Algebraic Structures on Smooth Vector Fields

Amnah A. Alkinani and Ahmad M. Alghamdi

1 Department of Basic Science, Adham University College, Umm Al-Qura University, Makkah 21955, Saudi Arabia
2 Mathematics Department, Faculty of Sciences, Umm Al-Qura University, P.O. Box 14035, Makkah 21955, Saudi Arabia; amghamdi@uqu.edu.sa

* Correspondence: aaakenane@uqu.edu.sa

Abstract: The aim of this work is to investigate some algebraic structures of objects which are defined and related to a manifold. Consider \( L \) to be a smooth manifold and \( \Gamma^\infty (TL) \) to be the module of smooth vector fields over the ring of smooth functions \( C^\infty (L) \). We prove that the module \( \Gamma^\infty (TL) \) is projective and finitely generated, but it is not semisimple. Therefore, it has a proper socle and nonzero Jacobson radical. Furthermore, we prove that this module is reflexive by showing that it is isomorphic to its bidual. Additionally, we investigate the structure of the Lie algebra of smooth vector fields. We give some questions and open problems at the end of the paper. We believe that our results are important because they link two different disciplines in modern pure mathematics.

Keywords: \( C^\infty \)-functions; vector fields; derivations; localization; Lie algebra; symmetric \( C^\infty \)-algebras

MSC: 26E10; 57R10; 57R25; 17A36; 13B30; 17B66; 05E05

1. Introduction

Smooth functions are functions that can be differentiable everywhere. Then, they are continuous. We consider the smooth manifold \( L \) (see Definition 4 below) and the set of smooth functions

\[
C^\infty (L) =: \{ f : L \rightarrow \mathbb{R} \mid f \text{ is a smooth function} \}.
\]

These are of fundamental importance in differential geometry. Here, \( \mathbb{R} \) means the field of real numbers. The set \( C^\infty (L) \) is an \( \mathbb{R} \)-vector space. This vector space has the structure as a ring by defining the addition and the multiplication pointwise of the functions. It satisfies the bilinearity condition that \( r(f_1 f_2) = (r f_1) f_2 = f_1 (r f_2) \) for all \( r \in \mathbb{R} \) and \( f_1, f_2 \in C^\infty (L) \). Then, it is an algebra over the real numbers \( \mathbb{R} \). Recall that by an algebra over a field, we mean a vector space over the field (or module over a commutative ring) endowed with a multiplication that satisfies the bilinearity condition. See [1] (page 131) for more details.

Let \( n \) be a natural number. For each smooth function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), there is an \( n \)-(operation) \( C_f : C^\infty (L)^n \rightarrow C^\infty (L) \) acting in accordance with the rule: \( C_f (g_1, g_2, \cdots, g_i, \cdots, g_n) = f (g_1(\cdot), g_2(\cdot), \cdots, g_i(\cdot), \cdots, g_n(\cdot)) \) where \( g_i \) is an element in \( C^\infty (L) \) for \( i = 1, 2, \cdots, n \) (as in [2,3]). The most important example of a \( C^\infty \)-ring is \( C^\infty (L) \).

A set of all smooth vector fields \( \Gamma^\infty (TL) \) on a manifold \( L \) has an algebraic structure: if \( X_1, X_2 \in \Gamma^\infty (TL) \) and \( f_1, f_2 \in C^\infty (L) \), then \( f_1 X_1 + f_2 X_2 \in \Gamma^\infty (TL) \). This structure implies that \( \Gamma^\infty (TL) \) is not only a vector space over the real numbers \( \mathbb{R} \) with infinite dimension but also a module over the ring of the smooth functions \( C^\infty (L) \). The most important property of the smooth vector fields is that they act as an \( \mathbb{R} \)-derivation of the algebra of smooth functions [6]. The smooth vector fields are important, since these help to describe the flow of objects in space. They are used in differential geometry, physics, engineering and even in computer graphics.
The purpose of this paper is to establish a new approach to link the smooth vector fields with the rings, the modules and the derivations. However to do so, many non-obvious algebraic identities need to be verified.

This study includes a description of the smooth vector fields as a module of the ring of smooth functions. This module is a finitely generated projective that is not semisimple. It has a proper socle and nonzero Jacobson radical. Furthermore, this module is reflexive because it is isomorphic to its bidual. Moreover, achieving the Lie algebra structure for these smooth vector fields. This study can be extended to future research, for instance, studying the Noetherian, Artinian, uniserial, injective and injective hull for this module. This enables us to examine invariant module structures that will be discussed in Section 6.

The paper is organized in the following order. There is a literature review subsection in this section. The second section provides the methodology for addressing this study. The third section recalls the definitions of a $C^\infty$-ring, its algebraic structure and its modules. The fourth section investigates the module structure of smooth vector fields. We prove that this module is a finitely generated projective, but it is not semisimple. It is reflexive due to its similarity to its bidual. In addition, it has a proper socle and nonzero Jacobson radical. The fifth section presents well-known examples of the Lie algebra structure for smooth vector fields and discusses them in proposition form. The sixth section presents some questions which are related to this work. These questions will lead to the study of invariant modules. We close the paper with Section 7, which contains the conclusions of our work.

**Literature Review of Studies of $C^\infty$-Rings and Vector Fields**

In the literature, it is well known that the space of continuous functions contains a subalgebra. This was the content of an old paper in 1956 by Rudin [7]. The $C^\infty$-ring was first introduced in synthetic differential geometry. It was developed first by Lawvere in 1960. Then, it was developed further by Dubuc in 1981 in the article [8]. Dubuc incorporated differential geometry into the field of application of algebraic techniques and tools. Additionally, he developed a model for synthetic differential geometry that is sufficiently general and has sufficient good characteristics to propose stronger axioms. This was followed by Moerijkad and Reyes [9] in 1991. They introduced $C^\infty$-rings, which are utilized in algebraic geometry as ordinary commutative rings with identity. On the other hand, they examined manifolds as $C^\infty$-rings, local $C^\infty$-rings as well as ideals of smooth functions. After that, in 2006, Kock [10] developed various types of rings. In the book by Kock, we observed synthetic differential geometry which started with the basic structure of the geometric line. In particular, we see vector fields and infinitesimal transformations as well as their commutators. Furthermore, that book offers an interesting approach to the derivatives. Recently, in 2020, a comprehensive treatment of smooth manifolds and observables was published, as can be seen in [5]. In particular, on page 37, we see an algebraic approach of the definition of smooth manifolds. There are recent publications in 2022 and 2023 that provide further directions and constructions, such as [11–13]. The interaction between the Legendrian satellite construction and the existence of exact, orientable Lagrangian cobordisms between Legendrian knots was observed in [11]. The derivative of a log-analytic function is log-analytic, as demonstrated in [12]. That study showed that log-analytic functions have strong quasianalytic properties. It also established the parametric version of Tamm’s theorem for log-analytic functions. Rainer showed in [13] that polynomials and differentiable functions defined on a convex body have many similarities. This had several consequences. We also mention [14], which used incompressible vector fields to characterize Killing vector fields. Particularly, this study showed that a nontrivial incompressible vector field on a compact Riemannian manifold can be classified as a Jacobi-type vector field if and only if it is Killing.
2. Methodology

Research on algebraic structures in smooth vector fields is a significant area of study. In reality, it helps us to understand their properties and relationships, which leads to advancements in different fields like physics, engineering and pure and applied mathematics. It allows us to develop new techniques and applications that can be applied for practical applications.

The methods and strategies that we have used in this study are standard. For example, utilizing and comparing previous results in the literature in this area to construct and investigate new problems and new theorems. This study requires the use of tools from linear algebra, Lie algebra and representation theory. The isomorphism theorem between smooth vector fields and derivations has demonstrated the significance of visualizing this study. Category theory is a crucial and important tool to investigate the results in our work.

3. Ring of Smooth Functions and Its Modules

There are three subsections in this section. The first one is designed for the ring of smooth functions. The second subsection deals with this ring structure as an \( \mathbb{R} \)-algebra. The third subsection focuses on the module structure over this ring.

3.1. The Ring of Smooth Functions

First, we study the ring of smooth functions, which is called the \( C^\infty \)-ring or smooth ring in classical algebra.

**Definition 1** ([2,3]). Let \( n \) and \( m \) be two non-negative integers. A \( C^\infty \)-ring is a non-empty set \( C \) together with operations \( C_f : C^n \to C \) for all smooth functions \( f : \mathbb{R}^n \to \mathbb{R} \). Here, the function \( f \) is an element of \( C^\infty(\mathbb{R}^n) \) with the following conditions:

1. If \( \pi_i : \mathbb{R}^n \to \mathbb{R} \) is the projection given by \( \pi_i(x_1, \ldots, x_n) = x_i \) for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \), then \( C_{\pi_i}(c_1, \ldots, c_n) = c_i \) for all \( (c_1, \ldots, c_n) \in \mathbb{C}^n \).
2. If \( f \) is an element of \( C^\infty(\mathbb{R}^n) \) and \( g_i \) is an element of \( C^\infty(\mathbb{R}^m) \) where \( g_i : \mathbb{R}^m \to \mathbb{R} \) with \( h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) \) is an element of \( C^\infty(\mathbb{R}^m) \) where \( i = 1, \ldots, n \), then

\[
C_h(c_1, \ldots, c_m) = C_f(C_{g_1}(c_1, \ldots, c_m), \ldots, C_{g_n}(c_1, \ldots, c_m))
\]  

for all \( c_1, \ldots, c_m \in C \).

**Remark 1.**

1. If \( n = 0 \), then we define \( C^0 \) to be the single point \( \{ \phi \} \), where \( \phi \) means the empty set.
2. A morphism of \( C^\infty \)-rings \( C \) and \( C' \) is a map \( \phi : C \to C' \) such that for all \( c_1, \ldots, c_n \in C \) and \( f \in C^\infty(\mathbb{R}^n) \), \( \phi(C_f(c_1, \ldots, c_n)) = C'_f(\phi(c_1), \ldots, \phi(c_n)) \).

The following definition is equivalent to the concept of the \( C^\infty \)-ring. Approach is called the categorical \( C^\infty \)-ring.

**Definition 2** ([2,3]). Write \( \text{Euc} \) to mean the category with objects \( \mathbb{R}^n \) for all \( n \in \mathbb{N} \cup \{0\} \). A categorical \( C^\infty \)-ring is a finite product preserving the functor \( \text{Euc} \mapsto \text{Sets} \).

**Example 1** ([2,3]). Let \( C^\infty(L) \) be the set of smooth functions from a smooth manifold \( L \) to the real numbers \( \mathbb{R} \). We define \( C_f : C^\infty(L)^n \to C^\infty(L) \) by \( (C_f(c_1, \ldots, c_n))(x) = f(c_1(x), \ldots, c_n(x)) \) for all \( f : \mathbb{R}^n \to \mathbb{R} \) smooth, \( c_1, \ldots, c_n \in C^\infty(L) \) and \( x \in L \). We see that \( C^\infty(L) \) and the operations \( C_f \) form a \( C^\infty \)-ring as follows:

1. The projection \( \pi_i : \mathbb{R}^n \to \mathbb{R} \) by \( \pi_i(x_1, \ldots, x_n) = x_i \) is given. Then, \( C_{\pi_i}(c_1, \ldots, c_n)(x) = \pi_i(c_1(x), \ldots, c_n(x)) = c_i(x) \).
2. If \( f \in C^\infty(\mathbb{R}^m), g_i \in C^\infty(\mathbb{R}^m) \), and
\( h(x_1, \cdots, x_m) = f( g_1(x_1, \cdots, x_m), \cdots, g_n(x_1, \cdots, x_m) ) \in C^\infty(\mathbb{R}^m) \) for all \( m, n \geq 0 \), then

\[
( C_h(c_1, \cdots, c_m) ) (x) = h( c_1(x), \cdots, c_m(x) )
\]

\[
= f( g_1(c_1(x), \cdots, c_m(x)), \cdots, g_n(c_1(x), \cdots, c_m(x)) )
\]

\[
= C_f( C_{g_1}(c_1(x), \cdots, c_m(x)), \cdots, C_{g_n}(c_1(x), \cdots, c_m(x)) )
\]

\[
= (C_f( C_{g_1}(c_1, \cdots, c_m), \cdots, C_{g_n}(c_1, \cdots, c_m)))(x)
\]

for all \( c_1, \cdots, c_m \in C^\infty(\mathbb{R}) \).

The simplest example of a \( C^\infty \)-ring is by taking the manifold \( L \) in the previous Example 1 as a point \(*\). In particular, \( C^\infty(*) = \mathbb{R} \) with operations:

\[
C_f : \mathbb{R}^n \longrightarrow \mathbb{R}
\]

\[
C_f(x_1, \cdots, x_n) = f(x_1, \cdots, x_n).
\]

3.2. Ring of Smooth Functions as an \( \mathbb{R} \)-Algebra

In this subsection, we follow Joyce’s paper and book [2,3]. In particular, the \( C^\infty \)-ring \( C \) has the structure of a commutative \( \mathbb{R} \)-algebra by defining the following operations in the special case \( n = 2 \).

- Addition on \( C \):
  for all \( c', c'' \in C \), \( C_f(c', c'') = c' + c'' \), where \( f : \mathbb{R}^2 \longrightarrow \mathbb{R} \) is \( f(x_1, x_2) = x_1 + x_2 \).
- Multiplication on \( C \):
  for all \( c', c'' \in C \), \( C_g(c', c'') = c'.c'' \), where \( g : \mathbb{R}^2 \longrightarrow \mathbb{R} \) is \( g(x_1, x_2) = x_1.x_2 \).
- Scalar multiplication:
  for all \( \lambda \in \mathbb{R} \), \( C_{\lambda'}(c') = \lambda c' \), where \( \lambda' : \mathbb{R} \longrightarrow \mathbb{R} \) is \( \lambda'(x) = \lambda x \).
- Elements 0 and 1 in \( C \):
  \( C_{0'}(\phi) = 0 \) and \( C_{1'}(\phi) = 1 \), where \( 0' : \mathbb{R}^0 \longrightarrow \mathbb{R} \) is \( 0'(\phi) = 0 \), \( 1' : \mathbb{R}^0 \longrightarrow \mathbb{R} \) is \( 1'(\phi) = 1 \) and \( \phi \) means the empty set.

Remark 2. 1. An ideal \( I \) in the \( C^\infty \)-ring \( C \) can be regarded as a commutative \( \mathbb{R} \)-algebra.
2. The ideal \( I \) in \( C^\infty(\mathbb{R}^n) \) is finitely generated if there exist \( f_1, \cdots, f_k \in C^\infty(\mathbb{R}^n) \), which generate \( I \). In this case, we write \( I = \langle f_1, \cdots, f_k \rangle \).

3.3. Modules over the Ring of Smooth Functions

In this subsection, we study the structure of modules over the \( C^\infty \)-ring.

Definition 3. A module \( M \) over a \( C^\infty \)-ring \( C \) is a module over \( C \) regarded as a commutative \( \mathbb{R} \)-algebra as in Section 3.2, and morphisms of \( C \)-modules are morphisms of \( \mathbb{R} \)-algebra modules.

This module consists of an abelian group \((M, +)\) and scalar multiplication operation:

\[
C \times M \longrightarrow M
\]

\[
(c, m) \longmapsto c.m
\]

such that for all \( c, c_i \in C \) and \( m, m_i \in M \) where \((i = 1, 2)\), we have

1. \( c(m_1 + m_2) = cm_1 + cm_2 \)
2. \( (c_1 + c_2)m = c_1m + c_2m \)
3. \( c_1(c_2m) = (c_1c_2)m \)
4. \( 1m = m \).

Hence \( C \)-modules form an abelian category, which is denoted as \( C \)-mod.

Example 2. 1. The \( C^\infty \)-ring \( C \) is a \( C \)-module.
2. If $V$ is any $\mathbb{R}$-vector space, then $C \otimes_{\mathbb{R}} V$ is a $C$-module with the action $c'(c \otimes_{\mathbb{R}} vs.) = c'c \otimes_{\mathbb{R}} vs.$ for all $c' \in C$ and $c \otimes_{\mathbb{R}} vs. \in C \otimes_{\mathbb{R}} V$.

As we will see in the next section, we consider the set of all smooth vector fields $X$ to be a module over the ring of smooth functions $C$.

4. Module Structure of Smooth Vector Fields

Our focus is on studying the structure of smooth vector fields as a module over $C$, where $C$ is the $C^\infty$-ring. We prove that this module is projective finitely generated, which is not semisimple. Furthermore, this module has a proper socle and nonzero Jacobson radical.

Let us first recall the definition of a manifold:

**Definition 4 ([14]).** We say that $L$ is a topological manifold if it has the following properties:

- $L$ is a Hausdorff space: for every pair of distinct points $p, q \in L$, there are disjoint open subsets $U, V \subseteq L$ such that $p \in U$ and $q \in V$.
- $L$ is second-countable: there exists a countable basis for the topology of $L$.
- $L$ is locally Euclidean space: each point of $L$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

It said to be of dimension $n$ if it is locally Euclidean of dimension $n$. A smooth manifold is a topological manifold $L$ together with a differentiable structure on $L$. An example of a smooth manifold is the sphere: $S^n \subseteq \mathbb{R}^{n+1}$.

4.1. Smooth Vector Fields; Basic Facts

Consider $L$ to be a smooth manifold. A **tangent vector** $X_p$ at a point $p \in L$ is a linear map $X_p : C^\infty(L) \to \mathbb{R}$ such that $X_p(fg) = X_p(f)p + f(p)X_p(g)$ for all $f, g \in C^\infty(L)$.

If $\dim(L) = d$, a **smooth vector bundle** of rank $r$ over $L$ is a smooth manifold $E$ with $\dim(E) = d + r$ and a smooth surjective map $\pi : E \to L$ such that:

1. for all $p \in L$, $E_p := \pi^{-1}(p)$ is a vector space with $\dim(E_p) = r$.
2. for all $p \in L$, there exists a neighborhood $U \subseteq L$ and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\pi = \pi_1 \circ \Psi$ where $\pi_1 : Y_1 \times Y_2 \to Y_1$ is the projection and the restriction $\Psi|_E : E_p \to \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$ is linear isomorphism.

**Remark 3.** 1. The tangent vector $X_p$ is an element of the dual space $C^\infty(L)^*$. 2. We say that $E$ is the total space, $\pi$ is the bundle projection and $\Psi$ is the local trivialization.

A tangent bundle is an example of a vector bundle. It is a manifold $TL$, which assembles all the tangent vectors in $L$. Define a smooth section of the vector bundle $\pi : E \to L$ to be a smooth map $s : L \to E$ such that $\pi \circ s = id_L$. The set of all smooth sections of $E$ is denoted by $\Gamma^\infty(E)$. In general, smooth vector fields $X$ are defined on smooth manifold $L$ as follows:

**Definition 5.** A smooth vector field $X$ on $L$ is a smooth map $X : L \to TL$ such that $\pi \circ X = id_L : L \to L$ where $\pi : TL \to L$ is the projection $\pi(X_p) = p$.

This means that the smooth vector field is the smooth section of the tangent bundle or the vector bundle. The set of all smooth vector fields on $L$ is denoted by $\Gamma^\infty(TL)$.

A **derivation** of the algebra $C^\infty(L)$ is a map $D : C^\infty(L) \to C^\infty(L)$ that satisfies the Leibniz rule $D(fg) = fD(g) + D(f)g$ for all $f, g \in C^\infty(L)$. The set of all derivations of the algebra $C^\infty(L)$ is denoted by $\text{Der}(C^\infty(L))$.

**Theorem 1 ([8] (Problem 19.12)).** Let $\Gamma^\infty(TL)$ be the set of all smooth vector fields and let $\text{Der}(C^\infty(L))$ be the set of all derivations of $C^\infty(L)$. Then, the map $\psi : \Gamma^\infty(TL) \to \text{Der}(C^\infty(L))$ is an isomorphism.
Remark 4. The above theorem means that $X(f) = D(f)$. In other words, every vector field is a derivation. Thus, we have an alternative definition of the smooth vector field $X$ as we will see in the next definition.

Definition 6. The smooth vector field $X$ on $L$ is a linear map $X : C^\infty(L) \to C^\infty(L)$ such that $X$ is a derivation. That is $X(fg) = fX(g) + X(f)g$ for all $f, g \in C^\infty(L)$.

Let $X_1, X_2 \in \Gamma^\infty(TL)$. We define the operations of vector fields addition and scalar multiplication as:

\begin{align*}
(X_1 + X_2)(x) &:= X_1(x) + X_2(x) \\
(fX)(x) &:= f(x)X(x).
\end{align*}

Then, $(\Gamma^\infty(TL), +)$ is an abelian group. Furthermore, $\Gamma^\infty(TL)$ is a module $M$ over the ring $C^\infty(L)$ via pointwise multiplication:

\[ f_x : x \mapsto f(x)X(x). \]

This operation fulfills the $C^\infty(L)$-module structure of $M = \Gamma^\infty(TL)$ as follows: for all $f, f_1, f_2 \in C^\infty(L)$ and $X, X_1, X_2 \in \Gamma^\infty(TL)$,

\begin{align*}
(f_1 + f_2)(X)(x) &=(f_1 + f_2)(x)X(x) \\
&=(f_1(x) + f_2(x))X(x) \\
&=(f_1 X + f_2 X)(x).
\end{align*}

\begin{align*}
((f_1f_2)X)(x) &= (f_1f_2)(x)X(x) \\
&= f_1(f_2(x))X(x) \\
&= (f_1(f_2X))(x) \\
(idX)(x) &= X(x).
\end{align*}

Theorem 2. With the above action, formula and notation, $\Gamma^\infty(TL)$ is a module over the ring of smooth functions $C^\infty(L)$.

From Theorem 2, $\Gamma^\infty(TL)$ is a module over $C^\infty(L)$. Then, there is a close connection between the structure of $C^\infty(L)$ and the structure of $\Gamma^\infty(TL)$. Certainly the most important $C^\infty(L)$-module is $C^\infty(L)$ itself with the module structure given by multiplication.

Remark 5. Since the smooth vector field is the smooth section of the vector bundle, the set of all smooth sections $\Gamma^\infty(E)$ is also a $C^\infty(L)$-module.

Example 3. The module of sections of vector bundles $\Gamma^\infty(E)$ is projective. This example is proven in Theorem 3, which extracted from [5] (Theorem 12.32).

4.2. The Module of Smooth Vector Fields Is Projective and Finitely Generated

In module theory, the study of finitely generated projective modules is a classical form. These modules connect with K-theory and algebraic geometry. In general, the Serre–Swan theorem identifies suitable modules for an algebra of functions with the modules of sections of vector bundles in space. In this regard, we would like to thank the reviewer for their
suggestions to study some additional structure (e.g., a Poisson or Jacobi bracket) as well as to consider the Lie algebra of vector field space and its relation to fluid dynamics. This will be a subject of a new paper. The Serre–Swan theorem has the following formula in differential geometry.

**Theorem 3** ([3] (Theorem 12.32)). If L is a connected smooth manifold and P is an $C^\infty(L)$-module, then P is isomorphic to the $C^\infty(L)$-module $\Gamma^\infty(E)$ if and only if P is finitely generated and projective.

Serre–Swan’s theorem states that $\Gamma^\infty(E)$ is a finitely generated and projective $C^\infty(L)$-module. Thus, from Remark 5, we conclude the following lemma.

**Lemma 1.** Let L be a connected smooth manifold with $\mathbb{R}$-algebra of smooth functions; there is an isomorphism between the smooth vector fields $\Gamma^\infty(TL)$ and finitely generated projective $C^\infty(L)$-modules.

**Remark 6.** If we omit the condition that the manifold is not connected, then the conclusion above is not true in general as in the following examples.

The examples below are somewhat well-known, but we include them for the sake of completeness.

**Example 4.** Suppose that the function

$$a(x) = \begin{cases} 1 & \text{if } x > 0 \\ e^{-\frac{1}{x^2}} & \text{if } x \leq 0 \end{cases}$$

(9)

is defined for every $x \in \mathbb{R}$, and $\beta(x) = a(x)(2 - x)$. Thus, $\beta_n(x) := \beta(x + n)$. We identify the ideal

$$I = \langle \{ \beta_n \mid n \in \text{the set of integers } \mathbb{Z} \} \rangle$$

(10)

such that all functions of I have the form $\gamma_1\beta_{n_1}, \gamma_2\beta_{n_2}, \ldots, \gamma_r\beta_{n_r}$, for some $\gamma_1, \ldots, \gamma_r \in C^\infty(\mathbb{R})$. Then, this form vanishes outside some big compact set, say $\{ |x| < n_1 + n_2 + \cdots + n_r \}$. Since the functions $\beta_n$ are exponential, they are smooth and positive in an open interval but zero everywhere else. So, these functions have a bump. Thus, we can choose $\beta_n$ with a bump over x for all $x \in \mathbb{R}$. If I is finitely generated by $f_1, \ldots, f_k$, then every function in I must vanish on $0(I) := \bigcap_{i=1}^k 0(f_i)$, where $0(f)$ means the set of zeros of f. However, there is no x in the zero set of all $\gamma \in I$. Hence, $0(I)$ is the empty set. This is a contradiction which implies that $I = \langle \{ \beta_n \mid n \in \text{the set of integers } \mathbb{Z} \} \rangle$ is not finitely generated by $f_1, \ldots, f_k$.

**Example 5.** The ideal

$$I = \{ f \mid f(x) = 0 \text{ for all } x \geq D \} \text{ in } C^\infty(\mathbb{R})$$

(11)

has no zeros because $f_D \in I$ for all $D \in \mathbb{R}$, where

$$f_D(x) = \begin{cases} \frac{1}{e^{x-D}} & \text{if } x < D \\ 0 & \text{if } x \geq D \end{cases}$$

(12)

If $f_1, \ldots, f_k \in I$ for some natural number k, then there exists $D_0 \in \mathbb{R}$ such that $f(x) = 0$ for all $f \in \langle f_1, \ldots, f_k \rangle$ and $x \geq D_0$. This means that the interval $[D_0, \infty) \subset 0((f_1, \ldots, f_k))$. Thus, $\langle f_1, \ldots, f_k \rangle \neq I$ because I has no zeros. So, I is not finitely generated.

We regard $M = \Gamma^\infty(TL)$ as a finitely generated projective module over the $C^\infty(L)$-ring, which is a module isomorphic to a direct summand in a free module of finite rank. We
The dual of $M$ is finitely generated projective module.

1. The dual module $M^*$ is the right $C^\infty(L)$-module

$$M^* := \text{Hom}_{C^\infty(L)}(M, C^\infty(L))$$ (13)

with $f, \mu : x \rightarrow \mu(x)f$ where $f \in C^\infty(L)$ and $\mu \in M^*$.

2. The bidual or double dual module $M^{**}$ is the left $C^\infty(L)$-module

$$M^{**} := \text{Hom}_{C^\infty(L)}(\text{Hom}_{C^\infty(L)}(M, C^\infty(L)), C^\infty(L)).$$ (14)

There is a well-known canonical homomorphism between $M$ and its bidual $M^{**}$

$$v : M \rightarrow M^{**}$$ (15)

with $v(x)(\mu) := \mu(x)$ for all $\mu \in M^*$.

The dual basis lemma below allows us to define a finitely generated projective module as follows:

**Lemma 2** ([15] (Lemma 2.9)). A left $R$-module $M$ is projective if and only if there exist $\{x_i\}_{i \in I} \subseteq M$ and $\{\mu_i\}_{i \in I} \subseteq M^*$ such that for all $x \in M$

1. $\mu_i(x) = 0$ for all but finitely many $i$.

2. $x = \sum_i \mu_i(x)x_i$.

**Definition 8.** A module $M$ is finitely generated projective if there exist $x_1, \cdots, x_n \in M$ and $\mu_1, \cdots, \mu_n \in M^*$ such that for all $x \in M$ we have $x = \sum_{i=1}^n \mu_i(x)x_i$.

The pair $\{x_i, \mu_i\}$ is called pair of dual bases.

**Proposition 1.** Let $M = \Gamma^\infty(TL)$ be a finitely generated projective $C^\infty(L)$-module. Then,

1. The dual of $M$ is finitely generated projective module.

2. The bidual of $M$ is isomorphic to $M$.

**Proof.** 1. Let $M = \Gamma^\infty(TL)$ be a finitely generated projective module; this means that there exists $n \in \mathbb{N}$ the set of integers $\mathbb{Z}$ and $C^\infty(L)$-module $M'$ such that $M \oplus M' = C^\infty(L)^n$. Since $\text{Hom}$-functor preserves direct summand and product, we have

$$M^* \bigoplus (M')^* = \text{Hom}_{C^\infty(L)}(M, C^\infty(L)) \bigoplus \text{Hom}_{C^\infty(L)}(M', C^\infty(L))$$

$$\cong \text{Hom}_{C^\infty(L)}(M \bigoplus M', C^\infty(L))$$

$$\cong \text{Hom}_{C^\infty(L)}(C^\infty(L)^n, C^\infty(L))$$

$$\cong \text{Hom}_{C^\infty(L)}(C^\infty(L), C^\infty(L))^n \cong C^\infty(L)^n$$ (16)

Since $M^*$ is a direct summand of a free module of finite rank, so it is finitely generated projective.

2. Let $\{x_i, \mu_i\}$ be a pair of dual bases for $M$, where $x_i \in M$ and $\mu_i \in M^*$ for all $i = 1, \cdots, n$. We claim that $\{\mu_i, v_i\}$ form a pair of dual bases for $M^*$. From Definition 17, $x = \sum_{i=1}^n \mu_i(x)x_i$ for all $x \in M$. Applying $\mu \in M^*$ yields

$$\mu(x) = \sum_{i=1}^n \mu_i(x)\mu_i = \sum_{i=1}^n v_i(\mu)\mu_i(x).$$ (17)

Thus, $\mu = \sum_{i=1}^n v_i(\mu)\mu_i$, and we conclude that $M \cong M^{**}$. □
Remark 7. From the above isomorphism between $M$ and its bidual, the module of smooth vector field $M = \Gamma^\infty(TL)$ is called reflexive.

4.3. The Module of Smooth Vector Fields Is Not Semisimple

An $C^\infty(L)$-module is called semisimple, if it is the direct sum of simple $C^\infty(L)$-modules.

Define a localization of a $C^\infty$-ring $C$ at a set $S$ as a $C^\infty$-ring $C\{S\}^{-1}$ and a morphism $\alpha : C \rightarrow C\{S\}^{-1}$ such that $\alpha(s)$ is invertible for all $s \in S$, with the universal property: if $C'$ is another $C^\infty$-ring and $\beta : C \rightarrow C'$ is another morphism such that $\beta(s)$ is invertible for all $s \in S$, then there exists a unique morphism $\phi : C\{S\}^{-1} \rightarrow C'$ such that $\phi \circ \alpha = \beta$. This localization always exists; see, for example, [2] (page 10) and [3] (page 13). A property $P$ of the ring $C^\infty(L)$ (or of $C^\infty(L)$-module $\Gamma^\infty(TL)$) is called a local property if $C^\infty(L)$ (or $\Gamma^\infty(TL)$) has $P$ if and only if $C^\infty(L_p)$ (or $\Gamma^\infty(TL)_p$) has $P$. It is known that if $R$ is a Noetherian ring, then $R\{S\}^{-1}$ is a Noetherian [16] (Theorem 4.IV). In fact, $C^\infty(L)$ is non-Noetherian ring. Thus, we conclude the following lemma.

Theorem 4. Let $M = \Gamma^\infty(TL)$ be the $C^\infty(L)$-module. If $L$ is a manifold of positive dimensional, then $M$ is not a semisimple module.

Proof. Consider $L$ to be a manifold with $\dim L > 0$. If $0 \neq f \in C^\infty(L)$, then the localization is denoted by $\Gamma^\infty(TL)_f$. Since $\Gamma^\infty(TL)$ is a free $C^\infty(L)$-module, $\Gamma^\infty(TL)_f$ is also a free $C^\infty(L)_f$-module with $\text{rank } (\Gamma^\infty(TL)_f) = \dim L$ from the local property. If $M = \Gamma^\infty(TL)$ is semisimple, then so is $\Gamma^\infty(TL)_f$. Thus, $C^\infty(L)_f$ is semisimple. However, a semisimple ring in particular is a Noetherian ring, and $C^\infty(L)$ is not Noetherian. This is a contradiction which implies that $M = \Gamma^\infty(TL)$ is not semisimple.

4.4. Socle and Radical of the Module of Smooth Vector Fields

Let $M = \Gamma^\infty(TL)$ be the $C^\infty(L)$-module. A submodule $N \subseteq M$ is called essential (small), if $N \cap H \neq 0_M (N + H \subseteq M)$ for any nonzero submodule $H$ of $M$ (for any $H \subseteq M$) and is denoted by $N \leq_e M (N \ll M)$. We note that $M \leq_e M$ and $0 \ll M$.

Definition 9. Let $M = \Gamma^\infty(TL)$ be a $C^\infty(L)$-module. Then,

1. The socle of $M$ is the submodule

$$\text{Soc}(M) = \sum\{N \leq M \mid N \text{ is simple}\},$$

(18)

2. The Jacobson radical of $M$ is the submodule

$$J(M) = \cap\{N \leq M \mid N \text{ is maximal}\}.$$  

(19)

Remark 8. 1. The socle of $M$ can be written as

$$\text{Soc}(M) = \cap\{N \leq M \mid N \leq_e M\}.$$  

(20)

2. The Jacobson radical of $M$ can be written as

$$J(M) = \sum\{N \leq M \mid N \ll M\}.  

(21)

It is clear from Definition 9 and Remark 8 that $M$ is semisimple if and only if $\text{soc}(M) = M$. Since $M = \Gamma^\infty(TL)$ is not semisimple, we conclude the following corollary.

Corollary 1. The socle of the $C^\infty(L)$-module $\Gamma^\infty(TL)$ is a proper submodule.

Furthermore, it is known that $M$ is semisimple if and only if $M$ is the only essential submodule of $M$ [17] (Example 5.1.2). Since $M = \Gamma^\infty(TL)$ is not semisimple and $0$ is not an essential submodule for any module, the following lemma is proven.
Lemma 3. The $C^\infty(L)$-module $\Gamma^\infty(TL)$ has a nontrivial proper essential submodule.

Theorem 5. The Jacobson radical of the $C^\infty(L)$-module $\Gamma^\infty(TL)$ is not the zero ideal.

Proof. We have a proposition stating that the Jacobson radical of any semisimple module is zero [17] (Theorem 9.2.1). From that proposition and Theorem 4, it follows that $J(\Gamma^\infty(TL)) \neq 0$. □

The Jacobson radical has many properties; see, for instance, [17]. These are some of them.
1. The Jacobson radical of the nonzero finitely generated module is not equal to that module.
2. The Jacobson radical of the projective module $P$ is $J(P) = PJ(P)$ for any ring $R$, and $P$ is regarded as an $R$-module.
3. If $K \leq M$, then $J(M/K) = (J(M) + K)/K$.

Thus, we decide from above properties the following corollary.

Corollary 2. Let $\Gamma^\infty(TL)$ be the $C^\infty(L)$-module. Then, $J(\Gamma^\infty(TL)) \neq \Gamma^\infty(TL)$ and $J(\Gamma^\infty(TL)) = \Gamma^\infty(TL)/J(\Gamma^\infty(L))$.

Remark 9. From the third property above and Corollary 2, we observe that $J(M J(M)) = J(M) + J(M)/J(M) = 0$. Hence, $M J(M)$ is semisimple such that $M J(M) = M/M(\Gamma^\infty(L))$.

5. Lie Algebra Structure of Smooth Vector Fields

This section is based on the Lie algebra structure of Lee [4] and Loring [6].

5.1. The Lie Bracket and the Lie Algebra

Let $X_1, X_2 \in \Gamma^\infty(TL)$ and $f \in C^\infty(L)$. We see that the operation from $C^\infty(L)$ to $C^\infty(L)$ which is defined by $f \mapsto X_1 X_2 f = X_1 (X_2 f)$ is not a derivation. The following example gives us the reason.

Example 6. Let $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$ be two smooth vector fields on the manifold $\mathbb{R}^2$. Let $f_1(x, y) = x$ and $f_2(x, y) = y$, where $f_1, f_2 \in C^\infty(\mathbb{R}^2)$. The direct computation illustrates that $X_1 X_2 (f_1 f_2) = 1$, while $f_1 (X_1 X_2 f_2) + f_2 (X_1 X_2 f_1) = 0$. Thus, $X_1 X_2$ is not a derivation on $C^\infty(\mathbb{R}^2)$.

Similarly, $f \mapsto X_2 X_1 f = X_2 (X_1 f)$ is not a derivation. Applying both of these operations to the smooth function $f$ and subtracting we obtain a derivation operator

$$D := X_1 X_2 - X_2 X_1.$$ (22)

In fact, vector fields provide all possible derivations of the algebra $C^\infty(L)$. This means that the map $\Gamma^\infty(TL) \rightarrow Der(C^\infty(L))$ is an isomorphism of $C^\infty(L)$-modules, see Theorem 1.

Definition 10. For all $X_1, X_2 \in \Gamma^\infty(TL)$, we call the commutator $[X_1, X_2] = X_1 X_2 - X_2 X_1$ the Lie bracket of $X_1$ and $X_2$.

The Lie bracket satisfies the following properties:

Proposition 2. For all $X, X_1, X_2, X_3 \in \Gamma^\infty(TL)$, the Lie bracket operation satisfies the following:

1. $[X, X] = 0$ (23)
1. Bilinearity: for all \( k \in \mathbb{R} \),
\[
r[X_1, X_2] = [rX_1, X_2] = [X_1, rX_2]
\] (24)

2. Bilinearity: for all \( r_1, r_2 \in \mathbb{R} \),
\[
[r_1X_1 + r_2X_2, X_3] = r_1[X_1, X_3] + r_2[X_2, X_3]
\] (25)
\[
[X_3, r_1X_1 + r_2X_2] = r_1[X_3, X_1] + r_2[X_3, X_2]
\] (26)

3. Antisymmetry:
\[
[X_1, X_2] = -[X_2, X_1]
\] (27)

4. Jacobi identity:
\[
[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.
\] (28)

**Proof.** It is straightforward from the definition and the direct computation. \( \square \)

Thus, the Lie bracket is clearly a bilinear antisymmetry form on \( \Gamma^\infty(TL) \) as \( \mathbb{R} \)-vector space, while over \( C^\infty(L) \)-ring we have the following lemma.

**Lemma 4.** The Lie bracket on the smooth vector fields \( \Gamma^\infty(TL) \) is not linear over \( C^\infty(L) \).

**Proof.** For all \( X_1, X_2 \in \Gamma^\infty(TL) \) and \( f_1, f_2 \in C^\infty(L) \), we have the following properties of the Lie bracket:

1. 
\[
[f_1X_1, X_2] = f_1X_1X_2 - X_2 f_1 X_1
\]
\[
= f_1X_1X_2 - (X_2 f_1) X_1 - f_1 X_2 X_1
\]
\[
= f_1 [X_1, X_2] - (X_2 f_1) X_1
\]

2. 
\[
[X_1, f_2 X_2] = -[f_2 X_2, X_1]
\]
(From antisymmetry property)
\[
= -(f_2 [X_2, X_1] - (X_1 f_2) X_2)
\]
(From 1. above)
\[
= f_2 [X_1, X_2] + (X_1 f_2) X_2
\]

3. 
\[
[f_1X_1, f_2 X_2] = f_1X_1 f_2 X_2 - f_2X_2 f_1 X_1
\]
\[
= f_1 (X_1 f_2) X_2 + f_1 f_2 X_1 X_2 - f_2 (X_2 f_1) X_1 - f_2 f_1 X_2 X_1
\]
\[
= f_1 f_2 [X_1, X_2] + f_1 (X_1 f_2) X_2 - f_2 (X_2 f_1) X_1
\]

We see that \( f_1 [X_1, X_2] = f_1 X_1 X_2 - f_1 X_2 X_1 \neq [f_1 X_1, X_2] \). Hence, \( \Gamma^\infty(TL) \) is not linear over \( C^\infty(L) \). \( \square \)

**Definition 11.** A Lie algebra is a vector space \( V \) over a field \( K \) with a Lie bracket operation:
\[
[\cdot, \cdot]: V \times V \rightarrow V
\] (29)
satisfying the following axioms:

1. Bilinearity: for all \( k_1, k_2 \in K \),
\[
[k_1V_1 + k_2V_2, V_3] = k_1 [V_1, V_3] + k_2 [V_2, V_3]
\] (30)
\[
[V_3, k_1V_1 + k_2V_2] = k_1 [V_3, V_1] + k_2 [V_3, V_2]
\] (31)
2. **Antisymmetry:**

\[ [V_1, V_2] = -[V_2, V_1] \]  \hfill (32)

3. **Jacobi identity:** for all \( V_1, V_2, V_3 \in V \),

\[ [V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]] = 0. \]  \hfill (33)

For example, consider \( \mathbb{R}^3 \) with the product

\[ (x_1, x_2, x_3) \ast (y_1, y_2, y_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \]  \hfill (34)

is a Lie algebra over the real numbers \( \mathbb{R} \).

In fact, for any finite dimensional vector space \( V \) over a field, we can consider all linear maps from \( V \) to \( V \), which is called the general linear algebra. As notation, we write this algebra as \( \text{gl}(V) \). Then, it has the structure of Lie algebra with the Lie bracket \( [x, y] = x \circ y - y \circ x \) for all \( x, y \in \text{gl}(V) \).

In general, any associative algebra \( A \) can be made to be a Lie algebra with the Lie bracket \( [x, y] = x.y - y.x \) for all \( x, y \in A \).

**Proposition 3.** The space of all smooth vector fields \( \Gamma^\infty(TL) \) is a Lie algebra over the field \( \mathbb{R} \).

**Proof.** First, we show that if \( X_1, X_2 \in \Gamma^\infty(TL) \), then the Lie bracket \( [X_1, X_2] \in \Gamma^\infty(TL) \). It suffices to show that \( [X_1, X_2] \) is a derivation of \( C^\infty(L) \) (see Definition 6). For all \( f_1, f_2 \in C^\infty(L) \),

\[
[X_1, X_2](f_1 f_2) = X_1 X_2 (f_1 f_2) - X_2 X_1 (f_1 f_2) \\
= X_1 (X_2 (f_1 f_2)) - X_2 (X_1 (f_1 f_2)) \\
= X_1 (X_2 f_1 f_2 + f_1 X_2 f_2) - X_2 (X_1 f_1 f_2 + f_1 X_1 f_2) \\
= X_1 X_2 f_1 f_2 + X_2 f_1 X_1 f_2 + X_1 f_1 X_2 f_2 + f_1 X_1 X_2 f_2 \\
\quad - X_2 X_1 f_1 f_2 - X_1 f_1 X_2 f_2 - X_2 f_1 X_1 f_2 - f_1 X_2 X_1 f_2 \\
= (X_1 X_2 - X_2 X_1) f_1 f_2 + f_1 (X_1 X_2 - X_2 X_1) f_2 \\
= [X_1, X_2] f_1 f_2 + f_1 [X_1, X_2] f_2.
\]

Now from Proposition 2, the bilinearity, antisymmetry and Jacobi identity are clear for all of \( X_1, X_2, X_3 \in \Gamma^\infty(TL) \). \( \square \)

**Lemma 5.** Let \( \Gamma^\infty(TL) \) be the Lie algebra over the field \( \mathbb{R} \) as in Proposition 3 above such that \( [X_1, X_2] \neq 0 \) for some \( X_1, X_2 \in \Gamma^\infty(TL) \). Then, the smooth vector fields \( X_1 \) and \( X_2 \) are linearly independent over the field \( \mathbb{R} \).

**Proof.** Let \( X_1, X_2 \in \Gamma^\infty(TL) \) such that \( [X_1, X_2] \neq 0 \). Suppose that \( X_1, X_2 \) are linearly dependent over \( \mathbb{R} \). Then, there exist \( r_1, r_2 \in \mathbb{R} \) with \( r_1 \neq 0 \) and \( r_2 \neq 0 \) such that

\[ r_1 X_1 + r_2 X_2 = 0. \]  \hfill (35)

That means

\[ r_1 X_1 = -r_2 X_2. \]  \hfill (36)

Let

\[ Y = [X_1, X_2]. \]  \hfill (37)

Then, from Proposition 2 (Property 2),

\[ r_1 Y = r_1 [X_1, X_2] = [r_1 X_1, X_2], \]  \hfill (38)

\[ -r_2 r_1 Y = -r_2 [r_1 X_1, X_2] = [r_1 X_1, -r_2 X_2]. \]  \hfill (39)
Since \( r_1 \neq 0, r_2 \neq 0 \) and \( Y \neq 0 \), Equation (39) does not equal zero. So,
\[
[r_1 X_1, -r_2 X_2] \neq 0.
\]
(40)

However, we have Equation (36) above. So, from Proposition 2 (Property 1) we have
\[
[r_1 X_1, -r_2 X_2] = 0.
\]
(41)

Thus, we contradict our hypothesis that \( X_1 \) and \( X_2 \) are linearly dependent. We conclude that they are linearly independent. \( \square \)

5.2. The Left Invariant Smooth Vector Fields

Consider \( V \) to be a Lie algebra over the field of real numbers \( \mathbb{R} \). A subspace \( W \subset V \) is called the Lie subalgebra of \( V \) if it is closed under the Lie bracket. That is, for all \( W_1, W_2 \in W \), \([W_1, W_2] \in W \). A subspace \( I \) of a Lie algebra \( V \) is called an ideal of \( V \) if \( x \in V, y \in I \) together imply \([x, y] \in I\); see [18] (page 6).

**Definition 12.** A Lie group \( G \) is a smooth manifold with a smooth group structure; that is, the multiplication map
\[
\mu : G \times G \longrightarrow G; \mu(g_1, g_2) = g_1 g_2
\]
(42)

and the inverse map
\[
i : G \longrightarrow G; i(g) = g^{-1}
\]
(43)
are smooth.

For \( h \in G \), we denote the operation of left (right) multiplication by \( h \triangleright \gamma : G \longrightarrow G; \triangleright h(g) = hg \) (\( \triangleright h : G \longrightarrow G; \triangleright h(g) = gh \)). It is called left (right) translation because \( \triangleright h \) (\( \triangleright \)) can be expressed as the composition of smooth maps. So, \( \triangleright h \) and \( \triangleright \) are smooth.

**Definition 13.** If \( F : L_1 \longrightarrow L_2 \) is a smooth map between manifolds. A vector field \( X_1 \) on \( L_1 \) and a vector field \( X_2 \) on \( L_2 \) are called \( F \)-related if
\[
X_1(f \circ F) = (X_2 f) \circ F \quad \text{for all } f \in C^\infty(L_2).
\]
(44)

If \( G \) is a Lie group, it is known that the tangent bundle \( TG \) of \( G \) is trivial \( TG \cong G \times \mathbb{R}^n \). We denote by \( d\triangleright \gamma : TG \longrightarrow TG \) the map induced by the left translation map \( \triangleright \gamma \).

**Definition 14.** A smooth vector field \( X \) on \( G \) is called left invariant if it is invariant under all left translations \( d\triangleright \gamma (X) = X \) for all \( g \in G \). This means that \( d\triangleright \gamma (X_h) = X_{gh} \) for all \( h \in G \). In other words, a smooth vector field \( X \) is left invariant if it is \( \triangleright \gamma \)-related to itself for all \( g \in G \).

**Proposition 4.** The space of left invariant smooth vector fields on \( G \) is a Lie subalgebra of the Lie algebra \( \Gamma^\infty(TG) \).

**Proof.** First, let \( F : L_1 \longrightarrow L_2 \) be a smooth map between manifolds and \( X_1, X_2 \in \Gamma^\infty(TL_1) \) be \( F \)-related to \( \hat{X}_1, \hat{X}_2 \in \Gamma^\infty(TL_2) \), respectively. Then, for all \( f \in C^\infty(L_2) \),
\[
[X_1, X_2](f \circ F) = X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F) \quad \text{(Definition of \([,\cdot]\))}
\]
\[
= X_1(\hat{X}_2 f) \circ F - X_2(\hat{X}_1 f) \circ F \quad \text{(Definition of vector fields \( F \)-related)}
\]
\[
= (\hat{X}_1 \hat{X}_2 f) \circ F - (\hat{X}_2 \hat{X}_1 f) \circ F
\]
\[
= (\hat{X}_1 X_2 - X_2 \hat{X}_1 f) \circ F
\]
\[
= [\hat{X}_1, \hat{X}_2] f \circ F.
\]

This means that \([X_1, X_2] \) on \( L_1 \) is \( F \)-related to \([\hat{X}_1, \hat{X}_2] \) on \( L_2 \).
Second, if \( X_1 \) is \( \mathcal{G}_g \)-related to itself and \( X_2 \) is \( \mathcal{G}_g \)-related to itself for all \( g \in G \), \( X_1 \) and \( X_2 \) are smooth left invariant vector fields. We conclude that the Lie bracket \([X_1, X_2]\) is \( \mathcal{G}_g \)-related to itself. Hence, \([X_1, X_2]\) are left invariant smooth vector fields. Thus, the space of left invariant smooth vector fields on \( G \) is closed under the Lie bracket. \( \square \)

6. Questions

The subject of this study is still in its infancy, with many topics in modern algebra that need to be studied. For instance, we make mention of some questions as follows.

1. Is \( M = \Gamma^\infty(TL) \) a Noetherian \( C^\infty(L) \)-module?
2. Is \( M \) an Artinian \( C^\infty(L) \)-module?

An \( R \)-module \( M \) is called a uniserial module if all submodules of \( M \) are totally ordered by inclusion.

**Problem 1.** Is \( M = \Gamma^\infty(TL) \) a uniserial \( C^\infty(L) \)-module?

Now, we will provide a brief description of some types of invariant modules over the associative ring \( R \) with identity as follows.

**Definition 15.** An \( R \)-module \( J \) is injective if for all \( R \)-module homomorphism \( f : A \rightarrow J \), and \( R \)-module monomorphism \( g : A \rightarrow B \), there exists \( R \)-module homomorphism \( h : B \rightarrow J \) such that \( h \circ g = f \).

For any submodule \( A \leq J \), and \( R \)-module homomorphism (monomorphism, respectively), there exists \( R \)-module endomorphism \( g : J \rightarrow J \) such that the restriction \( \text{Res}_J^J(g) = f \). This means that \( J \) is \( J \)-injective, which is called a quasi-injective (pseudo-injective, respectively) module.

**Definition 16.** The module \( J \) is called an automorphism-invariant if it satisfies one of the equivalent conditions of the following theorem.

**Theorem 6.** [19] (Theorem 2) The following are equivalent for an \( R \)-module \( J \):

1. \( f(J) \subseteq J \), for all \( f \in \text{Aut}(E(J)) \), where \( E(J) \) is injective hull of \( J \).
2. for all \( J_1, J_2 \leq J \), and for all \( R \)-module isomorphism \( f : J_1 \rightarrow J_2 \), there exists \( R \)-module endomorphism \( g : J \rightarrow J \) such that \( \text{Res}^J_{J_1}(g) = f \)
3. for all \( J_1, J_2 \leq J \), and \( R \)-module isomorphism \( f : J_1 \rightarrow J_2 \), there exists \( R \)-module automorphism \( g : J \rightarrow J \) such that \( \text{Res}^J_{J_1}(g) = f \).

Thus, the \( R \)-module \( J \) is quasi-injective (pseudo-injective, respectively) if it is invariant under all endomorphism (monomorphism, respectively) of injective hulls \( E(J) \).

Consider the following properties of \( R \)-module \( J \):

C1 Every submodule \( J' \) of \( J \) is essential in a direct summand of \( J \).
C2 If every submodule \( J' \) of \( J \) is isomorphic to a direct summand of \( J \), then \( J' \) is a direct summand of \( J \).
C3 If \( J_1, J_2 \) are direct summands of \( J \) such that \( J_1 \cap J_2 = 0 \), then \( J_1 \oplus J_2 \) is also a direct summand of \( J \).

An \( R \)-module \( J \) is called extending module if it satisfies C1. This module is called continuous (quasi-continuous, respectively) if it satisfies C1 and C2 (C1 and C3, respectively) properties.

The automorphism-extendable module, defined in the following definition, was studied by Tuganbaev in [20].
Definition 17. The R-module J is called an automorphism-extendable if all submodules J' of J and every automorphism of J' extend to an endomorphism of J.

Definition 17 means that for all J' \subseteq J, and R-module automorphism f : J' \rightarrow J', there exists R-module endomorphism g : J \rightarrow J such that g(j') = f(j') for all j' \in J'.

The automorphism-liftable module is a dual notion of automorphism-extendable module that is defined as the following.

Definition 18. The R-module J is called automorphism-liftable if all submodules J' of J and every automorphism of the quotient module J/J' lift to an endomorphism of J.

Definition 18 means that for all J' \subseteq J, R-module automorphism f : J/J' \rightarrow J/J', and R-module epimorphism h : J \rightarrow J/J', there exists R-module endomorphism g : J \rightarrow J such that f \circ h = h \circ g.

In addition, the question mentioned below is crucial for the study of invariant modules structure:

Problem 2. Is M = Γ∞(TL) an injective C∞(L)-module?

Therefore, we can also ask whether this module has an invariant module structure, such as quasi-injective, pseudo-injective, automorphism-invariant, automorphism-extendable, automorphism-liftable, continuous, quasi-continuous and extending modules. As far as we know, these questions are still unanswered.

Recall that a finite dimensional algebra A is called symmetric if there is an (A, A)-bimodule isomorphism A \cong A^*, where A^* is dual of A.

Problem 3. Is C∞(L) symmetric algebra over the real numbers \mathbb{R}?

Remark 10. Studying the symmetry of the smooth ring C∞(L) is significant because there is a fact that states that over symmetric algebra, the concept of projective modules is equivalent to the concept of injective modules; see, for instance, [21].

7. Conclusions

In this work, we have studied the ring of smooth functions, the C∞-ring. We regarded its structure as a commutative \mathbb{R}-algebra. The module structure over this ring was also investigated. In particular, we considered the smooth vector fields as a module over the C∞-ring. Furthermore, we have demonstrated the structures for this module as finitely generated projective modules. However, it is not semisimple. So, the Jacobson radical is not trivial. Moreover, we have investigated the Lie algebra structure of smooth vector fields. For future plans, we shall study the concept of finite group actions on surfaces; see [22]. Furthermore, our goal is to study C∞(L)-modules of automorphism-invariant, automorphism-extendable and automorphism-liftable modules in the future; see the references [23–26].

Author Contributions: Conceptualization, A.M.A.; data curation, A.A.A.; formal analysis, A.A.A.; investigation, A.M.A.; methodology, A.M.A.; supervision, A.M.A.; validation, A.M.A.; visualization, A.A.A.; writing—original draft preparation, A.A.A.; writing—review and editing. All authors discussed the results and contributed to the final manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data are included in this paper.
Acknowledgments: This work is a part of the first author’s PhD study at Umm Al-Qura University. The first author wishes to express her gratitude to Ahmad M. Alghamdi, her esteemed supervisor, for their invaluable advice, continuous support and patience during her study.

Conflicts of Interest: The authors declare no conflicts of interest.

References
14. Turki, N. A note on incompressible vector fields. Symmetry 2023, 15, 1479. [CrossRef]
16. Wright, L. Rings of Quotients and Localization. Master’s Thesis, Graduate School of the Texas Woman’s University, Denton, TX, USA, 1974.
19. Lee, T.; Zhou, Y. Modules which are invariant under automorphisms of their injective hulls. Algebra Appl. J. 2013, 12, 1250159. [CrossRef]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.