Article

On Some Combinatorial Properties of Oresme Hybrationals

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Abstract: In this paper, we study the Oresme hybrationals that generalize Oresme hybrid numbers and Oresme rational functions. We give a recurrence relation and a generating function for Oresme hybrationals. Moreover, we give some of their properties, among others, Binet formulas and general bilinear index-reduction formulas, through which we can obtain Catalan-, Cassini-, Vajda-, and d’Ocagne-type identities.

Keywords: Horadam numbers; Oresme numbers; complex numbers; hybrid numbers

1. Introduction

The Fibonacci sequence, \( \{F_n\} \) defined by \( F_0 = 0, F_1 = 1, \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2, \) is perhaps the most famous sequence in view of its connections with distinct areas of science, for example, Pascal’s triangle \([1–3]\), computer algorithm \([4,5]\), graph theory \([6]\), some problems of algebra \([7]\), theoretical physics \([8–10]\), and many other areas. This sequence has many interesting interpretations, applications, and generalizations, and based on them, a number of Fibonacci-like sequences were discovered. These generalizations have been given in many ways, some by preserving initial conditions or others by preserving the recurrence relations. In \([11]\), Horadam introduced a generalized Fibonacci sequence, named later as the Horadam sequence, as follows.

For the integer \( n \geq 0 \) Horadam sequence, \( W_n(W_0, W_1; p, q) \) is defined by the recurrence relation

\[
W_{n+2} = p \cdot W_{n+1} - q \cdot W_n,
\]

where \( W_0, W_1 \) are real, and \( p, q \) are integers.

For special values of \( W_0, W_1, p, q, \) Equation (1) gives well-known sequences also named as Fibonacci-type sequences; see Table 1.

Table 1. Special cases of Horadam sequences.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>( W_n(W_0, W_1; p, q) )</th>
<th>OEIS [12]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fibonacci</td>
<td>( W_n(0, 1; 1, -1) = F_n )</td>
<td>[11]</td>
</tr>
<tr>
<td>Lucas</td>
<td>( W_n(2, 1; 1, -1) = L_n )</td>
<td>[12]</td>
</tr>
<tr>
<td>Pell</td>
<td>( W_n(0, 1; 2, -1) = P_n )</td>
<td>[13]</td>
</tr>
<tr>
<td>Pell–Lucas</td>
<td>( W_n(2, 2; 2, -1) = Q_n )</td>
<td>[14]</td>
</tr>
<tr>
<td>Jacobsthal</td>
<td>( W_n(0, 1; 1, -2) = J_n )</td>
<td>[15]</td>
</tr>
<tr>
<td>Jacobsthal–Lucas</td>
<td>( W_n(2, 1; 1, -2) = j_n )</td>
<td>[16]</td>
</tr>
</tbody>
</table>

In the next decade, Horadam extended Equation (1) considering rational numbers \( W_0, \ W_1, \ p, \ q, \) see [13]. In particular, \( W_n(0, 1/2, 1, \ 1/4) \) gives the well-known Oresme sequence \( \{O_n\} \). The sequence \( \{O_n\} \) was introduced by Nicole Oresme in the 14th century, who found the sum of rational numbers formed by \( 0, 1, 2, 3/2, 5/4, 9/8, \ldots, \) These numbers form a sequence that can be defined by the recurrence relation \( O_n = O_{n-1} - \frac{1}{4} O_{n-2} \), for \( n \geq 2 \) with \( O_0 = 0, \ O_1 = \frac{1}{4} \).

Oresme numbers were generalized next by Cook in [14] in the following way. Let \( k \geq 2 \) and \( n \geq 0 \) be integers. Then, the \( k \)-Oresme numbers \( O_n^{(k)} \) are defined by

\[
O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k} O_{n-2}^{(k)},
\]

for \( n \geq 2 \) with \( O_0^{(k)} = 0, \ O_1^{(k)} = \frac{1}{k} \). Clearly, \( W_n(0, 1/2, 1, \ 1/4) = O_n^{(k)} \), and \( O_n^{(2)} = O_n \).

In [15], the sequence of \( k \)-Oresme numbers was extended to the sequence of rational functions \( \{O_n(x)\} \) adding to the definition of \( O_n^{(k)} \) a real variable \( x \) instead of an integer \( k \). These rational functions were named in [15] Oresme polynomials, but by their rational form, we call them Oresme rational functions.

Let \( x \in \mathbb{R} \setminus \{0\} \). The sequence \( \{O_n(x)\} \) of Oresme rational functions is recursively defined as follows:

\[
O_n(x) = \begin{cases} 
0 & \text{if } n = 0, \\
\frac{1}{n} & \text{if } n = 1, \\
O_{n-1}(x) - \frac{1}{n} O_{n-2}(x) & \text{if } n \geq 2.
\end{cases}
\]

The first six terms of the Oresme rational function sequence are

\[
\begin{align*}
O_0(x) &= 0 \\
O_1(x) &= \frac{1}{x} \\
O_2(x) &= \frac{1}{x} \\
O_3(x) &= \frac{x^2 - 1}{x^3} \\
O_4(x) &= \frac{x^2 - 2}{x^3} \\
O_5(x) &= \frac{x^4 - 3x^2 + 1}{x^5}.
\end{align*}
\]

Solving the characteristic Equation \( r^2 - r + \frac{1}{x^2} = 0 \) of the Oresme rational functions recurrence relation, we obtain the Binet formula

\[
O_n(x) = \frac{1}{\sqrt{x^2 - 4}} \left[ \left( \frac{x + \sqrt{x^2 - 4}}{2x} \right)^n - \left( \frac{x - \sqrt{x^2 - 4}}{2x} \right)^n \right]
\]

for \( x^2 - 4 > 0 \) (see [15]), and

\[
O_n(x) = \frac{i}{\sqrt{4 - x^2}} \left[ \left( \frac{x - \sqrt{4 - x^2} \, i}{2x} \right)^n - \left( \frac{x + \sqrt{4 - x^2} \, i}{2x} \right)^n \right]
\]

for \( x^2 - 4 < 0, \ x \neq 0 \), (see for details [16]).

The sequence \( \{O_n(x)\} \) is a generalization of the Oresme sequence because of \( O_n(2) = O_n \) and \( O_n(-2) = -O_n \).

Sequences and polynomials of the Fibonacci-type also have applications in the theory of hypercomplex numbers. Many authors have studied these sequences in the context of quaternions, hybrid numbers, and others. For example in the survey [7], the bibliography contains over 100 papers, and most of them have been published quite recently.
by these results, in this paper we use Oresme numbers and Oresme rational functions in the theory of hybrid numbers. We start with the necessary definitions.

In [17], Özdemir introduced a new noncommutative number system called hybrid numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined by

$$\mathbb{K} = \{z = a + bi + c\varepsilon + d\hbar: a, b, c, d \in \mathbb{R}\},$$

where

$$i^2 = -1, \varepsilon^2 = 0, \hbar^2 = 1, ih = -hi = \varepsilon + i.$$ (4)

Two hybrid numbers $z_1 = a_1 + b_1i + c_1\varepsilon + d_1\hbar$, $z_2 = a_2 + b_2i + c_2\varepsilon + d_2\hbar$ are equal if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$. The sum of two hybrid numbers is defined by $z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)i + (c_1 + c_2)\varepsilon + (d_1 + d_2)\hbar$. Using (4), we obtain the following multiplication table.

Using the rules given in Table 2, the multiplication of hybrid numbers can be made analogously to the multiplications of algebraic expressions. The set of hybrid numbers is a noncommutative ring with respect to the addition and multiplication operations.

Table 2. Multiplication rules for $i$, $\varepsilon$, and $\hbar$.

<table>
<thead>
<tr>
<th>$\cdot$</th>
<th>$i$</th>
<th>$\varepsilon$</th>
<th>$\hbar$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$-1$</td>
<td>$1 - \hbar$</td>
<td>$\varepsilon + i$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$1 + \hbar$</td>
<td>$0$</td>
<td>$-\varepsilon$</td>
</tr>
<tr>
<td>$\hbar$</td>
<td>$-(\varepsilon + i)$</td>
<td>$\varepsilon$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The Horadam hybrid sequence $\{H_n\}$, where $H_n = W_n + W_{n+1}i + W_{n+2}\varepsilon + W_{n+3}\hbar$, was defined and examined in [18]. The Oresme hybrid sequence $\{OH_n\}$, where $OH_n = O_n + O_{n+1}i + O_{n+2}\varepsilon + O_{n+3}\hbar$, was introduced independently in [16,19] (see also [19]), while the $k$-Oresme hybrid numbers $OH_n^{(k)}$, where $OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}i + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\hbar$, were introduced and studied in [16]. The results for Fibonacci-type hybrid numbers were next generalized on Fibonacci-type hybrinomials; see the list [7]. In [16], the authors defined a generalization of $k$-Oresme hybrid numbers and Oresme rational functions, named Oresme hybrational numbers. Properties and applications of $k$-Oresme hybrid numbers were studied in [20], published this year.

The results for Fibonacci-type hybrid numbers were next generalized on Fibonacci-type hybrinomials.

For $n \geq 0$ and $x \in \mathbb{R} \setminus \{0\}$, Oresme hybrational numbers are defined by

$$OH_n(x) = O_n(x) + O_{n+1}(x)i + O_{n+2}(x)\varepsilon + O_{n+3}(x)\hbar,$$ (5)

where $O_n(x)$ is the $n$th Oresme rational function and $i$, $\varepsilon$, and $\hbar$ are hybrid units. For $x = k$, we obtain the $k$-Oresme hybrid numbers.

This paper concerns Oresme hybrational numbers. We give a recurrence relation and a generating function for Oresme hybrational numbers. Moreover, we determine some identities for them, among others, Binet formulas and general bilinear index-reduction formulas. Using these formulas, we also obtain Catalan-, Cassini-, Vajda-, and d’Ocagne-type identities. It is worth mentioning that the domain for Oresme polynomials is symmetric with respect to 0. Moreover, due to the form of Binet’s formulas for Oresme polynomials, the two symmetric points 2 and $-2$ are also considered separately. Therefore, when determining identities for Oresme hybrational numbers, we consider three different cases.

2. Results

In this section, we give some combinatorial properties of Oresme hybrational numbers.
Let \( x \in \mathbb{R} \setminus \{0\} \). For future considerations we need initial conditions, which immediately follow from (5) and (2), of the form

\[
OH_0(x) = \frac{1}{x} + \frac{1}{x^3} + \frac{x^2 - 1}{x^3} h, \\
OH_1(x) = \frac{1}{x} + \frac{1}{x^3} + \frac{x^2 - 1}{x^3} \epsilon + \frac{x^2 - 2}{x^3} h. 
\] (6)

**Theorem 1.** Let \( x \in \mathbb{R} \setminus \{0\} \). Then,

\[
OH_n(x) = OH_{n-1}(x) - \frac{1}{x^2} OH_{n-2}(x) \text{ for } n \geq 2, 
\] (7)

with \( OH_0(x), OH_1(x) \) given by (6).

**Proof.** If \( n = 2 \), then from (2) and (5), we have

\[
OH_2(x) = O_2(x) + O_3(x) i + O_4(x) \epsilon + O_5 h \\
= \frac{1}{x} + \frac{x^2 - 1}{x^3} i + \frac{x^2 - 2}{x^3} \epsilon + \frac{x^4 - 3x^2}{x^3} + 1 h \\
= \frac{1}{x} + \frac{1}{x} + \frac{x^2 - 1}{x^3} i + \frac{x^2 - 2}{x^3} \epsilon - \frac{1}{x^3} + \frac{1}{x^5} h \\
= \frac{1}{x} + \frac{1}{x} + \frac{x^2 - 1}{x^3} i + \frac{x^2 - 2}{x^3} \epsilon - \frac{1}{x^3} + \frac{1}{x^5} h \\
= OH_1(x) - \frac{1}{x^2} OH_0(x). 
\]

So for \( n = 2 \), Equation (7) is true.

If \( n \geq 3 \), then using the definition of the Oresme rational functions, we have

\[
OH_n(x) = O_n(x) + O_{n+1}(x) i + O_{n+2}(x) \epsilon + O_{n+3}(x) h \\
= O_{n-1}(x) - \frac{1}{x^2} O_{n-2}(x) + \left( O_n(x) - \frac{1}{x^2} O_{n-1}(x) \right) i \\
+ \left( O_{n+1}(x) - \frac{1}{x^2} O_n(x) \right) \epsilon + \left( O_{n+2}(x) - \frac{1}{x^2} O_{n+1}(x) \right) h \\
= O_{n-1}(x) + O_n(x) i + O_{n+1}(x) \epsilon + O_{n+2}(x) h \\
- \frac{1}{x^2} \left[ O_{n-2}(x) + O_{n-1}(x) i + O_n(x) \epsilon + O_{n+1}(x) h \right] \\
= OH_{n-1}(x) - \frac{1}{x^2} OH_{n-2}(x), 
\]

which ends the proof. \( \square \)

An important technique for solving linear homogeneous recurrence equations is generating functions. They are typically used with linear recurrence relations with constant coefficients, but they can be applied also for nonconstant coefficients. The next theorem gives the generating function for Oresme hybrational.

**Theorem 2.** Let \( x \in \mathbb{R} \setminus \{0\} \). Then, the generating function for the Oresme hybrational sequence \( \{OH_n(x)\} \) is

\[
G(t, x) = \frac{\frac{1}{2} i + \frac{1}{x} \epsilon + \frac{x^2 - 1}{x^3} h + \left( \frac{1}{x} - \frac{1}{x^2} \epsilon - \frac{1}{x^3} h \right) t}{1 - t + \frac{1}{x^2} t}. 
\]
**Proof.** Assume that the generating function of the Oresme hybrational sequence \( \{OH_n(x)\} \)
has the form \( G(t, x) = \sum_{n=0}^{\infty} OH_n(x)t^n \). Then,
\[
G(t, x) = OH_0(x) + OH_1(x)t + OH_2(x)t^2 + \cdots .
\]

Multiplying the above equality on both sides by \(-t\) and then by \(\frac{1}{x^2}t^2\), we obtain the following
\[
- G(t, x)t = - OH_0(x)t - OH_1(x)t^2 - OH_2(x)t^3 + \cdots \\
\frac{1}{x^2}G(t, x)t^2 = \frac{1}{x^2}OH_0(x)t^2 + \frac{1}{x^2}OH_1(x)t^3 + \frac{1}{x^2}OH_2(x)t^4 + \cdots .
\]

By adding the three equalities above, we obtain
\[
G(t, x)\left(1 - t + \frac{1}{x^2}t^2\right) = OH_0(x) + (OH_1(x) - OH_0(x))t,
\]
since \(OH_{n+2}(x) = OH_{n+1}(x) - \frac{1}{x^2}OH_n(x)\) (see (7)), and the coefficients of \(t^n\) for \(n \geq 2\) are equal to zero. Moreover, \(OH_0(x) = \frac{1}{x}i + \frac{1}{x}\epsilon + \frac{1}{x^3}h\), \(OH_1(x) - OH_0(x) = \frac{1}{x} - \frac{1}{x^2}\epsilon - \frac{1}{x^3}h\).

**Properties of Oresme Hybrationals**

In this subsection, we give a number of identities for Oresme hybrationals. As the Binet formula for Oresme rational functions depends on the value of \(x\), some of them cannot be given for all \(x \in \mathbb{R} \setminus \{0\}\). First, we give equalities for all nonzero real variable \(x\), and next, we consider cases \(x^2 - 4 > 0\), \(x^2 - 4 < 0\), \(x = 2\), and \(x = -2\).

Note that the cases mentioned above contain sets that are symmetric with respect to 0. Due to the form of Binet’s formulas for Oresme polynomials, the two symmetric points 2 and \(-2\) are also considered separately.

Now, we describe the terms of the sequence \(\{OH_n(x)\}\) explicitly using a Binet-type formula. Because the Binet formula for Oresme rational functions depends on \(x\), we have to consider cases with respect to the sign of the expression \(x^2 - 4\).

**Theorem 3. (Binet-type formula)** Let \(n \geq 0\) be an integer. Then, for \(x^2 - 4 > 0\), we have
\[
OH_n(x) = C\alpha_1^n\hat{\alpha}_1^n - C\alpha_2^n\hat{\alpha}_2^n,
\]
where
\[
C = \frac{1}{\sqrt{x^2 - 4}} \\
\alpha_1 = \frac{x + \sqrt{x^2 - 4}}{2x} \\
\alpha_2 = \frac{x - \sqrt{x^2 - 4}}{2x} \\
\hat{\alpha}_1 = 1 + \alpha_1i + \alpha_1^2\epsilon + \alpha_1^3h \\
\hat{\alpha}_2 = 1 + \alpha_2i + \alpha_2^2\epsilon + \alpha_2^3h.
\]
Proof. By (3) and (5) we obtain
\[
\begin{align*}
OH_n(x) &= O_n(x) + O_{n+1}(x)i + O_{n+2}(x)\varepsilon + O_{n+3}(x)h \\
&= C_1^n - C_2^n + (C_1^{n+1} - C_2^{n+1})i + (C_1^{n+2} - C_2^{n+2})\varepsilon \\
&+ (C_1^{n+3} - C_2^{n+3})h \\
&= C_1^n \left(1 + a_1 i + a_1^2 \varepsilon + a_1^3 h\right) - C_2^n \left(1 + a_2 i + a_2^2 \varepsilon + a_2^3 h\right) \\
&= C_1^n \hat{\alpha}_1 - C_2^n \hat{\alpha}_2.
\end{align*}
\]
\[\square\]

Theorem 4. (General bilinear index-reduction formula) Let \(a \geq 0, b \geq 0, c \geq 0,\) and \(d \geq 0\) be integers such that \(a + b = c + d.\) Then, for \(x^2 - 4 > 0,\) we have
\[
\begin{align*}
OH_a(x) \cdot OH_b(x) - OH_c(x) \cdot OH_d(x) \\
&= \left(C_1 a_1^d a_2^d - C_2 a_1^d a_2^d\right) \hat{\alpha}_1 \cdot \hat{\alpha}_2 + \left(C_1 a_2^d a_1^d - C_2 a_2^d a_1^d\right) \hat{\alpha}_2 \cdot \hat{\alpha}_1,
\end{align*}
\]
where \(C, a_1, a_2, \hat{\alpha}_1,\) and \(\hat{\alpha}_2\) are given by (9).

Proof. Using (8), we have
\[
\begin{align*}
OH_a(x) \cdot OH_b(x) - OH_c(x) \cdot OH_d(x) \\
&= (C_1 a_1^d a_2^d - C_2 a_1^d a_2^d) \hat{\alpha}_1 \cdot \hat{\alpha}_2 \\
&\quad - (C_1 a_2^d a_1^d - C_2 a_2^d a_1^d) \hat{\alpha}_2 \cdot \hat{\alpha}_1 \\
&= -C_1 a_1^d a_2^d \hat{\alpha}_1 \hat{\alpha}_2 - C_2 a_2^d a_1^d \hat{\alpha}_1 \\
&\quad + C_1 a_1^d a_2^d \hat{\alpha}_1 \hat{\alpha}_2 + C_2 a_2^d a_1^d \hat{\alpha}_1 \\
&= \left(C_1 a_1^d a_2^d - C_2 a_1^d a_2^d\right) \hat{\alpha}_1 \cdot \hat{\alpha}_2 + \left(C_1 a_2^d a_1^d - C_2 a_2^d a_1^d\right) \hat{\alpha}_2 \cdot \hat{\alpha}_1,
\end{align*}
\]
which ends the proof. \(\square\)

From (10), we can derive Catalan-, Cassini-, Vajda-, and d’Ocagne-type identities. Before it, we need to prove the following lemma.

Lemma 1. Let \(\hat{\alpha}_1 = 1 + a_1 i + a_1^2 \varepsilon + a_1^3 h, \hat{\alpha}_2 = 1 + a_2 i + a_2^2 \varepsilon + a_2^3 h,\) where \(a_1 = \frac{x + \sqrt{x^2 - 4}}{2x}, \) \(a_2 = \frac{x - \sqrt{x^2 - 4}}{2x}.\) Then, for \(x^2 - 4 > 0,\) we have
\[
\begin{align*}
\hat{\alpha}_1 \cdot \hat{\alpha}_2 &= \frac{x^6 + 1}{x^6} + \frac{x^3 - \sqrt{x^2 - 4}}{x^3} i \\
&\quad + \frac{x^5 - 2x^3 - x^2 \sqrt{x^2 - 4} + \sqrt{x^2 - 4}}{x^5} \varepsilon + \frac{x^3 - 3x + \sqrt{x^2 - 4}}{x^3} h,
\end{align*}
\]
\[
\begin{align*}
\hat{\alpha}_2 \cdot \hat{\alpha}_1 &= \frac{x^6 + 1}{x^6} + \frac{x^3 + \sqrt{x^2 - 4}}{x^3} i \\
&\quad + \frac{x^5 - 2x^3 + x^2 \sqrt{x^2 - 4} - \sqrt{x^2 - 4}}{x^5} \varepsilon + \frac{x^3 - 3x - \sqrt{x^2 - 4}}{x^3} h.
\end{align*}
\]
Proof. By multiplication rules, we obtain
\[
\hat{a}_1 \cdot \hat{a}_2 = \left(1 + a_1 i + a_2^2 e + a_1^3 h\right) \cdot \left(1 + a_2 i + a_2^2 e + a_1^3 h\right)
\]
\[
= 1 + a_2 i + a_2^2 e + a_3^2 h + a_1 i + a_1 a_2 + a_1 a_2^2 (1 - h) + a_1 a_2^3 (e + i)
\]
\[
+ a_3^2 h + a_1^2 a_2 (1 + h) - a_1^2 a_2^2 e
\]
\[
+ a_3^2 h - a_1^2 a_2 (e + i) + a_3^2 a_2^2 e + a_3^3 a_2
\]
\[
= 1 - a_1 a_2 + a_1 a_2 (a_1 + a_2) + (a_1 a_2)^3
\]
\[
+ (a_1 + a_2 + a_1 a_2 (a_2 - a_1) (a_2 + a_1)) i
\]
\[
+ \left(a_1^2 + a_2^2 + a_1 a_2 (a_2 - a_1) (a_2 + a_1) + (a_1 a_2)^2 (a_1 - a_2)\right) e
\]
\[
+ \left(a_1^3 + a_2^3 + a_1 a_2 (a_2 - a_1)\right) h.
\]

Using the equalities
\[
a_1 \cdot a_2 = \frac{1}{x^2},
\]
\[
a_1 + a_2 = 1,
\]
\[
a_1 - a_2 = \frac{\sqrt{x^2 - 4}}{x},
\]
\[
a_1^2 + a_2^2 = (a_1 + a_2)^2 - 2a_1 a_2 = \frac{x^2 - 2}{x^2},
\]
\[
a_1^3 + a_2^3 = (a_1 + a_2)^3 - 3a_1 a_2 (a_1 + a_2) = \frac{x^2 - 3}{x^2},
\]
we obtain
\[
\hat{a}_1 \cdot \hat{a}_2 = \frac{x^6 + 1}{x^6} + \frac{x^3 - \sqrt{x^2 - 4}}{x^3}
\]
\[
+ \left(\frac{x^2 - 2}{x^2} - \frac{\sqrt{x^2 - 4}}{x^3} + \frac{\sqrt{x^2 - 4}}{x^3}\right) e
\]
\[
+ \left(\frac{x^2 - 3}{x^2} + \frac{\sqrt{x^2 - 4}}{x^3}\right) h
\]
\[
= \frac{x^6 + 1}{x^6} + \frac{x^3 - \sqrt{x^2 - 4}}{x^3}
\]
\[
+ \frac{x^5 - 2x^3 - x^2 \sqrt{x^2 - 4} + \sqrt{x^2 - 4}}{x^5} e
\]
\[
+ \frac{x^3 - 3x + \sqrt{x^2 - 4}}{x^3} h
\]
and we obtain the equality (11). The proof of (12) is analogous, so we omit it. \(\square\)

**Theorem 5.** (Catalan-type identity) Let \(n \geq 0\), \(r \geq 0\) be integers such that \(n \geq r\). Then, for \(x^2 - 4 > 0\), we have
\[
OH_{n+r}(x) \cdot OH_{n-r}(x) - (OH_n(x))^2
\]
\[
= \frac{4^r - \left(x + \sqrt{x^2 - 4}\right)^{2r}}{4^r x^{2n}(x^2 - 4)} \hat{a}_1 \cdot \hat{a}_2 + \frac{4^r - \left(x - \sqrt{x^2 - 4}\right)^{2r}}{4^r x^{2n}(x^2 - 4)} \hat{a}_2 \cdot \hat{a}_1
\]
where \(\hat{a}_1 \cdot \hat{a}_2, \hat{a}_2 \cdot \hat{a}_1\) are given by (11) and (12), respectively.

**Proof.** Setting \(a = n + r, b = n - r, c = d = n\) in (10), we have
\[
OH_{n+r}(x) \cdot OH_{n-r}(x) - (OH_n(x))^2
\]
\[
= C^2 \hat{a}_1^a \hat{a}_2^a \left(1 - \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^r\right) \hat{a}_1 \cdot \hat{a}_2 + C^2 \hat{a}_1^a \hat{a}_2^a \left(1 - \left(\frac{\hat{a}_2}{\hat{a}_1}\right)^r\right) \hat{a}_2 \cdot \hat{a}_1.
Moreover, putting
\[ C^2 = \frac{1}{x^4 - 4}, \quad (\alpha_1 \cdot \alpha_2)^n = x^{-2n}, \]
\[ \frac{\alpha_1}{\alpha_2} = \frac{1}{4} \left( x + \sqrt{x^2 - 4} \right)^2, \quad \frac{\alpha_2}{\alpha_1} = \frac{1}{4} \left( x - \sqrt{x^2 - 4} \right)^2, \]
the result follows. \( \Box \)

If \( r = 1 \), then we obtain the Cassini-type identity.

**Corollary 1.** (Cassini-type identity) Let \( n \geq 1 \) be an integer. Then, for \( x^2 - 4 > 0 \), we have
\[
OH_{n+1}(x) \cdot OH_{n-1}(x) - (OH_n(x))^2
= C^2 \alpha_1^n \alpha_2^n \left( 1 - \frac{\alpha_1}{\alpha_2} \right)^p \alpha_1 \cdot \alpha_2 + C^2 \alpha_1^n \alpha_2^n \left( 1 - \frac{\alpha_2}{\alpha_1} \right) \alpha_2 \cdot \alpha_1,
\]
where \( C, \alpha_1, \alpha_2, \hat{\alpha}_1, \hat{\alpha}_2 \) are given by (9), and \( \hat{\alpha}_1 \cdot \hat{\alpha}_2, \hat{\alpha}_2 \cdot \hat{\alpha}_1 \) are given by (11) and (12), respectively.

For special values of \( a, b, c, d \), we obtain the Vajda- and d’Ocagne-type identities, respectively.

**Corollary 2.** (Vajda-type identity) Let \( n \geq 0, m \geq 0, \) and \( p \geq 0 \) be integers such that \( n \geq p \). Then, for \( x^2 - 4 > 0 \), we have
\[
OH_{n+p}(x) \cdot OH_{n-p}(x) - OH_m(x) \cdot OH_n(x)
= C^2 \alpha_1^n \alpha_2^n \left( 1 - \left( \frac{\alpha_1}{\alpha_2} \right)^p \right) \alpha_1 \cdot \alpha_2 + C^2 \alpha_1^n \alpha_2^n \left( 1 - \left( \frac{\alpha_2}{\alpha_1} \right)^p \right) \alpha_2 \cdot \alpha_1,
\]
where \( C, \alpha_1, \alpha_2, \hat{\alpha}_1, \hat{\alpha}_2 \) are given by (9), and \( \hat{\alpha}_1 \cdot \hat{\alpha}_2, \hat{\alpha}_2 \cdot \hat{\alpha}_1 \) are given by (11), (12), respectively.

**Corollary 3.** (d’Ocagne-type identity) Let \( n \geq 0, m \geq 0 \) be integers such that \( n \geq m \). Then, for \( x^2 - 4 > 0 \), we have
\[
OH_n(x) \cdot OH_{m+1}(x) - OH_{n+1}(x) \cdot OH_m(x)
= C^2 \alpha_1^n \alpha_2^n (\alpha_1 - \alpha_2) \alpha_1 \cdot \hat{\alpha}_2 + C^2 \alpha_1^n \alpha_2^n (\alpha_2 - \alpha_1) \hat{\alpha}_2 \cdot \alpha_1,
\]
where \( C, \alpha_1, \alpha_2, \hat{\alpha}_1, \hat{\alpha}_2 \) are given by (9), and \( \hat{\alpha}_1 \cdot \hat{\alpha}_2, \hat{\alpha}_2 \cdot \hat{\alpha}_1 \) are given by (11) and (12), respectively.

Now, let \( x^2 - 4 < 0 \) and \( x \neq 0 \). Using the same methods, we can obtain analogous identities; so, we omit the proofs.

**Theorem 6.** (Binet-type formula) Let \( n \geq 0 \) be an integer. Then for \( x^2 - 4 < 0 \) and \( x \neq 0 \) we have
\[
OH_n(x) = D\beta_1^n \beta_1 - D\beta_2^n \beta_2,
\]
where

\[ D = \frac{i}{\sqrt{4 - x^2}}, \]
\[ \beta_1 = \frac{x - \sqrt{4 - x^2}}{2x}, \]
\[ \beta_2 = \frac{x + \sqrt{4 - x^2}}{2x}, \]
\[ \hat{\beta}_1 = 1 + \beta_1 i + \beta_1^2 \varepsilon + \beta_1^3 h, \]
\[ \hat{\beta}_2 = 1 + \beta_2 i + \beta_2^2 \varepsilon + \beta_2^3 h. \]  

**Theorem 7.** (General bilinear index-reduction formula) Let \( a \geq 0, b \geq 0, c \geq 0, d \geq 0 \) be integers such that \( a + b = c + d \). Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\text{OH}_a(x) \cdot \text{OH}_b(x) - \text{OH}_c(x) \cdot \text{OH}_d(x) = \left( D^2 \beta_1^3 \beta_2^4 - D^2 \beta_1^4 \beta_2^4 \right) \hat{\beta}_1 \cdot \hat{\beta}_2 + \left( D^2 \beta_1^3 \beta_2^4 - D^2 \beta_2^3 \beta_1^4 \right) \hat{\beta}_2 \cdot \hat{\beta}_1
\]

where \( D, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2 \) are given by (13).

**Lemma 2.** Let \( \hat{\beta}_1 = 1 + \beta_1 i + \beta_1^2 \varepsilon + \beta_1^3 h, \hat{\beta}_2 = 1 + \beta_2 i + \beta_2^2 \varepsilon + \beta_2^3 h \), where \( \beta_1 = \frac{x - \sqrt{4 - x^2}}{2x} \) and \( \beta_2 = \frac{x + \sqrt{4 - x^2}}{2x} \). Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\hat{\beta}_1 \cdot \hat{\beta}_2 = \frac{x^6 + 1 - x \sqrt{4 - x^2}}{x^6} + \frac{x^3 - \sqrt{4 - x^2} \varepsilon}{x^3} + \frac{x^3 - 3x^3 - x^2 \sqrt{4 - x^2} + \sqrt{4 - x^2} \varepsilon}{x^5},
\]
\[
\hat{\beta}_2 \cdot \hat{\beta}_1 = \frac{x^6 + 1 + x \sqrt{4 - x^2}}{x^6} + \frac{x^3 + \sqrt{4 - x^2} \varepsilon}{x^3} + \frac{x^3 - 3x^3 + x^2 \sqrt{4 - x^2} - \sqrt{4 - x^2} \varepsilon}{x^5}.
\]

**Proof.** Using the equalities

\[
\beta_1 \cdot \beta_2 = \frac{1}{x^2}, \]
\[
\beta_1 + \beta_2 = 1, \]
\[
\beta_2 - \beta_1 = \frac{\sqrt{4 - x^2}}{x}, \]
\[
\beta_1^2 + \beta_2^2 = (\beta_1 + \beta_2)^2 - 2\beta_1 \beta_2 = \frac{x^2 - 2}{x^2}, \]
\[
\beta_1^3 + \beta_2^3 = (\beta_1 + \beta_2)^3 - 3\beta_1 \beta_2 (\beta_1 + \beta_2) = \frac{x^2 - 3}{x^2}, \]
we obtain
\[
\hat{\beta}_1 \cdot \hat{\beta}_2 = \frac{x^6 + 1}{x^6} + \frac{x^3 + \sqrt{4 - x^2}}{x^3} i \\
+ \frac{x^5 - 2x^3 + x^2 \sqrt{4 - x^2}}{x^3} i - \sqrt{4 - x^2} i \\
+ \frac{x^3 - 3x - \sqrt{4 - x^2}}{x^3} h \\
= \frac{x^6 + 1 - x \sqrt{4 - x^2}}{x^6} + \frac{x^3 - \sqrt{4 - x^2}}{x^3} i \\
+ \frac{x^5 - 2x^3 - x^2 \sqrt{4 - x^2}}{x^5} i - \sqrt{4 - x^2} i h,
\]

and we obtain the equality (14). We omit the proof of (15). □

**Corollary 4.** (Catalan-type identity) Let \( n \geq 0, r \geq 0 \) be integers such that \( n \geq r \). Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\text{OH}_{n+r}(x) \cdot \text{OH}_{n-r}(x) - (\text{OH}_n(x))^2 = D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_1}{\beta_2} \right)^r \right) \hat{\beta}_1 \cdot \hat{\beta}_2 + D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_2}{\beta_1} \right)^r \right) \hat{\beta}_2 \cdot \hat{\beta}_1,
\]

where \( D, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2 \) are given by (13), and \( \beta_1 \cdot \hat{\beta}_2, \hat{\beta}_2 \cdot \hat{\beta}_1 \) are given by (14) and (15), respectively.

**Corollary 5.** (Cassini-type identity) Let \( n \geq 1 \) be an integer. Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\text{OH}_{n+1}(x) \cdot \text{OH}_{n-1}(x) - (\text{OH}_n(x))^2 = D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_1}{\beta_2} \right)^r \right) \beta_1 \cdot \beta_2 - D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_2}{\beta_1} \right)^r \right) \beta_2 \cdot \beta_1,
\]

where \( D, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2 \) are given by (13), and \( \beta_1 \cdot \hat{\beta}_2, \hat{\beta}_2 \cdot \hat{\beta}_1 \) are given by (14) and (15), respectively.

**Corollary 6.** (Vajda-type identity) Let \( n \geq 0, m \geq 0, \) and \( p \geq 0 \) be integers such that \( n \geq p \). Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\text{OH}_{m+p}(x) \cdot \text{OH}_{n-p}(x) - \text{OH}_m(x) \cdot \text{OH}_n(x) = D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_1}{\beta_2} \right)^r \right) \beta_1 \cdot \beta_2 + D^2 \beta_1^n \beta_2^n \left( 1 - \left( \frac{\beta_2}{\beta_1} \right)^r \right) \beta_2 \cdot \beta_1,
\]

where \( D, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2 \) are given by (13), and \( \beta_1 \cdot \hat{\beta}_2, \hat{\beta}_2 \cdot \hat{\beta}_1 \) are given by (14), (15), respectively.

**Corollary 7.** (d’Ocagne-type identity) Let \( n \geq 0, m \geq 0 \) be integers such that \( n \geq m \). Then, for \( x^2 - 4 < 0 \) and \( x \neq 0 \), we have

\[
\text{OH}_n(x) \cdot \text{OH}_{m+1}(x) - \text{OH}_{n+1}(x) \cdot \text{OH}_m(x) = D^2 \beta_1^n \beta_2^n \left( \beta_1 \beta_2 - \beta_2 \beta_1 \right) \beta_1 \cdot \beta_2 + D^2 \beta_1^n \beta_2^n \left( \beta_2 \beta_1 - \beta_1 \beta_2 \right) \beta_2 \cdot \beta_1,
\]

where \( D, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2 \) are given by (13), and \( \beta_1 \cdot \hat{\beta}_2, \hat{\beta}_2 \cdot \hat{\beta}_1 \) are given by (14), (15), respectively.

The next results concern special cases \( k = 2 \) or \( k = -2 \), respectively. Because of the fact that they follow by simple calculations, we omit the proofs.
Lemma 3 ([16,19]). (Binet-type formula) Let \( n \geq 0 \) be an integer. Then,
\[
OH_n = \frac{n}{2^n} + \frac{n+1}{2^{n+1}} i + \frac{n+2}{2^{n+2}} \epsilon + \frac{n+3}{2^{n+3}} h.
\]

Theorem 8. (General bilinear index-reduction formula) Let \( a \geq 0, b \geq 0, c \geq 0, \) and \( d \geq 0 \) be integers such that \( a + b = c + d \). Then, we have
\[
OH_a \cdot OH_b - OH_c \cdot OH_d
= \frac{65(ab - cd)}{2^{a+b+6}} + \left( \frac{ab - cd}{2^{a+b}} + \frac{d - b}{2^{a+b+2}} \right) i
+ \left( \frac{ab - cd}{2^{a+b+4}} + \frac{3(d - b)}{2^{a+b+6}} \right) \epsilon
+ \left( \frac{ab - cd}{2^{a+b+6}} + \frac{b - d}{2^{a+b+8}} \right) h.
\]

Corollary 8 ([16]). (Catalan-type identity) Let \( n \geq 0, r \geq 0 \) be integers such that \( n \geq r \). Then,
\[
OH_{n+r} \cdot OH_{n-r} - (OH_n)^2
= -65r^2 + \frac{-4r^2 + r}{4 \cdot 4^n \cdot i} + \frac{-8r^2 + 3r}{16 \cdot 4^n \cdot \epsilon} + \frac{-r^2 - r}{4 \cdot 4^n \cdot h}.
\]

Corollary 9 ([16]). (Cassini-type identity) Let \( n \geq 1 \) be an integer. Then,
\[
OH_{n+1} \cdot OH_{n-1} - (OH_n)^2
= \frac{-65}{64 \cdot 4^n} + \frac{-3}{4 \cdot 4^n \cdot i} + \frac{-5}{16 \cdot 4^n \cdot \epsilon} + \frac{-2}{4 \cdot 4^n \cdot h}.
\]

Corollary 10. (Vajda-type identity) Let \( n \geq 0, m \geq 0, \) and \( p \geq 0 \) be integers such that \( n \geq p \). Then,
\[
OH_{m+p} \cdot OH_{n-p} - OH_m \cdot OH_n
= \frac{65p(n - m - p)}{64 \cdot 2^{m+n}} + \frac{p(4n - 4m - 4p + 1)}{4 \cdot 2^{m+n}} i
+ \frac{p(8n - 8m - 8p + 3)}{16 \cdot 2^{m+n}} \epsilon + \frac{p(n - m - p - 1)}{4 \cdot 2^{m+n}} h.
\]

Corollary 11. (d’Ocagne-type identity) Let \( n \geq 0, m \geq 0 \) be integers such that \( n \geq m \). Then,
\[
OH_n \cdot OH_{m+1} - OH_{n+1} \cdot OH_m
= \frac{65(n - m)}{128 \cdot 2^{m+n}} + \frac{4n - 4m - 1}{8 \cdot 2^{m+n}} i + \frac{8n - 8m - 3}{32 \cdot 2^{m+n}} \epsilon + \frac{n - m + 1}{8 \cdot 2^{m+n}} h.
\]

Theorem 9. (Binet-type formula) Let \( n \geq 0 \) be an integer. Then,
\[
OH_n(-2) = \frac{O_n(-2) + (n+1)(-2) i + O_{n+2}(-2) \epsilon + O_{n+3}(-2) h}{-O_n - O_{n+1} i - O_{n+2} \epsilon - O_{n+3} h}
= \frac{-n}{2^n} + \frac{n+1}{2^{n+1}} i + \frac{n+2}{2^{n+2}} \epsilon + \frac{n+3}{2^{n+3}} h = -OH_n.
\]
Theorem 10. (General bilinear index-reduction formula) Let \( a \geq 0, b \geq 0, c \geq 0, \) and \( d \geq 0 \) be integers such that \( a + b = c + d \). Then, we have

\[
\begin{align*}
OH_a(-2) \cdot OH_b(-2) - OH_c(-2) \cdot OH_d(-2) \\
&= -OH_a \cdot (-OH_b) - (-OH_c) \cdot (-OH_d) \\
&= OH_a \cdot OH_b - OH_c \cdot OH_d \\
&= \frac{55(ab - cd)}{2^{a+b+6}} + \left( \frac{ab - cd}{2^{a+b}} + \frac{d - b}{2^{a+b+2}} \right) i \\
&+ \left( \frac{ab - cd}{2^{a+b+1}} + \frac{3(d - b)}{2^{a+b+4}} \right) \epsilon + \left( \frac{ab - cd}{2^{a+b+2}} + \frac{b - d}{2^{a+b+2}} \right) h.
\end{align*}
\]

3. Conclusions

In this paper, we considered Oresme hybrationals that generalize Oresme hybrid numbers and Oresme rational functions. We gave a recurrence relation and a generating function for Oresme hybrationals. We also obtained identities for them, in particular, Binet formulas and general bilinear index-reduction formulas. Using these formulas we determined Catalan-, Cassini-, Vajda-, and d’Ocagne-type identities.

Sequences and polynomials of the Fibonacci-type have applications in the theory of hypercomplex numbers. It could also be interesting to use Oresme numbers and Oresme rational functions in the theory of hypercomplex numbers and polynomials, for example, bihyperbolic numbers and bihypernomials.

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