

Certain Class of Close-to-Convex Univalent Functions

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Abstract: The purpose of this paper was to define a new class of close-to-convex function, denoted by $\mathcal{CV}(\delta, \alpha)$, which is a subclass of all functions that are univalent in $\mathbb{D}$ and have positive coefficients normalized by the conditions $f(0) = 0$, $f'(0) = 1$, if it satisfies such a condition that is dependent on positive real part. Furthermore, we proved how the power series distribution is essential for determining the sufficient and necessary condition on any function f in class $\mathcal{CV}(\delta, \alpha)$.

Keywords: close to convex function; univalent function; power series; Pascal distribution; Pascal probability

MSC: 30C45; 32A05

1. Introduction

Complex Function Theory (CFT) is a branch of mathematics that dates back to the eighteenth century. Its functions are complex-valued and analytic in a specific domain, and they can be of various variables. Furthermore, if a function's derivative exists at z_0 in its domain, it is said to be analytic (regular or holomorphic). Given that these functions are analytic, they have Taylor series developments in their domain, and can thus be expressed in a specific series form with centers at z_0 , and can be written as

$$f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 \dots$$

Several researchers, including Euler, Gauss, Riemann, Cauchy, and others, were interested in this branch because it has a wide range of applications in mathematics and science, and also has many interesting properties that real-valued functions do not have. As a result, Taylor series can be used to represent infinitely differentiable functions, holomorphic functions defined in the open unit disk, conformal functions that preserve angles when $f'(z) \neq 0$, line integrals, and other useful formulas.

Furthermore, the class of close-to-convex functions includes both the starlike and convex functions; they are also univalent and it is worth noting that O.M. Reade [1] studied the concept of classes after Kaplan [2] introduced them. Several exciting subclasses of the holomorphic function class and the univalent function class have been studied previously from various perspectives. For example, Altıntaş et al. [3], Owa et al. [4], Kowalczyk and Le-Bomba [5] (see [6]). Gao and Zhou [7], in particular, established a subclass of holomorphic functions, which is truly a subclass of close-to-convex functions.

In addition, many subclasses of the holomorphic and meromorphic functions have already been studied and analyzed in Geometric Function Theory (see [8–10]). Raghavendar and Swaminathan [11] investigated some basic properties of the q -close-to-convex functions, whereas Aral et al. [12] successfully investigated q -calculus, primarily in the Geometric Function Theory, and Srivastava [13] was also the first to develop the generalized q -hypergeometric function (see also [14–16]).



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Kanas and Wisniowska [17,18] conducted groundbreaking research to identify the subclasses of holomorphic functions associated with conic domains Ω_k where

$$\Omega_k = \left\{ u + iv : u^2 > k^2 \left\{ (u-1)^2 + v^2 \right\} \right\}, \quad u > 0, k \geq 0.$$

such that Ω_0 indicates right half plane, and Ω_1 is a domain associated with parabola, Ω_k , $0 < k < 1$ indicates hyperbola and an ellipse for $k > 1$.

Qing Hua is one of the researchers who is interested in the introduction and investigation of an exciting subclass of analytic and close-to-convex functions, deriving several properties such as coefficient bounds, distortion, and growth theorems [19].

In [2], it was established that one of the facts in starlikeness and convexity is dependent on the partial sums of the normalized univalent functions in the unit circle, allowing the univalent function to be replaced by a starlike or convex function with respect to the origin. Bernardi, S.D and Libera, R.J. [20,21] later added several integral operators to investigate the classes of starlike, convex, and close-to-convex functions.

The goal of this paper was to apply geometric function theory to a specific class of close-to-convex univalent functions, with a focus on the geometric properties of holomorphic functions. As a result, we must begin with a convex function $f(z)$ that conformally maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto a convex region, that is, the line segment that connects any two points in $f(\mathbb{D})$ lies entirely in $f(\mathbb{D})$ [22,23], and when we consider an holomorphic function $f(z)$ close-to-convex for $|z| < r$, if there exists a function $g(z)$ that is convex and univalent for $|z| < r$ and $\frac{f'(z)}{g'(z)}$ has a positive real part for $|z| < r$. This function $g(z)$ will be referred to as an associative function to the close-to-convex function $f(z)$, and we will refer to $f(z)$ as close-to-convex with respect to $g(z)$ when indicating an associate function [24].

Let $\mathcal{X}$ be the class that formed all holomorphic functions that are defined in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad (1)$$

Consider $\mathbb{V}$ to be a subclass of the class $\mathcal{X}$, which contains all functions that are univalent in $\mathbb{D}$, and without hurting generality, we suppose that functions have positive coefficients and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$, and is formed as follows

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad (2)$$

Each analytic functions in the class $\mathcal{X}$ map the unit disk conformally and one-to-one onto another simply connected domain in a complex plane, so that the image of the open unit disk specifies the geometric form of the function, such as starlike, convex, and close-to-convex [20,21].

The majority of the researchers discovered an intriguing criterion for analytic function univalence when they proved that if f is an analytic function and satisfies $\Re(f') > 0$, for all z in $\mathbb{D}$, then f is close-to-convex function and, thus, univalent in $\mathbb{D}$. Moreover, some of them investigated such connections with the classes of analytic univalent functions with positive coefficients in the open unit disk, while others defined the class of univalent, starlike functions defined from the unit disk onto the star-shaped region to obtain some results related to the upper bound of coefficients, growth theorems, and a sufficient condition to be in that class [17,25].

Duren established an analytical description of functions in the class of starlike, convex functions in [6], while Kaplan stated the class of close-to-convex functions and studied their properties.

Some definitions and previous work of [4,20,21,26] for starlike functions of specific order, as well as for convex functions.

The author [27] found the class of functions f , analytic in open unit disk normalized by the conditions $f(0) = 0$, $f'(0) = 1$, where α and γ are real numbers, with $\alpha > 1$, $\alpha \geq \gamma > 1$, and let

$$I(\alpha, f) = (1 - \alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right).$$

Then, f satisfies the condition below

$$\Re(I(\alpha, f)) = \Re \left((1 - \alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right) < \gamma$$

The author of [28] created a new idea by connecting some subclasses of analytic univalent functions with positive coefficients in the open unit disk using a convolution operator involving the Poisson distribution series.

The standard books [6,15] can be looked into for several interesting geometric properties of these classes.

2. Main Results

As a result, the following new a subclass of $\mathcal{X}$ is defined

Definition 1. For $0 \leq \delta < 1$; $0 \leq \alpha < 1$, we let

$$\mathcal{CV}(\delta, \alpha) = \left\{ f \in \mathcal{X} : \Re \left[\frac{1}{1-\delta} \left((1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) - \delta \right) \right] < p, z \in \mathbb{D} \right\}$$

This definition was used to describe our problem via the following necessary and sufficient criteria were satisfied for $f \in \mathcal{CV}(\delta, \alpha)$.

Theorem 1. A function $f \in \mathcal{V}$ given by (2) is in the class $\mathcal{CV}(\delta, \alpha)$ if and only if

$$\Re \left(\sum_{s=2}^{\infty} [2ps(\delta - 1) - 2\delta s + 2\alpha s] a_s \right) \geq 0 \quad (3)$$

where $0 \leq \delta < 1$; $0 \leq \alpha < 1$, and $0 < p < 1$.

Proof. Let f be of the form (2). So in order to show that $f \in \mathcal{CV}(\delta, \alpha)$, we see it suffices to prove that

$$\left| \frac{\frac{1}{1-\delta} \left((1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) - \delta \right)}{2(p-1) - \frac{1}{1-\delta} \left((1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) - \delta \right) - \delta} \right| < p \quad (4)$$

From (2), consider $f'(z) = 1 + \sum_{s=2}^{\infty} sa_s z^{s-1}$.

Therefore,

$$\begin{aligned} (1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) &= (1-\alpha) \left[1 + \sum_{s=2}^{\infty} sa_s z^{s-1} \right] + \alpha \left[\frac{z \left(\sum_{s=2}^{\infty} s(s-1)a_s z^{s-2} \right)}{1 + \sum_{s=2}^{\infty} sa_s z^{s-1}} + 1 \right] \\ &= \left(1 + \sum_{s=2}^{\infty} sa_s z^{s-1} \right) - \alpha \left(1 + \sum_{s=2}^{\infty} sa_s z^{s-1} \right) + \alpha \left[\frac{\left(\sum_{s=2}^{\infty} s(s-1)a_s z^{s-1} \right)}{1 + \sum_{s=2}^{\infty} sa_s z^{s-1}} + 1 \right] \end{aligned}$$

$$(1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) = \left(1 + \sum_{s=2}^{\infty} sa_s z^{s-1} \right) - \alpha \left(1 + \sum_{s=2}^{\infty} sa_s z^{s-1} \right) + \alpha \left[\frac{1 + \sum_{s=2}^{\infty} s^2 a_s z^{s-1}}{1 + \sum_{s=2}^{\infty} sa_s z^{s-1}} \right]$$

$$(1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) = \frac{(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})^2 - \alpha (1 + \sum_{s=2}^{\infty} sa_s z^{s-1})^2 + \alpha (1 + \sum_{s=2}^{\infty} s^2 a_s z^{s-1})}{1 + \sum_{s=2}^{\infty} sa_s z^{s-1}}$$

By substituting these values in the step that follows,

$$\left| \frac{\frac{1}{1-\delta} \left(\frac{(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 - \alpha(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 + \alpha(\sum_{s=2}^{\infty} s^2 a_s z^{s-1}) - \delta(1+\sum_{s=2}^{\infty} sa_s z^{s-1})}{1+\sum_{s=2}^{\infty} sa_s z^{s-1}} \right)}{2(p-1) - \frac{1}{1-\delta} \left(\frac{(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 - \alpha(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 + \alpha(\sum_{s=2}^{\infty} s^2 a_s z^{s-1}) - \delta(1+\sum_{s=2}^{\infty} sa_s z^{s-1})}{1+\sum_{s=2}^{\infty} sa_s z^{s-1}} \right)} - \delta \right| < p$$

Now we must carry out a short and simple computation yield to

$$\left| \frac{(1-\alpha)(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 + \alpha(\sum_{s=2}^{\infty} s^2 a_s z^{s-1}) - \delta(1+\sum_{s=2}^{\infty} sa_s z^{s-1})}{2(p-\delta p-1)(1+\sum_{s=2}^{\infty} sa_s z^{s-1}) + (\alpha-1)(1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2 - \alpha(\sum_{s=2}^{\infty} s^2 a_s z^{s-1}) + \delta(1+\sum_{s=2}^{\infty} sa_s z^{s-1})} \right| < p$$

As a result

$$\left| A^2 - \alpha A^2 + \alpha B - \delta A \right| \leq p \left| 2(p-\delta p-1)A + (\alpha-1)A^2 - \alpha B + \delta A \right|,$$

where $A = (1+\sum_{s=2}^{\infty} sa_s z^{s-1})^2$; $B = (1+\sum_{s=2}^{\infty} s^2 a_s z^{s-1})$.

We will continue to simplify in order to achieve our goal to obtain

$$\left| (1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right| \leq p \left| (2p-2\delta p-3+\delta) + \sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right|$$

Then,

$$\left| (1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right| - p \left| (2p-2\delta p-3+\delta) + \sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right| \geq 0 \quad (5)$$

As is commonly obvious $\Re(z) \leq |z|$, hence

$$\begin{aligned} & \Re \left((1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right) \\ & \leq \left| (1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right| \end{aligned} \quad (6)$$

In a similar direction, we have

$$\begin{aligned} & \Re \left((2p-2\delta p-3+\delta) + \sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right) \\ & \leq \left| (2p-2\delta p-3+\delta) + \sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right| \end{aligned} \quad (7)$$

From (6) and (7) with support of (5), we obtain

$$\begin{aligned} & \Re \left((1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right) \\ & - \Re \left((2p-2\delta p-3+\delta) + \sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha) s^2 a_s^2 z^{2s-2} \right) \geq 0 \end{aligned} \quad (8)$$

Beginning with (8), we compare the coefficients of z^{s-1} and z^{2s-2} for both sides of the inequality (5) as follows.

$$\Re \left(\sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s \right) - \Re \left(\sum_{s=2}^{\infty} [(2\alpha+\delta+2p-2\delta p)s - \alpha s^2] a_s \right) \geq 0$$

which implies

$$\Re \left(\sum_{s=2}^{\infty} \left(\left[(2+2\alpha-\delta) s + \alpha s^2 \right] - \left[(2\alpha+\delta+2p-2\delta p)s - \alpha s^2 \right] \right) a_s \right) \geq 0$$

Consequently,

$$\Re \left(\sum_{s=2}^{\infty} [2ps(\delta-1) - 2\delta s + 2\alpha s] a_s \right) \geq 0$$

Conversely, we only need to prove if $f \in \mathcal{CV}(\delta, \alpha)$ and z is real, then

$$\Re \left(\frac{1}{1-\delta} \left((1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) - \delta \right) \right) < p$$

Therefore, α is real and $|z| = 1$ and the coefficients a_s are non-negative, $\forall s \in \mathbb{N}$, and we obtain

$$\begin{aligned} \Re \left(\frac{(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})^2 - \alpha(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})^2 + \alpha(\sum_{s=2}^{\infty} s^2 a_s z^{s-1}) - \delta(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})}{(1-\delta)(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})} \right) < p \\ \Re \left(\frac{(1-\delta) + \sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1} + \sum_{s=2}^{\infty} (1+\alpha)s^2 a_s^2 z^{2s-2}}{(1-\delta)(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})} \right) < p \quad (9) \end{aligned}$$

Connecting (5)–(7) to find the simplifying for (9).

As a result, the following is a required criterion stated in the theorem.

$$\frac{\sum_{s=2}^{\infty} [(2+2\alpha-\delta)s + \alpha s^2] a_s z^{s-1}}{(1-\delta)(1 + \sum_{s=2}^{\infty} sa_s z^{s-1})} < p$$

The proof is complete. $\square$

Many authors have previously discovered coefficient bounds for the class of functions defined by subordination involving the conditions of starlikeness class or convexity, and the derivative of function itself or their ratios or product of powers of these definitions, or in terms of their summation or product [6,29–31].

In this paper, an attempt was made to utilize the expression of the Schwarz function as a bound condition for criteria of the close-to-convex class to study the coefficients of these types of functions, which has received little attention in previous studies.

Definition 2. A function f in $\mathbb{V}$ is said to be in the class $\mathcal{CV}_{\delta, \alpha}(\omega)$, if it satisfies

$$\Re \left[\frac{1}{1-\delta} \left((1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} - 1 \right) - \delta \right) \right] \prec \omega(z)$$

where $0 \leq \delta < 1; 0 \leq \alpha < 1$.

The previous definition is required in order to derive the following theorem.

Theorem 2. Let f be in the class $\mathcal{CV}_{\delta, \alpha}(\omega)$, and $\omega(z) = \sum_{s=1}^{\infty} \omega_s z^s$, Then, for any $\epsilon \in \mathbb{C}$. We have $|a_3 - \epsilon a_2^2| \leq \frac{m_1}{3(1-\alpha)} |\omega_2| + \frac{\alpha m_1}{3(1-\alpha)^2} |\omega_1| + \left(\frac{m_2}{3(1-\alpha)} + \frac{\epsilon m_1^2}{3(1-\alpha)^2} \right) |\omega_1|^2$.

Proof. Consider the condition

$$(1-\alpha)f'(z) + \alpha \left(\frac{zf''(z)}{f'(z)} - 1 \right) \prec T(\omega(z)),$$

which is a key component of the $\mathcal{CV}(\delta, \alpha)$ condition where $T(\omega(z))$ is a Schwarz function defined by the two holomorphic functions defined in the open unit disk

$$\omega(z) = \sum_{s=1}^{\infty} \omega_s z^s, \text{ with } T(z) = 1 + \sum_{s=1}^{\infty} m_s z^s,$$

To obtain the final form of the Schwarz function

$$T(\omega(z)) = 1 + m_1 \omega_1 z + (m_1 \omega_2 + m_2 \omega_1^2) z^2 + \dots$$

As a result, by satisfying the equality as shown below, the coefficient formula can be generated with a simple calculation. Let

$$(1 - \alpha) f'(z) + \alpha \left(\frac{z f''(z)}{f'(z)} - 1 \right) = T(\omega(z))$$

Then,

$$\begin{aligned} & (1 - \alpha) f'(z) + \alpha \left(\frac{z f''(z)}{f'(z)} - 1 \right) \\ &= (1 - \alpha) + \alpha \left[1 + \sum_{s=2}^{\infty} s a_s z^{s-1} \right] \left[\frac{\sum_{s=2}^{\infty} s(s-1) a_s z^{s-1}}{1 + \sum_{s=2}^{\infty} s a_s z^{s-1}} - 1 \right] \\ &= (1 - \alpha) [1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots] + \alpha \left[\frac{\sum_{s=2}^{\infty} s(s-1) a_s z^{s-1}}{1 + \sum_{s=2}^{\infty} s a_s z^{s-1}} - 1 \right] \end{aligned} \quad (10)$$

I will use the second term to simplify things.

$$\begin{aligned} \frac{\sum_{s=2}^{\infty} s(s-1) a_s z^{s-1}}{1 + \sum_{s=2}^{\infty} s a_s z^{s-1}} &= \frac{2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 + \dots}{1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots} \\ &= 2a_2 z^2 + (6a_3 - 4a_2^2) z^3 + (12a_4 - 6a_2 a_3) - 2a_2 (6a_3 - 4a_2^2) z^4 + \dots \\ &= 2a_2 z^2 + (6a_3 - 4a_2^2) z^3 + [12a_4 - 18a_2 a_3 + 8a_2^2] z^4 + \dots \end{aligned}$$

We must now replace the preceding term with (10) in order to obtain

$$\begin{aligned} & (1 - \alpha) f'(z) + \alpha \left(\frac{z f''(z)}{f'(z)} - 1 \right) \\ &= (1 - \alpha) [1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots] + \alpha \left[\frac{\sum_{s=2}^{\infty} s(s-1) a_s z^{s-1}}{1 + \sum_{s=2}^{\infty} s a_s z^{s-1}} - 1 \right] \\ &= -\alpha + 2a_2(1 - \alpha)z + [3(1 - \alpha)a_3 + 2\alpha a_2]z^2 + [4(1 - \alpha)a_4 + \alpha(6a_3 - 4a_2^2)]z^3 + \dots \end{aligned}$$

Already, we have

$$T(\omega(z)) = 1 + m_1 \omega_1 z + (m_1 \omega_2 + m_2 \omega_1^2) z^2 + \dots$$

So, as a result,

$$\begin{aligned} & -\alpha + 2a_2(1 - \alpha)z + [3(1 - \alpha)a_3 + 2\alpha a_2]z^2 + [4(1 - \alpha)a_4 + \alpha(6a_3 - 4a_2^2)]z^3 + \dots \\ &= 1 + m_1 \omega_1 z + (m_1 \omega_2 + m_2 \omega_1^2) z^2 + \dots \end{aligned}$$

We obtain

$$a_2 = \frac{m_1 \omega_1}{2(1 - \alpha)},$$

And

$$a_3 = \frac{1}{3(1 - \alpha)} \left(m_1 \omega_2 + m_2 \omega_1^2 - \frac{\alpha m_1 \omega_1}{(1 - \alpha)} \right)$$

Such that

$$\left| a_3 - \epsilon a_2^2 \right| \leq \frac{m_1}{3(1 - \alpha)} |\omega_2| + \frac{\alpha m_1}{3(1 - \alpha)^2} |\omega_1| + \left(\frac{m_2}{3(1 - \alpha)} + \frac{\epsilon m_1^2}{3(1 - \alpha)^2} \right) |\omega_1|^2$$

where $0 \leq \alpha < 1$, m_1 and m_2 are the coefficients of function $T(z)$. $\square$

3. Pascal Distribution Series

This section focuses on random variable distributions and how they play a fundamental role in statistics and probability, as well as how they are widely used to describe and model a wide range of real-world phenomena.

The Pascal distribution series is a current topic of study in Geometric Function Theory (see [32,33]), which uses hypergeometric functions to consider the effects on relationships between various subclasses of analytic and harmonic univalent functions.

Many researchers have examined some important features in geometric function theory in recent years, such as coefficient estimates, inclusion relations, and conditions of being in some known classes, using various probability distributions such as the Poisson, Pascal, Borel, Mittag-Leffler-type Poisson distribution, and so on (see [34–36]).

The main goal of discussing the significance of the power series in the study of geometric function theory is to concentrate on the association between the properties of power series functions and the geometric properties of univalent functions [2,20,21].

The remainder of our problem is presented in this paper to show the importance of the power series distribution for the holomorphic function in the class $\mathcal{GV}(\delta, \alpha)$, as well as the action of the Pascal distribution series on the class $\mathcal{GV}(\delta, \alpha)$.

As a result, one of the most exciting distributions in terms of power series is the Pascal distribution series, which is defined as

$$\mathcal{Q}(t, q, z) = z + \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} q^{s-1} (q-1)^t z^s \quad (11)$$

where $t \geq 1$, $0 \leq q \leq 1$, $z \in \mathbb{D}$. The main part in the form (11) is the function that generated by Pascal probability as follows

$$q(X = s) = \binom{s+t-2}{t-1} q^{s-1} (q-1)^t, \quad s \in \mathbb{N} = \{0, 1, 2, \dots\}, \quad (12)$$

where s, t are parameters [37,38].

Certain connections exist between the Pascal distribution series and these subclasses of analytic functions whose coefficients are Pascal distribution probabilities given necessary and sufficient conditions for the Pascal distribution power series [37].

The following theorems establish specific associations between the Pascal distribution series and subclasses of normalized analytic functions whose coefficients are Pascal transfer probabilities.

Theorem 3. *The necessary and sufficient condition for the function $\mathcal{Q}$ in the form (11) belongs in the class $\mathcal{GV}(\delta, \alpha)$, and it is*

$$\Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} [2ps(s-1) - 2\delta s + 2\alpha s] q^{s-1} (1-q)^t \right\} \geq 0$$

Proof. We must prove, according to Theorem 1, that

$$\Re \left(\sum_{s=2}^{\infty} [2ps(\delta-1) - 2\delta s + 2\alpha s] a_s \right) \geq 0$$

As a result, we allow the following equality by grouping the relation (3) and the implication (11).

$$\begin{aligned}
 \psi(s, q, \delta, \alpha) &= \Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} [2ps(s-1) - 2\delta s + 2\alpha s] q^{s-1} (1-q)^t \right\} \\
 &= \Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} [2ps(s-1) + 2s(\alpha - \delta)] q^{s-1} (1-q)^t \right\} \\
 &= \Re \left\{ 2p(\delta - 1) + 2(\alpha - \delta) \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} s q^{s-1} (1-q)^t \right\} \\
 \psi(s, q, \delta, \alpha) &= 2[p(\delta - 1) + (\alpha - \delta)] \Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} s q^{s-1} (1-q)^t \right\} \\
 &= 2[p(\delta - 1) + (\alpha - \delta)] \Re \left\{ t q (1-q)^t \sum_{s=2}^{\infty} \binom{s+t-2}{t} q^{s-2} \right\} \\
 &= 2[p(\delta - 1) + (\alpha - \delta)] \Re \left\{ (t+1) q (1-q)^t \sum_{s=0}^{\infty} \binom{s+t}{t} q^s \right\} \\
 \psi(s, q, \delta, \alpha) &= 2[p(\delta - 1) + (\alpha - \delta)] \Re \{(t+1)q\} \geq 0
 \end{aligned} \tag{13}$$

The proof is complete. $\square$

Theorem 4. If $\mathbb{L}(t, z) = \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} q^{s-1} (1-q)^t \frac{z^s}{s}$, then it belongs to class $\mathcal{CV}(\delta, \alpha)$ if, and only if, $2[p(s-1) - \delta + \alpha] \Re \{(t+1)q\} \geq 0$.

Proof. Since

$$\mathbb{L}(t, z) = \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} q^{s-1} (1-q)^t \frac{z^s}{s}$$

Then, by Theorem 1, we need only to show that

$$\Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} s [2p(s-1) - 2\delta + 2\alpha] \frac{1}{s} q^{s-1} (1-q)^t \right\} \geq 0$$

that is, let

$$\psi(q, \delta, \alpha) = 2 \Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} [p(s-1) - \delta + \alpha] q^{s-1} (1-q)^t \right\} \geq 0$$

Now, by writing $s = (s-1) + s$ and proceeding by Theorem 1, we obtain

$$\begin{aligned}
 &= 2 \Re \left\{ \sum_{s=2}^{\infty} \binom{(s-1)+t-1}{t-1} [p(s-1) - \delta + \alpha] q^{s-1} (1-q)^t \right\} \\
 &= 2[p(s-1) - \delta + \alpha] \Re \left\{ \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} q^{s-1} (1-q)^t \right\} \\
 &= 2[p(s-1) - \delta + \alpha] \Re \left\{ t q (1-q)^t \sum_{s=2}^{\infty} \binom{s+t-2}{t} q^{s-2} \right\} \\
 &= 2[p(s-1) - \delta + \alpha] \Re \left\{ (t+1) q (1-q)^t \sum_{s=0}^{\infty} \binom{s+t}{t} q^s \right\} \\
 &= 2[p(s-1) - \delta + \alpha] \Re \{(t+1)q\} \geq 0
 \end{aligned}$$

which is bounded by greater or than zero if, and only if, this theorem holds. $\square$

4. Conclusions

In this paper, for the first time, a new class of close-to-convex functions $\mathcal{CV}(\delta, \alpha)$ was introduced, which is a subclass of all functions that are univalent in $\mathbb{D}$ and have positive coefficients normalized by the conditions. $f(0) = 0$, $f'(0) = 1$; and it satisfies such condition

$$\Re \left(\sum_{s=2}^{\infty} [2ps(\delta - 1) - 2\delta s + 2\alpha s] a_s \right) \geq 0$$

where $0 \leq \delta < 1$; $0 \leq \alpha < 1$, depending on positive real part.

Furthermore, we showed a certain connection between the Pascal distribution series and subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution., which is defined as

$$\mathcal{Q}(t, q, z) = z + \sum_{s=2}^{\infty} \binom{s+t-2}{t-1} q^{s-1} (q-1)^t z^s$$

where $t \geq 1$, $0 \leq q \leq 1$, $z \in \mathbb{D}$, and the main part in the Pascal distribution series, is the function that was generated by Pascal probability as follows

$$q(X = s) = \binom{s+t-2}{t-1} q^{s-1} (q-1)^t, s \in \mathbb{N} = \{0, 1, 2, \dots\}$$

While other researchers defined and studied the other classes in geometric function theory having geometric properties as convex function, starlike function, and their orders as k -starlike function, when they proved that f is univalent function and $\Re(f'(z)) > 0$, when f satisfies the differential inequality

$$\Re \left[(1 - \alpha) f'(z) + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) \right] < \beta, z \in \mathbb{D}$$

and f belongs to class of all analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Ref. [26] and others investigated such connections between various subclasses of analytic univalent functions by applying certain convolution operator involving Poisson distribution series with positive coefficients in the open unit disk (cf. [27,36]).

The symmetry properties of the functions defined by an equation or inequality can be studied to obtain solutions with particular properties. The study of special functions by differential subordinations method could give interesting results about their symmetry properties. In a future paper, symmetry properties for different functions can be studied using the concept of quantum computing.

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