A Topological Approach to the Bézout’ Theorem and Its Forms

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Abstract: The interplays between topology and algebraic geometry present a set of interesting properties. In this paper, we comprehensively revisit the Bézout theorem in terms of topology, and we present a topological proof of the theorem considering $n$-dimensional space. We show the role of topology in understanding the complete and finite intersections of algebraic curves within a topological space. Moreover, we introduce the concept of symmetrically complex translations of roots in a zero-set of a real algebraic curve, which is called a fundamental polynomial, and we show that the resulting complex algebraic curve is additively decomposable into multiple components with varying degrees in a sequence. Interestingly, the symmetrically complex translations of roots in a zero-set of a fundamental polynomial result in the formation of isomorphic topological manifolds if one of the complex translations is kept fixed, and it induces repeated real roots in the fundamental polynomial as a component. A set of numerically simulated examples is included in the paper to illustrate the resulting manifold structures and the associated properties.

Keywords: topology; polynomial; zero-set; algebraic curve; manifolds

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1. Introduction

In algebraic geometry, the Bézout theorem is fundamental. Interestingly, the concepts of algebraic topology can play a very distinct role in the topological analysis of the Bézout theorem to expose new perspectives. In general, there are two approaches to understanding Bézout theorem: (1) abstract algebraic formulations and (2) topological formulations [1]. The purely algebraic approaches consider the commutative field $F$ generating a set of polynomials in $F[x_1,\ldots, x_n]$, where the field is considered to be algebraically closed in general $[2,3]$. On the other hand, the topological approaches to Bézout’s theorem consider the elements of algebraic topology and the topological vector spaces $[1,4,5]$. First, we present a brief discourse about the Bézout theorem in the classical form in Section 1.1 (i.e., Preliminaries section). In this paper, the algebraic zero-set of a polynomial $f$ is denoted as $Zr(f)$, representing the respective algebraic curve, and the line segment in an $n$-space is denoted as $L^n$. The set of integers is denoted as $Z$ and an arbitrary point in an $n$-space is denoted as $\langle a_i \rangle_{i=1}^n$. The set of intersection points between two algebraic curves $f, g$ is given by $S_{fg} = \{\langle (a_i) \rangle_{i=1}^n : t \in Z^+ \}$. The set of irreducible topological components of a polynomial $f$ is denoted as $C(f)$, which is a countable set. The intersections between any two algebraic curves are considered to be complete intersections preserving the standard conventions.

1.1. Preliminaries

Let $A^n(F)$ be an affine $n$-space over the commutative field $F$. The general (classical) version of the Bézout theorem in $A^n(F)$ is stated as follows $[6]$.

**Theorem 1.** If $f, g \in F[x_1,\ldots, x_n]$ are two irreducible polynomials and $f \ast g$ represents the intersections between $f, g$, then $\deg(f \ast g) = \deg(f) \cdot \deg(g)$, where $\deg(f \ast g)$ counts the degree
Theorem 2. If \( f, g \in F[x_1, \ldots, x_n] \) are irreducible polynomials and \( E \) is an irreducible component in \( \text{Zr}(f) \cap \text{Zr}(g) \) such that it maintains the condition given as \( \dim(E) = \dim(\text{Zr}(f) \cap \text{Zr}(g)) \), then

\[
\deg(f) \cdot \deg(g) = \deg(E) \cdot \sum_{E} \beta(\text{Zr}(f), \text{Zr}(g), E).
\]

The proof of Theorem 2 is detailed in [8] without resorting to the concepts of topology. However, another alternative version of the local Bézout theorem is presented in [9], which considers a set of \( n \)-polynomials with \( n \)-indeterminates. In this case, the coefficients of the polynomials are in the Hanselian local domain [10,11]. The abstract algebraic proof of the theorem is based on the structure of \( k \)-vector spaces without applying any concepts of topology within the respective topological vector spaces.

1.2. Motivations

The applications of algebraic curves encompass several domains in the mathematical sciences and in the computational sciences [12,13]. There are interplays between algebraic curves and the concepts of topology that determine various interesting algebraic as well as topological properties [12]. The various concepts of topology can be applied to the Bézout theorem to gain deeper understandings. Note that a generalized version of the Bézout theorem, which is called the local Bézout theorem, is given in classical form to accommodate the associated topological concepts. It states that if there is an isolated point \( p \) as a zero with multiplicity \( r \) of a set of algebraic curves with complete intersections, then the coefficients of the corresponding polynomials can be altered (i.e., inducing deformation to the algebraic curves) around a neighborhood of the point \( p \) [9,14]. The proof of the classical local Bézout theorem is formulated by employing the topological concepts and it considers a complex \( n \)-space [14,15]. Let us consider a projective topological (complex) vector space \( V \) containing a set of homogeneous polynomials given by \( P = \{ f_i : C^{n+1} \to C^n : i \in \mathbb{Z}^+ \} \) with the set of respective degrees \( \deg(P) = \{ d_i : 0 < i \leq n \} \). It is illustrated that, within a topologically projective space, the concepts of approximate zeros and actual zeros can be successfully introduced [1]. Moreover, the actual algebraic zero-set of a polynomial \( f_i \in P \) within the topological vector space \( V \) forms the homotopy paths given by \( H : [0, 1] \to V \times C^{n+1} \) such that the actual algebraic zero-sets of every \( f_i \in P \) are preserved by the algebraic curve \( H \) in the homotopy [1]. Evidently, the topological views enrich the understanding of the Bézout theorem in various forms. This motivates us to address the following questions: (a) Is it possible to address the Bézout theorem and its proof in the views of general topology? (b) What are the properties of extensions if the real roots are subjected to translations in complex spaces? (c) What are the properties of resulting topological manifolds? This paper addresses these questions in relative detail.

1.3. Contributions

The contributions made in this paper can be summarized as follows. We present the proof of the Bézout theorem by employing the elements of general topology considering...
\(n\)-dimensional space. In order to formulate the topological proof, we present the necessary definitions by combining the concepts of general topology and algebraic geometry. The presented topological proof is generalized in nature. Moreover, we present the concept of symmetrically complex translations of the real roots in a zero-set of a real algebraic curve and we show that the resulting complex algebraic curve is additively decomposable into multiple distinct components. It is illustrated that if we consider a real algebraic curve as a fundamental polynomial of degree 2 then the symmetrically complex translations of its roots in zero-set result in three additive components of a complex algebraic curve and the degrees of the components are formed in a sequence given as \(\langle 4, 3, 2 \rangle\). Interestingly, the fundamental polynomial itself becomes an additive component with the repeated real roots in the complex algebraic curve. Furthermore, the symmetrically complex translations of the fundamental polynomial generate isomorphic topological manifolds even if one of the complex translations is altered while keeping the other one fixed. The corresponding numerically simulated representative examples are included in the paper.

The rest of the paper is organized as follows. Section 2 presents a set of definitions and an example. Section 3 presents the topological proof of the \(B\) theorem in a general form. Section 4 presents the concept as well as definition of complex translation of zeros of a real algebraic curve and the associated properties. Section 5 illustrates the applicational aspects and topological comparisons, briefly. Finally, Section 6 concludes the paper.

2. Topology and Algebraic Curves

In this section, we present the formation of irreducible components of completely intersecting polynomials in the views of general topology. First, we present a definition.

Definition 1. Let \(A^n(F)\) be an affine (Hausdorff) \(n\)-space and \(F\) be an algebraic field such that two polynomials are given by \(f, g \in F[x_1, \ldots, x_n]\) in \(A^n(F)\). If any arbitrary point \(\langle a_i \rangle_{i=1}^n \in A^n(F)\) is an intersection point of \(f, g\), then \(Z_r(f) \setminus \{\langle a_i \rangle_{i=1}^n\}\) and \(Z_r(g) \setminus \{\langle a_i \rangle_{i=1}^n\}\) generate at least two topological components in each of the algebraic curves in \(\{f, g\}\).

Note that if the condition \(S_{fg} \neq \emptyset\) is maintained then it results in the conclusion that \(|C(f)| < +\infty\) and \(|C(g)| < +\infty\) in a compact topological subspace of \(A^n(F)\). Moreover, in a Hausdorff \(A^n(F)\), the component \(\{\langle a_i \rangle_{i=1}^n\}\) is closed for every \(t \in \mathbb{Z}^+\). It is not necessary to impose the additional topological condition that \(A^n(F)\) needs to be a simply connected (globally) contractible topological space. However, we consider, for simplicity, that the algebraic zero-sets \(Z_r(f) \subset A^n(F)\) and \(Z_r(g) \subset A^n(F)\) are compact in \(A^n(F)\).

Example 1. Suppose we consider \(f \in F[x_1, \ldots, x_n]\) and the corresponding intersection point is \(\langle a_i \rangle_{i=1}^n \in A^n(F)\) with \(g \in F[x_1, \ldots, x_n]\). If the polynomial \(f \in F[x_1, \ldots, x_n]\) maintains the topological property \(\text{Hom}(f, S^n)\), then \(Z_r(f) \setminus \{\langle a_i \rangle_{i=1}^n\}\) generates two topological components in \(f \in F[x_1, \ldots, x_n]\), which are given by \(C(f) = (Z_r(f) \setminus \{\langle a_i \rangle_{i=1}^n\}) \cup \{\langle a_i \rangle_{i=1}^n\}\). A similar result can be extended to \(g \in F[x_1, \ldots, x_n]\) if the condition \(\text{Hom}(g, S^n)\) is preserved by the respective intersecting polynomial. On the other hand, if both polynomials admit the topological properties \(\text{Hom}(f, L^n)\) and \(\text{Hom}(g, L^n)\), then it results in the condition that \(|S_{fg}| = 5\).

3. Topological Proof of Bézout’s Theorem

Let us consider two simple and irreducible polynomials \(f, g \in F[x_1, \ldots, x_n]\), which are finitely intersecting within the topological subspace \(B^n(F) \subset A^n(F)\). Suppose that \(S_{fg} = k\) such that \(k \in (0, +\infty)\), where \(\deg(f) = m > 0\) and \(\deg(g) = n > 0\). This indicates that \(\forall \langle (a_i)_{i=1}^n \rangle_{fg} \in S_{fg}\), and the following two conditions are admitted: (1) \((x_i)_{i=1}^n - (a_i)_{i=1}^n\) and (2) \((x_i)_{i=1}^n - (a_i)_{i=1}^n\). If we consider the polynomial \(f \in F[x_1, \ldots, x_n]\), then \(|C(f)| < +\infty\) and each \(p_u \in C(f) \setminus S_{fg}\) is irreducible with \(Z_r(p_u) \subset B^n(F)\). Thus, the polynomial \(f\) can be factored into the corresponding irreducible components such that \(f = \prod_{1 \leq i \leq k} (x_i)_{i=1}^n - (a_i)_{i=1}^n \prod_{0 < u < +\infty} p_u\). Following the same direc-
tion, we can infer that $|C(g)| < +\infty$ and $g = \prod_{1 \leq i \leq k} \langle (x_i)^n_i \rangle - \langle (a_i)_{i=1}^n \rangle \prod_{0 < u < +\infty} q_{uv}$, where every $q_{uv} \in C(g) \setminus S_{fg}$ is irreducible such that $Zr(q_{uv}) \subset B^n(F)$. Note that a polynomial can be formulated in $S_{fg}$ as $\lambda_{fg} = \prod_{1 \leq i \leq k} \langle (x_i)^n_i \rangle - \langle (a_i)_{i=1}^n \rangle$. Suppose the sets of various degrees of the sets of irreducible components $\{p_u\}$ and $\{q_{uv}\}$ are represented as $\{u_a : u_a, a \in Z^+\}$ and $\{w_b : w_b, b \in Z^+\}$, respectively. Thus, we can compute the degrees of polynomials as follows:

$$\deg(f) = \deg(\lambda_{fg}) + \sum_a u_a,$$

$$\deg(g) = \deg(\lambda_{fg}) + \sum_b w_b.$$  

Hence, this results in the conclusion that

$$[\deg(\lambda_{fg}) + \sum_a u_a] \cdot [\deg(\lambda_{fg}) + \sum_b w_b] \cdot (k + U)(k + W) = mn.$$  

Note that the values of $U, W \in (0, +\infty)$ are finite because $B^n(F) \subset A^n(F)$ within the corresponding topological affine space.

4. Complex Translations and Topological Properties

In this section, we present the concept of symmetrically complex translations of the roots of a zero-set of a real algebraic curve into the complex space and we analyze the associated topological properties of the resulting complex algebraic curve. In order to maintain simplicity, we consider a real algebraic curve of degree 2; however, the results can be extended to higher degree polynomials. First, we present the definition of the symmetrically complex translation of roots in a zero-set of a real algebraic curve, which is called a fundamental polynomial.

**Definition 2.** Let $A(F)$ be an affine (Hausdorff) 1-space and $F$ be a closed algebraic field such that a fundamental polynomial is given by $f \in F[x]$, where $\deg(f) = 1$. If any point $r \in R$ represents the real root of $f$, then $\{r + z_r, r + z*_r\}$ is a symmetrically complex translation such that $z_r, z*_r \in B_C \subset C$, where $B_C$ is a compact complex subspace and $z*_r$ is a complex conjugate of $z_r$.

This leads to the following Theorem.

**Theorem 3.** If $p(x)$ is a degree 2 fundamental polynomial with non-repeated real roots $\{r_1, r_2\}$, then the symmetrically complex translations of the set $\{r_1, r_2\}$ result in a complex polynomial $q(x)$, which is additively decomposable into three components $\{u(x), v(x), w(x)\}$ and the degrees of the components are in a sequence given by $(\deg(u) = 4, \deg(v) = 3, \deg(w) = 2)$.

**Proof.** Let $p(x)$ be a degree 2 fundamental polynomial with non-repeated real roots given in the set $\{r_1, r_2\}$. As a result, the polynomial can be represented as $p(x) = (x - r_1)(x - r_2)$. The resulting complex symmetric translations of set $\{r_1, r_2\}$ results in the following derivations:

$$q(x) = (x - (r_1 + z_1))(x - (r_1 + z_1^*))((x - (r_2 + z_2))(x - (r_2 + z_2^*),$$

$$\Rightarrow q(x) = \{(x - r_1)^2 - (x - r_1)(z_1 + z_1^*) + z_1z_1^*\} \{(x - r_2)^2 - (x - r_2)(z_2 + z_2^*) + z_2z_2^*\}. \tag{3}$$

Let us consider that $z_1z_1^* = a_1, z_2z_2^* = a_2$, where $\{a_1, a_2\} \subset R^+$, and $\{z_1 + z_1^*\} = z_11, (z_2 + z_2^*) = z_22$, where $\{z_11, z_22\} \subset C$. This results in a further derivation as follows.

$$q(x) = [(x - r_1)(x - r_2)]^2 - (x - r_1)^2(x - r_2)z_22 + a_2(x - r_1)^2 - (x - r_2)^2(x - r_1)z_11 + (x - r_1)(x - r_2)z_11z_22 - a_2z_11(x - r_1) + a_1(x - r_2)^2 - a_1z_22(x - r_2) + a_1a_2. \tag{4}$$
Thus, the polynomial \( q(x) \) can be represented as
\[
q(x) = (p(x))^2 - z_{22}(x - r_1)p(x) + a_2(x - r_1)^2 - z_{11}(x - r_2)p(x) + a_1(x - r_2)^2 - a_1z_{22}(x - r_2) + a_1a_2.
\]

Let us consider, for algebraic representations, that \( a_1a_2 = a_{12}, a_{12} \in R^+, a_2z_{11} = z_3, a_1z_{22} = z_4 \) and \( z_{11}z_{22} = z_{12} \), where \( \{z_3, z_4, z_{12}\} \subset C \).

This further leads to the following derivation.
\[
q(x) = (p(x))^2 + p(x)[z_{12} - z_{22}(x - r_1) - z_{11}(x - r_2)] + a_2(x - r_1)^2 - z_3(x - r_1) + a_1(x - r_2)^2 - z_4(x - r_2) + a_{12}.
\]

We can represent it in the following algebraically decomposed additive forms.
\[
q(x) = u(x) + v(x) + w(x),
\]
\[
u(x) = (p(x))^2, v(x) = z_B p(x)(x - z_A),
\]
\[
w(x) = a_2(x - r_1)^2 - z_3(x - r_1) + a_1(x - r_2)^2 - z_4(x - r_2) + a_{12},
\]
\[
z_B = -(z_{11} + z_{22}), z_A = (z_{12} + r_1z_{22} + r_2z_{11}) / |z_B|.
\]

Hence, we can conclude that the property \( \langle\deg(u) = 4, \deg(v) = 3, \deg(w) = 2\rangle \) is attained, and this completes the proof. \( \square \)

We present a set of numerically simulated examples representing the formation of resulting topological manifolds.

**Example 2.** A set of numerical examples illustrating the effects of symmetrically complex translations are presented. The fundamental polynomial \( p(x) \) is illustrated in Figure 1 with the set of real roots \( \{2, -7\} \).

![Figure 1](image-url)

**Figure 1.** The fundamental polynomial \( p(x) = (x - 2)(x + 7) \).
The topological manifolds generated by the polynomial due to the complex symmetric translations are illustrated in Figures 2 and 3. Note that the translations are performed at two different sets of complex roots in conjugate pairs. The translated polynomial in Figure 2 is given as follows:

\[ q(x) = (x - (2 + z1)) (x - (2 + z1^*)) (x + (7 + z2)) (x + (7 + z2^*)); z1 = (9 + 7i); z2 = (2 + 3i). \]

![Figure 2. Manifold of symmetrically complex translated polynomial.](image)

The translated polynomial in Figure 3 is given as follows:

\[ q(x) = (x - (2 + z1)) (x - (2 + z1^*)) (x + (7 + z2)) (x + (7 + z2^*)); z1 = (9 + 7i); z2 = (-2 - 3i). \]

![Figure 3. Manifold of symmetrically complex translated polynomial.](image)
It is interesting to note that the topological manifolds generated by polynomials in Figures 2 and 3 are topologically isomorphic, although the symmetrically complex translations of fundamental polynomials are carried out for different complex conjugate pairs. The additive components of the translated polynomial (translated at (9 + 7i) and (−2 − 3i) in the respective complex conjugate pairs) are illustrated in Figures 4 and 5.

Figure 4. Manifold of \( v(x) + w(x) \) of symmetrically complex translated \( p(x) \).

Figure 5. The component \( u(x) \) of symmetrically complex translated \( p(x) \).

It is important to note that the topological manifolds generated by the component polynomials \( \{v(x), w(x)\} \) of the symmetrically complex translated polynomial \( q(x) \) under
addition are homeomorphic to the topological manifolds generated by \( q(x) \). However, note that the orientations of the topological manifolds are different when exhibiting the influences of component \( u(x) \). Furthermore, we can infer two properties of the symmetrically complex translations as given in the following corollaries.

**Corollary 1.** The polynomial \( p(x) \) is a fundamental polynomial and real roots are repeated in the corresponding component \( u(x) \).

**Corollary 2.** The component \( v(x) \) has degree 3 and it induces the condition that \( \text{Im}(z_A) = 0 \).

**Proof.** The proof is relatively straightforward. Recall that \( \deg(p) = 2 \) with non-repeated real roots. This indicates that \( \text{Im}(z_A) = 0 \) because \( \deg(v) = \deg(p) + 1 \).

### 5. Applicational Aspects and Topological Comparisons

In this section, we briefly explain the application aspects of the presented concepts, and we illustrate the topological distinctions of the proposed formulations. In general, the concept of algebraic curves in any \( n \)-spaces can be applied to analyze the geometric as well as algebraic properties of the respective curves under induced topological deformations. For example, the braid paths in a topological space can be viewed as a set of topologically deformed non-intersecting algebraic curves retaining a set of algebraic as well as geometric properties [16]. Note that, in this paper, we have illustrated that the distributions of the zero-sets of an algebraic curve (i.e., a polynomial function) in a topological \( n \)-space and the displacements of the roots induce topological deformations in the resulting algebraic curve. Let us consider a real-valued continuous function \( f : \mathbb{R}^n \to \mathbb{R}^n \). It has a local Brouwer degree similar to the Brouwer-degree homeomorphic function \( g : S^{n-1} \to S^{n-1} \) if \( r > 0 \) is sufficiently small, and it admits the formulations of a version of Bézout’s theorem in the spaces of hypersurfaces [17,18]. In order to determine the Brouwer-degree homeomorphism of functions, it is necessary to retain the topological condition that \( f : \mathbb{R}^n \to \mathbb{R}^n \) preserves an isolated zero at origin (although it is not required that the function be smooth) [17]. Note that the formation of an isolated zero is essentially a topological concept and the formulations proposed in this paper do not require such a topological condition to maintain generality. In the projective spaces of hypersurfaces, the classical Bézout theorem allows integer-valued counting of intersection points to be achieved if the underlying algebraic field is closed and perfect [18]. The employment of a perfect algebraic field in the projective spaces of hypersurfaces indicates that irreducible components are topologically separable. The formulations proposed in this paper admit that the algebraically irreducible components are topologically separable; however, the associated conditions are different. This paper employs the algebraically closed field within an affine topological \( n \)-space such that irreducible components are topologically separable in another \( n \)-space under embeddings.

### 6. Conclusions

The topological views of algebraic curves expose several interesting properties. The algebraic curves comprise a set of topologically connected components and the Bézout theorem can be proved by employing the elements of topology in a connected topological space. The symmetrically complex translations (in different complex conjugate pairs) of the non-repeated real roots within the zero-set of a fundamental polynomial can be considered topological deformations and they generate isomorphic topological manifolds if one of the complex translations is fixed. In other words, the additive components of the decomposed form of a symmetrically complex translated fundamental polynomial generate isomorphic topological manifolds if the conjugate pairs of translations are varied. The numerically simulated topological manifolds derived from a fundamental polynomial in a lower dimension by inducing symmetrically complex translations illustrate that if the component with a repeated root is removed, then the remaining additive components form an oriented topological manifold that retains the property of homeomorphism.
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