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Self-Similar Solutions of a Bianchi Type-III Model with a Perfect Fluid and Cosmic String Cloud in Riemannian Geometry

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Abstract: A Bianchi type-III cosmological model with self-similarity symmetry is investigated in cases with perfect fluid distribution and a cosmic string cloud. We show that this model admits a non-trivial homothetic vector field which possess non-null homothetic bivectors. This homothetic vector field is not parallel to its source vector (current vector). We discussed cases where the homothetic vector is either orthogonal or parallel to the 4-velocity of a fluid element. We solved Einstein's field equations without making any assumptions on the geometry of space-time, only assuming that it admits self-similarity symmetry. We obtained new exact self-similar solutions for the Bianchi type-III model. We show that all obtained solutions are shear, and space-time is shear free in only the case where the homothetic vector field is parallel to the 4-velocity vector. The kinematical and physical properties of the obtained solutions are discussed.

Keywords: homothetic vector field; homothetic bivector vector field; Einstein's field equations; self-similar solutions; kinematical parameters; energy condition

MSC: 83C05; 83C15

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1. Introduction

The basic equation of the theory of general relativity is Einstein's field equation $G_{ab} = \kappa T_{ab}$, where the left-hand side represents the geometry of a given cosmological model and the right-hand side, the energy momentum tensor, represents the matter content in that model. These equations are nonlinear partial differential equations of a quadratic degree, and are usually difficult to handle unless some constraints are assumed on the geometry of a given model and there exist some conditions on the matter content. These geometric constraints assume some symmetry properties in space-time such as spherical symmetry, axial symmetry, cylindrical symmetry, stability, and spatial homogeneity. The assumed conditions on the stress energy–momentum tensor are that the matter is represented by dust, perfect fluid, cosmic strings cloud, bulk viscosity, etc. These constraints and conditions reduce the variables derived from the basic equations, and the partial differential equations are then reduced to ordinary differential equations with fewer variables which, in some special cases, can be solved more easily.

In addition to the above symmetry properties, different types of symmetries, such as isometry, homothetic, conformal, Ricci collineations, matter collineations, etc., are used to obtain exact solutions of Einstein's field equations. In the context of the theory of general relativity, these symmetries have been extensively studied [1–19]. Collinson and French [20], Katzin, Lavine and Davis [21], and Collinson [22] studied more general geometric symmetries. Furthermore, some of these symmetries have been studied in the context of the theory of teleparallel gravity [23–26].

These symmetries have attracted a lot of attention, not only because of their classical physical implications, but also because they simplify Einstein's field equations. One of these symmetries that plays an important role in cosmological situations and/or gravitational collapse is self-similarity symmetry. The main advantage of this symmetry is that it reduces the number of independent variables by introducing a self-similar variable and thus reduces the Einstein field equations. This variable is a dimensionless set of independent variables which are the coordinates of space and time.
Another important role of self-similar solutions is to describe asymptotic behaviors of more general non-self-similar solutions, which are dynamical and inhomogeneous solutions that are easy to obtain.

Spherically symmetric distributions of self-gravitating perfect fluid with self-similarity have been analyzed by Cahill and Taub [5]. Homotheties of plane-symmetric space-times underlining the physical significance of homotheties in general relativity studied Taub [27]. Godfrey [28] constructed all homothetic Wyel space-times. Shabbir and Khan [29] utilized algebraic and direct integration techniques to find self-similar vector fields in static spherically symmetric space-times including the orthogonal, parallel, and non-parallel non-tilted proper self-similar vector fields for a special choice of the metric functions. Sharif and Sehar [30,31] studied kinematic self-similar solutions of cylindrically and plane-symmetric space-times for perfect fluid and dust. A self-similar solution of a fluid with a spherical distribution was investigated in the context of general relativity in [32].

Self-similarity symmetry based on Lyra's geometry has been studied in [33–37]. The authors classified space-times according to admittance of such a symmetry. For the zero-displacement vector field, they obtained results consistent with those obtained previously in the theory of general relativity, based on Riemannian geometry. They also showed that in the case where the displacement vector field is constant, results obtained in the context of Lyra's geometry cannot be compared with those obtained in general relativity, using Riemannian geometry.

Our aim in the present work is to address the problem regarding the existence of a proper homothetic vector field (HVF) for the Bianchi type-III model and study the dynamical and physical implications of the obtained results in hopes of providing a way to obtain a better understanding of their asymptotic dynamics in the past and future.

As mentioned in [38], because of the fact that the Einstein tensor is invariant under homothetic transformations, the existence of solutions of Einstein's equations with this symmetry is naturally suggested.

Einstein's equations were solved for a cloud of string with heat flux in Bianchi type-III space-time by Yavuz and Yilmmaz [39]. They found five field equations that connect six unknown quantities, so they hypothesized that there is a relationship between the unknown quantities $C = A^n$.

2. Space-Time and Homothetic Vector Field

In the theory of general relativity, homothety or self-similarity is defined in terms of the homothetic vector $\mathbf{H} = H^a \frac{\partial}{\partial x^a} = H_a dx^a$. A vector field $\mathbf{H}$ on a space-time M is called homothetic vector field if one of the following conditions holds on a local chart :

$$\pounds_H g_{ab} = H_{a;b} + H_{b;a} = 2\psi g_{ab} \Leftrightarrow \qquad H_{a;b} = \psi g_{ab} + F_{ab}, \tag{1}$$

same in whole manuscipt. where $\pounds_H$ stands for the lie derivative over $\mathbf{H}$, ψ is a constant on M, and a semi-colon denotes a covariant derivative with respect to the metric connection.

If $\psi = 0$, $\mathbf{H}$ is a trivial homothetic vector field or Killing vector field on M; if $\psi \neq 0$, $\mathbf{H}$ is a nontrivial homothetic vector field. Thus,

$$F_{ab} = H_{[a;b]} = -F_{ba} \tag{2}$$

is the so-called homothetic bivector field (HBV). According to the definition of the Killing bivector given by [40,41], the homothetic bivector field can be interpreted as a test electromagnetic field [42,43]. The definition of the current vector, **J**, and the test electromagnetic field generated by the homothetic bivector, F_{ab}, are defined as follows [42]

$$F^{ab}_{;b} = -H^{b;a}_{;b} = R^{ab}H_b = 4\pi J^a, \tag{3}$$

where $R_{ab} = R^m_{abm}$ is the Ricci tensor. In the case of vacuum space-time, $R_{ab} = 0$, the current vector $\mathbf{J} = 0$ and the test electromagnetic field is then source-free. The source vector **J** given by Equation (3) satisfies

$$J^a_{;a} = 0.$$

For a geometric interpretation of Equation (1), we refer the reader to the references [3,5], and for physical properties, we refer reference [9].

We consider the diagonal form of the homogeneous and anisotropic space-time described by Bianchi type-III metric in the following form

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{-2\alpha x}dy^2 - C^2(t)dz^2, \tag{4}$$

with the convention $(x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z)$, α is a constant and $A(t), B(t)$ and $C(t)$ are the scale factors (metric components) and functions of the cosmic time t.

For a vector field $\mathbf{H} = (H^a(t,x,y,z)\frac{\partial}{\partial x^a})^4_{a=1}$, the homothetic Equation (1) for the model (4) is reduced to the following system of equations:

$$H^0_{,0} = \psi. \tag{5}$$

$$H^0_{,1} - A^2 H^1_{,0} = 0, \tag{6}$$

$$H^0_{,2} - B^2 e^{-2\alpha x} H^2_{,0} = 0, \tag{7}$$

$$H^0_{,3} - C^2 H^3_{,0} = 0, \tag{8}$$

$$H^1_{,1} + \frac{\dot{A}}{A} H^0 = \psi, \tag{9}$$

$$A^2 H^1_{,2} + B^2 e^{-2\alpha x} H^2_{,1} = 0, \tag{10}$$

$$A^2 H^1_{,3} + C^2 H^3_{,1} = 0, \tag{11}$$

$$H^2_{,2} - \alpha H^1 + \frac{\dot{B}}{B} H^0 = \psi, \tag{12}$$

$$C^2 H^3_{,2} + B^2 e^{-2\alpha x} H^2_{,3} = 0, \tag{13}$$

$$H^3_{,3} + \frac{\dot{C}}{C} H^0 = \psi, \tag{14}$$

The commas denote partial derivatives with respect to the coordinate indicated.

After straightforward calculation, the solutions of the above system of equations give the following differential constraint relations

$$\begin{array}{rcl} \frac{\dot{A}}{A} & = & \frac{a}{\psi t+c_0} = \frac{\psi}{\psi t+c_0}, \\ \frac{\dot{B}}{B} & = & \frac{b}{\psi t+c_0} = \frac{\psi}{\psi t+c_0} \\ \frac{\dot{C}}{C} & = & \frac{d}{\psi t+c_0}, \end{array} \tag{15}$$

where c_0, a, b, and d are arbitrary non-zero constants, and the contravariant components of the homothetic vector field are

$$H^0 = \psi t + c_0, \tag{16}$$

$$H^1 = c_1, \tag{17}$$

$$H^2 = \alpha c_1 y + c_2, \tag{18}$$

$$H^3 = (\psi - d)z + c_3, \tag{19}$$

where c_1, c_2, and c_3 are constants of integration.

The covariant components of the homothetic vector, $H_a = g_{ab}H^b$, are

$$H_0 = \psi t + c_0, \tag{20}$$

$$H_1 = -c_1 A^2, \tag{21}$$

$$H_2 = -(\alpha c_1 y + c_2)e^{-2\alpha x}B^2, \tag{22}$$

$$H_3 = -((\psi - d)z + c_3)C^2, \tag{23}$$

From Equations (16)–(19) and (20)–(23), we see that the obtained homothetic vector is non-null homothetic vector field ($H^a H_a \neq 0$).

Integrating Equation (15), we obtain the components of the metric as follows

$$\begin{array}{rcl} A(t) & = & n_1(\psi t + c_0), \\ B(t) & = & n_2(\psi t + c_0), \\ C(t) & = & n_3(\psi t + c_0)^{\frac{d}{\psi}}, \end{array} \tag{24}$$

where n_1, n_2, and n_3 are constants of integration.

According to the above discussion, we can state that the solutions with proper homothetic motion for the metric (4) are space-times with metric

$$ds^2 = dt^2 - (\psi t + c_0)^2(n_1^2 dx^2 + n_2^2 e^{-2\alpha x} dy^2 + n_3^2(\psi t + c_0)^{\frac{d}{\psi}-2} dz^2), \tag{25}$$

We choose, without loss of generality, $c_0 = c_2 = c_3 = 0$ and $c_1 = 1$. So, the space-time (4) admits the following non-null homothetic vector field

$$\mathbf{H} = H^a \frac{\partial}{\partial x^a} = \psi t \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + (\psi - d)z \frac{\partial}{\partial z}, \tag{26}$$

and the scale factors are

$$\begin{array}{rcl} A(t) & = & n_1 \psi t, \\ B(t) & = & n_2 \psi t, \\ C(t) & = & n_3 (\psi t)^{\frac{d}{\psi}}. \end{array} \tag{27}$$

In this case, the self-similar solutions of the metric (4) can be written in the following form

$$ds^2 = dt^2 - q_1^2 t^2 dx^2 - q_2^2 t^2 e^{-2\alpha x} dy^2 - q_3^2 t^{\frac{2d}{\psi}} dz^2, \tag{28}$$

where q_1, q_2, and q_3 are non-zero constants. Without loss of generality, we choose $n_1 = n_2 = n_3 = 1$; then, we have the following solution

$$\begin{aligned} A(t) &= \psi t, \\ B(t) &= \psi t, \\ C(t) &= (\psi t)^{\frac{d}{\psi}}, \end{aligned} \tag{29}$$

Homothetic Bivector Field

From Equation (2), the components of the covariant homothetic bivector of the model (4) are given by

$$F_{ab} = \begin{pmatrix} 0 & c_1 A\dot{A} & (c_1 \alpha y + c_2) B\dot{B} e^{-2\alpha x} & ((\psi - d) + c_3) C\dot{C} \\ -A\dot{A}c_1 & 0 & 0 & 0 \\ -(c_1 \alpha y + c_2) B\dot{B} e^{-2\alpha x} & 0 & 0 & 0 \\ -((\psi - d) + c_3) C\dot{C} & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

and the contravariant components are

$$F^{ab} = \begin{pmatrix} 0 & -c_1 \frac{\dot{A}}{A} & -(c_1 \alpha y + c_2) \frac{\dot{B}}{B} & -((\psi - d) + c_3) \frac{\dot{C}}{C} \\ c_1 \frac{\dot{A}}{A} & 0 & 0 & 0 \\ (c_1 \alpha y + c_2) \frac{\dot{B}}{B} & 0 & 0 & 0 \\ ((\psi - d) + c_3) \frac{\dot{C}}{C} & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

The non-vanishing components of the current vector **J** are

$$\begin{aligned} J^1 &= c_1 \frac{\ddot{A}}{A}, \\ J^2 &= (c_1 \alpha y + c_2) \frac{\ddot{B}}{B}, \\ J^3 &= ((\psi - d) z + c_3) \frac{\ddot{C}}{C}. \end{aligned} \tag{32}$$

From the components of the space-like homothetic vector (16)–(19) and the components of the above current vector **J**, we obtain

$$\mathbf{H} \times \mathbf{J} \neq 0$$

Then, the homothetic vector field **H** is not parallel to its source vector **J**. Consequently, Bianchi type III (4) admits a non-trivial homothetic vector field (see theorem 4.2 in [42]).

From Equations (30) and (31), we note that the Bianche type-III space-time (4) with a symmetry (homothetic symmetry) admits nontrivial homothetic vector fields which possess non-null homothetic bivectors.

3. Einstein's Field Equations and Dynamical Variables of the Model for Perfect Fluid Distribution

In this section, we study the gravitational effects of a perfect fluid. In this case, the energy momentum tensor is taken as

$$T_{ab} = (\rho + p) u_a u_b - p g_{ab}, \tag{33}$$

where p is the pressure, ρ is the energy density, **u** the 4-velocity vector, and it must verify $\pounds_H u_a = 0$. For the space-time (4) the 4-velocity vector is defined by $u^a = u_b = (1, 0, 0, 0)$, and it is verified $g_{ab} u^a u^b = 1$.

In natural units (G = 1 and c = 1), the Einstein field equations are

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{34}$$

where R_{ab} is the Ricci tensor, R is the Ricci scalar, and T_{ab} is the energy–momentum tensor given by Equation (33). For anisotropic Bianchi type-III space-time (4) in a comoving coordinate system, Einstein's field Equation (34) is read as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi p, \tag{35}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -8\pi p, \tag{36}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2} = -8\pi p, \tag{37}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2}{A^2} = 8\pi\rho, \tag{38}$$

$$\alpha(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) = 0, \tag{39}$$

Integration of Equation (39) using (24) yields

$$A = nB, \quad n = \frac{n_2}{n_1}. \tag{40}$$

where n is a constant of integration which can be taken as unity without loss of any generality. We notice that we obtain the same results if we use the components (24); so, for $n = 1$, we have

$$A = B \tag{41}$$

From Equations (35)–(38), using (24), we obtain the relations between the three constants ψ, α, and d as follows

$$d^2 = \psi^2 - \alpha^2, \tag{42}$$

and the physical variables p and ρ (pressure and density) are

$$p = \frac{\alpha^2 - \psi^2}{8\pi(\psi t + c_0)^2}, \tag{43}$$

and

$$\rho = \frac{\psi^2 + 2\psi d - \alpha^2}{8\pi(\psi t + c_0)^2}. \tag{44}$$

Without loss of generality, we choose $n_1 = n_2 = n_3 = 1$; then, we have the following new self-similar solution

$$ds^2 = dt^2 - (\psi t + c_0)^2(dx^2 + e^{-2\alpha x}dy^2 + (\psi t + c_0)^{\frac{\psi^2-\alpha^2}{\psi^2}-2}dz^2). \tag{45}$$

Again, without loss of generality, if we choose $c_0 = 0$, we obtain the following self-similar solution

$$ds^2 = dt^2 - (\psi t)^2(dx^2 + e^{-2\alpha x}dy^2 + (\psi t)^{\frac{\psi^2-\alpha^2}{\psi^2}-2}dz^2). \tag{46}$$

4. Two Special Cases

4.1. HVF Parallel to 4-Velocity Vector

In this case, the 4-velocity vector is proportional to the time component H^0, that is, $H^1, H^2, H^3 = 0$. The homothetic Equations (5)–(14) reduce to the following equations

$$H^0_{,0} = \psi. \tag{47}$$

and

$$\frac{\dot{A}}{A}H^0 = \psi, \quad \frac{\dot{B}}{B}H^0 = \psi \quad \frac{\dot{C}}{C}H^0 = \psi \tag{48}$$

From the above equations, we obtain

$$\frac{\dot{A}}{A} = \frac{\dot{B}}{B} = \frac{\dot{C}}{C} = \frac{\psi}{\psi t + c_0}, \tag{49}$$

which gives

$$A(t) = n_1(\psi t + c_0), B(t) = n_2(\psi t + c_0), C(t) = n_3(\psi t + c_0). \tag{50}$$

Without loss of generality, we choose $n_1 = n_2 = n_3 = 1$. Consequently, the self-similar solution assumes the form

$$ds^2 = dt^2 - (\psi t + c_0)^2(dx^2 + e^{-2\alpha x}dy^2 + dz^2). \tag{51}$$

If we choose $c_0 = 0$, the above self-similar solution leads to the following solution

$$ds^2 = dt^2 - (\psi t)^2(dx^2 + e^{-2\alpha x}dy^2 + dz^2). \tag{52}$$

4.2. HVF Orthogonal to 4-Velocity Vector Field

In this case, the component $H^0 = 0$ which implies that $\psi = 0$; then, the space-time (4) does not admit any self-similar solution.

5. Gravitational Effects of a Cosmic Strings Cloud

In this section, we will study the gravitational effects of the space-time cosmic string cloud (4) without making any assumptions between the scale components of the space-time under study. To execute this case, we consider the energy–momentum tensor T^a_b which takes the following form

$$T^a_b = \mu u^a u_b + \lambda X^a X_b, \tag{53}$$

where μ is the rest energy density for a string cloud with particles attached to it, and λ is the string cloud tension density.

Here, u^a is the four-velocity vector of particles and X_a is the unit space-like vector representing the direction of strings orthogonal to u^a. We choose X^a parallel to $\frac{\partial}{\partial z}$ ($X^0 = X^2 = X^3 = 0, X^1 \neq 0$). These two vectors u^a and X^a satisfy the following conditions

$$u_a u^a = 1 = -X_a X^a, \quad u_a X^a = 0. \tag{54}$$

For the space-time (4), in a comoving coordinate system, we obtain

$$u_a = u^a = (1, 0, 0, 0), X^a = (0, \frac{1}{A(t)}, 0, 0), X_a = (0, A(t), 0, 0)$$

If the particle density of the configuration is given by μ_p, then we have

$$\mu = \mu_p + \lambda. \tag{55}$$

Einstein's field Equation (34) for anisotropic Bianchi type-III space-time (4), in case of a cosmic strings cloud, read as

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=0, \tag{56}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=0, \tag{57}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{\alpha^2}{A^2}=-\lambda, \tag{58}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}-\frac{\alpha^2}{A^2}=\mu, \tag{59}$$

$$\alpha\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0, \tag{60}$$

Before solving the above Einstein's field equations, we notice that if we assume that the direction of strings parallel to $\frac{\partial}{\partial x}$ (or $\frac{\partial}{\partial y}$), the left-hand side of Equations (56) or (57) becomes equal λ; then, from Equation (24), we obtain $\lambda = 0$. So, the direction of strings was taken in the $\frac{\partial}{\partial z}$ direction.

Because all the scale factors A, B, C that appear in the left-hand sides of Einstein's field Equations (56)–(60) are functions of t alone, then λ and μ are function of t alone.

Using the components (24) in Equation (56), we obtain $d = 0$. Equations (58) and (59) give, respectively

$$\lambda=-\frac{\psi^2-\alpha^2}{(\psi t+c_0)^2}, \tag{61}$$

$$\mu=\frac{\psi^2-\alpha^2}{(\psi t+c_0)^2}, \tag{62}$$

According to the above discussion, the only self-similar solution for a Bianchi type-III model (4), with a cosmological strings cloud in $\frac{\partial}{\partial z}$ direction, is

$$ds^2=dt^2-(n_1^2(\psi t+c_0)^2dx^2+n_2^2(\psi t+c_0)^2e^{-2\alpha x}dy^2+n_3^2dz^2), \tag{63}$$

Without loss of generality, if $n_1 = n_2 = n_3 = 1$ and $c_0 = 0$, we obtain the following self-similar solution

$$ds^2=dt^2-(\psi t)^2dx^2+(\psi t)^2e^{-2\alpha x}dy^2+dz^2), \tag{64}$$

6. Physical and Kinematical Parameters of the Obtained Solutions

To discuss the physical behavior of the obtained self-similar models given by the metrics (45), (46), (51), (52), and (63), we find the following physical and kinematical properties of the solutions which are very important in the discussion of cosmology.
The following are the parameters for the metric (45)

1. The spatial volume is defined by

$$V=a^3(t)=A(t)B(t)C(t)=(\psi t+c_0)^{2+\frac{d}{\psi}},$$

where $a(t)$ is the average scale factor of the universe.

2. The scalar expansion, Θ, is

$$\Theta=u^a_{;a}=\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}=\frac{2\psi+d}{\psi t+c_0}.$$

3. The components of the shear tensor σ_i^j are given by

$$\sigma_0^0 = 0,$$
$$\sigma_1^1 = \frac{1}{3}\left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) = \frac{1}{3}\left(\frac{\psi}{\psi t + c_0} - \frac{d}{\psi t + c_0}\right),$$
$$\sigma_2^2 = \frac{1}{3}\left(\frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right) = \frac{1}{3}\left(\frac{\psi}{\psi t + c_0} - \frac{d}{\psi t + c_0}\right),$$
$$\sigma_3^3 = \frac{1}{3}\left(\frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = \frac{1}{3}\left(\frac{2d}{\psi t + c_0} - \frac{2\psi}{\psi t + c_0}\right),$$

4. The shear scalar σ is given by

$$\sigma = \left(\frac{1}{3}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2 - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{C}\dot{A}}{CA}\right]\right)^{\frac{1}{2}} = \frac{\psi - d}{\sqrt{3}(\psi t + c_0)}.$$

5. The average Hubble parameter is defined as

$$\bar{\mathbf{H}} = \frac{1}{3}\Theta = \frac{1}{3}(\bar{H}_1 + \bar{H}_2 + \bar{H}_3) = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = \frac{2\psi + d}{3(\psi t + c_0)},$$

6. The anisotropy parameter δ of the expansion is

$$\delta = \frac{1}{3}\Sigma_{i=1}^{3}\left(\frac{\bar{H}_i - \bar{H}}{\bar{H}}\right)^2 = \frac{3(\psi^2 + d^2)}{(2\psi + d)^2} - \frac{5}{3}$$

in which $\bar{H}_i, i = 1, 2, 3$ represent the Hubble parameter in the directions of x, y, and z, respectively.

7. Solution (46)

For the obtained solution (46), we obtain

$$V = a^3(t) = (\psi t)^{2 + \frac{d}{\psi}}.$$

$$\Theta = \frac{2\psi + d}{\psi t},$$
$$\sigma_0^0 = 0,$$
$$\sigma_1^1 = \sigma_2^2 = \frac{1}{3}\left(\frac{\psi}{\psi t} - \frac{d}{\psi t}\right),$$
$$\sigma_3^3 = \frac{1}{3}\left(\frac{2d}{\psi t} - \frac{2\psi}{\psi t}\right),$$
$$\sigma = \frac{\psi - d}{\sqrt{3}(\psi t)},$$
$$\bar{\mathbf{H}} = \frac{2\psi + d}{3\psi t},$$
$$\delta = \frac{3(\psi^2 + d^2)}{(2\psi + d)^2} - \frac{5}{3}.$$

8. Solution (51)

For the obtained solution (51), we have

$$V = (\psi t + c_0)^3.$$

$$\Theta = \frac{3\psi}{\psi t + c_0},$$
$$\sigma = 0.$$
$$\bar{\mathbf{H}} = \frac{\psi}{\psi t + c_0},$$
$$\delta = -\frac{2}{3}.$$

We notice from the above that the space-time (51) admits a time-like homothetic vector, $H^0 = \psi t + c_0$, which is parallel to the four-velocity vector. This space-time is shear free, i.e., $\sigma = 0$, and the homothetic factor $\psi = \frac{\Theta H^0}{3}$. This result agrees with the proposition given in [44], which states that a space-time admits a time-like homothetic vector field parallel to the unit time-like vector field u^a $(u^a u_a = 1)$ if and only if $\sigma_{ab} = 0$ and the homothetic factor ψ satisfies $\psi = \frac{\mathbf{H}\Theta}{3}$.

9. Solution (63)

For the obtained solution (46) (if $n_1 = n_2 = n_3 = 1$), we obtain

$$V = (\psi t + c_0)^2.$$

$$\Theta = \frac{2\psi}{\psi t + c_0},$$
$$\sigma_0^0 = 0,$$
$$\sigma_1^1 = \sigma_2^2 = \frac{\psi}{3(\psi t + c_0)},$$
$$\sigma_3^3 = -\frac{2\psi}{3(\psi t + c_0)},$$
$$\sigma = \frac{\psi}{\sqrt{3}(\psi t + c_0)},$$
$$\bar{\mathbf{H}} = \frac{2\psi}{3(\psi t + c_0)},$$
$$\delta = -\frac{1}{3}.$$

10. Conclusions

This work is devoted to the study of symmetries, in particular homothetic symmetry, of a Bianchi type-III model based on Riemannian geometry. We focused on this type of symmetry because space-time admitting it is recognized as stable from the point of view of a dynamical system; therefore, it is important from a physical point of view. For Bianchi type-III space-time, we have shown that such space-time admits non-trivial homothetic vector fields which possess non-null homothetic bivectors.

Exact solutions of Einstein's equations are of great importance in understanding the behavior of a variety of celestial phenomena. The literature abounds with many different techniques that have been used in an attempt to obtain new exact solutions for different configurations of matter.

In this paper, we studied Bianchi-type III space-time (4) and attempted to obtain exact solutions to Einstein's field Equation (34). To do this, in addition to self-similarity symmetry, we assumed that the matter is represented by a perfect fluid, as in Section 4, and by a cosmic string cloud, as in Section 5. In both cases, we obtained exact self-similar solutions for the space-time (4). Furthermore, we classified the space-time (4) under study according to admitting a homothetic vector field. For the obtained solutions (45), (51), and (63), we have the following: The spatial volume V increases as time increases, and it is finite when $t = 0$ and becomes infinite when $t \to \infty$. The expansion, shear, and average Hubble parameter decrease as time increases and tends to zero as $t \to \infty$. $\frac{\sigma}{\Theta}$ is a constant, that is, the expansion Θ in each model is proportional to the shear σ. The model (46) and (52) are began with a big bang at $t = 0$ and the expansion in the model decreases as time increases. These models have point-type singularity at $t = 0$. For the model (46), the energy density $\rho \to \infty$ when $t \to 0$ and $\rho \to 0$ when $t \to \infty$ provided $(\psi^2 + 2\psi d - \alpha^2) \neq 0$. The energy condition $\rho \geq 0$ requires that $(\psi^2 + 2\psi d - \alpha^2) \geq 0$.

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