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Applications of Horadam Polynomials for Bazilevič and λ-Pseudo-Starlike Bi-Univalent Functions Associated with Sakaguchi Type Functions

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Abstract: In this study, we introduce a new class of normalized analytic and bi-univalent functions denoted by $\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r)$. These functions are connected to the Bazilevič functions and the λ-pseudo-starlike functions. We employ Sakaguchi Type Functions and Horadam polynomials in our survey. We establish the Fekete-Szegö inequality for the functions in $\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r)$ and derive upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Additionally, we establish connections between our results and previous research papers on this topic.

Keywords: analytic functions; bi-univalent functions; Bazilevič functions; λ-pseudo-starlike functions; Sakaguchi Type Functions; Horadam polynomials; coefficient estimates; Fekete-Szegö problem

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1. Introduction and Preliminaries

In this research, we use the well-known notation $\mathcal{A}$ to represent analytic functions on the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. The analytic development for normalized type is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

and we define $\mathcal{S}$ the subfamily of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$.

We introduce the concept of Bazilevič functions in $\mathbb{U}$ (see [1]), characterized by the condition $\Re\left(\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}}\right) > 0$, and λ-pseudo-starlike functions in $\mathbb{U}$, defined by $\Re\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right) > 0$ (see [2]), also studies on λ-pseudo bi-univalent functions can be found [3,4].

Every function $f \in \mathcal{S}$ possesses an inverse f^{-1}, as is well known demonstrated by the Koebe one-quarter theorem, given by $f^{-1}\left(f(z)\right) = z$ and $f\left(f^{-1}(w)\right) = w$ for $|w| < r_0(f)$, where $r_0(f) \geqq \frac{1}{4}$. The inverse function is expressed as

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Denoted by Σ, the family of bi-univalent functions in $\mathbb{U}$ is given by (1). The research work by Srivastava et al. [5] is a crucial reference on this topic, presenting interesting examples of functions in the family Σ. Recently, there has been a renewed interest in the study of bi-univalent functions, as evident from various bi-univalent function surveys.

Recently, in mathematical literature, for functions in the class of univalent and bi-univalent functions associated with certain polynomials such as the Horadam polynomial, the coefficient estimates are found. Motivated in these sense, estimates on initial coefficients of the Taylor-Maclaurin series expansion and Fekete-Szegö inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials. The Horadam polynomials which are known to include, as special cases, many potentially useful polynomials as, for example, the Lucas polynomials, the Pell-Lucas polynomials, the Fibonacci polynomials, the Pell polynomials, and the Chebyshev polynomials of the second kind.Therefore, we provide through the entire paper relevant connections of our results with those considered in earlier investigations.

Because of its characteristics and uses in a variety of fields, including mathematical physics, engineering, and image processing, bi-univalent functions have been thoroughly researched in complex analysis and geometric function theory. The Bieberbach conjecture, which asserts that the Taylor coefficients of a bi-univalent function in the open unit disk fulfill specific inequalities, is one of the key findings pertaining to bi-univalent functions. That Louis de Branges confirmed the conjecture in 1985, and it has numerous significant ramifications for complex analysis. The Koebe function, the Bessel function, and several of their subclasses, such as starlike and convex functions, are instances of bi-univalent functions.

In this regard, let us recall some examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

It's important to note that the class Σ is not empty, although the Koebe function does not belong to Σ.

Furthermore, many authors have introduced and studied numerous subclasses of the bi-univalent function family Σ, analogously to the work by Srivastava et al. [5]. However, several recent papers have only provided non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin expansion (1) (see, for example, [6–13]). The general coefficient bounds $|a_n|$ for $n \in \mathbb{N}$ with $n \geqq 3$ for functions $f \in \Sigma$ have not been fully addressed for many subfamilies of Σ (see for example [14]).

Another well-known problem in the field of Geometric Function Theory is the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$. Its origin traces back to the disproof by Fekete and Szegö [15] of the Littlewood-Paley conjecture, which stated that the coefficients of odd univalent functions are bounded by unity. This functional has garnered significant attention, especially in the study of various subfamilies of univalent functions. Geometric function theory researchers are now actively exploring this subject [16–23].

Regarding the principle of subordination between analytic functions, suppose we have analytic functions f and g in $\mathbb{U}$. We say that the function f is subordinate to g if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ and satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. In such a case, we have $f(z) = g(\omega(z))$. This relationship is denoted as $f \prec g$ or equivalently, $f(z) \prec g(z)$ for $z \in \mathbb{U}$.

It is worth noting that if the function g is univalent in $\mathbb{U}$ [24], then the subordination condition $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. There are many subclasses of bi-univalent functions defined by the Horadam polynomials. In a recent publication, Hörçum and Koçer [25] conducted a study on the Horadam polynomials $h_n(r)$, defined by the recurrence relation as follows (also discussed in Horadam and Mahon [26]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R};\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{3}$$

where the initial conditions are given by $h_1(r) = a$ and $h_2(r) = br$, and a, b, p, and q are real constants. The recurrence relation leads to a characteristic equation:

$$t^2 - prt - q = 0.$$

Solving this quadratic equation yields two real roots:

$$\alpha = \frac{pr + \sqrt{p^2r^2 + 4q}}{2} \quad \text{and} \quad \beta = \frac{pr - \sqrt{p^2r^2 + 4q}}{2}.$$

Remark 1. *By choosing specific values for the parameters a, b, p and q, the Horadam polynomial $h_n(r)$ is reduced to certain known polynomials. Several special cases are recorded below:*

1. *Considering $a = b = p = q = 1$, we derive the Fibonacci polynomials $F_n(r)$.*
2. *Considering $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(r)$.*
3. *Considering $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(r)$.*
4. *Considering $a = b = p = 2$ and $q = 1$, we have the Pell-Lucas polynomials $Q_n(r)$.*
5. *Considering $a = b = 1$, $p = 2$ and $q = -1$, we find the Chebyshev polynomials $T_n(r)$ of the first kind.*
6. *Considering $a = 1$, $b = p = 2$ and $q = -1$, we deduce the Chebyshev polynomials $U_n(r)$ of the second kind.*

The aforementioned polynomials, including orthogonal polynomial families and other special polynomials, along with their extensions and generalizations, hold significant importance in various fields across numerous branches of science, particularly in statistical, mathematical, and physical sciences.

For further details regarding these polynomials, readers may refer to [26–28]. The generating function of the Horadam polynomials $h_n(r)$ is expressed as follows (see [25]):

$$\Pi(r,z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{4}$$

In a similar context, Srivastava et al. [29] explored analytic and bi-univalent functions in connection with the Horadam polynomials. This research was followed by further investigations conducted by Al-Amoush [30], Wanas and Alina [31], Abirami et al. [32], and other researchers (see, for example, [17,33–35]).

In our present investigation, we introduce and study a new class of normalized analytic and bi-univalent functions in the open unit disk by applying the Horadam polynomials. We establish forward the bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to the new introduced class.

2. Main Results

In this section, we begin by introducing a novel family of analytic functions denoted as $\mathcal{D}_\Sigma(\delta, \eta, \lambda, t, r)$. We derive estimates on the initial Taylor-Maclaurin coefficients and solve the Fekete-Szegö type inequalities for functions in this family.

Definition 1. *Let $0 \leqq \delta \leqq 1$, $\eta \geqq 0$, $\lambda \geqq 1$, $t \in \mathbb{C}$, $|t| \leq 1$ and $r \in \mathbb{R}$. A function $f \in \Sigma$ is said to be in the family $\mathcal{D}_\Sigma(\delta, \eta, \lambda, t, r)$ if it satisfies the following subordination:*

$$(1-\delta)\frac{((1-t)z)^{1-\eta} f'(z)}{(f(z) - f(tz))^{1-\eta}} + \delta \frac{(1-t)z(f'(z))^\lambda}{f(z) - f(tz)} \prec \Pi(r,z) + 1 - a$$

and

$$(1-\delta)\frac{((1-t)w)^{1-\eta} g'(w)}{(g(w) - g(tw))^{1-\eta}} + \delta \frac{(1-t)w(g'(w))^\lambda}{g(w) - g(tw)} \prec \Pi(r,w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (2).

Remark 2. *The class $\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r)$ of bi-univalent functions represents a generalization of several well-known families that have been extensively studied in previous research. Some of these families are being mentioned below for reference.*

1. *Considering $\delta = t = 0$, we get*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \mathcal{N}_\Sigma(\eta,r),$$
where $\mathcal{N}_\Sigma(\eta,r)$ is the bi-univalent function family studied recently by Wanas and Alb Lupas [31].
2. *Taking $\delta = \eta = t = 0$, we reobtain*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: S^*_\Sigma(r),$$
where $S^*_\Sigma(r)$ denote the bi-univalent function family studied by Srivastava et al. [29].
3. *For the values $\delta = t = 0$ and $\eta = 1$, we have*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \Sigma'(r),$$
where $\Sigma'(r)$ is the bi-univalent function family introduced by Alamoush [30].
4. *For conditions $\delta = t = 0$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow x$, we find*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \mathcal{B}^\eta_\Sigma(x),$$
where $\mathcal{B}^\eta_\Sigma(x)$ is the bi-univalent function family introduced by Bulut et al. [36].
5. *For $t = 0$, $\delta = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow x$, we have*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \mathcal{LB}_\Sigma(\lambda,x),$$
where $\mathcal{LB}_\Sigma(\lambda,x)$ is the bi-univalent function family investigated by Magesh and Bulut [37].
6. *For $\delta = t = \eta = 0$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow x$, we deduce*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: S_\Sigma(x),$$
where $S_\Sigma(x)$ is the bi-univalent function family given by Altınkaya and Yalçin [38].
7. *For condition $\delta = t = 0$, $\eta = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow x$, we derived*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \mathcal{B}_\Sigma(x),$$
where $\mathcal{B}_\Sigma(x)$ is the bi-univalent function family given by Bulut et al. [36].
8. *Second item;*
9. *Considering $\delta = t = 0$, $a = 1$, $b = p = 2$, $q = -1$, $r \longrightarrow x$ and $\Pi(x,z) = \left(\frac{1}{1-2xz+z^2}\right)^\alpha$, $0 < \alpha \le 1$, we have*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: P_\Sigma(\alpha,\eta),$$
where $P_\Sigma(\alpha,\eta)$ is the bi-univalent function family considered by Prema and Keerthi [39].
10. *Taking $t = 0$, $\delta = a = 1$, $b = p = 2$, $q = -1$, $r \longrightarrow x$ and $\Pi(x,z) = \left(\frac{1}{1-2xz+z^2}\right)^\alpha$, $0 < \alpha \le 1$, we have*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: \mathfrak{L}B^\lambda_\Sigma(\alpha),$$
where $\mathfrak{L}B^\lambda_\Sigma(\alpha)$ is the bi-univalent function family considered by Joshi et al. [40].
11. *For $\delta = \eta = t = 0$, $a = 1$, $b = p = 2$, $q = -1$, $r \longrightarrow x$ and $\Pi(x,z) = \left(\frac{1}{1-2xz+z^2}\right)^\alpha$, $0 < \alpha \le 1$, we have*
$$\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r) =: S^*_\Sigma(\alpha),$$
where $S^*_\Sigma(\alpha)$ is the bi-univalent function family introduced by Brannan and Taha [41].

12. *Considering* $\delta = t = 0$, $\eta = a = 1$, $b = p = 2$, $q = -1$, $r \longrightarrow x$ *and* $\Pi(x,z) = \left(\frac{1}{1-2xz+z^2}\right)^{\alpha}$, $0 < \alpha \leq 1$, *we have*

$$\mathcal{D}_{\Sigma}(\delta,\eta,\lambda,t,r) =: \mathcal{H}_{\Sigma}^{\alpha},$$

where $\mathcal{H}_{\Sigma}^{\alpha}$ *is the bi-univalent function family considered by Srivastava et al. [5].*

Theorem 1. *Let* $0 \leqq \delta \leqq 1$, $\eta \geqq 0$, $\lambda \geqq 1$ *and* $r \in \mathbb{R}$. *If* $f \in \mathcal{A}$ *is in the family* $\mathcal{D}_{\Sigma}(\delta,\eta,\lambda,t,r)$, *then*

$$|a_2| \leqq \frac{|br|\sqrt{|br|}}{\sqrt{|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b - p\Delta(\delta,\eta,\lambda,t)]br^2 - qa\Delta(\delta,\eta,\lambda,t)|}}$$

and

$$|a_3| \leqq \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)} + \frac{b^2r^2}{[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]^2},$$

where

$$\Phi(\delta,\eta,\lambda,t) = (1-\delta)\Big(3-(1-\eta)\Big(t^2+t+1\Big)\Big)+\delta\Big(3\lambda-t^2-t-1\Big), \tag{5}$$

$$\Psi(\delta,\eta,\lambda,t) = (1-\delta)(1-\eta)(t+1)\left(\frac{1}{2}(2-\eta)(t+1)-2\right)+\delta\Big((t+1)^2-2\lambda(t-\lambda+2)\Big) \tag{6}$$

and

$$\Delta(\delta,\eta,\lambda,t) = [(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]^2. \tag{7}$$

Proof. Considering $f \in \mathcal{D}_{\Sigma}(\delta,\eta,\lambda,t,r)$. Then there are two analytic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{8}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{9}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{U}),$$

such that

$$(1-\delta)\frac{((1-t)z)^{1-\eta} f'(z)}{(f(z)-f(tz))^{1-\eta}} + \delta\,\frac{(1-t)z(f'(z))^{\lambda}}{f(z)-f(tz)} = \Pi(r,u(z)) - a$$

and

$$(1-\delta)\frac{((1-t)w)^{1-\eta} g'(w)}{(g(w)-g(tw))^{1-\eta}} + \delta\,\frac{(1-t)w(g'(w))^{\lambda}}{g(w)-g(tw)} = \Pi(r,v(w)) - a.$$

or, equivalently, that

$$(1-\delta)\frac{((1-t)z)^{1-\eta} f'(z)}{(f(z)-f(tz))^{1-\eta}} + \delta\,\frac{(1-t)z(f'(z))^{\lambda}}{f(z)-f(tz)} = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \tag{10}$$

and

$$(1-\delta)\frac{((1-t)w)^{1-\eta} g'(w)}{(g(w)-g(tw))^{1-\eta}} + \delta\,\frac{(1-t)w(g'(w))^{\lambda}}{g(w)-g(tw)} = 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots. \tag{11}$$

Combining (8), (9), (10) and (9), we deduce that

$$(1-\delta)\frac{((1-t)z)^{1-\eta}f'(z)}{(f(z)-f(tz))^{1-\eta}}+\delta\,\frac{(1-t)z(f'(z))^{\lambda}}{f(z)-f(tz)}=1+h_2(r)u_1z+\Big[h_2(r)u_2+h_3(r)u_1^2\Big]z^2+\cdots \tag{12}$$

and

$$(1-\delta)\frac{((1-t)w)^{1-\eta}g'(w)}{(g(w)-g(tw))^{1-\eta}}+\delta\,\frac{(1-t)w(g'(w))^{\lambda}}{g(w)-g(tw)}=1+h_2(r)v_1w+\Big[h_2(r)v_2+h_3(r)v_1^2\Big]w^2+\cdots. \tag{13}$$

If

$$\max\{|u(z)|,|v(w)|\}<1 \qquad (z,w\in\mathbb{U}),$$

it is well-known that,

$$\left|u_j\right|\leqq 1 \qquad \text{and} \qquad \left|v_j\right|\leqq 1 \qquad (\forall\, j\in\mathbb{N}). \tag{14}$$

Further, by comparing the corresponding coefficients in (12) and (13), and after some simplification, we have

$$[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]\,a_2=h_2(r)u_1, \tag{15}$$

$$\begin{aligned}&\Big[(1-\delta)\Big(3-(1-\eta)\Big(t^2+t+1\Big)\Big)+\delta\Big(3\lambda-t^2-t-1\Big)\Big]\,a_3\\&+\Big[(1-\delta)(1-\eta)(t+1)\Big(\frac{1}{2}(2-\eta)(t+1)-2\Big)+\delta\Big((t+1)^2-2\lambda(t-\lambda+2)\Big)\Big]\,a_2^2\\&=h_2(r)u_2+h_3(r)u_1^2,\end{aligned} \tag{16}$$

$$-[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]\,a_2=h_2(r)v_1 \tag{17}$$

and

$$\begin{aligned}&\Big[(1-\delta)\Big(3-(1-\eta)\Big(t^2+t+1\Big)\Big)+\delta\Big(3\lambda-t^2-t-1\Big)\Big]\,\Big(2a_2^2-a_3\Big)\\&+\Big[(1-\delta)(1-\eta)(t+1)\Big(\frac{1}{2}(2-\eta)(t+1)-2\Big)+\delta\Big((t+1)^2-2\lambda(t-\lambda+2)\Big)\Big]\,a_2^2\\&=h_2(r)v_2+h_3(r)v_1^2.\end{aligned} \tag{18}$$

We deduce from (15) and (17) that

$$u_1=-v_1 \tag{19}$$

and

$$2[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]^2\,a_2^2=h_2^2(r)(u_1^2+v_1^2). \tag{20}$$

If we add (16) to (18), we get that

$$\begin{aligned}&2\Big\{\Big[(1-\delta)\Big(3-(1-\eta)\Big(t^2+t+1\Big)\Big)+\delta\Big(3\lambda-t^2-t-1\Big)\Big]\\&+\Big[(1-\delta)(1-\eta)(t+1)\Big(\frac{1}{2}(2-\eta)(t+1)-2\Big)+\delta\Big((t+1)^2-2\lambda(t-\lambda+2)\Big)\Big]\Big\}\,a_2^2\\&=h_2(r)(u_2+v_2)+h_3(r)(u_1^2+v_1^2).\end{aligned} \tag{21}$$

Now, substituting the value of $u_1^2+v_1^2$ from (20) into the right-hand side of (21), we obtain that

$$a_2^2=\frac{h_2^3(r)(u_2+v_2)}{2\big[h_2^2(r)(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))-h_3(r)\Delta(\delta,\eta,\lambda,t)\big]}, \tag{22}$$

where $\Phi(\delta,\eta,\lambda,t)$, $\Psi(\delta,\eta,\lambda,t)$ and $\Delta(\delta,\eta,\lambda,t)$ are given by (5), (6) and (7), respectively. Using (3), (14) and (22), by further computations, we have

$$|a_2| \leqq \frac{|br|\sqrt{|br|}}{\sqrt{|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b - p\Delta(\delta,\eta,\lambda,t)]br^2 - qa\Delta(\delta,\eta,\lambda,t)|}}.$$

Next, if we subtract (18) from (16), we can easily see that

$$2\Big[(1-\delta)\Big(3-(1-\eta)\Big(t^2+t+1\Big)\Big)+\delta\Big(3\lambda-t^2-t-1\Big)\Big]\ \Big(a_3-a_2^2\Big) = h_2(r)(u_2-v_2)+h_3(r)(u_1^2-v_1^2). \quad (23)$$

We deduce from (23), in view of (19) and (20), that

$$a_3 = \frac{h_2(r)(u_2-v_2)}{2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]} + \frac{h_2^2(r)(u_1^2+v_1^2)}{2[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]^2}.$$

By applying (3), we get

$$|a_3| \leqq \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)} + \frac{b^2r^2}{[(1-\delta)(2-(1-\eta)(t+1))+\delta(2\lambda-t-1)]^2}.$$

This completes the proof of Theorem 1. □

In the subsequent theorem, we establish the Fekete-Szegö inequality for the class $\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r)$.

Theorem 2. *Consider* $0 \leqq \delta \leqq 1$, $\eta \geqq 0$, $\lambda \geqq 1$ *and* $r,\mu \in \mathbb{R}$. *If* $f \in \mathcal{A}$ *is in the class* $\mathcal{D}_\Sigma(\delta,\eta,\lambda,t,r)$, *then we obtain*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)} \\ \left(|\varphi-1| \leqq \frac{|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)|}{b^2r^2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]}\right) \\ \frac{|br|^3|\mu-1|}{|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)|} \\ \left(|\varphi-1| \geqq \frac{|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)|}{b^2r^2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]}\right). \end{cases}$$

Proof. In view of (22) and (23) we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{h_2(r)(u_2-v_2)}{2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]}+(1-\mu)a_2^2 \\ &= \frac{h_2(r)(u_2-v_2)}{2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]} \\ &+ \frac{h_2^3(r)(u_2+v_2)(1-\mu)}{2\big[h_2^2(r)(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))-h_3(r)\Delta(\delta,\eta,\lambda,t)\big]} \\ &= \frac{h_2(r)}{2}\Bigg[\left(\varphi(\mu,r)+\frac{1}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)}\right)u_2 \\ &+\left(\varphi(\mu,r)-\frac{1}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)}\right)v_2\Bigg], \end{aligned}$$

where

$$\varphi(\mu,r)=\frac{h_2^2(r)(1-\mu)}{h_2^2(r)(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))-h_3(r)\Delta(\delta,\eta,\lambda,t)}.$$

It follows from (3) that

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)} \\ \left(0\leqq|\varphi(\mu,r)|\leqq\frac{1}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)}\right) \\ |br|.|\varphi(\mu,r)| \\ \left(|\varphi(\mu,r)|\geqq\frac{1}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)}\right), \end{cases}$$

which, after simple computation, yields

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)} \\ \left(|\varphi-1|\leqq\frac{\left|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)\right|}{b^2r^2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]}\right) \\ \frac{|br|^3|\mu-1|}{\left|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)\right|} \\ \left(|\varphi-1|\geqq\frac{\left|[(\Phi(\delta,\eta,\lambda,t)+\Psi(\delta,\eta,\lambda,t))b-p\Delta(\delta,\eta,\lambda,t)]br^2-qa\Delta(\delta,\eta,\lambda,t)\right|}{b^2r^2[(1-\delta)(3-(1-\eta)(t^2+t+1))+\delta(3\lambda-t^2-t-1)]}\right). \end{cases}$$

These complete the proof of Theorem 2. □

3. Special Cases and Consequences

In this section, we explore specific cases and implications of our main results, namely Theorems 1 and 2. Certain special consequences regarding our new statements are established. We deal here with the initial Taylor-Maclaurin coefficient inequalities and the Fekete-Szegö inequalities.

We choose to select below an example in which, by setting $\mu=1$ in Theorem 2, we are led to the following corollary.

Corollary 1. *Let $0 \leqq \delta \leqq 1$, $\eta \geqq 0$, $\lambda \geqq 1$ and $r \in \mathbb{R}$. If $f \in \mathcal{A}$ is in the class $\mathcal{D}_{\Sigma}(\delta, \eta, \lambda, t, r)$, then the following inequality holds.*

$$\left|a_3 - a_2^2\right| \leqq \frac{|br|}{(1-\delta)(3-(1-\eta)(t^2+t+1)) + \delta(3\lambda - t^2 - t - 1)}.$$

Remark 3. *Choosing certain values for the parameters t, δ, η, a, b, p and q, in our main results (Theorems 1 and 2), one can be deduced several known results. Several special cases are recorded below.*

1. *If we let $\delta = t = 0$ in our Theorems, we have the results for well-known class $\mathcal{N}_{\Sigma}(\eta, r)$ of bi-Bazilevič functions which was recently investigated by Wanas and Lupas [31].*
2. *Considering $\delta = t = \eta = 0$ in our Theorems, we obtain the results for the class $S^*_{\Sigma}(r)$ which was obtained recently by Srivastava et al. [29].*
3. *Taking $\delta = t = 0$ and $\eta = 1$ in our Theorems, we get the results for the well-known class $\Sigma'(r)$ which was studied recently by Alamoush [30].*
4. *If we let $\delta = t = 0$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, one can obtains the results for the class $\mathcal{B}^{\eta}_{\Sigma}(t)$ of bi-Bazilevič functions which was discussed recently by Bulut et al. [36].*
5. *Considering $t = 0$, $\delta = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, we derive the results for the family $\mathcal{LB}_{\Sigma}(\lambda, t)$ of bi-pseudo-starlike functions which was recently investigated by Magesh and Bulut [37].*
6. *If we put $\delta = t = \eta = 0$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, we obtain the results for the family $S_{\Sigma}(t)$ of bi-starlike functions which was considered recently by Altınkaya and Yalçin [38].*
7. *If we let $\delta = t = 0$, $\eta = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, we find the results for the class $\mathcal{B}_{\Sigma}(t)$ which was discussed recently by Bulut et al. [36].*

4. Conclusions

In this study, our primary objective was to define a novel class of bi-univalent functions denoted as $\mathcal{D}_{\Sigma}(\delta, \eta, \lambda, t, r)$. This class was established through the utilization of Bazilevič functions and λ-pseudo-starlike functions. To investigate the properties of this class, we employed Sakaguchi Type Functions and the Horadam polynomial $h_n(r)$, which is governed by the recurrence relation (3), as well as the generating function $\Pi(r, z)$ given in (4).

Throughout our research, we obtained upper bounds for the first two coefficients of the power series for functions belonging to the $\mathcal{D}_{\Sigma}(\delta, \eta, \lambda, t, r)$ family. Additionally, we addressed the well-known Fekete-Szegö problem and established interesting connections between our findings and previous results from related studies.

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