangleright (l_1' \otimes m_1) = (l_1 \otimes l_1') \otimes \ast $$

and the operator $\sigma_2 : (L \otimes M)_2 \rightarrow (L \otimes M)_1$ is given on generators by

$$ \sigma_2(l_2 \otimes \ast) = \sigma_2(l_2) \otimes \ast \quad \text{and} \quad \sigma_2(\ast \otimes m_2) = \ast \otimes \sigma_2(m_2) $$

$$ \sigma_2(l_1 \otimes m_1) = (l_1 \otimes \ast)(\ast \otimes m_1)(-l_1 \otimes \ast)(\ast \otimes -m_1) $$

Dimension 3: $(L \otimes M)_p$ is the coproduct of $(L \otimes M)_1$-modules $N_i$, $i = 0, 1, \ldots, p$. Each $N_i$ is defined from a $(L \otimes M)_1$-module $N'_i$ by imposing the relation $\sigma_2(n_2) \triangleright n = n \triangleright \sigma_2(n_2)$ for $n_2 \in (L \otimes M)_2$ and $n \in N'_i$. These modules are given by the induced $(L \otimes M)_1$-module $L'_p$ and $M'_p$ via the canonical morphisms $L_1 \rightarrow (L \otimes M)_1$ and $M_1 \rightarrow (L \otimes M)_1$. The generators will be written as $l_p \otimes \ast$ and $\ast \otimes m_p$, for $l_p \in L_p$ and $m_p \in M_p$, and their relations are

$$ (l_p \otimes \ast) + (l_p' \otimes \ast) = (l_p + l_p') \otimes \ast \quad \text{and} \quad (\ast \otimes m_p) + (\ast \otimes m_p') = (\ast \otimes m_p + m_p') $$

$$ (\ast \otimes m_1) \triangleright (\ast \otimes m_p) = (\ast \otimes m_1) \triangleright (\ast \otimes m_1) = (\ast \otimes m_1) \triangleright (\ast \otimes m_1) $$

To summarize, the crossed complex $L \otimes M$ over $\ast$ is generated by elements $l \otimes m, l \otimes \ast, \ast \otimes m$, where $l \in L, m \in M$ with the following relations:

1. $|l \otimes m| = |l| + |m|$, $|l \otimes \ast| = |l|, |\ast \otimes m| = |m|$

2. For $|l| \geq 1, |m| \geq 2, |m'| = 1$

$$ (l \otimes m \triangleright m' = (l \otimes m) \triangleright (\ast \otimes m') $$

$$ (l \otimes m') \triangleright m = (\ast \otimes m') \triangleright (l \otimes m) $$

3. For $|m| \geq 2, |m'| = 1$

$$ (\ast \otimes m \triangleright m' = (\ast \otimes m) \triangleright (\ast \otimes m') $$

$$ (\ast \otimes m') \triangleright m = (\ast \otimes m') \triangleright (\ast \otimes m) $$
4. For $|m| \geq 1, |l| \geq 2, |l'| = 1$
   \[(l \triangleright l' \otimes m) = (l \otimes *) \triangleright (l \otimes m)\]
   \[(l' \triangleright l \otimes m) = (l' \otimes *) \triangleright (l \otimes m)\]

5. For $|l| \geq 2, |l'| = 1$
   \[(l' \triangleright l \otimes *) = (l' \otimes *) \triangleright (l' \otimes *)\]
   \[(l' \triangleright l \otimes *) = (l' \otimes *) \triangleright (l \otimes *)\]

6. For $|m| \geq 1, |l| \geq 1, k \in k$
   \[(l + l') \otimes m = (l \otimes m) + (l' \otimes m)\]
   \[l \otimes (m + m') = (l \otimes m) + (l \otimes m')\]
   \[(l + l') \otimes * = (l \otimes *) + (l' \otimes *)\]
   \[* \otimes (m + m') = (* \otimes m) + (* \otimes m')\]
   \[(kl + l') \otimes m = k(l \otimes m) + (l' \otimes m)\]
   \[l \otimes (km + m') = k(l \otimes m) + (l \otimes m')\]

7. For $|m| \geq 1, |l| = |l'| = 1$
   \[[l, l'] \otimes m = (l \otimes *) \triangleright (l' \otimes m) + (l \otimes m) \triangleright (l' \otimes *)\]
   and for $|l| \geq 1, |m| = |m'| = 1$
   \[l \otimes [m, m'] = (l \otimes m) \triangleright (* \otimes m') + (* \otimes m) \triangleright (l \otimes m')\]

8. For $|l| \geq 2$
   \[\sigma(l \otimes *) = \sigma(l) \otimes *\]
   for $|m| \geq 2$
   \[\sigma(* \otimes m) = * \otimes \sigma(m)\]

   for $|m| = 2, |l| = 1$,
   \[\sigma((* \otimes m) \triangleright (l \otimes *)) = [((* \otimes \sigma(m)), (l \otimes *)]\]
   \[\sigma((l \otimes *) \triangleright (* \otimes m)) = [(l \otimes *), (* \otimes \sigma(m))]\]

   for $|l| \geq 2, |m| \geq 2$
   \[\sigma(l \otimes m) = \sigma(l) \otimes m + (-1)^{|l|}(l \otimes \sigma(m))\]

   for $|l| = 1, |m| \geq 2$
   \[\sigma(l \otimes m) = -(l \otimes \sigma(m)) - (* \otimes m) \triangleright (l \otimes *) \triangleright (* \otimes m)\]

   for $|m| = 1, |l| \geq 2$
   \[\sigma(l \otimes m) = (-1)^{|l|}(l \otimes m) \triangleright (l \otimes *) \triangleright (* \otimes m) + (\sigma(l) \otimes m)\]

   for $|m| = |l| = 1$ there are two possibilities depending on the action
   \[\sigma(l \otimes m) = -(l \otimes m) \triangleright (l \otimes *) + (l \otimes *) \triangleright (* \otimes m)\]
   \[\sigma(l \otimes m) = -(l \otimes m) \triangleright (l \otimes *) + (l \otimes *) \triangleright (* \otimes m)\]

   where $|l|$ is the degree of an element $l \in L$ that is if $|l| = p$ if $l \in L_p$. The sum $l + l'$ and $[l, l']$ of elements $l, l' \in L$ are defined only in case $|l| = |l'|$.

**Proposition 1.** The assignment $(L, M) \mapsto (L \otimes M)$ defines a functor

\[\text{Crs}_L \times \text{Crs}_L \to \text{Crs}_L\]

**Proof.** The boundaries $\sigma$ must be consistent with the relations and the criterion $\sigma^2 = 0$ must be met in order to demonstrate that $(L \otimes M)$ is a well-defined crossed complex of Lie algebras. For some of the cases, we will check that $\sigma^2 = 0$. 


For \( l_2 \in L_2 \) and \( m_1 \in M_1 \),
\[
\sigma_2\sigma_3(l_2 \otimes m_1) = \sigma_2((\ast \otimes m_1) \triangleright (l_2 \otimes \ast) + (l_2 \otimes \ast) \triangleright (\ast \otimes m_1) + \sigma_2 l_2 \otimes m_1) \\
= ((\ast \otimes m_1) \triangleright (\sigma_2 l_2 \otimes \ast) + (\sigma_2 l_2 \otimes \ast) \triangleright (\ast \otimes m_1) + \sigma_2 l_2 \otimes \sigma_2 m_1) \\
= ((\ast \otimes m_1) \triangleright (\sigma_2 l_2 \otimes \ast) + (\sigma_2 l_2 \otimes \ast) \triangleright (\ast \otimes m_1)) + ((\ast \otimes m_1) \triangleright (\sigma_2 l_2 \otimes \ast) - (\sigma_2 l_2 \otimes \ast) \triangleright (\ast \otimes m_1)) \\
= 0
\]

For \( l_1 \in L_1 \) and \( m_2 \in M_2 \),
\[
\sigma_2\sigma_3(l_1 \otimes m_2) = \sigma_2(-l_1 \otimes c_2 m_2 - (\ast \otimes m_2) \triangleright (l_1 \otimes \ast) + (l_1 \otimes \ast) \triangleright (\ast \otimes m_2)) \\
= -([\ast \otimes \sigma_2 m_2] \triangleright (l_1 \otimes \ast)) + ([\ast \otimes \sigma_2 m_2] \triangleright (\ast \otimes m_2)) \\
= 0
\]

For \( l_3 \in L_3 \) and \( m_1 \in M_1 \),
\[
\sigma_3\sigma_4(l_3 \otimes m_1) = \sigma_3(-[(\ast \otimes m_1) \triangleright (l_3 \otimes \ast) + (l_3 \otimes \ast) \triangleright (\ast \otimes m_1) + \sigma_3 l_3 \otimes m_1]) \\
= -([\ast \otimes \sigma_3 m_1] \triangleright (l_3 \otimes \ast)) + ([\ast \otimes \sigma_3 m_1] \triangleright (\ast \otimes m_1)) \\
= 0
\]

For \( l_1 \in L_1 \) and \( m_3 \in M_3 \),
\[
\sigma_3\sigma_4(l_1 \otimes m_3) = \sigma_3(-[l_1 \otimes \sigma_3 m_3 - (\ast \otimes m_3) \triangleright (l_1 \otimes \ast) + (l_1 \otimes \ast) \triangleright (\ast \otimes m_3)]) \\
= -([\ast \otimes \sigma_3 m_3] \triangleright (l_1 \otimes \ast)) + ([\ast \otimes \sigma_3 m_3] \triangleright (\ast \otimes m_3)) \\
= 0
\]

\( \square \)

For morphisms \( F : L \to L' \) and \( G : M \to M' \), the induced map
\[
F \otimes G : L \otimes M \to L' \otimes M'
\]
is defined by \( F \otimes G(l \otimes m) = F(l) \otimes G(m) \).

In the category of crossed complexes of Lie algebras, where the zero object \( \ast \) is 0 in each degree, we obtain
\[
L \otimes \ast \cong L \cong \ast \otimes L.
\]

The inclusions \( i_L : L \to L \otimes M, i_M : M \to L \otimes M \) and projections \( p_L : L \otimes M \to L, p_M : L \otimes M \to M \) are defined by \( i_L(l) = l \otimes \ast, i_M(m) = \ast \otimes m, p_L(l \otimes \ast) = l, p_M(\ast \otimes m) = m \), and \( p_L(l \otimes m) = 0, p_M(l \otimes m) = 0 \).

**Proposition 2.** For any two crossed complexes of Lie algebras, the interchange map
\[
T : L \otimes M \to M \otimes L
\]
defined as \( l \otimes m \mapsto (-1)^{|l|+|m|}(m \otimes l) \) commutes with the boundary maps.
Proof. For \( l_2 \in L_2 \) and \( m_1 \in M_1 \), we obtain
\[
T\sigma_3(l_2 \otimes m_1) = (-1)^2((\ast \otimes m_1) \triangleright (l_2 \otimes \ast) + (l_2 \otimes \ast) \triangleleft (\ast \otimes m_1)) + \sigma_2l_2 \otimes m_1
\]
\[
= \sigma_3(-(m_1 \otimes l_2)) = \sigma_3(T(l_2 \otimes m_1))
\]

For \( l_1 \in L_1 \) and \( m_2 \in M_2 \), we obtain
\[
T\sigma_3(l_1 \otimes m_2) = (\sigma_2(d) \otimes l_1) + (m_2 \otimes \ast) \triangleleft (\ast \otimes l_1) - (\ast \otimes l_1) \triangleright (m_2 \otimes \ast)
\]
\[
= \sigma_3(-(m_2 \otimes l_1)) = \sigma_3(T(l_1 \otimes m_2))
\]

In an analogous way, the verification of the remaining relations can be demonstrated. \( \square \)

4. Crossed Differential Graded Lie Algebras

Tonks [14] introduced theories of crossed differential graded algebras and coalgebras. In this section, using the tensor product \( \otimes \) for the category of crossed complexes of Lie algebras, we define a multiplication map \( \mu : L \otimes L \to L \) as a bimorphism in the category of crossed complexes of Lie algebras. Let

\[
L : \cdots \xrightarrow{\sigma_{n+1}} L_n \xrightarrow{\sigma_n} L_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_1} C_1
\]

be a crossed complex of Lie algebras.

Definition 10. A multiplication map \( \mu : L \otimes L \to L \) for this crossed complex is a family of maps \( \mu_{ij} : L_i \otimes_k L_j \to L_{i+j} \) satisfying the following conditions, where \( l_i \in L_i \) and \( l_j \in L_j \):

**M1.** For all \( i, j \geq 2 \) and \( l_1 \in L_1 \):

\[
\mu_{ij}(l_1 \triangleright l_1 \otimes l_j) = l_1 \triangleright \mu_{ij}(l_1 \otimes l_j)
\]
\[
\mu_{ij}(l_1 \triangleleft l_1 \otimes l_j) = \mu_{ij}(l_1 \otimes l_j) \triangleleft l_1 = \mu_{ij}(l_1 \otimes l_1 \triangleleft l_j)
\]

for \( i = j = 1 \) and \( l_1, l_1', l_1'' \in L_1 :

\[
\mu_{11}([l_1, l_1'] \otimes l_1'') = l_1 \triangleright \mu_{11}(l_1' \otimes l_1'') + \mu_{11}(l_1 \otimes l_1'') \triangleleft l_1'
\]
\[
\mu_{11}(l_1 \otimes l_1' l_1'') = \mu_{11}(l_1 \otimes l_1') \triangleleft l_1'' + l_1' \triangleright \mu_{11}(l_1 \otimes l_1'')
\]

**M2.** for \( i \geq 1, j \geq 1 \) and \( k \in k \),

\[
\mu_{ij}((kl_i + l_i') \otimes l_j) = k\mu_{ij}(l_i \otimes l_j) + \mu_{ij}(l_i' \otimes l_j)
\]
\[
\mu_{ij}(l_i \otimes (kl_j + l_j')) = k\mu_{ij}(l_i \otimes l_j) + \mu_{ij}(l_i \otimes l_j')
\]

**M3.** for \( i \geq 2, j \geq 2, \)

\[
\sigma_{i+j}(\mu_{ij}(l_i \otimes l_j)) = \mu_{i-1,j}(\sigma_i(l_i) \otimes l_j) + (-1)^i \mu_{i,j-1}(l_i \otimes \sigma_j(l_j))
\]

for \( i = 1, j \geq 2 \)

\[
\sigma_{i+j}(\mu_{ij}(l_1 \otimes l_j)) = -\mu_{1,j-1}(l_1 \otimes \sigma_{i-1}(l_j)) - l_j \triangleleft l_1 + l_1 \triangleright l_j
\]

for \( i \geq 2, j = 1, \)

\[
\sigma_{i+1}(\mu_{1j}(l_1 \otimes l_j)) = (-1)^i l_1 \triangleright l_j - l_j \triangleleft l_1 + \mu_{i-1,j}(\sigma_i l_j) \otimes l_1,
\]

and for \( i = j = 1, l_1, l_1' \in L_1 \)

\[
\sigma_2(\mu_{11}(l_1 \otimes l_1')) = [l_1, l_1'] - [l_1', l_1]
\]
A crossed complex of Lie algebras \( L \), together with the multiplication \( m : L \otimes L \to L \) that meets the specified conditions, constitutes a crossed differential graded algebra (dgla) of Lie algebras. We denote such a crossed dgla by \((L, \mu)\).

A \( k \)-truncation of \((L, \mu)\) is a crossed complex \( L \) of Lie algebras together with \( L_n = \{0\} \) for \( n \geq k+1 \) and the maps \( \mu_{ij} \) are zero maps for \( i+j \geq k+1 \).

**Theorem 1.** For \( k = 2 \). A 2-truncated crossed dgla

\[
\delta_r(L, \mu) : \cdots \to \{0\} \to \{0\} \xrightarrow{\sigma_2} L_2 \xrightarrow{\sigma_2} L_1
\]

together with a non-zero map \( \mu_{11} : L_1 \otimes L_1 \to L_2 \) and zero maps \( \mu_{ij} : L_i \otimes L_j \to L_{i+j} \) for \( i+j \geq 3 \) is a braided crossed module of Lie algebras.

**Proof.** Take \( \delta_2 = \sigma_2 \) and \( \{-, -\} = \mu_{11} \). Then, from axiom M2, the braiding map is \( k \)-bilinear.

\[
\mu_{11}((kl_1 + l'_1) \otimes l''_1) = k\mu_{11}(l_1 \otimes l''_1) + \mu_{11}(l'_1 \otimes l''_1)
\]

\[
\mu_{11}(l_1 \otimes (kl'_1 + l''_1)) = \mu_{11}(l_1 \otimes l'_1) + \mu_{11}(l_1 \otimes l''_1)
\]

From axiom M1, we have

B5. \( \mu_{11}(l_1 l'_1 \otimes l''_1) = l_1 \bullet l_1 \bullet l_1 \bullet l''_1 \)

B6. \( \mu_{11}(l_1 \otimes l'_1 l''_1) = \mu_{11}(l_1 \otimes l'_1 \otimes l''_1) + \mu_{11}(l_1 \otimes l'_1 \otimes l''_1) \)

From axiom M3, we have \( \sigma_3(\mu_{12}(l_1 \otimes l_2)) = 0 \) and, thus,

\[
0 = -\mu_{11}(l_1 \otimes \sigma_2(l_2)) - l_2 \triangleright l_1 + l_1 \triangleright l_2
\]

then, we obtain

B2. \( \mu_{11}(l_1 \otimes \sigma_2(l_2)) = l_1 \triangleright l_2 - l_2 \triangleright l_1 \)

Since \( \sigma_3(\mu_{21}(l_2 \otimes l_1)) = 0 \) and \( 0 = l_1 \triangleright l_2 - l_2 \triangleright l_1 + \mu_{11}(\sigma_2(l_2) \otimes l_1) \), we obtain

B1. \( \sigma_2\mu_{11}(l_1 \otimes l'_1) = [l_1, l'_1] - [l'_1, l_1] \)

B3. \( \mu_{11}(\sigma_2(l_2) \otimes l_1) = l_2 \triangleright l_1 - l_1 \triangleright l_2 \)

B4. \( \mu_{11}(\sigma_2(l_2) \otimes \sigma_2(l''_2)) = [l_2, l''_2] - [l''_2, l_2] \)

for all \( l_i \in L_i \) for \( i = 1, 2 \) and \( k \in \mathbb{k} \). \( \square \)

5. Braided Crossed Modules and Reduced Simplicial Algebras

In this section, as an application for braided crossed modules of Lie algebras, we will construct a functor from the category of braided crossed modules of Lie algebras to that of reduced simplicial Lie algebras. We recall the construction of the inverse of this functor given in [5].

A simplicial Lie algebra \( L \) consists of a family of Lie algebras \( L_n \) together with homomorphisms of Lie algebras \( d^n_i = E_n \to E_{n-1} \) \( 0 \leq i \leq n \) \((n \neq 0)\) and \( s^n_i = E_n \to E_{n+1} \) \( 0 \leq i \leq n \) called, respectively, face and degeneracy maps, which satisfy the usual simplicial identities given in [15]. A simplicial Lie algebra that has zero as its initial component is said to be reduced.

Given a simplicial Lie algebra \( LE \), the Moore complex \((NL, \partial)\) of this simplicial Lie algebra is the chain complex defined by

\[
(NL)_n = \ker d^n_0 \cap \ker d^n_1 \cap \cdots \cap \ker d^n_{n-1}
\]

with the boundaries \( \partial_n : NL_n \to NL_{n-1} \) induced from \( d^n_0 \) by the restriction. The length of the Moore complex \( NL \) is \( k \) if for all \( n > k \), \( NL_n = \{0\} \).
Akça and Arvasi defined the functions $C_{\alpha, \beta}$ for simplicial Lie algebras. They defined an ideal $L_n$ in $L_n$ for each $n \geq 1$, generated by elements $C_{\alpha, \beta}(x_\alpha \otimes y_\beta)$ for $x_\alpha \in NL_{n-#\alpha}$ and $y_\beta \in NL_{n-#\beta}$.

The $k$-bilinear morphisms
\[
\{ C_{\alpha, \beta} : NL_{n-#\alpha} \otimes NL_{n-#\beta} \to NL_n : (\alpha, \beta) \in P(n), n \geq 0 \}
\]
and
\[
\{ C_{\beta, \alpha} : NL_{n-#\beta} \otimes NL_{n-#\alpha} \to NL_n : (\alpha, \beta) \in P(n), n \geq 0 \}
\]
are given by $C_{\alpha, \beta}(x_\alpha \otimes y_\beta) = (1 - s_{n-1}d_{n-1})... (1 - s_0d_0)([s_\alpha(x_\alpha), s_\beta(y_\beta)])$, and similarly, $C_{\beta, \alpha}(y_\beta \otimes x_\alpha) = (1 - s_{n-1}d_{n-1})... (1 - s_0d_0)([s_\beta(y_\beta), s_\alpha(x_\alpha)])$ where $s_\alpha : NL_{n-#\alpha} \to L_n, s_\beta : NL_{n-#\beta} \to L_n$, $p : L_n \to NL_n$ are defined by composite projections $p = p_{n-1}...p_0$ with $p_j = 1 - sjd_j$ for $j = 0, 1, ..., n - 1$.

**Definition 11.** Let $L$ be a simplicial Lie algebra. For $n \geq 0$, let $M_n$ be the ideal of $L_n$ generated by degenerate elements, and suppose that $L_n = M_n$. For $n > 0$, $I_n$ is the ideal of $L_n$ generated by the elements of the forms $C_{\alpha, \beta}(x_\alpha \otimes y_\beta)$, and $C_{\beta, \alpha}(y_\beta \otimes x_\alpha)$ with $(\alpha, \beta) \in P(n)$.

**Example 3.** For $n = 2$ and $L_2 = M_2$. Thus, $\alpha = (0)$ and $\beta = (1)$. For $x, y \in NL_1 = \ker d_0$, the ideal $I_2$ of $NL_2$ is generated by the elements of the forms;
\[
C_{(1),(0)}(x \otimes y) = p_1p_0[s_1x, s_0y] = p_1[s_1x, s_0y] - p_1[s_0d_0s_1x, s_0d_0s_0y]
\]
\[
= p_1[s_1x, s_0y] (\because x \in \ker d_0)
\]
\[
= [s_1x, s_0y] - [s_1d_1s_1x, s_1d_1s_0y]
\]
\[
= [s_1x, s_0y] - [s_1s_1x, s_1y]
\]
\[
= [s_1x, s_0y] - s_1y
\]

and
\[
C_{(0),(1)}(y \otimes x) = p_0p_1[s_0y, s_1x] = p_0[s_0y, s_1x] - p_0[s_1d_1s_0y, s_1x]
\]
\[
= p_0(s_0y, s_1x) - [s_1y, s_1x]
\]
\[
= [s_0y, s_1x] - [s_1y, s_1x] - [s_0d_0s_1y, s_1x] = [s_0d_0s_1y, s_1x]
\]
\[
= [s_0y, s_1x] - [s_1y, s_1x] (\because x \in \ker d_0)
\]
\[
= [s_0y - s_1y, s_1x]
\]

**Remark 1.** $C_{(0),(1)}(y \otimes x) \neq C_{(1),(0)}(x \otimes y)$.

For a simplicial Lie algebra $L$ with a Moore complex of length 1,
\[
\partial_1 : NL_1 \to NL_0
\]
is a crossed module of Lie algebras [8].

**Proposition 3.** Let $L$ be a reduced simplicial Lie algebra and $L_n = M_n$ for $n > 0$. Then the map
\[
\overline{\partial}_2 : NL_2/\partial_3(NL_3) \to NL_1
\]
induced from $d_2^L$ by restriction becomes a braided crossed module of Lie algebras together with the braiding map
\[
\{- \otimes -\} : NL_1 \otimes NL_1 \to NL_2/\partial_3(NL_3)
\]
\[
(x \otimes y) \mapsto \overline{C}_{(0),(1)}(y \otimes x) - \overline{C}_{(1),(0)}(x \otimes y)
\]
for all $x, y \in NL_1$, where the right hand side denotes a coset in $NL_2/\partial_3 NL_3$, represented by an element in $NL_2$, and where

$$C_{(0),(1)}(y \otimes x) = C_{(1),(0)}(x \otimes y) = s_1 x (s_1 y - s_0 y) - (s_1 y - s_0 y) s_1 x.$$ 

**Proof.** B1. For $x, y \in NE_1$, we have

$$\partial_2 \{ x \otimes y \} = d_2 \left( \{ s_1 x, s_1 y - s_0 y \} \right) - \left[ \{ s_1 y - s_0 y, s_1 x \} \right] = [x, y] - [x, s_0 d_1 y] + [s_0 d_1 y, x].$$

Since $L$ is a reduced simplicial algebra, we have $d_1 x = d_1 y = 0$, and then we obtain

$$\partial_2 \{ x \otimes y \} = [x, y] - [x, y].$$

B2. From $\partial_3(C_{(2),(1)}(y \otimes a)) = [s_1 y, s_0 d_2 a - s_1 d_2 a] + [s_1 y, a] \in \partial_3(NL_3)$, we obtain

$$s_1 y \uparrow (s_1 d_2 a - s_0 d_2 a) \equiv [s_1 y, a] \mod \partial_3(NL_3)$$

and from $\partial_3(C_{(0),(2)}(a \otimes y)) = (s_0 d_2 a - s_1 d_2 a) \uparrow s_1 y + [a, s_1 y] \in \partial_3(NL_3)$, we obtain

$$(s_0 d_2 a - s_1 d_2 a) s_1 y \equiv -[a, s_1 y] \mod \partial_3(NL_3)$$

then for $a \in NL_2$ and $y \in NL_1$, we have

$$\{ y \otimes d_2 a \} = [s_1 y, s_1 d_2 a] - [s_1 y, s_0 d_2 a] - [s_1 d_2 a, s_1 y] + [s_0 d_2 a, s_1 y] \equiv [s_1 y, a] - [a, s_1 y] \mod \partial_3(NL_3)$$

$$= y \uparrow a - a \uparrow y.$$

B3. For $b \in NL_2$ and $x \in NL_1$, from $\partial_3(C_{(1),(0)}(x \otimes b)) = [s_0 x - s_1 x, s_1 d_2 b] - [s_0 x - s_1 x, b] \in \partial_3(NL_3)$, we have

$$[s_0 x - s_1 x, s_1 d_2 b] \equiv [s_0 x - s_1 x, b] \mod \partial_3(NL_3),$$

and from $\partial_3(C_{(2),(1)}(b, x)) = [s_1 d_2 b, s_0 x - s_1 x] - [b, s_0 x - s_1 x] \in \partial_3(NL_3)$, we have

$$[s_1 d_2 b, s_0 x - s_1 x] \equiv [b, s_0 x - s_1 x] \mod \partial_3(NL_3).$$

That is, for $b \in NL_2$ and $x \in NL_1$, we have

$$\{ \partial_2 b \otimes x \} = [s_1 d_2 b, s_0 x - s_1 x] - [s_0 x - s_1 x, s_1 d_2 b] \equiv [b, s_1 x] - [s_1 x, b] \mod \partial_3(NL_3)$$

$$= b \downarrow x - x \downarrow b.$$

B4. From $\partial_3(C_{(1),(0)}(a \otimes b)) = [s_1 d_2 a, s_0 d_2 b] - [s_1 d_2 a, s_1 d_2 b] - ab \in \partial_3(NL_3)$, we have

$$[s_1 d_2 a, (s_1 d_2 b - s_0 d_2 b)] \equiv [a, b] \mod \partial_3(NL_3),$$

and from $\partial_3(C_{(0),(1)}(b \otimes a)) = [s_1 d_2 b, s_0 d_2 a] - [s_1 d_2 b, s_1 d_2 a] - ba \in \partial_3(NL_3)$, we have

$$[s_1 d_2 b - s_0 d_2 b, s_1 d_2 a] \equiv [b, a] \mod \partial_3(NL_3).$$

Thus, we have

$$\{ \partial_2 a \otimes \partial_2 b \} = [s_1 d_2 a, s_1 d_2 b - s_0 d_2 b] - [s_1 d_2 b - s_0 d_2 b, s_1 d_2 a] \equiv [a, b] - [b, a] \mod \partial_3(NL_3).$$

Axioms B5. and B6. can be shown similarly. □
Theorem 2. The category of reduced simplicial Lie algebras with the Moore complex of length 2 is equivalent to that of the braided crossed modules of Lie algebras.

Proof. Let $L$ be a reduced simplicial Lie algebra with the Moore complex of length 2. In the previous proposition, a braided crossed module of Lie algebras $\partial_2 : NL_2 \to N L_1$ is obtained. Therefore, we obtain a 2-truncated reduced simplicial crossed module of Lie algebras, using the action of $K \times M$ on $K$, such that

$$(k, m)(k', m') = (m \triangleright k' + k \triangleright m' + [k, k'], [m, m'])$$

We can define an action of $m' \in M$ on $(k, m) \in K \times M$ by $m \triangleright (k, m') = (m \triangleright k, [m, m'])$ and $(k, m') \triangleright m = (k \triangleright m, [m, m'])$. Using this action, we can define $L_2 = (K \times M) \times M$ with the multiplication

$$(k, m_1, m_2)(k', m'_1, m'_2) = (m_2 \triangleright k' + k \triangleright m'_2 + m_1 \triangleright k' + k \triangleright m'_1 + [k, k'], [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1], [m_2, m'_2])$$

where $m'_2 \triangleright k' = k' \triangleright m_2 - \partial(k') \otimes m_2$ and $k' \triangleright m_1 = \partial(k') \otimes m_1$ for all $k, k' \in K$ and $m_1, m_2, m'_1, m'_2 \in M$. The maps between $L_2$ and $L_1$ are

$$d_0(k, m_1, m_2) = m_2, \quad d_1(k, m_1, m_2) = m_1 + m_2, \quad d_2(k, m_1, m_2) = \partial(k) + m_1 + m_2$$

and

$$s_0(m) = (0, m, 0), \quad s_1(m) = (0, 0, m).$$

Now, we will show that these maps are homomorphisms of Lie algebras. For $x = (k, m_1, m_2)$ and $y = (k', m'_1, m'_2)$, we show that they preserve the Lie bracket.

$$d_0([x, y]) = d_0((m_2 \triangleright k' + k \triangleright m'_2 + m_1 \triangleright k' + k \triangleright m'_1 + [k, k'], [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1], [m_2, m'_2]))$$

$$= [m_2, m'_2]$$

$$d_1([x, y]) = d_1((m_2 \triangleright k' + k \triangleright m'_2 + m_1 \triangleright k' + k \triangleright m'_1 + [k, k'], [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1], [m_2, m'_2]))$$

$$= [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1] + [m_2, m'_2]$$

$$= [m_1 + m_2], (m'_1 + m'_2)$$

$$= [d_1(x), d_1(y)]$$

and

$$d_2([x, y]) = d_2((m_2 \triangleright k' + k \triangleright m'_2 + m_1 \triangleright k' + k \triangleright m'_1 + [k, k'], [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1], [m_2, m'_2]))$$

$$= [m_2, \partial k'] + [\partial k, m'_2] + [m_1, \partial k'] + [\partial k, m'_1] + [\partial k, \partial k']$$

$$+ [m_2, m'_1] + [m_1, m'_2] + [m_1, m'_1] + [m_2, m'_2]$$

$$= ([\partial(k) + m_1 + m_2], (\partial k' + m'_1 + m'_2))$$

$$= [d_2(x), d_2(y)].$$

Similar way maps $s_1$ and $s_0$ are Lie algebra homomorphisms. These homomorphisms also satisfy the simplicial identities. Therefore, we obtain a 2-truncated reduced simplicial Lie algebra $\{L_0, L_1, L_2\}$. There is a $\text{Cosk}_2$ functor from the category of 2-truncated reduced
simplicial algebras to that of reduced simplicial Lie algebras with the Moore complex of length 2.

\[
\begin{align*}
\Delta : \text{BXMod} & \rightarrow \text{ReSimpLAlg}_{\leq 2} \\
\text{Cosk}_2 & \quad \text{TrReSimpLAlg}
\end{align*}
\]

That is, there exists a functor

\[
\Delta : \text{BXMod} \rightarrow \text{ReSimpLAlg}_{\leq 2}.
\]

With this conclusion, it is clear that the Moore complex of reduced simplicial Lie algebras has a dimension of \( n \leq 2 \). Arvasi and Akça [15] demonstrate this using pairs of hyper-crossed complexes. Using the equality

\[
\partial_3(NL_3 \cap D_3) = \sum [K_I, K_J]
\]

we need to obtain

\[
\partial_3(NL_3 \cap D_3) = 0
\]

Using simplicial operators \( s_i \) and \( d_j \), we have the following results:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \beta )</th>
<th>( \partial_3 C_{a,b}(x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>2</td>
<td>((-k+m \triangleright k, 0_M, 0_M))</td>
</tr>
<tr>
<td>(2,0)</td>
<td>1</td>
<td>((0_K, 0_M, 0_M))</td>
</tr>
<tr>
<td>(2,1)</td>
<td>0</td>
<td>((-m \triangleright k, 0_M, 0_M))</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>((0_K, 0_M, 0_M))</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>((0_K, 0_M, 0_M))</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>((0_K, 0_M, 0_M))</td>
</tr>
</tbody>
</table>

This means that the size of the Moore complex is \( n \leq 2 \). In this construction, for a braided crossed module \( \partial : K \rightarrow M \), we obtain

\[
\begin{align*}
\Delta(\partial : K \rightarrow M) := L : (K \times M) & \quad \begin{array}{c}
\rightarrow \hspace{1cm} d_2 \\
\rightarrow \hspace{1cm} d_1 \\
\leftrightarrow \hspace{1cm} s_0 _{\text{lin}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
K & \quad d_0 \\
M & \quad \{0\}
\end{align*}
\]

Using the definitions of \( d_0, d_1 \) and \( d_2 \), we obtain \( NL_2 = \ker d_0 \cap \ker d_1 \cong K \) and \( NL_1 = M, NL_0 = \{0\} \). Thus, we have

\[
N\Delta(\partial : K \rightarrow M) := \partial : K \rightarrow M
\]

and similarly, for a reduced simplicial algebra \( L \) with a Moore complex of length 2, we obtain \( \Delta(N(L)) := L \) \( \Box \)

6. Conclusions

In this paper, we provide equivalent definitions for Lie algebras in the setting of whiskered structures, bimorphisms, crossed complexes, crossed differential graded algebras, and tensor products. We further demonstrate that a 2-truncation of the crossed differential graded Lie algebra obtained from our adapted definitions gives rise to a braided crossed module of Lie algebras, which was previously defined by Ulualan, and validates the reliability and accuracy of our formulations. Furthermore, the functor we construct
from the braided crossed module of Lie algebras to simplicial Lie algebras enables systematic mapping between different categories of Lie algebras. This construction shows the validity of our adapted definitions and also establishes their compatibility with related categories of Lie algebras. Additionally, the features of the tensor products we obtained further underscore the accuracy and precision of our formulations.

Applications for Lie algebras and related algebraic structures may be found in a wide variety of studies, including physics, mathematical biology, and extensions to other algebraic structures. Mock-Lie algebras are vector spaces equipped with a bilinear product that is both commutative and satisfies the Jacobi identity, similar to Lie algebras. Indeed, mock-Lie algebras are critically important to the study of integrable systems, quantum groups, and the topological quantum field theory, among various areas of mathematics and theoretical physics [16,17]. Exploring the features, structures, and potential uses of mock-Lie algebras is significant in regard to the relationship between Lie algebras and mock-Lie algebras. This research could involve investigating mock-Lie algebra representations, analyzing how Lie algebraic categories interact with them, and looking at the functors between them.

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