Article

Baire Category Soft Sets and Their Symmetric Local Properties

Zanyar A. Ameen 1,* and Mesfer H. Alqahtani 2

1 Department of Mathematics, College of Science, University of Duhok, Duhok 42001, Iraq
2 Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322, Saudi Arabia; m_halqahtani@ut.edu.sa
* Correspondence: zanyar@uod.ac

Abstract: In this paper, we study soft sets of the first and second Baire categories. The soft sets of the first Baire category are examined to be small soft sets from the point of view of soft topology, while the soft sets of the second Baire category are examined to be large. The family of soft sets of the first Baire category in a soft topological space forms a soft $\sigma$-ideal. This contributes to the development of the theory of soft ideal topology. The main properties of these classes of soft sets are discussed. The concepts of soft points where soft sets are of the first or second Baire category are introduced. These types of soft points are subclasses of non-cluster and cluster soft sets. Then, various results on the first and second Baire category soft points are obtained. Among others, the set of all soft points at which a soft set is of the second Baire category is soft regular closed. Moreover, we show that there is symmetry between a soft set that is of the first Baire category and a soft set in which each of its soft points is of the first Baire category. This is equivalent to saying that the union of any collection of soft open sets of the first Baire category is again a soft set of the first Baire category. The last assertion can be regarded as a generalized version of one of the fundamental theorems in topology known as the Banach Category Theorem. Furthermore, it is shown that any soft set can be represented as a disjoint soft union of two soft sets, one of the first Baire category and the other not of the first Baire category at each of its soft points.

Keywords: soft nowhere dense set; soft set of the first category; soft set of the second category; soft codense; Banach Category Theorem; ideal soft topology

MSC: 54A99; 54F65; 03E99

1. Introduction

In the modern world, it is becoming more and more crucial to mathematically describe and manipulate different types of uncertainty in order to solve complicated problems in a variety of disciplines, including economics, environmental science, engineering, medical, and social sciences. Although other theories like theory of the fuzzy set [1], theory of the rough set [2], and the probability theory are well-known and useful methods for treating uncertainty and ambiguity, each has its own set of restrictions. One major weakness shared by these mathematical techniques is the lack of parametrization equipment.

The soft set theory was developed in 1999 by Molodtsov [3] as a solution to the problems associated with the aforementioned theories for managing uncertainty. A collection of parameterized universe possibilities known as soft sets was proposed. The interpretation of sets for modeling uncertainty has been discussed in [4] and then developed in [5]. A standardized framework for modeling uncertain data is provided by the nature of parameter sets linked to soft sets. This results in the quick development of soft set theory and related fields in a short period of time and offers several applications of soft sets in practical settings (see [6–15]).

Several scholars have used soft set theory to study mathematical structures such as soft group theory [16], soft ring theory [17], soft convex structures [18], soft ideals [19], etc.
Khameneh and Kilicman [20] introduced the idea of $\sigma$-algebras in soft set configurations. They explored some fundamental characteristics of soft $\sigma$-algebras. This concept’s other basic features were covered in [21].

Soft topology is one of the structures introduced by Shabir and Naz [22] and Çağman et al. [23] as a novel generalization of the classical topology. The work in [22,23] was crucial to the development of the field of soft topology. Then, many classical concepts in topology have been generalized and extended in soft set settings, including soft separation axioms [24,25], soft connected spaces [26], soft compact spaces [27], soft separable spaces [25], soft paracompact spaces [26], and soft extremally disconnected spaces [28].

It is known that soft open sets are the building blocks of soft topology, but other classes of soft sets can contribute to the growth of soft topology, namely soft dense [29], soft codense [30], soft somewhat open [31], soft nowhere dense [29], soft meager (first-category soft set) [29], and soft somewhere dense open [32].

Another fruitful area of study is determining ways to produce soft topologies over a common universal set. Terepeta provided two exceptional methods for constructing soft topologies from crisp topologies in her paper [33]. The soft topology produced by one of the formulas was then demonstrated by Al-shami [34] to be identical to the enriched soft topology. Alcantud [35] enhanced Terepeta’s formulas such that they could be used to create a soft topology from a system of crisp topologies. Jafari et al. proposed a method of inducing soft topologies by soft relations [36]. Al-shami et al. [37] proposed a method of generating a new soft topology, named a primal soft topology, via a mathematical structure called soft primal. Primal soft topologies can be regarded as a generalization of the soft ideal topology studied in [19] under the name of $^\ast$-soft topology. In [38], the authors introduced methods of generating soft topologies by various soft set operators, soft closure and soft derived set operators among them. The cluster soft set [39] can be seen as a generalized version of the latter soft set operators. The set of soft points at which a soft set is of the second Baire category is a special case of a cluster soft set.

The motivations for writing the present work are:

- To study several classes of generalized soft open sets in soft topological spaces, which is an active path of research;
- To introduce the so-called soft points of the first or second category. The latter concept is a verbal generalization of soft limit points and soft closure points. Also, a soft point where a soft set is of the second category is a special case of the cluster soft point. The set of soft points at which a soft set is of the second category plays a crucial role in developing soft ideal topological spaces, which is another method of generating soft topologies.

The following is how the paper’s material is organized: Section 2 provides a review of some definitions, properties, and operation of soft set and soft topology. Section 3 investigates some of the features of specific soft sets in soft topological spaces. Section 4 discusses the soft sets of first and second categories along with some operations. It also defines the concepts of soft points at which soft sets are of the first or second Baire category. Then, several properties of these classes of soft points are obtained. Section 5 ends the work with a brief conclusion.

2. Preliminaries

We let $2^X$ be the family of all subsets of an initial universe $X$, $T$ be the associated set of parameters, and $I$ be any index set.

**Definition 1 ([3]).** Given a set-valued mapping $A : T \to 2^X$ and $T \subseteq T$, an ordered pair $(A, T) = \{(t, A(t)) : t \in T\}$ is stated to be the soft set over $X$. That is, a soft set $(A, T)$ over $X$ is expressed by

$$(A, T) = \{(t, A(t)) : t \in T, A(t) \in 2^X\}.$$
The collection of all soft subsets of $X$ associated with $T$ (resp. $T^c$) is meant by $SS(X_T)$ (resp. $SS(X_{T^c})$).

**Remark 1.** We can easily extend the soft set $(A, T)$ to the soft set $(A, T')$ by assuming $A(t) = \emptyset$ for any $t \in T - T'$.

**Definition 2 ([40]).** A soft set $(A, T)$ is said to be null along with $E$, $\Phi_T$, if $A(t) = \emptyset$ for each $t \in T$ and is called absolute along with $E$, $X_T$, if $A(t) = X$ for each $t \in T$. The absolute and null soft sets are denoted by $\Phi_T$ and $X_T$, respectively.

**Definition 3 ([8]).** A soft set $(A, T)^c = (A^c, T)$ is called the soft complement of a soft set $(A, T)$, where $A^c : T \to 2^X$ is a set-valued mapping such that $A^c(t) = X - A(t)$ for all $t \in T$.

Evidently, $((A, T)^c)^c = (A, T)$, $(\Phi_T)^c = X_T$, and $(X_T)^c = \Phi_T$.

**Definition 4 ([41]).** A soft element denoted by $x_i$ is a soft set $(A, T)$ over $X$ whenever $A(t) = \{x\}$ and $F(t') = \emptyset$ for all $t' \in T$ with $t \neq t'$, where $t \in T$ and $x \in X$. The soft element is called a soft point in [42]. We prefer to use the concept of soft point in the sequel. By statement $x_i \in (A, T)$, we mean $x \in A(t)$. By $SP(X_T)$ we denote the set of all soft points over $X$ along with $T$.

**Definition 5 ([43]).** Let $T_1, T_2 \subseteq T$. It is said that $(A_1, T_1)$ is a soft subset of $(A_2, T_2)$, written by $(A_1, T_1) \subseteq (A_2, T_2)$, if $T_1 \subseteq T_2$ and $A_1(t) \subseteq A_2(t)$ for all $t \in T_1$. And $(A_1, T_1)$ is said to be equal to $(A_2, T_1)$, written by $(A_1, T_1) = (A_2, T_2)$, if $(A_1, T_1) \subseteq (A_2, T_2)$ and $(A_2, T_2) \subseteq (A_1, T_1)$.

**Definition 6 ([33,40]).** Let $\{ (A_i, T) : i \in I \}$ be a family of soft sets over $X$.
1. The soft intersection of $(A_i, T)$, for $i \in I$, is a soft set $(A, T)$ such that $A(t) = \bigcap_{i \in I} A_i(t)$ for all $t \in T$ and is denoted by $(A, T) = \bigcap_{i \in I} (A_i, T)$.
2. The soft union of $(A_i, T)$, for $i \in I$, is a soft set $(A, T)$ such that $A(t) = \bigcup_{i \in I} A_i(t)$ for all $t \in T$ and is denoted by $(A, T) = \bigcup_{i \in I} (A_i, T)$.

**Definition 7 ([39,40]).** Let $(A, T), (B, T) \in SS(X_T)$. The soft set difference $(A, T)$ and $(B, T)$ is defined to be the soft set $(C, T) = (A, T) - (B, T)$, where $C(t) = A(t) - B(t)$ for all $t \in T$.

**Definition 8 ([44]).** A soft set $(A, T)$ is countable (resp. finite) if $A(t)$ is countable (resp. finite) for each $t \in T$. Otherwise, it is uncountable (resp. infinite).

In what follows, by two different soft points $x_i, y_j$, we intend either $x \neq y$ or $t \neq t'$, and by two disjoint soft sets $(A, T), (B, T)$ over $X$ we mean $(A, T) \cap (B, T) = \emptyset$.

**Definition 9 ([22]).** Family $\Theta \subseteq SS(X_T)$ is said to be a soft topology over $X$ if
1. $\Phi_T, X_T \in \Theta$,
2. $(A_1, T), (A_2, T) \in \Theta$ implies $(A_1, T) \cap (A_2, T) \in \Theta$, and
3. $\{ (A_i, T) : i \in I \} \subseteq \Theta$ implies $\bigcup_{i \in I} (A_i, T) \in \Theta$.

$(X, \Theta, T)$ means a soft topological space. Soft open sets are the elements of $\Theta$, and their complements are soft closed sets. The family of all soft closed sets is meant to be $\Theta^c$. The set of soft topologies over $X$ forms a lattice and it is denoted by $\Theta(X_T)$ (see [45,46]).

**Definition 10 ([22]).** Let $(Y, T) \neq \Phi_T$ be a soft subset of $(X, \Theta, T)$. Then, $\Theta_{(Y, T)} = \{ (B, T)^\cap (Y, T) : (B, T) \in \Theta \}$ is called a relative soft topology over $Y$, and $(Y, \Theta_{(Y, T)}, T)$ is a soft subspace of $(Y, \Theta, T)$.

**Lemma 1 ([22]).** Let $(Y, \Theta_{(Y, T)}, T)$ be a soft subspace of $(Y, \Theta, T)$ and let $(A, T)^\cap (Y, T) \in \Theta$. Then, $(A, T) \in \Theta_{(Y, T)}$ if and only if $(A, T) \in \Theta$.
Definition 11 ([41]). Let \((B, T) \in SS(X_T)\) and \(\Theta \in \Theta(X_T)\). Then, \((B, T)\) is called a soft neighborhood of \(x_i \in SP(X_T)\) if there exists \((W, T) \in \Theta(x_i)\) such that \(x_i \in (W, T) \subseteq (B, T)\), where \(\Theta(x_i)\) is the family of all elements of \(\Theta\) that contain \(x_i\).

Definition 12 ([46]). A soft topology generated by family \(\mathcal{F} \subseteq SS(X_T)\) is the intersection of all soft topologies over \(X\) including \(\mathcal{F}\).

Definition 13 ([23]). A (countable) soft base for soft topology \(\Theta\) is a (countable) subfamily \(\mathcal{B} \subseteq \Theta\) such that members of \(\Theta\) are unions of members of \(\mathcal{B}\).

Lemma 2 ([22]). For soft topological space \((X, \Theta, T)\), collection \(\Theta(t) = \{(A(t) : (A, T) \in \Theta)\}\) forms a (crisp) topology on \(X\) for each \(t \in T\).

Definition 14 ([22]). Let \((B, T) \in SS(X_T)\) and \(\Theta \in \Theta(X_T)\). The soft boundary of \((A, T)\) is given by \(b(A, T) = cl(A, T) - int(A, T)\).

Definition 15 ([38,47]). Let \((A, T) \in SS(X_T)\) and \(\Theta \in \Theta(X_T)\). The soft boundary of \((A, T)\) is given by \(b(A, T) = cl(A, T) - int(A, T)\).

Definition 16 ([23]). Let \((B, T) \in SS(X_T)\) and \(\Theta \in \Theta(X_T)\). A soft point \(x_i \in SP(X_T)\) is called a soft limit point of \((B, T)\) if \((B, T) \cap (B, T) - \{x_i\} \neq \Phi_T\) for all \((B, T) \in \Theta(x_i)\). The family of all soft limit points is denoted by \(D(B, T)\).

Then, \(cl(A, T) = (A, T) \cup D(A, T)\) (see Theorem 5 in [23]).

Definition 17 ([19]). A non-null class \(I \subseteq SS(X_T)\) is termed a soft ideal over \(X\) if \(I\) satisfies the following conditions:

1. If \((A, T), (B, T) \in I\), then \((A, T) \subseteq (B, T) \in I\), and
2. If \((B, T) \in I\) and \((A, T) \subseteq (B, T)\), then \((A, T) \in I\).

\(I\) is called a soft \(\sigma\)-ideal if (1) holds for countably many soft sets. We denote the family of soft ideals over \(X\) by \(\mathcal{I}(X_T)\).

Definition 18 ([39]). Let \((A, T) \in SS(X_T), \Theta \in \Theta(X_T), and \(I \in \mathcal{I}(X_T)\). A soft point \(x_i \in SP(X_T)\) is a cluster soft point of \((A, T)\) if \((A, T) \cap (W, T) \notin I\) for each \((W, T) \in \Theta(x_i)\).

The set of all the cluster soft points of \((A, T)\) is called the cluster soft set of \((A, T)\) and is denoted by \(\zeta_{\Theta}(A, T)\) or shortly \(\zeta(A, T)\).

Family \(\Theta_x(I) = \{(A, T) \in SS(X_T) : \zeta((A, T) \subseteq (A, T)) \in I\}\) is a soft topology over \(X\) and is called the cluster soft topology (or soft ideal topology). Lemma 11 and Theorem 6 in [38] guarantee that the cluster soft topology is equivalent to soft \(\Theta^*\)-topology constructed differently in [19].

Remark 2 ([39]). Given \((A, T) \in SS(X_T), \Theta \in \Theta(X_T), and \(I \in \mathcal{I}(X_T)\). We remark that

1. If \(I = \{\Phi_T\}\), then \(\zeta(A, T) = cl(A, T)\). That is, the soft cluster points are identical to the soft closure points of \((A, T)\).
2. If \(I = \{(J, T) : (J, T) \in SS(X_T), (J, T) \text{ is finite}\}\), then \(\zeta(A, T) = D(A, T)\). That is, the soft cluster points are identical to the soft limit points of \((A, T)\).

Now, we present some properties of soft cluster sets.
Proposition 1 ([39]). Let \((A, T),(B, T) \in SS(X_T), \Theta \in \Theta(X_T), \) and \(I \in I(X_T)\). The properties listed below are true:
1. If \((A, T) \in I, \) then \(c(A, T) = \bar{\Phi}\).
2. If \((A, T) \subset (B, T), \) then \(c(A, T) \subset c(B, T)\).
3. \(c((A, T) \tilde{\cap} (B, T)) = c(A, T) \tilde{\cap} c(B, T)\).
4. \(c((A, T) \tilde{\cup} (B, T)) = c(A, T) \tilde{\cup} c(B, T)\).
5. \(c(A, T) - c(B, T) \subset c((A, T) - (B, T))\).

For any family \(\mathcal{F}\) of finite subsets of \(I\), we have the following statements:

Proposition 2 ([39]). Let \((A_i, T) \in SS(X_T) \) for \(i \in I, \Theta \in \Theta(X_T), \) and \(I \in I(X_T)\). The properties listed below are true:
1. \(c\left(\bigcup_{i \in N}(A_i, T)\right) = \bigcup_{i \in N}c(A_i, T)\), where \(N \in \mathcal{F}\).
2. \(\bigcup_{i \in I}c(A_i, T) \subset c\left(\bigcup_{i \in I}(A_i, T)\right)\).
3. \(c\left(\bigcap_{i \in I}(A_i, T)\right) \subset c\left(\bigcap_{i \in I}(A_i, T)\right)\).
4. \(c\left(\bigcup_{i \in I}(A_i, T)\right) = \bigcup_{i \in I}c(A_i, T) \cap \left[\bigcap_{n \in N}c\left(\bigcup_{i \in N}(A_i, T)\right)\right]\).

Lemma 4 ([39]). Let \((A, T) \in SS(X_T), \Theta \in \Theta(X_T), \) and \(I \in I(X_T)\). Then,
1. \(c(A, T) \subset c\left(\bigcup_{i \in N}(A_i, T)\right)\).
2. \(c(A, T) \in \Theta\).
3. \(c(c(A, T)) \subset c(A, T)\).

Definition 19. Let \((A, T),(B, T) \in SS(X_T)\) and let \(\Theta \in \Theta(X_T)\). Then, \((A, T)\) is called
1. soft regular closed [48] if \(c(int(A, T)) = (A, T)\).
2. soft dense in \((B, T)\) [29,30] if \((B, T) \subset c(int(A, T))\).
3. soft codense [30] if \(int(A, T) = \Phi_T\).
4. soft somewhat open [31] if \(int(C, T) \neq \Phi_T\).
5. soft \(C_T\) [49] if \((A, T) = \bigcap_{n=1}^{\infty}(W_n, T), \) where \((W_n, T) \in \Theta\).
6. soft \(F_T\) [30] if \((A, T) = \bigcup_{n=1}^{\infty}(A_n, T), \) where \((A_n, T) \in \Theta\).
7. soft nowhere dense [29] if \(c(int(A, T)) \subset c((B, T))\).
8. soft somewhere dense open [32] if \(c(int(A, T)) \neq \Phi_T\).

3. Some Properties of Soft Nowhere Dense and Soft Codense Sets
In this section, we start by showing several properties of soft codense and soft nowhere dense sets that are applied in the remainder of the paper. The family of all soft nowhere dense subsets of \(X_T\) is denoted by \(\mathcal{N}(X_T)\). If we consider the family of all soft nowhere dense subsets of a soft set \((A, T),\) say, the notion is changed to \(\mathcal{N}(A, T)\), and this is seen for all other types of families of soft sets. The set of all soft codense subsets of \(X_T\) is denoted by \(C(X_T)\).

Lemma 5. Let \((N, T) \in SS(X_T)\) and let \(\Theta \in \Theta(X_T)\). The next arguments are equivalent:
1. \((N, T) \in \mathcal{N}(X_T)\).
2. \(X_T \setminus cl(N, T)\) is soft dense in \(X_T\).
3. For each \(\Phi_T \neq (B, T) \in \Theta,\) there exists \((C, T) \in \Theta\) such that \(\Phi_T \neq (C, T) \subset (B, T)\) and \((C, T) \cap (N, T) = \Phi_T\).

Proof. (1) \(\Rightarrow\) (2) Let \(\Phi_T \neq (D, T) \in \Theta\). Since \(int(cl(N, T)) = \Phi_T,\) then \(X_T \setminus cl(N, T) \cap (D, T) \neq \Phi_T\). Hence, \(X_T \setminus cl(N, T)\) is soft dense in \(X_T\).

(2) \(\Rightarrow\) (3) Let \(\Phi_T \neq (B, T) \in \Theta\). Set \((C, T) = X_T \setminus cl(N, T) \cap (B, T)\). Then, \(\Phi_T \neq (C, T) \cap (N, T) \neq \Phi_T\).
(3) $\implies$ (1) If $(N, T) \notin \mathcal{N}(X_T)$, then there exists $(B, T) \in \Theta$ such that $\Phi_T \neq (B, T) \subseteq \text{cl}(N, T)$. Therefore, $(B, T) \cap \text{cl}(N, T) \neq \Phi_T$ and so $(B, T) \cap (N, T) \neq \Phi_T$. Thus, there exists no $(C, T) \in \Theta$ with $\Phi_T \neq (C, T) \subseteq (B, T)$ such that $(C, T) \cap (N, T) = \Phi_T$. □

**Lemma 6.** Let $(A, T) \in \text{SS}(X_T)$ and let $\Theta \in \Theta(X_T)$. The next arguments are equivalent:

1. $(A, T) \in C(X_T)$.
2. $(A, T)^c$ is soft dense in $X_T$.
3. $(A, T) \subseteq \text{cl}(A, T)$.

**Proof.** (1) $\implies$ (2) If $(A, T) \in C(X_T)$, then $\text{int}(A, T) = \Phi_T$. By Lemma 3, $\text{cl}((A, T)^c) = X_T$ and thus $(A, T)^c$ is soft dense in $X_T$.

(2) $\implies$ (3) Suppose $(A, T)^c$ is soft dense in $X_T$. Then, $\text{cl}((A, T)^c) = X_T$. Therefore, $(A, T) \subseteq \text{cl}(A, T) \cap X_T = \text{cl}(A, T) \cap (A, T)^c = \text{b}(A, T)$.

(3) $\implies$ (1) If $(A, T) \subseteq \text{cl}(A, T)$, then $(A, T)^c = \text{cl}(A, T) - \text{int}(A, T)$. Since $A \subseteq \text{int}(A)$, we must have $\text{int}(A, T) = \Phi_T$. Hence, $(A, T) \in C(X_T)$. □

**Lemma 7.** Let $(N, T) \in \text{SS}(X_T)$ and let $\Theta \in \Theta(X_T)$. If $(N, T) \in \mathcal{N}(X_T)$, then $\text{cl}(N, T) \in \mathcal{N}(X_T)$.

**Proof.** Suppose $(N, T) \in \mathcal{N}(X_T)$. Then, $\text{cl}(X_T - \text{cl}(N, T)) = X_T$. Since $\text{cl}(\text{cl}(N, T)) = \text{cl}(N, T),\text{cl}(X_T - \text{cl}(\text{cl}(N, T))) = \text{cl}(X_T - \text{cl}(N, T)) = X_T$. Thus, $\text{cl}(N, T) \in \mathcal{N}(X_T)$. □

**Proposition 3.** Let $(M, T), (Y, T) \in \text{SS}(X_T)$ such that $(M, T) \subseteq (Y, T)$ and let $\Theta \in \Theta(X_T)$. If $(Y, T) \in \Theta$, then $(N, T) \in \mathcal{N}(Y, T)$ if and only if $(N, T) \in \mathcal{N}(X_T)$.

**Proof.** Let $(N, T) \in \mathcal{N}(Y, T)$ and let $\Phi_T \neq (B, T) \in \Theta$ such that $(B, T) \cap (Y, T) \neq \Phi_T$. Then there exists $\Phi_T \neq (C, T) \in \Theta_{(Y, T)}$ such that $(C, T) \subseteq (B, T) \cap (Y, T)$ and $(C, T) \cap (N, T) = \Phi_T$. Now, we have $(D, T) \in \Theta$ such that $(C, T) = (D, T) \cap (Y, T)$. Therefore, $(D, T) \subseteq (B, T)$ and $(D, T) \cap (N, T) = \Phi_T$. Hence, $(N, T) \in \mathcal{N}(X_T)$.

Conversely, let $(Y, T) \in \Theta$ and $(N, T) \in \mathcal{N}(X_T)$. Let $\Phi_T \neq (C, T) \in \Theta_{(Y, T)}$. Then, $(C, T) \in \Theta$. By assumption, there exists $(B, T) \in \Theta$ such that $\Phi_T \neq (B, T) \subseteq (C, T)$ and $(B, T) \cap (N, T) = \Phi_T$. Since, also, $(B, T) \in \Theta_{(Y, T)}$; hence, $(N, T) \in \mathcal{N}(Y, T)$. □

**Remark 3.** In the above proposition, condition $(Y, T) \in \Theta$ can be replaced by $(Y, T)$, which is soft dense in $X_T$, and the proof remains the same.

**Proposition 4.** Let $\Theta \in \Theta(X_T)$ and let $(M, T), (N, T), (Y, T), (Z, T) \in \text{SS}(X_T)$ such that $(M, T) \subseteq (N, T) \subseteq (Y, T) \subseteq (Z, T)$. If $(N, T) \in \mathcal{N}(Y, T)$, then $(M, T) \in \mathcal{N}(Z, T)$.

**Proof.** Suppose $(N, T) \in \mathcal{N}(Y, T)$. By the first part of the proof of Proposition 3, $(N, T) \in \mathcal{N}(Z, T)$. Since $(M, T) \subseteq (N, T), (M, T) \in \mathcal{N}(Z, T)$. □

**Proposition 5.** Let $\Theta \in \Theta(X_T)$ and let $(M, T), (N, T), (Y, T), (Z, T) \in \text{SS}(X_T)$ such that $(M, T) \subseteq (N, T) \subseteq (Y, T) \subseteq (Z, T)$. If $(M, T) \in \mathcal{N}(Z, T)$, $(N, T)$ is soft dense in $(Y, T)$, and $(Y, T)$ is soft open in $(Z, T)$, then $(N, T) \in \mathcal{N}(N, T)$.

**Proof.** Suppose $(M, T) \in \mathcal{N}(Z, T)$, $(N, T)$ is soft dense in $(Y, T)$, and $(Y, T)$ is soft open in $(Z, T)$. Let $\Phi_T \neq (B, T) \in \Theta_{(N, T)}$. Then there exists $(C, T) \in \Theta_{(Z, T)}$ such that $(B, T) = (C, T) \cap (N, T)$. Evidently, the soft set $(D, T) = (C, T) \cap (Y, T) \neq \Phi_T$. Since $(M, T) \in \mathcal{N}(Z, T)$, there exists $(W, T) \in \Theta_{(Z, T)}$ such that $\Phi_T \neq (W, T) \subseteq (D, T)$ and $(W, T) \cap (M, T) = \Phi_T$. Set $(V, T) = (W, T) \cap (N, T) \in \Theta_{(N, T)}$, such that $(V, T) \subseteq (B, T)$ and $(V, T) \cap (M, T) = \Phi_T$. Since $\Phi_T \neq (W, T) \in \Theta_{(Z, T)}$ and $(N, T)$ is soft dense in $(Y, T)$, $(V, T) \neq \Phi_T$. Thus, $(M, T) \in \mathcal{N}(N, T)$. □
Definition 20. Let \( \Theta \in \Theta(X_T) \) and let \((A, T) \in SS(X_T)\). Then, \((A, T)\) is called soft nowhere dense (resp. soft codense) at a soft point \(x_t \in SP(X_T)\) if there exists \((B, T) \in \Theta(x_t)\) such that \((B, T) \triangleleft \hat{(A, T)} \in \mathcal{N}(X_T)\) (resp. \((B, T) \triangleleft \hat{(A, T)} \in \mathcal{C}(X_T)\)).

The set of soft points at which \((A, T)\) is soft nowhere dense (resp. soft codense) is denoted by \(C_N(A, T)\) (resp. \(C_0(A, T)\)).

Definition 21. We say \((A, T)\) is not soft nowhere dense (or soft somewhere dense) at a soft point \(x_t \in SP(X_T)\) if for each \((B, T) \in \Theta(x_t)\) such that \((B, T) \triangleleft \hat{(A, T)} \notin \mathcal{N}(X_T)\). And it is not soft codense (or soft somewhat open) at a soft point \(x_t \in SP(X_T)\) if for each \((B, T) \in \Theta(x_t)\) such that \((B, T) \triangleleft \hat{(A, T)} \notin \mathcal{C}(X_T)\).

The set of soft points at which \((A, T)\) is soft somewhere dense (resp. soft somewhat open) is denoted by \(C_{sd}(A, T)\) (resp. \(C_{sw}(A, T)\)).

Lemma 8. Let \( \Theta \in \Theta(X_T) \), \((A, T) \in SS(X_T)\), and \(x_t \in SP(X_T)\). Then, \((A, T)\) is soft nowhere dense at \(x_t\) if and only if \(cl(A, T)\) is soft codense at \(x_t\).

Proof. Suppose \((A, T)\) is not soft nowhere dense at \(x_t\). Then, for each \((C, T) \in \Theta(x_t)\), we have \((C, T) \triangleleft \hat{(A, T)} \notin \mathcal{N}(X_T)\). Therefore, there exists \((B, T) \in \Theta\) such that \(\Phi_T \neq (B, T) \triangleleft \hat{(C, T)} \triangleleft \hat{(A, T)}\). This implies that \((B, T) = (B, T) \triangleleft \hat{(C, T)} \triangleleft \hat{(A, T)} = \hat{(C, T)} \triangleleft \hat{(A, T)}\), and hence \((C, T) \triangleleft \hat{(B, T)} \neq \Phi_T\). Since \((B, T) \triangleleft \hat{(C, T)} \triangleleft \hat{(A, T)}\), and hence \((C, T) \triangleleft \hat{(B, T)} \neq \Phi_T\). This proves that \((C, T) \triangleleft \hat{(A, T)}\) is not soft codense and thus \(cl(A, T)\) is not soft codense at \(x_t\).

Conversely, if \(cl(A, T)\) is not soft codense at \(x_t\), then for each \((B, T) \in \Theta(x_t)\), we have \((B, T) \triangleleft \hat{(C, T)} \triangleleft \hat{(A, T)}\) is not soft codense. It follows that there exists \((C, T) \in \Theta\) such that \(\Phi_T \neq (C, T) \triangleleft \hat{(B, T)} \triangleleft \hat{(A, T)}\). Therefore, \((C, T) \triangleleft \hat{(B, T)} \triangleleft \hat{(A, T)} \neq \Phi_T\). This means that \((B, T) \triangleleft \hat{(A, T)}\) is not soft nowhere dense, and hence, \((A, T)\) is not soft nowhere dense at \(x_t\). □

Proposition 6. Let \( \Theta \in \Theta(X_T) \) and \((A, T) \in SS(X_T)\). Then,

1. \(C_{sw}(A, T) = cl(int(A, T))\).
2. \(C_{sd}(A, T) = cl(int(cl(A, T)))\).

Proof. (1) Let \(x_t \in SP(X_T)\). Suppose that \(x_t \in cl(int(A, T))\) and \((C, T) \in \Theta(x_t)\), then \((C, T) \triangleleft \hat{cl(int(A, T))} = \Phi_T\). This means \(\Phi_T \neq (C, T) \triangleleft \hat{cl(int(A, T))}\). Thus, \(C_{sw}(A, T)\).

Conversely, suppose \(x_t \notin cl(int(A, T))\). Then \(x_t \in [cl(int(A, T))]^c\). Set \(\Phi_T = [cl(int(A, T))]^c\) and \((C, T) \in \Theta(x_t)\) such that \((C, T) \triangleleft \hat{(A, T)} \in \mathcal{C}(X_T)\) because \(int((C, T)) \triangleleft \hat{(A, T)} \in \mathcal{C}(X_T)\). This implies that \(x_t \notin C_{sw}(A, T)\). Consequently, \(C_{sw}(A, T) = cl(int(A, T))\).

(2) It follows from (1) and Lemma 8. □

Proposition 7. Let \( \Theta \in \Theta(X_T) \) and \((A, T) \in SS(X_T)\). Then,

1. \(C_0(A, T) \in \mathcal{C}(X_T)\).
2. \(C_N(A, T) \in \mathcal{N}(X_T)\).

Proof. (1) Obviously, \(C_0(A, T) = (A, T) - C_{sw}(A, T) = (A, T) - cl(int(A, T)) \triangleleft \hat{(A, T)} - int(A, T)\). Since \((A, T) \triangleleft int(A, T) \in \mathcal{C}(X_T)\) and \(C_0(A, T) \triangleleft \hat{(A, T)} \triangleleft \hat{int(A, T)} \triangleleft \hat{int(A)} \in \mathcal{C}(X_T)\).

(2) Similar to (1). □
4. First- and Second-Category Soft Sets

**Definition 22.** Let $\Theta \in \Theta(X_T)$. Soft set $(M, T)$ over $X$ is called of the first Baire category or simply first category if

$$(M, T) = \bigcup_{n=1}^{\infty} (N_n, T),$$

where $(N_n, T) \in \mathcal{N}(X_T)$ for each $n$. The set of all soft sets of the first category over $X$ is denoted by $\mathcal{M}(X_T)$.

If $(M, T)$ does not possess the representation in equation (1), it said to be of the second Baire category or simply second category. The set of all soft sets of the first category over $X$ is denoted by $S(X_T)$.

The definition of soft sets of the first category appeared in the literature under the name of soft meager sets (see [29,50]).

We now have possible connections between the soft sets defined earlier.

first-category soft set $\iff$ soft nowhere dense $\implies$ soft condense

The reverse of the above arrows is not generally possible.

**Example 1.** Let $\Theta$ be the soft topology on the set of real number $\mathbb{R}$ generated by $\{(t_1, F(t_1)), (t_2, F(t_2))\} : F(t_1) = (r_1, s_1); F(t_2) = (r_2, s_2); r_1, r_2, s_1, s_2 \in \mathbb{R}, r_1 < s_1$, where $T = \{t_1, t_2\}$ is a set of parameters and $i = 1, 2$. Soft set $(A, T) = \{(t_1, C), (t_2, \emptyset)\}$ is soft nowhere dense, which is consequently soft condense and a soft set of the first category, where $C$ is the ternary Cantor set. Soft set $(B, T) = \{(t_1, \mathbb{Q}), (t_2, \mathbb{R})\}$ is of the first category and soft condense but not soft nowhere dense, where $\mathbb{Z}, \mathbb{Q}$ are sets of integers and rationals, respectively. Soft set $(C, T) = \{(t_1, \mathbb{Q}^c), (t_2, \emptyset)\}$ is soft condense but not of the first category.

We let $(Y, T) = \{(t_1, \mathbb{Q}), (t_2, \emptyset)\}$ and $\Theta(Y, T)$ be the relative soft topology on $(Y, T)$. Given any soft subset $(R, T)$ of $(Y, T)$. Then, we have $(R, T) = \bigcup_{y \in \Theta(Y, T)} \{(t_1, \{y\}), (t_2, \emptyset)\}$. Evidently, each soft set $\{(t_1, \{y\}), (t_2, \emptyset)\}$ is soft nowhere dense in $\Theta(Y, T)$ implies $(R, T)$ is of the first category. Thus, we show that any soft set of $(Y, T)$ is of the first category. Hence, if $(U, T) \in \Theta(Y, T)$, then $(U, T)$ is of the first category but not soft condense.

**Remark 4.** We can apparently conclude that $\mathcal{M}(X_T)$ is closed under soft subsets and countable soft unions. This means that $\mathcal{M}(X_T)$ forms a soft $\sigma$-ideal over $X$.

**Proposition 8.** Let $(M, T) \in SS(X_T)$ and let $\Theta \in \Theta(X_T)$. If $(M, T) \in \mathcal{M}(X_T)$, then $(M, T) \subset (A, T)$, where $(A, T)$ is a soft $F_\sigma$ set in $\mathcal{M}(X_T)$.

**Proof.** Directly follows from Lemma 7. $\qed$

**Proposition 9.** Let $(M, T) \in SS(X_T)$ and let $\Theta \in \Theta(X_T)$. If $X_T \in \mathcal{S}(X_T)$, then every soft dense $G_\delta$ subset of $X_T$ belongs to $\mathcal{S}(X_T)$.

**Proof.** $X_T \in \mathcal{S}(X_T)$ and let $(B, T)$ be soft dense $G_\delta$ set in $\mathcal{S}(X_T)$. Then $(B, T) = \cap_{n=1}^{\infty} (W_n, T)$, where $(W_n, T) \in \Theta$. Therefore, $X_T - (B, T) = X_T - (\cap_{n=1}^{\infty} (W_n, T)) = \cup_{n=1}^{\infty} (X_T - (W_n, T)) \in \mathcal{M}(X_T)$. Thus, $(B, T)$ must be in $\mathcal{S}(X_T)$, otherwise $X_T$ would be in $\mathcal{M}(X_T)$, a contradiction. $\qed$

**Proposition 10.** Let $\Theta \in \Theta(X_T)$ and let $(M, T), (Y, T) \in SS(X_T)$ such that $(M, T) \subset (Y, T)$. If $(Y, T) \in \Theta$ or soft dense, then $(M, T) \in \mathcal{M}(X_T)$ if and only if $(M, T) \in \mathcal{M}(X_T)$.

**Proof.** Apply Proposition 3, Remark 3, and equation (1). $\qed$
Proposition 11. Let \( \Theta \in \Theta(X_T) \) and let \( (M, T), (N, T), (Y, T), (Z, T) \in SS(X_T) \) such that \( (M, T) \subseteq (N, T) \subseteq (Y, T) \subseteq (Z, T) \). If \( (N, T) \) is soft dense in \( (Y, T) \) and \( (Y, T) \) is soft open in \( (Z, T) \), then \( (M, T) \in M(Z, T) \) if and only if \( (M, T) \in M(N, T) \).

Proof. Apply Propositions 4 and 5, and equation (1). \( \square \)

Theorem 1. Let \( \Theta \in \Theta(X_T) \) and let \( (M, T) \in SS(X_T) \). If \( (M, T) \in M(X_T) \), then \( (M, T) \) can be written as the disjoint soft union of a soft nowhere dense set and a soft set of the first category in itself.

Proof. Let \( (M, T) \in M(X_T) \). Consider the following decomposition:

\[
(M, T) = \left[ (M, T) - \text{int} (\text{cl} (M, T)) \right] \bigcup \left[ (M, T) \cap \text{int} (\text{cl} (M, T)) \right].
\]

(2)

We first show that \( (M, T) \cap \text{int} (\text{cl} (M, T)) \in N(X_T) \). We let \( \Phi_T \neq (C, T) \in \Theta \). We suppose for each \( (B, T) \in \Theta \) such that \( \Phi_T \neq (B, T) \subseteq (C, T) \); we have \( (B, T) \cap \text{cl} (M, T) \neq \Phi_T \). This means that each soft point in \( (C, T) \) is a soft limit point of \( (M, T) \) and so is a soft point in \( \text{cl} (M, T) \). Therefore, \( (C, T) \subseteq \text{int} (\text{cl} (M, T)) \), and hence \( (C, T) \cap \left[ (M, T) - \text{int} (\text{cl} (M, T)) \right] = \Phi_T \). Thus, for that \( \Phi_T \neq (C, T) \in \Theta \), either \( (C, T) \cap \left[ (M, T) - \text{int} (\text{cl} (M, T)) \right] = \Phi_T \) or there exists \( (B, T) \in \Theta \) such that \( \Phi_T \neq (B, T) \subseteq (C, T) \) and \( (B, T) \cap \text{cl} (M, T) = \Phi_T \). The latter case is impossible, so \( (C, T) \cap \left[ (M, T) - \text{int} (\text{cl} (M, T)) \right] = \Phi_T \). This proves that \( (M, T) \cap \text{int} (\text{cl} (M, T)) \in N(X_T) \).

We now prove that \( (M', T) = (M, T) \cap \text{int} (\text{cl} (M, T)) \) is of the first category in its own soft dense in \( \text{int} (\text{cl} (M, T)) \). We can easily check that \( (M', T) \) is also soft dense in \( \text{int} (\text{cl} (M, T)) \). Now, we apply Proposition 11 to \( (M, T) = (M', T), (N, T) = (M', T), (Y, T) = \text{int} (\text{cl} (M, T)), \) and \( (Z, T) = X_T \); we obtain that \( (M', T) \in M(X_T) \). \( \square \)

Definition 23. Let \( (A, T) \in SS(X_T) \) and \( \Theta \in \Theta(X_T) \). We say \( (A, T) \) is of the second category at a soft point \( x_i \in SP(X_T) \) if \( (A, T) \cap \text{int}(W, T) \notin M(X_T) \) for each \( (W, T) \in \Theta(x_i) \). Otherwise, we say \( (A, T) \) is of the first category at \( X_T \). The set of all soft points at which \( (A, T) \) is of the first (resp. second) category is denoted by \( C_1(A, T) \) (resp. \( C_2(A, T) \)).

Remark 5. Given \( \Theta \in \Theta(X_T) \), evidently, \( (A, T) \in SS(X_T) \) is of the first category at a soft point \( x_i \in SP(X_T) \) if and only if there exists \( (W, T) \in \Theta(x_i) \) such that \( (A, T) \cap \text{int}(W, T) \notin M(X_T) \).

Theorem 2 (Generalized Banach Category Theorem). Let \( \Theta \in \Theta(X_T) \) and \( (A, T) \in SS(X_T) \). The next arguments are equivalent:

1. If \( (A, T) \in S(X_T) \), then \( (A, T) \) is of the second category at each soft point in some \( \Phi_T \neq (B, T) \in \Theta \).
2. If \( (A, T) \) is of the first category at each of its soft points, then \( (A, T) \in M(X_T) \).
3. If \( \{(A_i, T) : i \in I\} \subseteq SS(X_T) \) in which \( (A_i, T) \in M(X_T) \) and \( (A_i, T) \in \Theta(A, T) \), where \( (A, T) = \bigcup_{i \in I} (A_i, T) \), then \( (A, T) \in M(X_T) \).
4. If \( \{(A_i, T) : i \in I\} \subseteq SS(X_T) \) in which \( (A_i, T) \in M(X_T) \) and \( (A_i, T) \in \Theta(A, T) \), then \( \bigcup_{i \in I} (A_i, T) \in M(X_T) \).

Proof. (1) \( \Rightarrow \) (2) Suppose (1) holds. If \( (A, T) \in S(X_T) \), then, by (1), \( (A, T) \) is of the second category at each soft point in some \( \Phi_T \neq (B, T) \in \Theta \). Pick any soft point \( x_i \in (B, T) \), then \( (A, T) \cap \text{int}(B, T) \in S(X_T) \) and so \( (A, T) \cap \text{int}(B, T) \neq \Phi_T \). This means that \( x_i \) is a soft point in \( (A, T) \) at which \( (A, T) \) is not of the first category, a contradiction.

(2) \( \Rightarrow \) (3) Let \( x_i \in (A, T) \). Then \( x_i \) is in some \( (A_0, T) \subseteq (A, T) \). Since \( (A_0, T) \) is soft open in \( (A, T) \), then there exists a soft open \( (B, T) \) in \( X_T \) containing \( x_i \) such that \( (A_0, T) = (B, T) \cap (A, T) \). Since \( (A_0, T) \in M(X_T) \), so \( (A, T) \) is of the first category at each of its soft points. By (2), \( (A, T) \) is of the first category.

(3) \( \Rightarrow \) (4) Directly follows from (3).
(4) \implies (1) Assume \((A, T)\) is not of the second category at a soft point of any non-null soft open. Therefore, each \(\Phi_T \neq (B, T) \in \Theta\) includes \(\Phi_T \neq (C, T) \in \Theta\) for which \((A, T) \cap (C, T) \in \mathcal{M}(X_T)\). We prove that \((A, T) \in \mathcal{M}(X_T)\). Let \(H = \{(C, T) : \Phi_T \neq (C, T) \in \Theta, (A, T) \cap (C, T) \in \mathcal{M}(X_T)\}\). Clearly, \((D, T) \cap (A, T) \neq \Phi_T\) for each \(\Phi_T \neq (D, T) \in \Theta\), where \((D, T) \subseteq (C, T)\) and \((C, T) \in \mathcal{H}\). Set \((V, T) = \widehat{\bigcup (H \cap (A, T))}\). Then, for each \((C, T) \in \mathcal{H}\), \((C, T) \cap (A, T) = \Phi_T\). This implies that \((C, T) \cap (A, T)\) is soft open in \((V, T)\). Applying Proposition 11 to \((M, T) = (C, T), (N, T) = (V, T), (Y, T) = \widehat{\mathcal{H}}\) and \((Z, T) = X_T\), we see that \((C, T) \cap (A, T) \in \mathcal{M}(V_T)\) for each \((C, T) \in \mathcal{H}\). By (4), \((V, T) \in \mathcal{M}(X_T)\).

Let \((B, T) \in \Theta\). Suppose \((C, T) \in \Theta\) such that \(\Phi_T \neq (C, T) \subseteq (B, T)\) and \((A, T) \cap (C, T) \in \mathcal{M}(X_T)\). If \(C, T \notin \mathcal{H}\), there exists \((D, T) \in \Theta\) such that \(\Phi_T \neq (D, T) \subseteq (C, T)\) and \((D, T) \cap (A, T) = \Phi_T\). and \((D, T) \cap (A, T) \notin (V, T) = \Phi_T\). If \((C, T) \in \mathcal{H}\), then \((C, T) \cap (X_T - \mathcal{H}) = \Phi_T\). Therefore, \((C, T) \cap (A, T) \notin (V, T) = \Phi_T\). This proves that \((A, T) \cap (V, T) \in \mathcal{M}(X_T)\). Consequently, \((A, T) \cap (V, T) = \Phi_T\). \(\square\)

**Proposition 12.** Let \((A, T), (B, T) \in SS(X_T)\) and \(\Theta \in \Theta(X_T)\). The properties listed below are true:

1. If \((A, T) \subseteq (B, T)\), then \(C_2(A, T) \subseteq C_2(B, T)\).
2. \(C_2((A, T) \cap (B, T)) \subseteq C_2(A, T) \cap C_2(B, T)\).
3. \(C_2((A, T) \cup (B, T)) = C_2(A, T) \cup C_2(B, T)\).
4. \(C_2(A, T) - C_2(B, T) \subseteq C_2((A, T) - (B, T))\).
5. \(C_2(A, T) \subseteq cl(A, T)\).
6. \(C_2(A, T) \subseteq \Theta^e\).
7. \(C_2(C_2(A, T)) \subseteq C_2(A, T)\).

**Proof.** It follows from Proposition 1 and Lemma 4. \(\square\)

**Proposition 13.** Let \((A_i, T) \in SS(X_T), for i \in I, and \Theta \in \Theta(X_T)\). The properties listed below are true:

1. \(C_2\left(\bigcup_{i \in N}(A_i, T)\right) = \bigcup_{i \in N}C_2(A_i, T), \text{ where } N \in \mathcal{S} \text{ and } \mathcal{S} \text{ is a collection of finite subsets of } I\).
2. \(\bigcup_{i \in I}C_2(A_i, T) \subseteq C_2\left(\bigcup_{i \in I}(A_i, T)\right)\).
3. \(C_2\left(\bigcap_{i \in I}(A_i, T)\right) \subseteq \bigcap_{i \in I}C_2(A_i, T)\).
4. \(C_2\left(\bigcap_{i \in I}(A_i, T)\right) = \bigcup_{i \in I}C_2(A_i, T) \bigcap \bigcap_{N \in \mathcal{S}}C_2\left(\bigcup_{i \in 1-N}(A_i, T)\right)\).

**Proof.** It follows from Proposition 2. \(\square\)

**Proposition 14.** Let \(\Theta \in \Theta(X_T)\) and \((A, T), (B, T) \in SS(X_T)\). If \((B, T) \in \Theta\), then \(C_2(A, T) \cap (B, T) \cap (A, T)\).

**Proof.** Let \(x_i \in SP(X_T)\). If \(x_i \in (B, T) \cap (A, T)\), for each \((C, T) \in \Theta(x_i)\), we have \((C, T) \cap (B, T) \cap (A, T) \notin \mathcal{M}(X_T)\). Since \((C, T) \cap (B, T) \cap (A, T) \in \mathcal{M}(X_T)\), \(x_i \in C_2((B, T) \cap (A, T))\). Therefore, \((B, T) \cap (A, T) \subseteq (B, T) \cap (A, T)\). For the reverse, since \((B, T) \cap (A, T) \subseteq (B, T) \cap (A, T)\), by Proposition 12 (1), \(C_2((B, T) \cap (A, T)) \subseteq (B, T) \cap (A, T)\) and therefore, \((B, T) \cap (A, T) \subseteq (B, T) \cap (A, T)\) \subseteq \((B, T) \cap (A, T)\). Consequently, \((B, T) \cap (A, T) = (B, T) \cap (A, T)\). \(\square\)

**Lemma 9.** Let \(\Theta \in \Theta(X_T)\) and \((A, T) \in SS(X_T)\). Then \((A, T) \in \mathcal{M}(X_T)\) if and only if \(C_2(A, T) \cap (A, T) = \Phi_T\) if and only if \(C_2(A, T) = \Phi_T\).

**Proof.** If \((A, T) \in \mathcal{M}(X_T)\) implies \(C_2(A, T) = \Phi_T\) and so \(C_2(A, T) \cap (A, T) = \Phi_T\). Conversely, if \(C_2(A, T) = \Phi_T\), then \(C_2(A, T) \cap (A, T) = \Phi_T\) and thus \((A, T) \in \mathcal{M}(X_T)\) from Theorem 2. \(\square\)
Lemma 10. Let \( \Theta \in \Theta(X_T) \) and \((A, T), (B, T) \in SS(X_T)\). If \((B, T) \in M(X_T)\), then \(C_2([A, T] \cup (B, T)] = C_2(A, T) = C_2(A, T - (B, T)]\).

**Proof.** Applying Proposition 12 (3), we have \(C_2([A, T] \cup (B, T)] = C_2(A, T) \cup C_2(B, T)\). Since \((B, T) \in M(X_T)\), so \(C_2(B, T) = \Phi_T\). Therefore, \(C_2([A, T] \cup (B, T)] = C_2(A, T)\).

On the other hand, by Proposition 12 (4), we have \(C_2(A, T) - C_2(B, T) \subseteq C_2([A, T] - (B, T)]\). Since \(C_2(B, T) = \Phi_T\), then \(C_2(A, T) \subseteq C_2([A, T] - (B, T)]\). Since \((A, T) - (B, T) \subseteq (A, T)\), by Proposition 12 (1), \(C_2([A, T] - (B, T]) \subseteq C_2(A, T)\). Consequently, \(C_2([A, T] - (B, T)] = C_2(A, T)\). Hence, \(C_2([A, T] \cup (B, T)] = C_2(A, T) = C_2([A, T] - (B, T)]\). \(\square\)

Theorem 3. Let \( \Theta \in \Theta(X_T) \) and \((A, T) \in SS(X_T)\). Then

1. \(C_2(C_1(A, T)) = \Phi_T\), i.e., \(C_2((A, T) - C_2(A, T)) = \Phi_T\)
2. \(C_2(C_2(A, T)) = C_2(A, T)\).
3. \(C_2(A, T) = cl(int(C_2(A, T)))\).

**Proof.** (1) Since \((A, T) - C_2(A, T) \subseteq (A, T), by Proposition 12 (1), \(C_2(C_1(A, T)) = C_2((A, T) - C_2(A, T))^c \subseteq C_2(A, T)\). Therefore, \((A, T) - C_2(A, T) \cap C_1(C_2(A, T)) = (A, T) - C_2(A, T)\) \(\cap C_2((A, T) - C_2(A, T))^c \subseteq (A, T) - C_2(A, T)\). Thus, we have \(C_2(C_1(A, T)) = \Phi_T\).

(2) Since, from (1), \(C_2((A, T) - C_2(A, T)) = \Phi_T\), and by Proposition 12 (4), \(C_2(A, T) - C_2(C_2(A, T)) \subseteq \Phi_T\), therefore, \(C_2(C_2(A, T)) \subseteq \Phi_T\) implies \(C_2(A, T) \subseteq \Phi_T\). The converse of the inclusion holds true by Proposition 12 (7). Hence, \(C_2(C_2(A, T)) = C_2(A, T)\).

(3) Since \(\{\Phi_T\} \subseteq N(X_T) \subseteq M(X_T)\), by Definition 18 and Proposition 6 (2), it follows that

\[C_2(A, T) \subseteq cl(int(cl(A, T))) \subseteq cl(A, T).\] (3)

Substitute \(C_2(A, T)\) in expression (3) and since \(C_2(A, T)\) is soft closed, by Proposition 12 (6), we obtain that

\[C_2(C_2(A, T)) \subseteq cl(int(cl(A, T))) \subseteq C_2(A, T).\]

Since, by (2), \(C_2(A, T) = C_2(C_2(A, T))\), hence \(C_2(A, T) = cl(int(C_2(A, T)))\). \(\square\)

Theorem 4. Let \( \Theta \in \Theta(X_T) \) and \((A_n, T) \in SS(X_T)\) for \(n = 1, 2, \ldots\). Then

\[C_2(\bigcup_{n=1}^{\infty} (A_n, T)) - \bigcap_{n=1}^{\infty} C_2(A_n, T) \in N(X_T).\]

**Proof.** Let \( \Phi_T \neq (W, T) \in \Theta \). We find \((V, T) \in \Theta\) such that

\[\Phi_T \neq (W, T) \subseteq (W, T) \quad \text{and} \quad C_2(\bigcup_{n=1}^{\infty} (A_n, T)) - \bigcap_{n=1}^{\infty} C_2(A_n, T) \cap (V, T) = \Phi_T.\]

For each \(n\), if \((W, T) \cap C_2(A_n, T) = \Phi_T\), then \((W, T) \cap C_2(A_n, T) \subseteq (A_n, T) - C_2(A_n, T)\) and hence \((W, T) \cap (\bigcup_{n=1}^{\infty} (A_n, T)) \subseteq \bigcup_{n=1}^{\infty} [(A_n, T) - C_2(A_n, T)]\). It follows from Theorem 3 (1) that \((W, T) \cap (\bigcup_{n=1}^{\infty} (A_n, T)) \in M(X_T)\). By Lemma 9, \(C_2([W, T) \cap (\bigcup_{n=1}^{\infty} (A_n, T)] = \Phi_T\) and by Proposition 14, \((W, T) \cap C_2[\bigcup_{n=1}^{\infty} (A_n, T)] = \Phi_T\). By equating \((V, T)\) with \((W, T)\) we have \((V, T) \cap C_2[\bigcup_{n=1}^{\infty} (A_n, T)] = \Phi_T\).

On the other hand, if for some \(n\), \((W, T) \cap C_2(A_n, T) \neq \Phi_T\). Set \((V, T) = (W, T) \cap int(C_2(A_n, T))\). By Theorem 3 (3), \((V, T) \neq \Phi_T\). Since \((V, T) \subseteq C_2(A_n, T)\), we must obtain that \((V, T) \cap C_2[\bigcup_{n=1}^{\infty} (A_n, T)] = \Phi_T\). This finishes the proof. \(\square\)

Theorem 5. Let \( \Theta \in \Theta(X_T) \) and \((A, T) \in SS(X_T)\). Then \((A, T)\) can be written as a soft union of two soft sets, one of the first category and the other not of the first category at any of its soft points.
Proof. Consider the following decomposition

\[(A, T) = ((A, T) \cap C_2(A, T)) \cup \sim((A, T) \setminus C_2(A, T)).\]

Evidently, \((A, T) \cap C_2(A, T)\) and \((A, T) \setminus C_2(A, T)\) are disjoint. By Theorem 3 (1), \((A, T) \setminus C_2(A, T) \in \mathcal{M}(X_T)\). On the other hand, since, by Lemma 10, \(C_2((A, T) \cap C_2(A, T)) = C_2(A, T)\), so \((A, T) \cap C_2(A, T) \subset C_2(A, T) = C_2((A, T) \cap C_2(A, T))\). This means that \((A, T) \cap C_2(A, T)\) is not of the first category at any soft point \(x_t\) in itself. □

5. Conclusions and Future Work

The ongoing introduction of new classes of topological spaces, examples, and their properties and relations has contributed to the development of topology. As a result, it is critical to expand the area of soft topological spaces in the same manner. We made a new contribution to the field of soft topology by studying certain classes of soft sets in soft topological spaces. The study starts with investigating some important results and operations on soft nowhere dense and soft codense sets. We introduced the concepts of soft points at which a soft set is soft nowhere dense or soft codense followed by their relationships. Correspondingly, we introduced the conceptions and studied soft points at which a soft set is soft somewhere dense or soft somewhat open. We showed that the set of soft points at which a soft set is soft somewhere open is soft regular closed. Then, we recalled the definitions of soft sets of the first and second categories. The main properties of these classes of soft sets were obtained with some operations. In particular, we proved that the set of soft points at which a soft set is soft somewhere closed, and no soft point in a soft set of the first category can be of the second category. We generalized the so-called “Banach Category Theorem” on soft topologies that states that any soft union of soft open sets of the first category is of the first category. We end this work with a representation that states a soft set can be written as a disjoint soft union of two soft sets, one of the first category and the other not of the first category.

From a categorical point of view, this study might be seen as a valuable source for advancing the field of soft topology. The results produced in this research can be used to study soft sets and soft functions of the Baire property, and many advanced topics in soft topology.

Author Contributions: Conceptualization, Z.A.A. and M.H.A.; Methodology, Z.A.A.; Formal Analysis, Z.A.A. and M.H.A.; Investigation, Z.A.A. and M.H.A.; Writing Original Draft Preparation, Z.A.A.; Writing Review and Editing, Z.A.A. and M.H.A.; Funding Acquisition, M.H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References


34. Al-shami, T.M.; El-Shafei, M.E. Generating soft topologies via soft set operators. Symmetry 2022, 14, 914.


42. Al-shami, T.M.; El-Shafei, M.E. Generating soft topologies via soft set operators. Symmetry 2022, 14, 914.


44. Al-shami, T.M.; El-Shafei, M.E. Generating soft topologies via soft set operators. Symmetry 2022, 14, 914.


47. Al-shami, T.M.; El-Shafei, M.E. Generating soft topologies via soft set operators. Symmetry 2022, 14, 914.


Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.