Third Hankel Determinant for Subclasses of Analytic and \( m \)-Fold Symmetric Functions Involving Cardioid Domain and Sine Function

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Abstract: In this research, we define a few subclasses of analytic functions which are connected to sine functions and the cardioid domain in the unit disk. We investigate initial coefficient bounds, the Fekete–Szegö problem and second and third Hankel determinants for the functions \( f \) belonging to these newly defined classes. We also define the class of \( m \)-fold symmetric functions related with the sine function and then investigate the bounds of the third Hankel determinant for twofold symmetric and threefold symmetric functions.

Keywords: analytic functions; cardioid domain starlike functions; Hankel determinant; subordination; \( m \)-fold symmetric functions

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1. Introduction

Let \( H(U) \) represent the class of analytic functions in the open unit disk \( U = \{ \gamma : \gamma \in \mathbb{C} \text{ and } |\gamma| < 1 \} \). Also, let \( A \) define a subclass of analytic functions \( f \) in \( H(U) \), satisfying the normalization conditions

\[
    f(0) = f'(0) - 1 = 0,
\]

and each \( f \in A \) has the following series of the form:

\[
    f(\gamma) = \gamma + \sum_{n=2}^{\infty} a_n \gamma^n.
\]

The analytic function \( f \) is called univalent in the open unit disk \( U \), if there exists one-to-one correspondence between \( U \) and its image under \( f \). It means for all \( \gamma_1, \gamma_2 \in U \), if \( f(\gamma_1) = f(\gamma_2) \) implies \( \gamma_1 = \gamma_2 \).

\( S \) represents the subclass of \( A \) containing univalent functions and \( C, S^*, R \) and \( K \) are the common subclasses of \( S \) whose members are convex, starlike, bounded turning and close-to-convex, respectively.

In [1], Chichra initiated the class of functions \( R_\alpha \) that hold the condition

\[
    \text{Re}\{f'(\gamma) + \alpha f''(\gamma)\} > 0, \quad (\alpha \geq 0, \, \gamma \in U),
\]
for \( f \in A \). He also proved that if \( f \in R_\alpha \) then \( \text{Re} f'(\gamma) > 0, \gamma \in U \) and hence \( f \) is univalent in \( U \). Singh et al. [2] showed that if \( f \in R, (\alpha = 1) \) then \( f \) is also starlike in \( U \). Furthermore, Singh et al. proved some other interesting results related to the functions class \( R \). Also in [3], an example was given by Krzyz, in which he showed that the class \( C \) is not a super set of the class \( R \). After that, Ali introduced and studied the class \( R(\beta) \) for \( \beta < 1 \) as follows:

\[
\text{Re} \left\{ f'(\gamma) + \gamma f''(\gamma) \right\} > \beta, \quad \gamma \in U.
\]

for \( f \in A \). Recently, Noor et al. generalized this class using the idea of multivalent functions. For detail study about the class \( R \) and its various extensions, see [4,5].

Furthermore, if a function \( f \) maps \( U \) on to a starshaped domain, then we say that \( f \) is a starlike function and is denoted by \( S^* \). Analytically

\[
\text{Re} \left( \frac{\gamma f'(\gamma)}{f(\gamma)} \right) > 0, \quad \gamma \in U.
\]

Moreover, the function \( w \) which is analytic under the conditions given below is called a Schwarz function,

\[
w(0) = 0,
\]

and

\[
|w(\gamma)| < 1.
\]

For the analytic functions \( f \) and \( g \), the function \( f \) is said to subordinate to the function \( g \) and is denoted by

\[
f \prec g
\]

if the above defined Schwarz function \( w \) occurs, such that

\[
f(\gamma) = g(w(\gamma)).
\]

The class of Caratheodory functions is given in [6] and is denoted by \( \mathcal{P} \), so for every \( p \in \mathcal{P} \) hold the following requirement:

\[
p(0) = 1 \quad \text{and} \quad \text{Re}(p(\gamma)) > 0, \quad \gamma \in U,
\]

and has the series of the form:

\[
p(\gamma) = 1 + \sum_{n=1}^{\infty} c_n \gamma^n.
\]

The geometric function theory was founded in the eighteenth century, but in 1985, de Branges [7] settled the well-known problem in the univalent function theory by proving the Bieberbach conjecture [8] for the coefficient estimates of the class \( S \) of univalent functions. This hypothesis provided a new direction for investigation in this field of research, particularly related to coefficient bounds. A number of subclasses of the class \( S \) were studied by many distinguished researchers from different viewpoints and perspectives, which involved different kinds of domains and functions. The Fekete-Szego inequality is one of the inequalities Fekete and Szego (1933) found for the coefficients of univalent analytic functions and associated it to the Bieberbach conjecture [8]. The classes \( S^* \) of starlike and \( C \) of convex functions are the basic subclasses of the class \( S \) of univalent functions. Ma and Minda [9] contributed a significant contribution in 1992 by defining the basic structure of families of univalent functions as below:

\[
S^*(\varphi) = \left\{ f \in A : \frac{\gamma f'(\gamma)}{f(\gamma)} \prec \varphi(\gamma) \right\},
\]

where the function \( \varphi \) involved in the right-hand side of (4) is analytic, with the conditions
\[ \varphi(0) > 0 \text{ and } \Re(\varphi(\gamma)) > 0 \text{ in } U.\]

On the other hand, if we assume \( \varphi(\gamma) = \frac{1 + \gamma}{1 - \gamma} \), in the right hand side of (4), we can have the class of starlike functions such as:

\[ S^* = \left\{ f \in A : \frac{g_f'(\gamma)}{g(\gamma)} < \frac{1 + \gamma}{1 - \gamma} \right\}. \]

As a special example of \( S^*(\varphi) \) a variety of sub-families of the generalized analytic functions have been examined in recent years. For example, Janowski [10] studied the following class of starlike functions associated with the Janowski functions as:

\[ S^*(\gamma, M) = \left\{ f \in A : \frac{g_f'(\gamma)}{g(\gamma)} < \frac{1 + \gamma \gamma}{1 + \gamma \gamma}, (-1 \leq M < L \leq 1) \right\}. \]

Furthermore, by choosing \( L = (1 - 2\alpha) \) and \( M = -1 \), we obtain the well-known class of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), as follows:

\[ S^*(\alpha) = \left\{ f \in A : \frac{g_f'(\gamma)}{g(\gamma)} < \frac{1 + (1 - 2\alpha) \gamma}{1 - \gamma} \right\}. \]

Sokół and Stankiewicz [11] set \( \varphi(\gamma) = \sqrt{1 + \gamma} \) and defined the family of functions \( S^*_L \) as given below:

\[ S^*_L = \left\{ f \in A : \frac{g_f'(\gamma)}{g(\gamma)} < \sqrt{1 + \gamma} \right\}. \]

Recently in [12], Arif et al. chose \( \varphi(\gamma) = 1 + \sin \gamma \) and defined the following classes of convex, starlike and bounded turning functions as follows:

\[ C_{\sin} = \left\{ f \in A : 1 + \frac{g_f''(\gamma)}{g_f'(\gamma)} < 1 + \sin \gamma \right\} \]
\[ S^*_{\sin} = \left\{ f \in A : \frac{g_f'(\gamma)}{g(\gamma)} < 1 + \sin \gamma \right\} \]
\[ R^*_{\sin} = \left\{ f \in A : f'(\gamma) < 1 + \sin \gamma \right\}. \]

For these classes they investigated initial bounds, Fekete–Szegö problems and the third Hankel determinant. Sharma et al. [13] introduced the class of starlike functions whose image has a cardioid form under an open unit disk. Mendiratta et al. [14] studied the function class \( S^*_L = S^*(e^\gamma) \) of starlike functions by using exponential function applying the technique of subordination. Recently, Srivastava et al. [15] generalized this class and determined an upper bound of the third-order Hankel determinant.

For \( f \in A \), the \( j \)th Hankel determinant is defined by

\[ \mathcal{H}_{j,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix}, \tag{5} \]

where \( n, j \in \mathbb{N} \) and \( a_1 = 1 \).

For different values of \( j \) and \( n \), the \( \mathcal{H}_{j,n}(f) \) has the different form:

(i) For \( j = 2 \) and \( n = 1 \), we obtain the Fekete–Szegö functional that is:

\[ \mathcal{H}_{2,1}(f) = \left| a_3 - a_2^2 \right|, \]

and its modified form is:

\[ \left| a_3 - \mu a_2^2 \right|. \]
where \( \mu \) is real or complex number (see [16]).

(ii) Janteng [17] presented the second Hankel determinant in the form indicated below:
\[
H_{2,2}(f) = \begin{vmatrix}
    a_2 & a_3 \\
    a_3 & a_4
\end{vmatrix} = |a_2a_4 - a_3^2|
\]

and several academics (see, for example [18–22]) subsequently examined it for a few new classes of analytic functions.

(iii) For \( j = 3 \) and \( n = 1 \), the third Hankel determinant form is indicated below:
\[
H_{3,1}(f) = \begin{vmatrix}
    1 & a_2 & a_3 \\
    a_2 & a_3 & a_4 \\
    a_3 & a_4 & a_5
\end{vmatrix} = a_3\left(a_2a_4 - a_3^2\right) - a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right).
\]

In 1966, Pommerenke [23] explored research on the Hankel determinants for univalent starlike functions. Ehrenborg [24] studied the Hankel determinants related to exponential functions. The class of close-to-convex functions was examined and studied by Noor in her article [25] in 1983; she found the Hankel determinants for her defined functions class. Following this work, Janteng et al. [17] studied starlike and convex functions and for these classes, they found Hankel determinants. For the absolute constant \( \lambda \), and for \( f \in S \), Hayman [26] investigated sharp inequality which is given by
\[
|H_{2,n}(\eta)| \leq \lambda \sqrt{n}.
\]

In 2021, for the same class, Obradović, and Tuneski [27] found Hankel determinants of second and third order
\[
|H_{2,2}(f)| \leq \lambda, \quad 1 \leq \lambda \leq \frac{11}{3}
\]
and
\[
|H_{3,1}(f)| \leq \lambda, \quad \frac{4}{9} \leq \lambda \leq \frac{32 + \sqrt{285}}{15}.
\]

Recently, different researchers have been active in finding the sharp bounds of Hankel determinants for different families of functions. Including for example, the second Hankel determinant for certain subclasses of the class \( S \) has been given by Cho et al. [28,29]. Also in their investigation of classes of starlike functions of order \( \alpha \) and strongly starlike functions, Janteng [17,30] developed the Hankel determinant and demonstrated that \( |H_{2,2}(f)| \) is constrained by \((1 - \beta)^2\) and \( \beta^2 \).

The computation of the third Hankel determinant is tough compared to the second Hankel determinant. In 2010, Babalola [31] computed third Hankel determinant for some classes of univalent functions. By using the same approach, a number of other well-known authors have given the bounds for the third Hankel determinant for different kind of subclasses of analytic and bi-univalent functions. For example, a different approach has been used by Zaprawa [32] and for the classes of starlike and convex functions; thus, the bounds for the third Hankel determinant have been obtained as:
\[
|H_{3,1}(f)| \leq \begin{cases} 
    1 & \text{if } f \in S^* \\
    \frac{1}{49} & \text{if } f \in C
\end{cases}.
\]

Zaprawa’s result was then improved by Kwon et al. [33] in 2018; they proved that
\[
|H_{3,1}(f)| \leq \frac{8}{9}, \quad f \in S^*.
\]

Recently, the above bounds were improved again by Zaprawa et al. [34] in their article, and they proved that
\[
|H_{3,1}(f)| \leq \frac{5}{7}, \quad f \in S^*.
\]
Hankel determinants are quite useful for studying power series with integral coefficients and singularities. The Hankel determinants have been applied in a wide range of technical research, especially those that depend significantly on mathematical techniques. Readers who are interested in learning how Hankel determinants are used in the solutions of the aforementioned problems may read [35–37]. For instance, they are used in the theory of Markov processes and then we see their applications in the solutions of non-stationary signals in the Hamburger moment problem.

Examples regarding the sharp bounds of the third Hankel determinant for particular subclasses of starlike functions, a number of recent developments from the year 2023 are mentioned in [38] and the references therein.

Let \( m \in \mathbb{N} = \{1, 2, \ldots\} \). If a rotation around the origin through an angle \( \frac{2\pi}{m} \) carries on itself, the domain is said to be \( m \)-fold symmetric. It could be seen that

\[
f(\gamma) = e^{2\pi i/m}f(\gamma), \quad (\gamma \in \mathbb{U}).
\]

By \( \mathcal{S}(m) \), we mean the set of \( m \)-fold symmetric univalent functions having the following Taylor series form

\[
f(\gamma) = \gamma + \sum_{k=1}^{\infty} a_{mk+1} \gamma^{mk+1}, \quad (\gamma \in \mathbb{U}). \tag{6}
\]

For \( m \)-fold symmetric functions, the class of \( \mathcal{P}^m \) is defined as:

\[
\mathcal{P}^m = \left\{ p \in \mathcal{P}, \quad p(\gamma) = 1 + \sum_{k=1}^{\infty} a_{mk} \gamma^{mk} \quad (\gamma \in \mathbb{U}) \right\}. \tag{7}
\]

Inspired by the aforementioned work, we want to contribute the following to the literature on inequalities related to analytic functions.

- To introduce the novel classes \( \mathcal{R}_{\sin} \) and \( \mathcal{R}_{\text{card}} \) of bounded turning functions which are subordinated by sine function and cardioid domain.
- To introduce a new class \( \mathcal{R}^m_{\sin} \) by using \( m \)-fold symmetric functions.
- To find the sharp coefficient bounds for functions of the classes \( \mathcal{R}_{\sin} \) and \( \mathcal{R}_{\text{card}} \).
- To find the Fekete Szegö functional for the classes \( \mathcal{R}_{\sin} \) and \( \mathcal{R}_{\text{card}} \).
- To find the upper bounds of the third order Hankel determinant for the class \( \mathcal{R}^m_{\sin} \).

To continue with the mentioned above, we now establish the following:

**Definition 1.** A function \( f \in \mathcal{R}_{\sin} \), where \( f \) is of the form (1) if

\[
f'(\gamma) + \gamma f''(\gamma) \prec 1 + \sin \gamma. \tag{8}
\]

The graph of \( f \) over \( \mathbb{U} \) is given in Figure 1a.

**Definition 2.** A function \( f \in \mathcal{R}_{\text{card}} \), where \( f \) is of the form (1) if

\[
f'(\gamma) + \gamma f''(\gamma) \prec 1 + \frac{4}{3} \gamma + \frac{2}{3} \gamma^2. \tag{9}
\]

The graph of \( f \) over \( \mathbb{U} \) is given in Figure 1b.

**Definition 3.** An analytic function \( f \) of the form (6) belongs to the family \( \mathcal{R}^m_{\sin} \) if and only if

\[
f'(\gamma) + \gamma f''(\gamma) \prec 1 + \sin \gamma. \tag{10}
\]
Figure 1. Mappings of subordinating functions over $U$.

2. A Set of Lemmas

Lemma 1. Let the function $p(\gamma)$ of the form (3), then

\begin{align*}
|c_n| &\leq 2 \text{ for } n \geq 1, \quad (11) \\
|c_{n+k} - \mu c_n c_k| &< 2, \text{ for } 0 \leq \mu \leq 1 \quad (12) \\
|c_m c_n - c_l c_j| &\leq 4 \text{ for } n + m = k + l, \quad (13) \\
|c_{n+2k} - \mu c_n c_k^2| &\leq 2(1 + 2\mu) \text{ for } \mu \in \mathbb{R}. \quad (14) \\
|c_2 - \frac{c_1^2}{2}| &\leq 2 - \frac{|c_1^2|}{2}, \quad (15)
\end{align*}

for the complex number $\mu$, we have

\begin{equation}
|c_2 - \mu c_1^2| \leq \max\{2, 2|\mu - 1|\}. \quad (16)
\end{equation}

We may refer the interested to see [39] for the results in (11)–(15). Also, for the inequality (16) see [40].

Lemma 2. Let the function $p \in \mathcal{P}$ be given by (3), then

\begin{equation}
|Jc_1^3 - Kc_1 c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|. \quad (17)
\end{equation}

Proof. To prove (17), first let us consider the left-hand side of (17), and then, after some suitable simplifications and using (11) and (12), along with the result $|c_1^3 - c_1 c_2 + c_3| \leq 2$ which is due to [41], we have

\begin{align*}
|Jc_1^3 - Kc_1 c_2 + Lc_3| &= |J(c_1^3 - 2c_1 c_2 + c_3) - (K - 2J)(c_1 c_2 - c_3) + (J - K + L)c_3| \\
&\leq |J||c_1^3 - 2c_1 c_2 + c_3| + |K - 2J||c_1 c_2 - c_3| + |J - K + L||c_3| \\
\end{align*}

We divided our paper into four parts. The history of analytic functions, the Hankel determinant, and $m$-fold symmetric functions were introduced in Section 1. Next, we used this work as inspiration to create new classes of analytic and $m$-fold symmetric functions related to sine functions. In Section 2, we explored certain established lemmas that will
help to the proof of the article’s main results. In Section 3, for functions in the classes $R_{\sin}$ and $R_{\text{card}}$, we first calculated the initial coefficient bound, Fekete–Szegö problems and the second and third Hankel determinants. Then, we quickly determined the third Hankel determinant for functions $f$ in the class $R_{\sin}$ for both twofold and threefold symmetry. Finally, some last thoughts are discussed in Section 4.

3. Main Results

Our first result is related to find bounds for the functions $f$ to be in the class $R_{\sin}$.

**Theorem 1.** If $f \in R_{\sin}$ is of the form (8), then

\[
|a_2| \leq \frac{1}{4}, \quad (18)
\]

\[
|a_3| \leq \frac{1}{9}, \quad (19)
\]

\[
|a_4| \leq \frac{17}{192}, \quad (20)
\]

\[
|a_5| \leq \frac{97}{1800}. \quad (21)
\]

The first two bounds are the best possible.

**Proof.** Let $f \in R_{\sin}$, then the relation (8) leads us to

\[
f'(\gamma) + \gamma f''(\gamma) = 1 + \sin(w(\gamma)). \quad (22)
\]

where $w$ indicates the Schwarz function. Now assume a function $p$ such that

\[
p(\gamma) = \frac{1 + w(\gamma)}{1 - w(\gamma)} = 1 + c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots, \quad (23)
\]

then $p \in \mathcal{P}$. This implies that

\[
w(\gamma) = \frac{p(\gamma) - 1}{p(\gamma) + 1} = \frac{c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots}{2 + c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots}.
\]

Using the left hand side of (1) and (22), we can write

\[
f'(\gamma) + \gamma f''(\gamma) = 1 + 4a_2 \gamma + 9a_3 \gamma^2 + 16a_4 \gamma^3 + 25a_5 \gamma^4 + \ldots \quad (24)
\]

Using some simplifications, we have

\[
1 + \sin(w(\gamma)) = 1 + w(\gamma) - \frac{w^3(\gamma)}{3!} + \frac{w^5(\gamma)}{5!} - \frac{w^7(\gamma)}{7!} + \ldots, \quad (25)
\]

\[
= 1 + \frac{1}{2}c_1 \gamma + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) \gamma^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{5c_1^3}{48}\right) \gamma^3 + \ldots \quad (26)
\]

From (24) and (26), it follows that

\[
a_2 = \frac{c_1}{8}, \quad (27)
\]

\[
a_3 = \frac{1}{9} \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right), \quad (28)
\]

\[
a_4 = \frac{1}{16} \left(-\frac{5c_1^2}{48} + \frac{c_3}{2} - \frac{c_1 c_2}{2}\right), \quad (29)
\]
\[ a_5 = \frac{1}{25} \left( c_4 + \frac{5c_2^2c_2}{16} - \frac{c_2^2}{4} - \frac{c_1c_3}{2} - \frac{c_4^4}{32} \right). \] (30)

From (27) and (28) in conjunction with (11), we have
\[ |a_2| \leq \frac{1}{4} \quad \text{and} \quad |a_3| \leq \frac{1}{9}, \]
and by rearranging (29), it gives
\[ |a_4| = \frac{1}{8} \left| 1 - \frac{7}{24} \right| \leq \frac{17}{192}. \]

From (30), it follows that
\[ |a_5| = \frac{1}{25} \left| \frac{1}{4} \left( c_4 - \frac{c_1c_2}{2} \right) + \frac{3c_2^2}{16} \left( c_2 - \frac{c_1^2}{6} \right) - \frac{c_2}{18} \left( c_2 + \frac{9c_1^2}{4} \right) \right|. \]

Using triangle inequality along with (11), (12) and (15), we obtain
\[ |a_5| \leq \frac{1}{25} \left[ \frac{1}{2} + \frac{3|x^2|}{16} \left( 2 - \frac{|x^2|}{6} \right) - \frac{1}{9} \left( 2 - \frac{9|x^2|}{4} \right) \right]. \]

Suppose that \( |c_1| = x \) and \( x \in [0, 2] \); therefore
\[ |a_5| = \frac{1}{25} \left[ 1 + \frac{3|x^2|}{16} \left( 2 - \frac{|x^2|}{6} \right) - \frac{1}{9} \left( 2 - \frac{9|x^2|}{4} \right) \right]. \]

The above function attains its maximum value at \( x = \sqrt{2} \), hence
\[ |a_5| \leq \frac{97}{1800}. \]

This is our required bound.

Equality for the bounds given in (18) and (19) is obtained by taking
\[ f'_n(\gamma) + \gamma f''_n(\gamma) = 1 + \sin(w(\gamma))^{n-1}. \]

For the given functions
\[
\begin{align*}
f_2(\gamma) &= \gamma + \frac{1}{4}\gamma^2, \quad n = 2, \\
f_3(\gamma) &= \gamma + \frac{1}{9}\gamma^3, \quad n = 3.
\end{align*}
\]

\[ \square \]

**Theorem 2.** Let \( f \in \mathcal{R}_\sin \). Then, for a complex number \( \delta \)
\[ |a_3 - \delta a_2^2| \leq \max \left\{ \frac{1}{9}, \frac{1}{18}|3\delta - 2| \right\}. \] (31)
Proof. Using (27) and (28), one may write

$$|a_3 - \delta a_2^2| = \left| \frac{c_2}{18} - \frac{c_2^2}{24} \right| = \left[ \frac{1}{18} \left( c_2 - \left( \frac{1}{2} \right) \left( \frac{3\delta}{2} \right) c_1^2 \right) \right].$$

Application of relation (16) gives

$$|a_3 - \delta a_2^2| \leq \max \left\{ \frac{1}{9}, \frac{1}{18} |3\delta - 2| \right\}.$$ 

This completes the proof of our result. \[\square\]

If we take $\delta = 1$ in the above Theorem 2, we have the following result.

Corollary 1. If $f \in R_{\sin}$. Then

$$|a_3 - a_2^2| \leq \frac{1}{9}. \quad (32)$$

This inequality is sharp for the function

$$f_3(\gamma) = \gamma + \frac{1}{9} \gamma^3$$

Theorem 3. Let $f \in R_{\sin}$. Then

$$|a_2 a_3 - a_4| \leq \frac{301}{1152}. \quad (33)$$

Proof. From (27)–(29) we have

$$|a_2 a_3 - a_4| = \left| \frac{11c_1 c_2}{288} + \frac{7c_3^3}{2304} - \frac{c_3}{32} \right|.$$ 

After some simple calculations using the applications of Lemma 2, this leads us to

$$|a_2 a_3 - a_4| \leq \frac{301}{1152}.$$ 

The proof of the result is now completed. \[\square\]

Theorem 4. Let $f \in R_{\sin}$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{121}{10,368}. \quad (34)$$

Proof. From (27)–(29), we have

$$|a_2 a_4 - a_3^2| = \left| \frac{c_1 c_3}{256} - \frac{17c_1^2 c_2}{20,736} - \frac{c_2^2}{324} - \frac{155c_4^4}{165,888} \right|$$

After applying triangle inequality along with (11) and (12), we obtain

$$|a_2 a_4 - a_3^2| \leq \left\{ \frac{4}{256} - \frac{4}{324} - \frac{155(16)}{165,888} \right\} \leq \frac{121}{10,368}.$$
The proof of the result is now completed. □

**Theorem 5.** Let \( f \in \mathcal{R}_{\text{sin}} \). Then

\[
H_{3,1}(f) \leq \frac{4,541,527}{149,299,200}.
\]

**Proof.** It can be seen from (5) that

\[
H_{3,1}(f) = a_3 \left( a_2 a_4 - a_3^2 \right) + a_4 (a_2 a_3 - a_4) + a_5 \left( a_3 - a_2^2 \right).
\]

Here \( a_1 = 1 \). Taking modulus on both sides of the above equation and applying triangle inequality, we have

\[
|H_{3,1}(f)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|.
\]

By using (19)–(21) and (32)–(34), we obtain

\[
|H_{3,1}(f)| \leq \left( \frac{1}{9} \right) \left( \frac{121}{10,368} \right) + \left( \frac{17}{192} \right) \left( \frac{301}{1152} \right) + \left( \frac{97}{1800} \right) \left( \frac{1}{9} \right) = \frac{4,541,527}{1,492,992}.
\]

□

These are the main findings of the functions class \( \mathcal{R}_{\text{card}} \).

**Theorem 6.** If \( f \in \mathcal{R}_{\text{card}} \) and has the form given in (9), then

\[
|a_2| \leq \frac{1}{3},
\]

\[
|a_3| \leq \frac{4}{27},
\]

\[
|a_4| \leq \frac{1}{12},
\]

\[
|a_5| \leq \frac{2}{25}.
\]

The first bound is the best possible.

**Proof.** Let \( f \in \mathcal{R}_{\text{card}} \) and then, by the definition of subordination, there exists a Schwarz function \( w(\gamma) \) with the properties that

\[
w(0) = 1 \quad \text{and} \quad w(\gamma) < 1.
\]

such that

\[
f'(\gamma) + \gamma f''(\gamma) = 1 + \frac{4}{3} \gamma + \frac{2}{3} \gamma^2.
\]

Define the function

\[
p(\gamma) = \frac{1 + w(\gamma)}{1 - w(\gamma)} = 1 + c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots,
\]

clearly with \( p \in \mathcal{P} \). This implies that

\[
w(\gamma) = \frac{p(\gamma) - 1}{p(\gamma) + 1} = \frac{c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots}{2 + c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + \ldots}.
\]

Now, from (39), we have

\[
f'(\gamma) + \gamma f''(\gamma) = 1 + 4a_2 \gamma + 9a_3 \gamma^2 + 16a_4 \gamma^3 + 25a_5 \gamma^4 + \ldots
\]
and
\[ 1 + \frac{4}{3} w(\gamma) + \frac{2}{3} w(\gamma)^2 = 1 + \frac{2}{3} c_1 \gamma + \left( \frac{2c_2}{3} - \frac{c_1^2}{6} \right) \gamma^2 + \left( \frac{2c_3}{3} - \frac{c_1 c_2}{3} \right) \gamma^3 + \left( \frac{c_4}{2} + \frac{5c_2^2 c_2}{16} - \frac{c_2^4}{16} - \frac{c_1 c_3}{2} - \frac{c_1^4}{32} \right) \gamma^4 \ldots \] (41)

Comparing (40) and (41), we have
\[ a_2 = \frac{c_1}{6}, \] (42)
\[ a_3 = \frac{1}{9} \left( \frac{2c_2}{3} - \frac{c_1^2}{6} \right), \] (43)
\[ a_4 = \frac{1}{16} \left( \frac{2c_3}{3} - \frac{c_1 c_2}{3} \right), \] (44)
\[ a_5 = \frac{1}{25} \left( \frac{c_4}{2} + \frac{5c_2 c_2}{16} - \frac{c_2^4}{16} - \frac{c_1 c_3}{2} - \frac{c_1^4}{32} \right). \] (45)

From (42) and (43) in conjunction with (11), we have
\[ |a_2| \leq \frac{1}{3} \quad \text{and} \quad |a_3| \leq \frac{4}{27}. \]

By rearranging (44), this gives
\[ |a_4| = \frac{1}{16} \left| \frac{2}{3} c_3 - \frac{c_1 c_2}{2} \right|. \]

Using triangle inequality along with (12) and (14), we have
\[ |a_4| \leq \frac{1}{12}. \]

Now, again, we take modulus and applying triangle inequality on (45) along with (11), (15) and (12) to obtain
\[ |a_5| \leq \frac{1}{25} \left[ \frac{2}{3} + \frac{2}{3} + \frac{|c_1|}{6} \left( 2 - \frac{|c_1^2|}{4} \right) \right]. \]

Suppose that \(|c_1| = x\) and \(x \in [0, 2] \); therefore
\[ |a_5| \leq \frac{1}{25} \left[ \frac{2}{3} + \frac{2}{3} + \frac{|x^2|}{6} \left( 2 - \frac{|x^2|}{4} \right) \right]. \]

The maximum value of the above function can be attained at \(x = \sqrt{4} = 2.\)

Thus, we have
\[ |a_5| \leq \frac{2}{25}. \]

This is our desired bound.

Equality for the bound given in (35) is obtained by taking
\[ f_2(\gamma) = \gamma + \frac{1}{3} \gamma^2. \]

\[ \square \]
Theorem 7. Let \( f \in \mathcal{R}_{\text{card}} \). Then for a complex number \( \delta \)
\[ |a_3 - \delta a_2^2| \leq \max \left\{ \frac{4}{27}, \frac{1}{27} |5\delta - 2| \right\}. \] (46)

Proof. Using (42) and (43), one may write
\[ |a_3 - \delta a_2^2| = \left| \frac{2c_2}{27} - \frac{5c_1^2}{54} \right| = \left[ \frac{2}{27} \left( c_2 - \left( \frac{1}{2} \right) \left( \frac{5\delta}{2} \right) c_1^2 \right) \right]. \]

Application of relation (16) gives
\[ |a_3 - \delta a_2^2| \leq \max \left\{ \frac{4}{27}, \frac{1}{27} |5\delta - 2| \right\}. \]

This completes the proof of our result. \( \square \)

If we take \( \delta = 1 \) in the above Theorem, we have the following result.

Corollary 2. If \( f \in \mathcal{R}_{\text{card}} \). Then
\[ |a_3 - a_2^2| \leq \frac{1}{9}. \] (47)

Theorem 8. Let \( f \in \mathcal{R}_{\text{card}} \). Then
\[ |a_2 a_3 - a_4| \leq \frac{1}{12}. \] (48)

Proof. From (42)–(44), we have
\[ |a_2 a_3 - a_4| = \left| \frac{43c_1 c_2}{1296} - \frac{c_1^3}{234} - \frac{c_3}{24} \right|. \]

Using Lemma 2, we have
\[ |a_2 a_3 - a_4| \leq \frac{1}{12}. \]

This completes the proof of our result. \( \square \)

Theorem 9. Let \( f \in \mathcal{R}_{\text{card}} \). Then
\[ |a_2 a_4 - a_3^2| \leq \frac{161}{2916}. \] (49)

Proof. From (42)–(44), we may write
\[ |a_2 a_4 - a_3^2| = \left| \frac{c_1}{144} \left( c_3 - \frac{17c_1 c_2}{162} \right) - \frac{c_2}{729} - \frac{c_4}{2916} \right|. \]

Now, using triangle inequality along with (11) and (12), we have
\[ |a_2 a_4 - a_3^2| \leq \left\{ \frac{4}{144} + \frac{16}{729} + \frac{16}{2916} \right\} \leq \frac{161}{2916}. \]

This completes the proof of our result. \( \square \)
Theorem 10. Let \( f \in \mathcal{R}_{\text{card}} \). Then
\[
H_{3,1}(f) \leq \frac{337,589}{1,968,300}.
\]

Proof. It can be seen from (5) that
\[
H_{3,1}(f) = a_3 \left( a_2 a_4 - a_2^3 \right) + a_4 (a_2 a_5 - a_4) + a_5 \left( a_3 - a_2^2 \right).
\]
Here \( a_1 = 1 \), and applying modulus along with triangle inequality, we have
\[
|H_{3,1}(f)| \leq |a_3||a_2 a_4 - a_2^3| + |a_4||a_2 a_5 - a_4| + |a_5||a_3 - a_2^2|.
\]
By using (36)–(38) and (47)–(49), we obtain
\[
|H_{3,1}(f)| \leq \frac{4}{27} \left( \frac{161}{2916} \right) + \frac{1}{12} \left( \frac{1}{12} \right) + \frac{2}{25} \left( \frac{1}{12} \right) = \frac{337,589}{1,968,300}.
\]

Bounds of \(|H_{3,1}(f)|\) for Twofold Symmetric and Threefold Symmetric Functions

Theorem 11. Let \( f \in \mathcal{R}_{\text{sin}}^{(2)} \) be of the form (6). Then
\[
|H_{3,1}(f)| \leq \frac{1}{225}.
\]

Proof. Since \( f \in \mathcal{R}_{\text{sin}}^{(2)} \), therefore, there exists a function \( p \in \mathcal{P}^{(2)} \) such that
\[
f'(\gamma) + \gamma f''(\gamma) = 1 + \sin \left( \frac{p(\gamma) - 1}{p(\gamma) + 1} \right).
\]
For \( f \in \mathcal{R}_{\text{sin}}^{(2)} \) using the series form (6) and (7), when \( m = 2 \) in the above relation, we can write
\[
a_3 = \frac{c_2}{18}, \quad a_5 = \frac{c_4}{50} - \frac{c_2^2}{100}.
\]
It is clear that for \( f \in \mathcal{R}_{\text{sin}}^{(2)} \)
\[
H_{3,1}(f) = a_3 a_5 - a_3^3.
\]
Therefore,
\[
H_{3,1}(f) = \frac{c_2}{900} \left( c_4 - \frac{53}{81} \frac{c_2^2}{2} \right).
\]
Now, making use of triangle inequality along with (12), we have
\[
|H_{3,1}(f)| \leq \frac{1}{225}.
\]
The proof of our theorem is completed.

Theorem 12. If \( f \in \mathcal{R}_{\text{sin}}^{(3)} \), then
\[
|H_{3,1}(f)| \leq \frac{1}{256}.
\]

Proof. Since \( f \in \mathcal{R}_{\text{sin}}^{(3)} \), therefore, there exists a function \( p \in \mathcal{P}^{(3)} \) such that
\[
f'(\gamma) + \gamma f''(\gamma) = 1 + \sin \left( \frac{p(\gamma) - 1}{p(\gamma) + 1} \right).
\]
For \( f \in \mathcal{R}_{\text{sin}}^{(3)} \), using the series form (6) and (7), when \( m = 3 \) in the above relation, we can write
\[
a_4 = \frac{c_3}{32}.
\]
It is easy to see that
\[
H_{3,1}(f) = -a_4^2,
\]
therefore
\[
H_{3,1}(f) = \frac{c_3^2}{1024}.
\]
Using coefficient estimates for class \( \mathcal{P} \) and triangle inequality, we obtain
\[
|H_{3,1}(f)| = \frac{1}{256}.
\]
Hence the proof is completed. \( \square \)

4. Conclusions

In this study, we studied three new subclasses \( \mathcal{R}_{\text{sin}}, \mathcal{R}_{\text{card}} \) and the class of \( m \)-fold symmetric functions \( \mathcal{R}_{\text{sin}}^m \). All three subclasses are defined by using the technique of subordination connected with sine functions and cardioid domain. Many researchers have defined the subclasses of analytic functions including starlike and convex functions, but here, we considered the class of bounded turning associated with sine functions and the cardioid domain. We found the Fekete–Szegö functional, second- and third-order Hankel determinants. Also, we investigated the third Hankel determinant for twofold and threefold symmetric functions.

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