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Some Analysis of the Coefficient-Related Problems for Functions of Bounded Turning Associated with a Symmetric Image Domain

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Abstract: In the last few years, numerous subfamilies of univalent functions, whether directly or indirectly associated with exponential functions, have been introduced and thoroughly investigated. Among these, the families $\mathcal{S}_e^*$, $\mathcal{C}_e$ and $\mathcal{R}_e$ defined by subordination to e^z have been intensively investigated. While the coefficient problem on the class $\mathcal{S}_e^*$ and $\mathcal{C}_e$ has been solved in many cases, in this paper, we mainly intend to compute the sharp estimates of some initial coefficients, the Feketo–Szegö inequality, and the sharp bounds of second- and third-order Hankel determinants for functions belonging to the class $\mathcal{R}_e$. This work has the potential to significantly enrich and enhance the exploration of univalent functions in conjunction with exponential functions, making the field more comprehensive and robust.

Keywords: Hankel determinant; bounded turning functions; exponential function

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1. Introduction and Definitions

Let $\mathcal{H}(\mathbb{D})$ represent the family of analytic functions defined in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$, the normalized functions taking the form of

$$f(z) = z + \sum_{s=2}^{\infty} b_s z^s, \quad z \in \mathbb{D}, \tag{1}$$

belong to the class $\mathcal{A}$. Let $\mathcal{S} \subset \mathcal{A}$ be the set of all univalent functions in $\mathbb{D}$. We denote by $\mathcal{P}$ the set of all analytic functions in $\mathbb{D}$ in which the function $p(z) \in \mathcal{P}$ satisfies the conditions $\Re(p(z)) > 0$ and

$$p(z) = 1 + \sum_{n=1}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

Such functions are also known as the Carathéodory functions [1]. A basic relationship in geometry function theory is subordination. We write $g \prec \tilde{g}$ to illustrate that g is subordinate to $\tilde{g}$. It is explained that, for a given two functions $g, \tilde{g} \in \mathcal{H}(\mathbb{D})$, a Schwarz function ω exists such that $g(z) = \tilde{g}(\omega(z))$ for $z \in \mathbb{D}$. Once $\tilde{g}$ is univalent in $\mathbb{D}$, then this relation is equivalent to saying that

$$g(z) \prec \widetilde{g}(z), \quad (z \in \mathbb{D}) \quad \Longleftrightarrow \quad g(0) = \widetilde{g}(0) \quad \text{and} \quad g(\mathbb{D}) \subset \widetilde{g}(\mathbb{D}).$$

In 1916, Bieberbach [2] gave the most prominent conjecture in function theory, known as the "Bieberbach conjecture", which states that, for any $s \geq 2$, $|b_s| < s$ if $f \in \mathcal{S}$. He himself proved this for $s = 2$. This conjecture remained an unsolved problem for a long time, and, finally, in 1985, de-Branges [3] proved it for every $s \geq 2$ by using hypergeometric functions. In an effort to resolve this conjecture between 1916 and 1985, several researchers produced a variety of other exciting results, which ultimately strengthened geometric function theory research. A number of these included estimating the nth coefficient bounds for a variety of subfamilies within the family of univalent functions; these include, but are not limited to, starlike $\mathcal{S}^*$, convex $\mathcal{C}$, bounded turning $\mathcal{R}$, and many more.

In 1992, Ma and Minda [4] constructed a family of univalent analytic functions (say) $q^*(z)$, which maps D onto the star-shaped domain with respect to $q^*(0) = 1$ in the right half-plane and is symmetric about the real axis. The Ma and Minda families of $\mathcal{C}(q^*)$, $\mathcal{S}^*(q^*)$ and $\mathcal{R}(q^*)$ are defined in set-builder form, respectively, as

$$\begin{aligned} \mathcal{C}(q^*) &= \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec q^*(z), \ (z \in \mathbb{D}) \right\}, \\ \mathcal{S}^*(q^*) &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q^*(z), \ (z \in \mathbb{D}) \right\}, \\ \mathcal{R}(q^*) &= \left\{ f \in \mathcal{A} : f'(z) \prec q^*(z), \ (z \in \mathbb{D}) \right\}. \end{aligned}$$

The researchers concentrated on a few fundamental but significant findings, all of which were based on the geometrical properties of these functions. A few of these include covering, growth, and distortion theorems. Moreover, it is to be noted from the literature that several subfamilies have been intensively investigated recently as particular choices of the above-defined classes. As is evident, all of these particularly selected functions in the below-provided classes exhibit a close relationship with the exponential function.

(i). $\mathcal{S}^*_{SG} \equiv \mathcal{S}^*\left(\frac{2}{1+e^{-z}}\right)$ and $\mathcal{C}_{SG} \equiv \mathcal{C}\left(\frac{2}{1+e^{-z}}\right)$ [5], $\mathcal{R}_{SG} \equiv \mathcal{R}\left(\frac{2}{1+e^{-z}}\right)$ [6],
(ii). $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos(z))$ [7], $\mathcal{S}^*_{car} \equiv \mathcal{S}^*(1+ze^z))$ [8], $\mathcal{C}_{car} \equiv \mathcal{C}(1+ze^z)$ [9],
(iii). $\mathcal{R}_{car} \equiv \mathcal{R}(1+ze^z)$ [10], $\mathcal{S}^*_{\sin} \equiv \mathcal{S}^*(1+\sin(z))$ [11], $\mathcal{S}^*_{pet} \equiv \mathcal{S}^*\left(1+\sinh^{-1} z\right)$ [12].

The determinant $H_{\iota,n}(f)$, where $\iota,\ n \in \mathbb{N} = \{1,2,\cdots\}$, is known as the Hankel determinant and was contributed by Pommerenke [13,14]. It is formed by the coefficients of the function $f \in \mathcal{S}$ and is defined as

$$H_{\iota,n}(f) = \begin{vmatrix} b_n & b_{n+1} & \ldots & b_{n+\iota-1} \\ b_{n+1} & b_{n+2} & \ldots & b_{n+\iota} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n+\iota-1} & b_{n+q} & \ldots & b_{n+2\iota-2} \end{vmatrix}. \tag{3}$$

The significance of the Hankel determinant is evident in the field of singularity theory, and it was shown in [15] to be an efficient approach for the examination of power series with integral coefficients. There are relatively few publications in the literature that give the bounds of the Hankel determinant for functions of general class $\mathcal{S}$. The best estimate for $f \in \mathcal{S}$ was determined by Hayman in [16] and is $|H_{2,n}(f)| \leq |\eta|$, where η is a constant. Additionally, for $f \in \mathcal{S}$, it was shown in [17] that the second-order Hankel determinant $|H_{2,2}(f)| \leq \eta$ for $0 \leq \eta \leq 11/3$. The two determinants $H_{2,1}(f)$ and $H_{2,2}(f)$ have been extensively studied in the literature for various subfamilies of univalent functions. The work done by the authors [18–21], where they determined sharp bounds for the second determinant, is particularly noteworthy.

In comparison to the sharp bound of the second-order Hankel determinant, the sharp bound of the third-order Hankel determinant $H_{3,1}(f)$ for any analytic or univalent function is much more difficult to find. This is why there are, in the literature, very few articles in which sharp bounds of the determinant $H_{3,1}(f)$ have been obtained. The sharp bounds of this determinant were obtained recently for the classical classes $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{R}$ in the articles [22], [23], and [24], respectively. These sharp bounds are

$$|H_{3,1}(f)| \leq \begin{cases} \frac{4}{135}, & \text{for} \quad f \in \mathcal{C}, \\ \frac{4}{9}, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{1}{4}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

These were not easy tasks, as the articles [25,26] show that there have been many attempts before. Moreover, by using a simple technique, Lecko et al. [27] and Kowalczyk et al. [28] determined the sharp bounds of $H_{3,1}(f)$ for functions belonging to the families $\mathcal{S}^*(1/2)$ and $\mathcal{C}(-1/2)$, respectively. Furthermore, in Table 1, we give more sharp bounds for this determinant for some specific subfamilies of $\mathcal{S}$.

Table 1. Sharp bounds on $|H_{3,1}(f)|$ for some subfamilies of $\mathcal{S}$.

Author/s	Class	Sharp Bound	Year	Reference
Barukab et al.	$\mathcal{R}_{pet}$	1/16	2021	[29]
Riaz et al.	$\mathcal{S}^*_{SG}$	1/36	2022	[9]
Shi et al.	$\mathcal{S}^*_{\sin}$	1/9	2022	[30]
Riaz et al.	$\mathcal{C}_{SG}$	1/576	2022	[9]
Shi et al.	$\mathcal{R}_{\sin}$	1/16	2022	[30]
Arif et al.	$\mathcal{R}_{SG}$	1/64	2022	[6]
Wang et al.	$\mathcal{S}^*_{pet}$	1/9	2023	[31]
Neha and Kumar	$\mathcal{S}^*_{car}$	1/9	2023	[32]
Shi et al.	$\mathcal{R}_{car}$	1/16	2023	[10]

In 2015, Mendiratta et al. [33] considered the exponential function and observed that the function $q^*(z) = e^z$ has a positive real part. Using this particular function, they introduced the classes $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ and $\mathcal{C}_e \equiv \mathcal{C}(e^z)$. The structural formula, inclusion relations, coefficient estimates, growth and distortion results, subordination theorems, and various radii constants for functions in the class $\mathcal{S}^*_e$ were obtained in the same article. Later, in 2022, Shi et al. [34] introduced and studied a subfamily of bounded turning functions $\mathcal{R}_e$ defined by

$$\mathcal{R}_e = \left\{ f \in \mathcal{S} : f'(z) \prec e^z, \ \ z \in \mathbb{D} \right\}.$$

The goal of the present article is to compute the sharp bounds of the third-order Hankel determinant $H_{3,1}(f)$ for the family $\mathcal{R}_e$ of bounded turning functions associated with an exponential function. In addition to this, we also obtain sharp bounds for certain coefficient-related problems that include the first four initial coefficients, Fekete–Szegö type inequality, and the second-order Hankel determinant for such a class.

2. A Set of Lemmas

We use the following lemmas to obtain our main results.

Lemma 1 ([35])**.** *Assume $p \in \mathcal{P}$ as the form of* (2)*. Then*

$$2\mu_2 = \mu_1^2 + \kappa\left(4 - \mu_1^2\right), \tag{4}$$

$$4\mu_3 = \mu_1^3 + 2\left(4 - \mu_1^2\right)\mu_1\kappa - \mu_1\left(4 - \mu_1^2\right)\kappa^2 + 2\left(4 - \mu_1^2\right)\left(1 - |\kappa|^2\right)\delta, \tag{5}$$

$$8\mu_4 = \mu_1^4 + \left(4 - \mu_1^2\right)\kappa\left[c_1^2\left(\kappa^2 - 3\kappa + 3\right) + 4\kappa\right] - 4\left(4 - \mu_1^2\right)\left(1 - |\kappa|^2\right)$$
$$\left[\mu_1(\kappa - 1)\delta + \overline{\kappa}\delta^2 - \left(1 - |\delta|^2\right)\varrho\right], \tag{6}$$

for some $\kappa, \delta, \varrho \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 2 ([1]). *If p belongs to class $\mathcal{P}$ and has the form (2), then*

$$|\mu_n| \le 2 \ for \ \ n \ge 1. \tag{7}$$

Lemma 3 ([36]). *Let $p \in \mathcal{P}$. If $\tau \in [0,1]$ and $\tau(2\tau - 1) \le \upsilon \le \tau$, we have*

$$\left|\mu_3 - 2\tau\mu_1\mu_2 + \upsilon\mu_1^3\right| \le 2. \tag{8}$$

Lemma 4 ([37]). *If $p \in \mathcal{P}$ is taking the form of (2), then*

$$\left|\gamma\mu_1^4 + a\mu_2^2 + 2\alpha\mu_1\mu_3 - \frac{3}{2}\beta\mu_1^2\mu_2 - \mu_4\right| \le 2$$

for all α, β, γ, and a fulfilling the conditions that $0 < \alpha < 1$, $0 < a < 1$ and

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1-\alpha)^2(1-a). \tag{9}$$

Lemma 5 ([38]). *If p belongs to class $\mathcal{P}$ and of the form (2), we obtain*

$$|\mu_{n+k} - \nu\mu_n\mu_k| \le 2\max\{1, |2\nu - 1|\} = \begin{cases} 2 & for \quad 0 \le \nu \le 1; \\ 2|2\nu - 1| & \quad otherwise. \end{cases} \tag{10}$$

3. Main Results

We begin by finding the bounds on the first four coefficients for functions belonging to class $\mathcal{R}_e$.

Theorem 1. *Let $f \in \mathcal{R}_e$ and has the form (1). Then,*

$$|b_2| \le \frac{1}{2}, \quad |b_3| \le \frac{1}{3}, \quad |b_4| \le \frac{1}{4}, \quad |b_5| \le \frac{1}{5}.$$

These bounds are best possible.

Proof. Let $f \in \mathcal{R}_e$. Then, by definition, there exists a Schwarz function such as

$$f'(z) = e^{\omega(z)}, \quad z \in \mathbb{D}.$$

Suppose that

$$q(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \cdots, \tag{11}$$

which is equivalent to

$$\omega(z) = \frac{q(z)-1}{q(z)+1} = \frac{\mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \cdots}{2 + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \cdots}, \tag{12}$$

and it is known that $q \in \mathcal{P}$. Based on (1), we see that

$$f'(z) = 1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3 + 5b_5 z^4 + \cdots. \tag{13}$$

By simplification and using the series expansion of (12), it is found that

$$\begin{aligned} e^{\omega(z)} &= 1 + \left(\tfrac{1}{2}\mu_1\right)z + \left(\tfrac{1}{2}\mu_2 - \tfrac{1}{8}\mu_1^2\right)z^2 + \left(\tfrac{1}{2}\mu_3 + \tfrac{1}{48}\mu_1^3 - \tfrac{1}{4}\mu_1\mu_2\right)z^3 \\ &\quad + \left(\tfrac{1}{2}\mu_4 - \tfrac{1}{8}\mu_2^2 + \tfrac{1}{384}\mu_1^4 + \tfrac{1}{16}\mu_1^2\mu_2 - \tfrac{1}{4}\mu_1\mu_3\right)z^4 + \cdots. \end{aligned} \tag{14}$$

Comparing the coefficients in (13) and (14) leads to

$$b_2 = \frac{1}{4}\mu_1, \tag{15}$$

$$b_3 = \frac{1}{3}\left(\frac{1}{2}\mu_2 - \frac{1}{8}\mu_1^2\right), \tag{16}$$

$$b_4 = \frac{1}{4}\left(\frac{1}{2}\mu_3 + \frac{1}{48}\mu_1^3 - \frac{1}{4}\mu_1\mu_2\right), \tag{17}$$

$$b_5 = \frac{1}{5}\left(\frac{1}{2}\mu_4 - \frac{1}{8}\mu_2^2 + \frac{1}{384}\mu_1^4 + \frac{1}{16}\mu_1^2\mu_2 - \frac{1}{4}\mu_1\mu_3\right). \tag{18}$$

For b_2, implementing Lemma 2, we obtain

$$|b_2| \leq \frac{1}{2}.$$

For b_3, reordering (16), we obtain

$$b_3 = \frac{1}{6}\left(\mu_2 - \frac{1}{4}\mu_1\mu_1\right).$$

Using Lemma 5, we have

$$|b_3| \leq \frac{1}{3}.$$

For b_4, we can write (17) as

$$|b_4| = \frac{1}{8}\left|\left(\mu_3 - 2\left(\frac{1}{4}\right)\mu_1\mu_2 + \frac{1}{24}\mu_1^3\right)\right|.$$

From (8), we have

$$0 \leq \tau = \frac{1}{4} \leq 1, \quad \tau \geq v = \frac{1}{24},$$

and

$$\tau(2\tau - 1) = -\frac{1}{8} \leq v.$$

Thus, by applying Lemma 3, we obtain

$$|b_4| \leq \frac{1}{4}.$$

For b_5, we can rewrite (18) as

$$\begin{aligned} b_5 &= -\frac{1}{10}\left(-\frac{1}{192}\mu_1^4 + \left(\frac{1}{4}\right)\mu_2^2 + 2\left(\frac{1}{4}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{1}{12}\right)\mu_1^2\mu_2 - \mu_4\right). \\ &= \frac{1}{10}\left(\gamma\mu_1^4 + a\mu_2^2 + 2\alpha\mu_1\mu_3 - \frac{3}{2}\beta\mu_1^2\mu_2 - \mu_4\right), \end{aligned} \tag{19}$$

where

$$\gamma = -\frac{1}{192}, \quad a = \frac{1}{4}, \quad \alpha = \frac{1}{4}, \quad \beta = \frac{1}{12},$$

are such that

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a),$$

with $0 < \alpha < 1, 0 < a < 1$. Hence, by using Lemma 4 in (19), we have

$$|b_5| \leq \frac{1}{5}.$$

On the sharpness, it is noted that the equalities can be achieved by the following function:

$$f_n'(z) = e^{(z^n)},$$

where $n = 1, 2, 3, 4$. Thus, we have

$$f_1(z) = e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots, \tag{20}$$

$$f_2(z) = \int_0^z e^{t^2}\,dt = z + \frac{1}{3}z^3 + \frac{1}{10}z^5 + \cdots, \tag{21}$$

$$f_3(z) = \int_0^z e^{t^3}\,dt = z + \frac{1}{4}z^4 + \frac{1}{14}z^7 + \cdots, \tag{22}$$

$$f_4(z) = \int_0^z e^{t^4}\,dt = z + \frac{1}{5}z^5 + \frac{1}{18}z^9 + \cdots. \tag{23}$$

□

Now, we consider the Fekete–Szegö inequality for $f \in \mathcal{R}_e$.

Theorem 2. *Suppose that ν is a constant complex number. Then, for $f \in \mathcal{R}_e$, we have*

$$\left|b_3 - \nu b_2^2\right| \leq \max\left\{\frac{1}{3}, \frac{3|\nu| - 2}{12}\right\}.$$

The equality is attained on the function defined by (20) and (21).

Proof. Making use of (15) and (16), it is found that

$$\left|b_3 - \nu b_2^2\right| = \left|\frac{\mu_2}{6} - \frac{\mu_1^2}{24} - \nu\frac{\mu_1^2}{16}\right|.$$

By rearranging, it yields

$$\left|b_3 - \nu b_2^2\right| = \frac{1}{6}\left|\mu_2 - \left(\frac{3\nu + 2}{8}\right)\mu_1^2\right|.$$

The application of Lemma 5 leads us to

$$\left|b_3 - \nu b_2^2\right| \leq \frac{2}{6}\max\left\{1, \left|\left(\frac{3\nu - 2}{4}\right)\right|\right\}.$$

After the simplification, we obtain

$$\left|b_3 - \nu b_2^2\right| \leq \max\left\{\frac{1}{3}, \frac{3|\nu| - 2}{12}\right\}.$$

The required proof is accomplished. □

Placing $\nu = 1$ in Theorem 2, we deduce the following corollary.

Corollary 1. *Let $f \in \mathcal{R}_e$. Then,*

$$\left|b_3 - b_2^2\right| \leq \frac{1}{3}.$$

This inequality is sharp for the function f_2 given in (21).

On the class of univalent functions $\mathcal{S}$, a coefficient problem was proposed by Zalcman in 1960. It is conjectured that, for $f \in \mathcal{S}$, we have

$$\left|b_n^2 - b_{2n-1}\right| \leq (n-1)^2, \quad n \geq 2. \tag{24}$$

The generalized form of the Zalcman inequality is

$$|b_{n+m-1} - b_n b_m| \leq (n-1)(m-1), \quad m, n \in \mathbb{N}, m \geq 2, n \geq 2. \tag{25}$$

We intend to consider the cases of Zalcman inequalities for a special choice of m and n with respect to $f \in \mathcal{R}_e$ in the following.

Theorem 3. *Assume that $f \in \mathcal{R}_e$ and has the series expansion (1). Then,*

$$|b_2 b_3 - b_4| \leq \frac{1}{4}.$$

This equality is attained with the extremal function defined in (22).

Proof. Using (15)–(17), we have

$$|b_2 b_3 - b_4| = \frac{1}{8}\left|\mu_3 - 2\left(\frac{5}{12}\right)\mu_1\mu_2 + \frac{1}{8}\mu_1^3\right|.$$

From Lemma 3, we have

$$0 \leq \tau = \frac{5}{12} \leq 1, \qquad \tau \geq v = \frac{1}{8},$$

and

$$\tau(2\tau - 1) = -\frac{5}{72} \leq v.$$

Applying triangle inequality along with Lemma 3 leads us to

$$|b_2 b_3 - b_4| \leq \frac{1}{4}.$$

Thus, the required proof is completed. □

Theorem 4. *Let $f \in \mathcal{R}_e$ be the series expansion (1). Then,*

$$|b_5 - b_2 b_4| \leq \frac{1}{5}.$$

This inequality is sharp.

Proof. From (16) and (18), we obtain

$$|b_5 - b_2 b_4| = \left|-\frac{1}{1280}\mu_1^4 - \frac{1}{40}\mu_2^2 - \frac{13}{160}\mu_1\mu_3 + \frac{9}{320}\mu_1^2\mu_2 + \frac{1}{10}\mu_4\right|.$$

After simplifying, we have

$$|b_5 - b_2 b_4| = -\frac{1}{10}\left|\frac{1}{128}\mu_1^4 + \frac{1}{4}\mu_2^2 + 2\left(\frac{13}{32}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{3}{16}\right)\mu_1^2\mu_2 - \mu_4\right|. \tag{26}$$

Comparing the right side of (26) with

$$\left|\gamma\mu_1^4 + a\mu_2^2 + 2\alpha\mu_1\mu_3 - \frac{3}{2}\beta\mu_1^2\mu_2 - \mu_4\right|, \tag{27}$$

where

$$\gamma = \frac{1}{128}, \quad a = \frac{1}{4}, \quad \alpha = \frac{13}{32}, \quad \beta = \frac{3}{16},$$

are such that

$$8a(1-a)\Big((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\Big) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a),$$

with $0 < \alpha < 1, 0 < a < 1$. Therefore, by applying Lemma 4 in (27), we have

$$|b_5 - b_2 b_4| \leq \frac{1}{5}.$$

The required inequality is sharp for the function f_4 given in (23). □

Theorem 5. *Let $f \in \mathcal{R}_e$ has the series form* (1). *Then,*

$$\left|b_5 - b_3^2\right| \leq \frac{1}{5}.$$

This inequality is sharp.

Proof. Using (16) and (18), we obtain

$$\begin{aligned}\left|b_5 - b_3^2\right| &= \left|-\frac{7}{5760}\mu_1^4 - \frac{19}{360}\mu_2^2 - \frac{1}{20}\mu_1\mu_3 + \frac{19}{720}\mu_1^2\mu_2 + \frac{1}{10}\mu_4\right| \\ &:= \left|\gamma\mu_1^4 + a\mu_2^2 + 2\alpha\mu_1\mu_3 - \frac{3}{2}\beta\mu_1^2\mu_2 - \mu_4\right|,\end{aligned}$$

where

$$\gamma = \frac{7}{576}, a = \frac{19}{36}, \alpha = \frac{1}{4}, \beta = \frac{19}{108}.$$

By virtue of

$$8a(1-a)\Big((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\Big) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 < 4a\alpha^2(1-\alpha)^2(1-a),$$

and by using Lemma 4, we have

$$\left|b_5 - b_3^2\right| \leq \frac{1}{5}.$$

The required inequality is sharp for f_4 given in (23). □

Next, we will give direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \leq 2^{p(n-1)} - n^p,$$

over the class $\mathcal{R}_e$ for the choice of $n = 4$, $p = 1$, and for $n = 5$, $p = 1$. Krushkal introduced and proved this inequality for the whole class of univalent functions in [39].

Theorem 6. *If $f \in \mathcal{R}_e$ and is of the form* (1). *Then*

$$\left|b_4 - b_2^3\right| \leq \frac{1}{4}.$$

This outcome is sharp.

Proof. Using (15) and (17), we have

$$\left|b_4 - b_2^3\right| = \frac{1}{8}\left|\mu_3 - 2\left(\frac{1}{4}\right)\mu_1\mu_2 + \left(-\frac{1}{12}\right)\mu_1^3\right|.$$

From Lemma 3, we have

$$0 \le \tau = \frac{1}{4} \le 1, \qquad \tau \ge v = -\frac{1}{12},$$

and

$$\tau(2\tau - 1) = -\frac{1}{8} \le v.$$

Now, with the application of the triangle inequality along with (8), we obtain

$$\left|b_4 - b_2^3\right| \le \frac{1}{4}.$$

This outcome is sharp for the function f_3 given in (22). □

Theorem 7. *If $f \in \mathcal{R}_e$ and it has the series form* (1)*, then*

$$\left|b_5 - b_2^4\right| \le \frac{1}{5}.$$

The above outcome is best possible.

Proof. From (15) and (18), we obtain

$$\left|b_5 - b_2^4\right| = \left|-\frac{13}{3840}\mu_1^4 - \frac{1}{40}\mu_2^2 - \frac{1}{20}\mu_1\mu_3 + \frac{1}{80}\mu_1^2\mu_2 + \frac{1}{10}\mu_4\right|.$$

After simplifying, we have

$$\left|b_5 - b_2^4\right| = -\frac{1}{10}\left|\frac{13}{384}\mu_1^4 + \frac{1}{4}\mu_2^2 + 2\left(\frac{1}{4}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{1}{12}\right)\mu_1^2\mu_2 - \mu_4\right|. \tag{28}$$

Comparing the right side of (28) with

$$\left|\gamma\mu_1^4 + a\mu_2^2 + 2\alpha\mu_1\mu_3 - \frac{3}{2}\beta\mu_1^2\mu_2 - \mu_4\right|, \tag{29}$$

where

$$\gamma = \frac{13}{384}, \quad a = \frac{1}{4}, \quad \alpha = \frac{1}{4}, \quad \beta = \frac{1}{12},$$

are such that

$$8a(1-a)\left((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2\right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4a\alpha^2(1-\alpha)^2(1-a),$$

with $0 < \alpha < 1, 0 < a < 1$. Thus, by virtue of Lemma 4 in (29), we have

$$\left|b_5 - b_2^4\right| \le \frac{1}{5}.$$

The required inequality is sharp for the function f_4 given in (23). □

Finally, we determine the bounds of the second and third Hankel determinants for $f \in \mathcal{R}_e$.

Theorem 8. *Let $f \in \mathcal{R}_e$ have the representation* (1)*. Then,*

$$|H_{2,2}(f)| = \left|b_2 b_4 - b_3^2\right| \le \frac{1}{9}.$$

The result is sharp and equality obtained by the extremal function defined in (21).

Proof. From (15)–(17), we have

$$H_{2,2}(f) = -\frac{1}{2304}\mu_1^4 - \frac{1}{576}\mu_1^2\mu_2 + \frac{1}{32}\mu_1\mu_3 - \frac{1}{36}\mu_2^2.$$

By the rotation-invariant property for the class $\mathcal{R}_e$ and the functional $|H_{2,2}(f)|$, we can assume that $\mu_1 = \mu \in [0,2]$. Using Lemma 1 to express μ_2 and μ_3, we obtain

$$|H_{2,2}(f)| = \left| -\frac{1}{2304}\mu^4 + \frac{1}{1152}\mu^2\left(4-\mu^2\right)\kappa - \frac{1}{128}\mu^2\left(4-\mu^2\right)\kappa^2 - \frac{1}{144}\left(4-\mu^2\right)^2\kappa^2 \right. \\ \left. + \frac{1}{64}\mu\left(4-\mu^2\right)\left(1-|\kappa|^2\right)\delta \right|.$$

Using the triangle inequality along with the fact that $|\delta| \le 1$ and $|\kappa| = t \le 1$, we see

$$|H_{2,2}(f)| \le \frac{1}{2304}\mu^4 + \frac{1}{128}\mu^2\left(4-\mu^2\right)t^2 + \frac{1}{144}\left(4-\mu^2\right)^2 t^2 + \frac{1}{1152}\mu^2\left(4-\mu^2\right)t \\ + \frac{1}{64}\mu\left(4-\mu^2\right)\left(1-t^2\right) =: F(\mu,t).$$

It is easy to show that $\frac{\partial F}{\partial t} \ge 0$ on $[0,1]$; thus, we have $F(\mu,t) \le F(\mu,1)$. Taking $t=1$ gives

$$|H_{2,2}(f)| \le \frac{1}{2304}\mu^4 + \frac{5}{576}\mu^2\left(4-\mu^2\right) + \frac{1}{144}\left(4-\mu^2\right)^2 =: l(\mu).$$

As $l'(\mu) < 0$, it is known that l is a decreasing function and $l(\mu) \le l(0)$. Hence, we have

$$|H_{2,2}(f)| \le \frac{1}{9}.$$

□

Theorem 9. *Let $f \in \mathcal{R}_e$ be given the series form (1). Then*

$$|H_{2,3}(f)| \le \frac{1}{16}.$$

This result is the best possible.

Proof. By placing (16)–(18) with $\mu_1 = \mu$ into $H_{2,3}(f) = b_3 b_5 - b_4^2$, we obtain

$$\begin{aligned} H_{2,3}(f) &= \tfrac{1}{552{,}960}\big(-27\mu^6 + 120\mu^4\mu_2 + 9216\mu_2\mu_4 - 2304\mu_2^3 - 432\mu^2\mu_2^2 \\ &\quad + 4032\mu\mu_2\mu_3 - 2304\mu^2\mu_4 + 432\mu^3\mu_3 - 8640\mu_3^2\big). \end{aligned} \tag{30}$$

Using $\lambda = 4-\mu^2$ in (4)–(6) of Lemma 1, we obtain

$$\begin{aligned} H_{2,3}(f) = \frac{1}{552,960}\Big\{ &-3\mu^6 + 288\mu^4\kappa^3\lambda - 1296\mu^4\kappa^2\lambda + 36\mu^2\kappa^4\lambda^2 - 72\mu^2\kappa^3\lambda^2 - 396\mu^2\kappa^2\lambda^2 \\ &+1152\mu^2\kappa^2\lambda - 12\mu^4\kappa\lambda + 2304\kappa^3\lambda^2 - 288\kappa^3\lambda^3 - 1152\mu^2\lambda\overline{\kappa}\left(1-|\kappa|^2\right)\delta \\ &-1152\mu^3\kappa\lambda\left(1-|\kappa|^2\right)\delta + 216\mu^3\lambda\left(1-|\kappa|^2\right)\delta - 144\mu\kappa^2\lambda^2\left(1-|\kappa|^2\right)\delta \\ &\left(1-|\kappa|^2\right)\delta^2 - 2160\lambda^2\left(1-|\kappa|^2\right)^2\delta^2 - 2304\lambda^2|\kappa|^2\left(1-|\kappa|^2\right)\delta^2 - 1008\mu\kappa\lambda^2 \\ &+2304\kappa\lambda^2\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho + 1152\mu^2\lambda\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho\Big\}. \end{aligned}$$

Clearly, we can write it in the form of

$$H_{2,3}(f) = \frac{1}{552,960}\Big[\zeta_1(\mu,\kappa) + \zeta_2(\mu,\kappa)\delta + \zeta_3(\mu,\kappa)\delta^2 + \Phi(\mu,\kappa,\delta)\varrho\Big].$$

Here, $\rho, \kappa, \delta \in \overline{\mathbb{D}}$ and

$$\begin{aligned}
\zeta_1(\mu,\kappa) &= -3\mu^6 + \Big(4-\mu^2\Big)\Big[\Big(4-\mu^2\Big)\Big(216\mu^2\kappa^3 + 36\mu^2\kappa^4 + 1152\kappa^3 - 396\mu^2\kappa^2\Big) \\
&\quad +288\mu^4\kappa^3 - 396\mu^4\kappa^2 + 1152\mu^2\kappa^2 - 12\mu^4\kappa\Big], \\
\zeta_2(\mu,\kappa) &= 72\Big(4-\mu^2\Big)\Big(1-|\kappa|^2\Big)\Big[\Big(4-\mu^2\Big)\Big(-14\mu\kappa - 2\mu\kappa^2\Big) - 16\mu^3\kappa + 3\mu^3\Big], \\
\zeta_3(\mu,\kappa) &= 144\Big(4-\mu^2\Big)\Big(1-|\kappa|^2\Big)\Big[\Big(4-\mu^2\Big)\Big(-|\kappa|^2 - 15\Big) - 8\mu^2\overline{\kappa}\Big], \\
\Phi(\mu,\kappa,\delta) &= 576\Big(4-\mu^2\Big)\Big(1-|\kappa|^2\Big)\Big(1-|\delta|^2\Big)\Big[2\mu^2 + 4\kappa\Big(4-\mu^2\Big)\Big].
\end{aligned}$$

By making $|\kappa| = x$, $|\delta| = y$ along with $|\varrho| \le 1$, it is noted that

$$\begin{aligned}
|H_{2,3}(f)| &\le \tfrac{1}{552,960}\left[|\zeta_1(\mu,x)| + |\zeta_2(\mu,x)|y + |\zeta_3(\mu,x)|y^2 + |\Phi(\mu,x,\delta)|\right]. \\
&\le \tfrac{1}{552,960}\left[\Gamma(\mu,x,y)\right],
\end{aligned} \tag{31}$$

where we set

$$\Gamma(\mu,x,y) = r_1(\mu,x) + r_2(\mu,x)y + r_3(\mu,x)y^2 + r_4(\mu,x)\Big(1-y^2\Big),$$

with

$$\begin{aligned}
r_1(\mu,x) &= 3\mu^6 + \Big(4-\mu^2\Big)\Big[\Big(4-\mu^2\Big)\Big(216\mu^2x^3 + 36\mu^2x^4 + 1152x^3 + 396\mu^2x^2\Big) \\
&\quad +288\mu^4x^3 + 396\mu^4x^2 + 1152\mu^2x^2 + 12\mu^4x\Big], \\
r_2(\mu,x) &= 72\Big(4-\mu^2\Big)\Big(1-x^2\Big)\Big[\Big(4-\mu^2\Big)\Big(14\mu x + 2\mu x^2\Big) + 16\mu^3x + 3\mu^3\Big], \\
r_3(\mu,x) &= 144\Big(4-\mu^2\Big)\Big(1-x^2\Big)\Big[\Big(4-\mu^2\Big)\Big(x^2+15\Big) + 8\mu^2x\Big], \\
r_4(\mu,x) &= 576\Big(4-\mu^2\Big)\Big(1-\mu^2\Big)\Big[2\mu^2 + 4x\Big(4-\mu^2\Big)\Big].
\end{aligned}$$

Then, our task is to find the maximum value of Γ in the closed domain defined by $\Theta := [0,2]\times[0,1]\times[0,1]$. In light of $\Gamma(0,0,1) = 34,560$, it is seen that

$$\max_{(\mu,x,y)\in\Theta}\{\Gamma(\mu,x,y)\} \ge 34,560. \tag{32}$$

Now, we aim to illustrate that the maximum value of Γ with $(\mu,x,y)\in\Theta$ is equal to 34,560.

When $x = 1$, it reduces to

$$\Gamma(\mu,1,y) = 3\mu^6 + \Big(4-\mu^2\Big)\Big[\Big(4-\mu^2\Big)\Big(1152 + 648\mu^2\Big) + 1152\mu^2 + 696\mu^4\Big].$$

As

$$\frac{\partial\Gamma}{\partial\mu} = -270\mu^5 - 9600\mu^3 + 11,520\mu,$$

placing $\frac{\partial\Gamma}{\partial\mu} = 0$, we obtain the critical point $\mu \approx 1.0780$; thus, $\max\Gamma(\mu,1,y) \approx 21,813.93 < 34,560$. If $\mu = 2$, $\Gamma(2,x,y) \equiv 192 < 34,560$. Thus, we also assume $\mu < 2$ and $x < 1$. Let $(\mu,x,y) \in [0,2)\times[0,1)\times(0,1)$. Then,

$$\frac{\partial\Gamma}{\partial y} = 72\left(4-\mu^2\right)(1-x^2)\left\{4y(x-1)\left[\left(4-\mu^2\right)(x-15)+8\mu^2\right]\right.$$
$$\left.+\mu\left[2x\left(4-\mu^2\right)(7+x)+\mu^2(16x+3)\right]\right\}.$$

Inserting $\frac{\partial\Gamma}{\partial y}=0$ yields

$$y_0 = \frac{\mu\left[2x(4-\mu^2)(7+x)+\mu^2(16x+3)\right]}{4(x-1)[(4-\mu^2)(15-x)-8\mu^2]}.$$

If $y_0 \in (0,1)$, then we must have the following inequalities:

$$\mu^3(16x+3)+2\mu x\left(4-\mu^2\right)(7+x)+4(1-x)\left(4-\mu^2\right)(15-x) < 32\mu^2(1-x), \tag{33}$$

$$\mu^2 > \frac{4(15-x)}{23-x}. \tag{34}$$

It is not difficult to prove that the inequality in Equation (33) is false for $x \in \left[\frac{1}{2},1\right)$. Therefore, for the existence of a critical point (μ_0,x_0,y_0) with $y_0 \in (0,1)$, we have $t_0 < \frac{1}{2}$. Let $g(t)=\frac{4(15-t)}{23-t}$. By observing that g is decreasing on $(0,1)$, we have $\mu_0^2 > g\left(\frac{1}{2}\right)=\frac{116}{45}$. As $x_0 < \frac{1}{2}$, we know

$$r_1(\mu_0,x_0) \le r_1\left(\mu_0,\frac{1}{2}\right) =: \phi_1(\mu_0). \tag{35}$$

Using $1-x_0^2<1$ and $x_0<\frac{1}{2}$, we obtain

$$r_j(\mu_0,x_0) \le \frac{4}{3}r_j\left(\mu_0,\frac{1}{2}\right) =: \phi_j(\mu_0) \quad j=2,3,4. \tag{36}$$

Therefore, we deduce that

$$\Gamma(\mu_0,x_0,y_0) \le \phi_1(\mu_0)+\phi_4(\mu_0)+\phi_2(\mu_0)y_0+[\phi_3(\mu_0)-\phi_4(\mu_0)]y_0^2 =: \Psi(\mu_0,y_0).$$

In light of $\phi_3(\mu_0)-\phi_4(\mu_0)=36\left(4-\mu_0^2\right)\left(116-45\mu_0^2\right)\le 0$, it follows that $\frac{\partial^2\psi}{\partial y_0^2}\le 0$ for $y_0\in(0,1)$. Thus, we have

$$\frac{\partial\Psi}{\partial y_0} \ge \frac{\partial\Psi}{\partial y_0}\Big|_{y_0=1} = \phi_2(\mu_0)+2[\phi_3(\mu_0)-\phi_4(\mu_0)] \ge 0, \quad \mu_0 \in \left(\sqrt{\frac{116}{45}},2\right).$$

This means that

$$\Psi(\mu_0,y_0) \le \psi(\mu_0,1) = \phi_1(\mu_0)+\phi_2(\mu_0)+\phi_3(\mu_0) =: \widetilde{\phi}(\mu_0).$$

Because $\widetilde{\phi}$ takes a maximum value 16,368.92, we have $\Gamma(\mu_0,x_0,y_0) < 34,560$. Next, we prove that the maximum value of Γ is less than 34,560 when $y=0$. Actually,

$$\Gamma(\mu,x,0) = r_1(\mu,x)+r_4(\mu,x). \tag{37}$$

In the case of $x<\frac{7}{10}$, we have

$$r_1(\mu,x) \le r_1\left(\mu,\frac{7}{10}\right) =: \tau_1(\mu) \tag{38}$$

and

$$r_4(\mu,x) \le \frac{100}{51}r_4\left(\mu,\frac{7}{10}\right) =: \tau_2(\mu). \tag{39}$$

Then
$$\Gamma(\mu,x,0) \le \tau_1(\mu) + \tau_2(\mu) =: \tau(\mu). \tag{40}$$
By virtue of τ having its maximum value 32,126.98 at $\mu = 0$, we obtain
$$\Gamma(\mu,x,0) \le \max_{\mu\in[0,2)} \{\tau(\mu)\} < 34,560, \quad x \in \left[0, \frac{7}{10}\right). \tag{41}$$
If $x \ge \frac{7}{10}$, then $r_1(\mu,x) \le r_1(\mu,1)$. This leads to
$$\Gamma(\mu,x,0) \le r_1(\mu,1) + r_4(\mu,x) =: \varpi(\mu,x). \tag{42}$$
It is found that $\frac{\partial\varpi}{\partial x} \le 0$ for $x > \frac{7}{10}$. Hence,
$$\varpi(\mu,x) \le \varpi\left(\mu, \frac{7}{10}\right). \tag{43}$$
Combining (42) and (43), we obtain the conclusion that
$$\Gamma(\mu,x,0) \le \varpi\left(\mu, \frac{7}{10}\right) =: \eta(\mu). \tag{44}$$

As η achieves its maximum value of about 31,860.76 at $\mu \approx 0.5912$, we obtain $\Gamma(\mu,x,0) < 34,560$ on $x \in \left[\frac{7}{10}, 1\right)$. Based on the above discussion, we see that the maximum value of Γ on $y = 0$ is less than 34,560.

At this time, the problem reduces to finding the maximum value of Γ when $y = 1$. Indeed,
$$\begin{aligned}\Gamma(\mu,x,1) = 3\mu^6 &+ \left(4-\mu^2\right)^2\Big[36\left(\mu^2-4\mu-4\right)x^4 + 72\left(3\mu^2-14\mu+16\right)x^3 \\ &+36\left(11\mu^2+4\mu-60\right)x^2 + 1008\mu x + 2160\Big] \\ &+ \left(4-\mu^2\right)\Big[288\mu^2\left(\mu^2-4\mu-4\right)x^3 + 36\mu^2\left(11\mu^2-6\mu+32\right)x^2 \\ &+12\mu^2\left(\mu^2+96\mu+96\right)x + 216\mu^3\Big] =: \Omega(\mu,x).\end{aligned}$$
By observing that $\mu^2 - 4\mu - 4 \le 0$ for $\mu \in [0,2)$, we find
$$\begin{aligned}\Omega(\mu,x) \le 3\mu^6 &+ \left(4-\mu^2\right)^2\Big[72(3\mu^2-14\mu+16)x^3 + 36\left(11\mu^2+4\mu-60\right)x^2 \\ &+1008\mu x + 2160\Big] + \left(4-\mu^2\right)\Big[36\mu^2\left(11\mu^2-6\mu+32\right)x^2 \\ &+12\mu^2\left(\mu^2+96\mu+96\right)x + 216\mu^3\Big] =: Q(\mu,x).\end{aligned}$$

Furthermore, using $3\mu^2 - 14\mu + 16 \ge 0$, $x^3 \le x^2 \le x \le 1$ and some basic calculations, it leads to
$$\begin{aligned}Q(\mu,x) \le 3\mu^6 &+ 36\left(4-\mu^2\right)^2\Big[\left(17\mu^2-24\mu-28\right)x^2 + 28\mu x + 60\Big] \\ &+ \left(4-\mu^2\right)\left(408\mu^4 + 1152\mu^3 + 2304\mu^2\right) =: W(\mu,x).\end{aligned}$$

Suppose that
$$\begin{aligned}R(\mu,x) &= \left(17\mu^2-24\mu-28\right)x^2 + 28\mu x + 60 \\ &=: Ax^2 + Bx + C,\end{aligned}$$

where $A = 17\mu^2 - 24\mu - 28$, $B = 28\mu$ and $C = 60$. Clearly, we have $A \leq 0$ for $\mu \in [0, 2)$. Assuming that

$$\tilde{x}_0 = -\frac{B}{2A} = -\frac{14\mu}{17\mu^2 - 24\mu - 28}. \tag{45}$$

Let $\hat{\mu}_0 = \frac{5+\sqrt{501}}{17} \approx 1.6108$ be the only root of the equation $17\mu^2 - 10\mu - 28 = 0$ that lies in $(0, 2)$. For $\mu \geq \hat{\mu}_0$, we have $\tilde{x}_0 \geq 1$. Hence, $R(\mu, x) \leq R(\mu, 1)$, which leads to

$$W(\mu, x) \leq 3\mu^6 + 36\left(4 - \mu^2\right)^2 R(\mu, 1) + \left(4 - \mu^2\right)\left(408\mu^4 + 1152\mu^3 + 2304\mu^2\right) =: \vartheta(\mu).$$

Since ϑ obtains its maximum of 24,950.52 on $\mu = \hat{\mu}_0$, we obtain

$$\Gamma(\mu, x, 1) < 34,560, \quad (\mu, x) \in [\hat{\mu}_0, 2) \times [0, 1). \tag{46}$$

For $\mu < \hat{\mu}_0$, we see that $\tilde{x}_0 \in [0, 1)$. Then,

$$R(\mu, x) \leq C - \frac{B^2}{4A} = 60 + \frac{196\mu^2}{28 + 24\mu - 17\mu^2} \leq 60 + \frac{196}{\min_{\mu \in [0, \hat{\mu}_0)}\{28 + 24\mu - 17\mu^2\}}\mu^2.$$

As $\min_{\mu \in [0, \hat{\mu}_0)} 28 + 24\mu - 17\mu^2 \geq 22$, it follows that

$$R(\mu, x) \leq 60 + \frac{196}{22}\lambda^2 \leq 60 + 9\mu^2. \tag{47}$$

Therefore, we obtain

$$W(\mu, x) \leq 3\mu^6 + \left(4 - c\mu^2\right)^2\left(2160 + 324\mu^2\right) + \left(4 - \mu^2\right)\left(408\mu^4 + 1152\mu^3 + 2304\mu^2\right) =: \tilde{\varpi}(c).$$

It is calculated that $\tilde{\varpi}$ achieves its maximum value of 34,560 at $\mu = 0$ for all $\mu \in [0, \hat{\mu}_0)$. Therefore, we find that $\Gamma(\mu, x, y) \leq 34,560$ on the domain Θ, which leads to

$$|H_{2,3}(f)| \leq \frac{1}{16} = 0.0625.$$

It is sharp for the function f_3 given in (22). □

Theorem 10. *Let $f \in \mathcal{R}_e$ have the series representation (1). Then*

$$|H_{3,1}(f)| \leq \frac{1}{16}.$$

This inequality is sharp.

Proof. From the definition, we see that $H_{3,1}(f)$ can be written as

$$H_{3,1}(f) = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_3b_5 - b_2^2b_5.$$

By virtue of the rotation invariance for $f \in \mathcal{R}_e$, we suppose that $\mu_1 = \mu \in [0, 2]$. By placing (15)–(18) in the above relation, we obtain

$$H_{3,1}(f) = \frac{1}{552,960}\Big(-65\mu^6 + 168\mu^4\mu_2 + 720\mu^3\mu_3 - 528\mu^2\mu_2^2 - 5760\mu^2\mu_4$$
$$+9792\mu\mu_2\mu_3 - 4864\mu_2^3 + 9216\mu_2\mu_4 - 8640\mu_3^2\Big). \tag{48}$$

Suppose that $\lambda = 4 - \mu^2$. Then, by (4)–(6) of Lemma 1, we obtain

$$H_{3,1}(f) = \frac{1}{552,960}\Big\{-\mu^6 + 2304\lambda^2\kappa^3 - 608\lambda^3\kappa^3 - 576\mu^2\lambda\kappa^2 - 144\mu^4\lambda\kappa^3 + 108\mu^4\lambda\kappa^2$$
$$+12\mu^4\lambda\kappa + 36c^2\lambda^2\kappa^4 - 792\mu^2\lambda^2\kappa^3 - 2160\lambda^2\left(1-|\kappa|^2\right)^2\delta^2 + 60\mu^2\lambda^2\kappa^2$$
$$-576\mu^2\lambda\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho + 576\mu^3\lambda\kappa\left(1-|\kappa|^2\right)\delta + 576\mu^2\lambda\overline{\kappa}\left(1-|\kappa|^2\right)\delta^2$$
$$+72\mu^3\lambda\left(1-|\kappa|^2\right)\delta - 144\mu\lambda^2\kappa^2\left(1-|\kappa|^2\right)\delta - 2304\lambda^2|\kappa|^2\left(1-|\kappa|^2\right)\delta^2$$
$$+432\mu\lambda^2\kappa\left(1-|\kappa|^2\right)\delta + 2304\lambda^2\kappa\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\varrho\Big\}.$$

Now, we can write it as

$$H_{3,1}(f) = \frac{1}{552,960}\Big[d_1(\mu,\kappa) + d_2(\mu,\kappa)\delta + d_3(\mu,\kappa)\delta^2 + \chi(\mu,\kappa,\delta)\varrho\Big].$$

Here, $\varrho, \kappa, \delta \in \overline{\mathbb{D}}$ and

$$d_1(\mu,\kappa) = -\mu^6 + \left(4-\mu^2\right)\Big[\left(4-\mu^2\right)\left(-128\kappa^3 - 184\mu^2\kappa^3 + 36\mu^2\kappa^4 + 60\mu^2\kappa^2\right)$$
$$-576\mu^2\kappa^2 - 144\mu^4\kappa^3 + 108\mu^4\kappa^2 + 12\mu^4\kappa\Big],$$
$$d_2(\mu,\kappa) = 72\left(4-\mu^2\right)\left(1-|\kappa|^2\right)\Big[\left(4-\mu^2\right)\left(6\mu\kappa - 2\mu\kappa^2\right) + 8\mu^3\kappa + \mu^3\Big],$$
$$d_3(\mu,\kappa) = 144\left(4-\mu^2\right)\left(1-|\kappa|^2\right)\Big[\left(4-\mu^2\right)\left(-|\kappa|^2 - 15\right) + 4\mu^2\overline{\kappa}\Big],$$
$$\chi(\mu,\kappa,\delta) = 576\left(4-\mu^2\right)\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\Big[-\mu^2 + 4\kappa\left(4-\mu^2\right)\Big].$$

Setting $|\kappa| = x, |\delta| = y$ and using $|\varrho| \le 1$, we obtain

$$\begin{aligned} |H_{3,1}(f)| &\le \tfrac{1}{552,960}\left[|d_1(\mu,x)| + |d_2(\mu,x)|y + |d_3(\mu,x)|y^2 + |\chi(\mu,x,\delta)|\right]. \\ &\le \tfrac{1}{552,960}[\Lambda(\mu,x,y)], \end{aligned} \tag{49}$$

where

$$\Lambda(\mu,x,y) = v_1(\mu,x) + v_2(\mu,x)y + v_3(\mu,x)y^2 + v_4(\mu,x)\left(1-y^2\right),$$

with

$$v_1(\mu,x) = \mu^6 + \left(4-\mu^2\right)\Big[\left(4-\mu^2\right)\left(128x^3 + 184\mu^2x^3 + 36\mu^2x^4 + 60\mu^2x^2\right)$$
$$+576\mu^2x^2 + 144\mu^4x^3 + 108\mu^4x^2 + 12\mu^4x\Big],$$
$$v_2(\mu,x) = 72\left(4-\mu^2\right)\left(1-x^2\right)\Big[\left(4-\mu^2\right)\left(6\mu x + 2\mu x^2\right) + 8\mu^3x + \mu^3\Big],$$
$$v_3(\mu,x) = 144\left(4-\mu^2\right)\left(1-x^2\right)\Big[\left(4-\mu^2\right)\left(x^2+15\right) + 4\mu^2x\Big],$$
$$v_4(\mu,x) = 576\left(4-\mu^2\right)\left(1-x^2\right)\Big[c^2 + 4x\left(4-\mu^2\right)\Big].$$

We are now able to obtain the maximum value of Λ with (μ,x,y) still restricted in Θ. As it is observed that

$$v_j(\mu,x) \le r_j(\mu,x), \quad (j = 1,2,3,4), \tag{50}$$

a conclusion can be made that $\Lambda(\mu,x,y) \le \Gamma(\mu,x,y) \le 34,560$ on $[0,2]\times[0,1]\times[0,1]$. Therefore, according to (49), we obtain

$$|H_{3,1}(f)| \le \frac{1}{552,960}[\Lambda(\mu,x,y)] \le \frac{1}{16}.$$

It is sharp for the function f_3 given in (22). □

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