Article

On Circular q-Rung Orthopair Fuzzy Sets with Dombi Aggregation Operators and Application to Symmetry Analysis in Artificial Intelligence

Zeeshan Ali 1 and Miin-Shen Yang 2,*

1 Department of Mathematics and Statistics, Riphah International University, Islamabad 44000, Pakistan; zeeshan.ali@riphah.edu.pk
2 Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taoyuan 32023, Taiwan
* Correspondence: msyang@cycu.edu.tw

Abstract: Circular q-rung orthopair fuzzy sets (FRSs) were recently considered as an extension of q-rung orthopair sets (q-ROFSs), circular intuitionistic FSSs (Cir-IFSs), and circular Pythagorean sets (Cir-PFSs). However, they are only considered for some simple algebraic properties. In this paper, we advance the work on circular q-ROFSs (Cirq-ROFSs) in Dombi aggregation operators (AOs) with more mathematical properties of algebraic laws. These include the circular q-rung orthopair fuzzy (Cirq-ROF) Dombi weighted averaging (Cirq-ROFDWA), Cirq-ROF Dombi ordered weighted averaging (Cirq-ROFDOWA), Cirq-ROF Dombi weighted geometric (Cirq-ROFDWG), and Cirq-ROF Dombi ordered weighted geometric (Cirq-ROFDOWG) operators. Additionally, we present the properties of idempotency, monotonicity, and boundedness for the proposed operators. In the context of artificial intelligence, symmetry analysis plays a significant and efficient role that can refer to several aspects. Thus, to compute the major aspect, we identify the multi-attribute decision-making (MADM) techniques based on the proposed operators for Cirq-ROF numbers (Cirq-ROFNs) to enhance the worth of the evaluated operators. Finally, we use some existing techniques for comparison to our results to show the validity and supremacy of the proposed method.

Keywords: fuzzy sets; circular q-rung orthopair fuzzy sets; Dombi averaging/geometric aggregation operators; symmetry analysis in artificial intelligence; multi-attribute decision-making

1. Introduction

Symmetry analysis in artificial intelligence [1] is a multi-dimensional technique that spans algorithm design, knowledge representation, data manipulation, and fairness selections [2]. Utilizing symmetry analysis methods can improve the capability, fairness, supremacy, and interpretability of artificial intelligence across many domains. Furthermore, these concepts have been utilized in many fields—for instance, machine learning, game theory, artificial intelligence, neural networks, data mining, and decision-making—under the consideration of the classical set theory [3]. Unfortunately, these procedures have lost a lot of information because of limited options with only zero and one. Therefore, a stronger idea is required to maintain more information. Zadeh [4] initiated a fuzzy set (FS) where the truth grade $\mathcal{M}_{q}(\mathcal{x})$ of an FS becomes $\mathcal{M}_{q}(\mathcal{x}) \in [0,1]$, and this FS theory has been applied in various fields [5–7]. Furthermore, Atanassov [8] modified the FS theory by adding another grade of falsity $\mathcal{N}_{q}(\mathcal{x})$ and initiated the idea of intuitionistic FS (IFS) with the condition that $0 \leq \mathcal{M}_{q} + \mathcal{N}_{q} \leq 1$. IFS has since been studied in different works [9–11]. However, in the presence of the pair $(0.6,0.5)$, the idea of IFS has not worked prominently. Therefore, Yager and Abbasov [12] presented the Pythagorean FS (PFS) with the new condition that $0 \leq \mathcal{M}_{q}^{2} + \mathcal{N}_{q}^{2} \leq 1$, resulting in the
PFS having different applications [13–15]. However, the PFS still fails in the presence of the pair (0.9,0.8) due to its limited features. Yager [16] then presented the idea of q-rung orthopair FSs (q-ROFs) with a prominent and dominant condition of \( 0 \leq \mathcal{M}_{q_{cc}}^{asc} + \mathcal{N}_{q_{cc}}^{asc} \leq 1, \; q_{cc} \geq 1 \). These q-ROFs have been applied in many areas [17–19]. Some specifications of q-ROFs are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Methods</th>
<th>Conditions</th>
<th>Mathematical Forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_{sc} = 1, N_{qcc} = 0 )</td>
<td>Fuzzy sets (FSs)</td>
<td>( 0 \leq M_{qcc} \leq 1 )</td>
<td>( \left( \mathcal{M}<em>{q</em>{cc}}(\mathcal{x}), \mathcal{N}<em>{q</em>{cc}}(\mathcal{x}) \right) : \mathcal{x} \in \mathcal{X}_{UNI} )</td>
</tr>
<tr>
<td>( q_{sc} = 1 )</td>
<td>Intuitionistic FSs</td>
<td>( 0 \leq M_{qcc} + N_{qcc} \leq 1 )</td>
<td>( \left( \mathcal{M}<em>{q</em>{cc}}(\mathcal{x}), \mathcal{N}<em>{q</em>{cc}}(\mathcal{x}) \right) : \mathcal{x} \in \mathcal{X}_{UNI} )</td>
</tr>
<tr>
<td>( q_{sc} = 2 )</td>
<td>Pythagorean FSs</td>
<td>( 0 \leq M_{qcc}^{2} + N_{qcc}^{2} \leq 1 )</td>
<td>( \left( \mathcal{M}<em>{q</em>{cc}}(\mathcal{x}), \mathcal{N}<em>{q</em>{cc}}(\mathcal{x}) \right) : \mathcal{x} \in \mathcal{X}_{UNI} )</td>
</tr>
<tr>
<td>( q_{sc} \geq 1 )</td>
<td>q-Rung orthopair FSs</td>
<td>( 0 \leq M_{qcc}^{asc} + N_{qcc}^{asc} \leq 1 )</td>
<td>( \left( \mathcal{M}<em>{q</em>{cc}}(\mathcal{x}), \mathcal{N}<em>{q</em>{cc}}(\mathcal{x}) \right) : \mathcal{x} \in \mathcal{X}_{UNI} )</td>
</tr>
</tbody>
</table>

Table 1 contains the mathematical forms of FSs, IFSs, PFSs, and q-ROFs. The IFS theory and its applications have been widely used in many areas because the computed structure of an FS is very reliable and dominant due to its features. However, finding the radius between the truth and the falsity grade is very complicated because no one can do it yet. After a long assessment, we noticed that the idea of circular IFS (Cir-IFS) was presented by Atanassov [20], where the truth grade, the falsity grade, and the radius of a circle around the point of both grades are also the major parts of Cir-IFS. Furthermore, Bozyigit et al. [21] proposed the circular PFS (Cir-PFS) and then gave some applications in [22–24]. Recently, Yusoff et al. [25] extended Cir-PFS to circular q-ROFs (Cirq-ROFs), which have a condition of \( 0 \leq M_{qcc}^{asc} + N_{qcc}^{asc} \leq 1, q_{cc} \geq 1 \) with a radius of a circle around the point \( \left( \mathcal{M}_{q_{cc}}(\mathcal{x}), \mathcal{N}_{q_{cc}}(\mathcal{x}) \right) \) on the plane. Cir-IFSs and Cir-PFSs had some applications in [22–24].
Table 2. Specifications of Cirq-ROFSs.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Conditions</th>
<th>Mathematical Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_{sc} = 1, N_{Qq} = 0, R_{Qq} = 0)</td>
<td>(0 \leq M_{Qq} \leq 1)</td>
<td>({ \bar{x}, (M_{Qq}(\bar{x}), N_{Qq}(\bar{x}), R_{Qq}) } : \bar{x} \in X_{uni} )</td>
</tr>
<tr>
<td>(q_{sc} = 1, R_{Qq} = 0)</td>
<td>(0 \leq M_{Qq} + N_{Qq} \leq 1)</td>
<td>({ \bar{x}, (M_{Qq}(\bar{x}), N_{Qq}(\bar{x}), R_{Qq}) } : \bar{x} \in X_{uni} )</td>
</tr>
<tr>
<td>(q_{sc} = 1)</td>
<td>(0 \leq M_{Qq}^2 + N_{Qq}^2 \leq 1, R_{Qq} \in [0,1])</td>
<td>({ \bar{x}, (M_{Qq}(\bar{x}), N_{Qq}(\bar{x}), R_{Qq}) } : \bar{x} \in X_{uni} )</td>
</tr>
<tr>
<td>(q_{sc} = 2, R_{Qq} = 0)</td>
<td>(0 \leq M_{Qq}^{esc} + N_{Qq}^{esc} \leq 1)</td>
<td>({ \bar{x}, (M_{Qq}(\bar{x}), N_{Qq}(\bar{x}), R_{Qq}) } : \bar{x} \in X_{uni} )</td>
</tr>
<tr>
<td>(q_{sc} \geq 1, R_{Qq} = 0)</td>
<td>(0 \leq M_{Qq}^{esc} + N_{Qq}^{esc} \leq 1, R_{Qq} \in [0,1])</td>
<td>({ \bar{x}, (M_{Qq}(\bar{x}), N_{Qq}(\bar{x}), R_{Qq}) } : \bar{x} \in X_{uni} )</td>
</tr>
</tbody>
</table>

The demonstration in Tables 1 and 2 with different parameters and conditions shows the advantages and disadvantages of these extended types of FSs. Thus, the major contributions of this paper are listed below:

1. Identification of the novel technique of Cirq-ROFSs with their flexible properties, such as algebraic laws and Dombi laws.
2. Proposal of the Cirq-ROFDWA, Cirq-ROFDOWA, Cirq-ROFDWG, and Cirq-ROFDOWG operators with their properties, such as idempotency, monotonicity, and boundedness.
3. Creation of the multi-attribute decision-making (MADM) technique based on the proposed operators for Cirq-ROF numbers (Cirq-ROFNs) with application to the symmetry analysis in artificial intelligence to compute its major aspects.
4. Use of some existing techniques for comparison to our results to show the validity and supremacy of the proposed method.

The remainder of this paper is as follows. In Section 2, we review the DTN and DTCN with q-ROFSs and their operational laws for any universal set. In Section 3, we identify the novel technique of Cirq-ROFSs and their flexible properties, such as algebraic laws and Dombi laws. In Section 4, we propose the Cirq-ROFDWA, Cirq-ROFDOWA, Cirq-ROFDWG, and Cirq-ROFDOWG operators with some basic properties. In the context of artificial intelligence, symmetry analysis can refer to several aspects. Therefore, in Section 5, to compute the major aspect, we create the MADM technique based on the initiated operators for Cirq-ROFNs to enhance the worth of the proposed operators. In Section 6, we illustrate some numerical examples. In Section 7, we use existing techniques for comparison to our results to show the validity and supremacy of the presented theory. Some concluding remarks are stated in Section 8.

2. Preliminaries

This section discusses the DTN and DTCN with q-ROFSs and their operational laws for any universal set.

**Definition 1** ([27]). Let \(f_{KN}, g_{KN} \in [0,1]\) with \(\sigma_{SC} \geq 1\). Then

\[
DOM(f_{KN}, g_{KN}) = \frac{1}{1 + \left(\frac{1 - f_{KN}}{f_{KN}}\right)^{\sigma_{SC}} \left(\frac{1 - g_{KN}}{g_{KN}}\right)^{\sigma_{SC}}}
\]

The demonstration in Tables 1 and 2 with different parameters and conditions shows the advantages and disadvantages of these extended types of FSs. Thus, the major contributions of this paper are listed below:

1. Identification of the novel technique of Cirq-ROFSs with their flexible properties, such as algebraic laws and Dombi laws.
2. Proposal of the Cirq-ROFDWA, Cirq-ROFDOWA, Cirq-ROFDWG, and Cirq-ROFDOWG operators with their properties, such as idempotency, monotonicity, and boundedness.
3. Creation of the multi-attribute decision-making (MADM) technique based on the proposed operators for Cirq-ROF numbers (Cirq-ROFNs) with application to the symmetry analysis in artificial intelligence to compute its major aspects.
4. Use of some existing techniques for comparison to our results to show the validity and supremacy of the proposed method.

The remainder of this paper is as follows. In Section 2, we review the DTN and DTCN with q-ROFSs and their operational laws for any universal set. In Section 3, we identify the novel technique of Cirq-ROFSs and their flexible properties, such as algebraic laws and Dombi laws. In Section 4, we propose the Cirq-ROFDWA, Cirq-ROFDOWA, Cirq-ROFDWG, and Cirq-ROFDOWG operators with some basic properties. In the context of artificial intelligence, symmetry analysis can refer to several aspects. Therefore, in Section 5, to compute the major aspect, we create the MADM technique based on the initiated operators for Cirq-ROFNs to enhance the worth of the proposed operators. In Section 6, we illustrate some numerical examples. In Section 7, we use existing techniques for comparison to our results to show the validity and supremacy of the presented theory. Some concluding remarks are stated in Section 8.
Symmetry 2024, 16, 260

\[ \text{DOM}^c(f_{RN}, g_{RN}) = 1 - \frac{1}{1 + \left( \frac{f_{RN}}{1-g_{RN}} \right)^{\frac{1}{\sigma_{sc}}} + \left( \frac{g_{RN}}{1-f_{RN}} \right)^{\frac{1}{\sigma_{sc}}}} \]

Thus, algebraic norms are special cases of the above norms. We next talk about the existing techniques, called q-ROFSs, and discuss their operational laws.

**Definition 2** ([16]). Let \( X_{\text{UNI}} \) be a universe of discourse. The q-ROFS is defined below:

\[ Q_{CQ} = \left\{ \left( x, (M_{CQ}(x), N_{CQ}(x)) : x \in X_{\text{UNI}} \right) \right\} \]

where the truth grade is represented by \( M_{CQ}(x) \) and the falsity grade is shown by \( N_{CQ}(x) \) with \( 0 \leq M_{CQ}(x) + N_{CQ}(x) \leq 1 \), \( q_{sc} \geq 1 \). Further, the refusal grade is represented by \( \mathcal{R}_{CQ}(x) = \left( 1 - (M_{CQ}^{q_{sc}} + N_{CQ}^{q_{sc}}) \right)^{1/q_{sc}} \), where the mathematical shape of q-ROFN is stated by \( Q_j = (M_{Q_j}, N_{Q_j}) \), \( j = 1, 2, \ldots, n \).

**Definition 3** ([16]). Let \( Q_j = (M_{Q_j}, N_{Q_j}) \), \( j = 1, 2, \ldots, n \) be a family of q-ROFNs. Then

\[ Q_1 \oplus Q_2 = (M_{Q_1} + M_{Q_2} - M_{Q_1} M_{Q_2}^{1/q_{sc}}, N_{Q_1} N_{Q_2}) \]

\[ Q_1 \otimes Q_2 = \left( M_{Q_1} M_{Q_2}, (N_{Q_1} + N_{Q_2} - N_{Q_1} N_{Q_2})^{1/q_{sc}} \right) \]

\[ q_{sc} Q_j = \left( 1 - \left( 1 - M_{Q_j}^{q_{sc}} \right) \frac{1}{q_{sc}}, N_{Q_j}^{q_{sc}} \right) \]

\[ Q_j^{q_{sc}} = \left( M_{Q_j}^{q_{sc}}, \left( 1 - \left( 1 - N_{Q_j}^{q_{sc}} \right) \frac{1}{q_{sc}} \right) \right) \]

**Definition 4** ([16]). Let \( Q_j = (M_{Q_j}, N_{Q_j}) \), \( j = 1, 2, \ldots, n \) be a family of q-ROFNs. Then, the score value \( \text{SCO}(Q_j) \) and the accuracy value \( \text{ACC}(Q_j) \) are defined as:

\[ \text{SCO}(Q_j) = (M_{Q_j})^{q_{sc}} - \left( N_{Q_j} \right)^{q_{sc}} \in [-1, 1] \]

\[ \text{ACC}(Q_j) = (M_{Q_j})^{q_{sc}} + \left( N_{Q_j} \right)^{q_{sc}} \in [0, 1] \]

The score value and accuracy value have the following properties:

(1) If \( \text{SCO}(Q_1) > \text{SCO}(Q_2) \), then \( Q_1 > Q_2 \).

(2) If \( \text{SCO}(Q_1) < \text{SCO}(Q_2) \), then \( Q_1 < Q_2 \).

(3) If \( \text{SCO}(Q_1) = \text{SCO}(Q_2) \), then we have:

(i) If \( \text{ACC}(Q_1) > \text{ACC}(Q_2) \), then \( Q_1 > Q_2 \).

(ii) If \( \text{ACC}(Q_1) < \text{ACC}(Q_2) \), then \( Q_1 < Q_2 \).

3. Circular q-ROFSs with Their Dombi Operational Laws

In this section, we first review the definition of circular q-ROFSs (Cirq-ROFSs). We then consider the novel techniques of Cirq-ROFSs with their operational laws. Moreover, we also evaluate some Dombi operational laws for Cirq-ROFNs with some properties.

**Definition 5** ([26]). Let \( X_{\text{UNI}} \) be a universe of discourse. The Cirq-ROFS is defined as
\[ Q_{cQ} = \{ (\bar{x}, (M_{Q_{cQ}}(\bar{x}), N_{Q_{cQ}}(\bar{x}), R_{Q_{cQ}}(\bar{x}))) : \bar{x} \in X_{UNI} \} \]

where the truth grade is represented by \( M_{Q_{cQ}}(\bar{x}) \) and the falsity grade is shown by \( N_{Q_{cQ}}(\bar{x}) \) with
\[ 0 \leq M_{Q_{cQ}}^{ASC} + N_{Q_{cQ}}^{ASC} \leq 1, \ q_{SC} \geq 1, \] and \( R_{Q_{cQ}}(\bar{x}) \) represents the radius of the circle around the point \((M_{Q_{cQ}}(\bar{x}), N_{Q_{cQ}}(\bar{x})) \) on the plane. The refusal grade is presented by \( R_{Q_{cQ}}(\bar{x}) = \left(1 - (M_{Q_{cQ}}^{ASC} + N_{Q_{cQ}}^{ASC})\right)^{1/q_{SC}} \), where the mathematical shape of the Cirq-ROFN is stated by \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}) \), \( j = 1, 2, ..., n \).

We mention that Definition 5 is more general than the original Cirq-ROFS defined in Yusoff et al. [26]. In Defintion 5, we consider that the radius \( R_{Q_{cQ}}(\bar{x}) \) depends on \( \bar{x} \in X_{UNI} \). In fact, when \( R_{Q_{cQ}}(\bar{x}) = R_{Q_{cQ}} \) (a constant) for all \( \bar{x} \in X_{UNI} \), it is the same Cirq-ROFS as defined in Yusoff et al. [26].

**Definition 6.** Let \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}) \), \( j = 1, 2, ..., n \) be a family of Cirq-ROFN. Then
\[ Q_1 \oplus_{ec} Q_2 = \left( (M_{Q_1}^{ASC} + M_{Q_2}^{ASC} - M_{Q_1}^{ASC} M_{Q_2}^{ASC})^{1/q_{SC}}, N_{Q_1} N_{Q_2}, (R_{Q_1}^{ASC} + R_{Q_2}^{ASC} - R_{Q_1}^{ASC} R_{Q_2}^{ASC})^{1/q_{SC}} \right); \]
\[ Q_1 \oplus_{c} Q_2 = \left( (M_{Q_1}^{ASC} + M_{Q_2}^{ASC} - M_{Q_1}^{ASC} M_{Q_2}^{ASC})^{1/q_{SC}}, N_{Q_1} N_{Q_2}, R_{Q_1} R_{Q_2} \right); \]
\[ Q_1 \otimes_{ec} Q_2 = \left( (N_{Q_1}^{ASC} + N_{Q_2}^{ASC} - N_{Q_1}^{ASC} N_{Q_2}^{ASC})^{1/q_{SC}}, (R_{Q_1}^{ASC} + R_{Q_2}^{ASC} - R_{Q_1}^{ASC} R_{Q_2}^{ASC})^{1/q_{SC}} \right); \]
\[ Q_{sC} Q_{j}^{ec} = \left( (1 - (1 - M_{Q_j}^{ASC})^{1/q_{SC}}), N_{Q_j}^{ASC}, (1 - (1 - R_{Q_j}^{ASC})^{1/q_{SC}}) \right); \]
\[ Q_{sC} Q_{j}^{c} = \left( (1 - (1 - M_{Q_j}^{ASC})^{1/q_{SC}}), N_{Q_j}^{ASC}, R_{Q_j}^{ASC} \right); \]
\[ (Q_{j}^{ec})^{ASC} = \left( M_{Q_j}^{ASC}, (1 - (1 - N_{Q_j}^{ASC})^{1/q_{SC}}), (1 - (1 - R_{Q_j}^{ASC})^{1/q_{SC}}) \right); \]
\[ (Q_{j}^{c})^{ASC} = \left( M_{Q_j}^{ASC}, (1 - (1 - N_{Q_j}^{ASC})^{1/q_{SC}}), (1 - (1 - R_{Q_j}^{ASC})^{1/q_{SC}}) \right). \]

These defined equations in Definition 6 are the newly proposed basic operators for Cirq-ROFS. We next propose some new Dombi operational laws for Cirq-ROFNs in Definition 7 as follows.

**Definition 7.** Let \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}) \), \( j = 1, 2, ..., n \) be a family of Cirq-ROFN. Then, Dombi operational laws are as follows:
\[ \mathcal{Q}_1 \oplus \mathcal{Q}_2 = \left( 1 - \left( {1 \over 1 + \left( \frac{\mathcal{M}_{q_{21}}^{\text{SC}}}{1 - \mathcal{M}_{q_{21}}^{\text{SC}}} \right)^{\sigma_{\text{SC}}} \left( \frac{\mathcal{M}_{q_{22}}^{\text{SC}}}{1 - \mathcal{M}_{q_{22}}^{\text{SC}}} \right)^{1 \over \sigma_{\text{SC}}} } \right) \right)^{{1 \over \sigma_{\text{SC}}}}; \]

\[ \mathcal{Q}_1 \ominus \mathcal{Q}_2 = \left( 1 \left( \frac{\mathcal{M}_{q_{21}}^{\text{SC}}}{1 - \mathcal{M}_{q_{21}}^{\text{SC}}} \right)^{\sigma_{\text{SC}}} \left( \frac{\mathcal{M}_{q_{22}}^{\text{SC}}}{1 - \mathcal{M}_{q_{22}}^{\text{SC}}} \right)^{1 \over \sigma_{\text{SC}}} \right)^{{1 \over \sigma_{\text{SC}}}}; \]

\[ \mathcal{Q}_1 \otimes \mathcal{Q}_2 = \left( 1 \left( \frac{1 - \mathcal{M}_{q_{21}}^{\text{SC}}}{\mathcal{M}_{q_{21}}^{\text{SC}}} \right)^{\sigma_{\text{SC}}} \left( \frac{1 - \mathcal{M}_{q_{22}}^{\text{SC}}}{\mathcal{M}_{q_{22}}^{\text{SC}}} \right)^{1 \over \sigma_{\text{SC}}} \right)^{{1 \over \sigma_{\text{SC}}}}; \]
\[
Q_1 \otimes Q_2 = \left( \begin{array}{c}
\left( 1/1 + \left( \frac{1 - \mathcal{M}_Q^{qsc}}{\mathcal{M}_Q^{qsc}} \right)^{\sigma_{sc}} + \left( \frac{1 - \mathcal{M}_Q^{qsc}}{\mathcal{M}_Q^{qsc}} \right)^{\frac{1}{qsc}} \right)
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\left( 1 - \left( 1/1 + \left( \frac{1 - \mathcal{N}_Q^{qsc}}{\mathcal{N}_Q^{qsc}} \right)^{\sigma_{sc}} + \left( \frac{1 - \mathcal{N}_Q^{qsc}}{\mathcal{N}_Q^{qsc}} \right)^{\frac{1}{qsc}} \right) \right)^{\sigma_{sc}} \right)
\end{array} \right)
\]

\[
\mathcal{Q}_{sc}Q_1^{sc} = \left( \begin{array}{c}
\left( 1 - \left( 1/1 + \left( \mathcal{Q}_{sc} \left( \frac{1 - \mathcal{N}_Q^{qsc}}{\mathcal{N}_Q^{qsc}} \right) \right)^{\sigma_{sc}} \right) \right)^{\frac{1}{qsc}} \right)
\end{array} \right)
\]

\[
\mathcal{Q}_{sc}Q_1^{sc} = \left( \begin{array}{c}
\left( 1 - \left( 1/1 + \left( \mathcal{Q}_{sc} \left( \frac{1 - \mathcal{N}_Q^{qsc}}{\mathcal{N}_Q^{qsc}} \right) \right)^{\sigma_{sc}} \right) \right)^{\frac{1}{qsc}} \right)
\end{array} \right)
\]
Theorem 1. Let $Q_j = \left( M_{Q_j}, N_{Q_j}, R_{Q_j} \right)$, $j = 1, 2, \ldots, n$ be a family of Cirq-ROFNs. Then, we have the following results:

1. $Q_1 \otimes^{tc} Q_2 = Q_2 \otimes^{tc} Q_1$.
2. $Q_1 \otimes^{tc} Q_2 = Q_2 \otimes^{tc} Q_1$.
3. $\sigma_{\text{SC}} Q_1 \otimes^{tc} \sigma_{\text{SC}} Q_2 = \sigma_{\text{SC}} (Q_1 \otimes^{tc} Q_2)$.
4. $(Q_1)^{\text{esc}} \otimes^{tc} (Q_2)^{\text{esc}} = (Q_1 \otimes^{tc} Q_2)^{\text{esc}}$.
5. $\sigma_{\text{SC}} Q_{-1} \otimes^{tc} \sigma_{\text{SC}} Q_{-2} = (\sigma_{\text{SC}} Q_{-1} + \sigma_{\text{SC}} Q_{-2}) Q_1$.
6. $(Q_1)^{\text{esc}} \otimes^{tc} (Q_2)^{\text{esc}} = (Q_1)^{\text{esc}} + Q_{-1} Q_{-2}$.

Proof.

(1) Based on Definition 7, we obtain:
\( \mathcal{Q}_1 \oplus^{tc} \mathcal{Q}_2 = \left\{ \begin{array}{l}
1 \left( 1 + \left( \frac{M_{q_{1,1}}^{q_{1}}}{1 - M_{q_{1,1}}^{q_{1}}} \right) \sigma_{SC} + \left( \frac{M_{q_{2,1}}^{q_{1}}}{1 - M_{q_{2,1}}^{q_{1}}} \right) \sigma_{SC} \frac{1}{\sigma_{SC}} \right) \right\}, \\
\left( \frac{1}{1 + \left( \frac{1 - N_{q_{1,1}}^{q_{1}}}{N_{q_{1,1}}^{q_{1}}} \right) \sigma_{SC} + \left( \frac{1 - N_{q_{2,1}}^{q_{1}}}{N_{q_{2,1}}^{q_{1}}} \right) \sigma_{SC} \frac{1}{\sigma_{SC}} \right) \right\}, \\
\left( 1 - \left( 1 + \left( \frac{R_{q_{2,1}}^{q_{1}}}{1 - R_{q_{2,1}}^{q_{1}}} \right) \sigma_{SC} + \left( \frac{R_{q_{1,1}}^{q_{1}}}{1 - R_{q_{1,1}}^{q_{1}}} \right) \sigma_{SC} \frac{1}{\sigma_{SC}} \right) \right\).
\end{array} \right.

(2) Omitted because it is similar to step 1.
(3) Based on Definition 7, we obtain:
\[ q_{sc} \mathcal{Q}_1 \oplus^{ic} q_{sc} \mathcal{Q}_2 \]

\[
\left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_1}}{1 - M^{qsc}_{q_1}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right) \left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_2}}{1 - M^{qsc}_{q_2}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right)
\]

\[
= \left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_1}}{1 - M^{qsc}_{q_1}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right) \left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_2}}{1 - M^{qsc}_{q_2}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right)
\]

\[
= \left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_1}}{1 - M^{qsc}_{q_1}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right) \left( \left( 1 - \frac{1}{1 + \left(q_{sc} \left( \frac{M^{qsc}_{q_2}}{1 - M^{qsc}_{q_2}} \right)^{\frac{1}{q_{sc}}} \right)} \right) \right)
\]

(4) Omitted because it is similar to step 3.

(5) Based on Definition 7, we obtain:
\[
\begin{align*}
\rho_{s_{c-1}}\mathcal{Q}_1 \oplus^c \rho_{s_{c-2}}\mathcal{Q}_1 & = \left(1 - \frac{1}{1 + \left(\rho_{s_{c-1}} \left(\frac{\mathcal{M}_{Q_1}^{\rho_{s_{c-1}}}}{1 - \mathcal{M}_{Q_1}^{\rho_{s_{c-2}}}}\right)^{\frac{1}{\rho_{s_{c-1}}}}\right)}\right)^{\frac{1}{\rho_{s_{c-1}}}}, \\
\rho_{s_{c-1}}\mathcal{Q}_1 \oplus^c \rho_{s_{c-2}}\mathcal{Q}_1 & = \left(1 + \left(\rho_{s_{c-2}} \left(\frac{\mathcal{M}_{Q_1}^{\rho_{s_{c-2}}}}{1 - \mathcal{M}_{Q_1}^{\rho_{s_{c-2}}}}\right)^{\frac{1}{\rho_{s_{c-2}}}}\right)\right)^{\frac{1}{\rho_{s_{c-2}}}}.
\end{align*}
\]

(6) Omitted because it is similar to step 5. □

**Theorem 2.** Let \(\mathcal{Q}_j = (\mathcal{M}_{Q_j}, N_{Q_j}, \mathcal{R}_{Q_j}), j = 1, 2, \ldots, n\) be a family of Circ-RoFNs. Then

1. \(\mathcal{Q}_1 \oplus^t \mathcal{Q}_2 = \mathcal{Q}_2 \oplus^t \mathcal{Q}_1\).
2. \(\mathcal{Q}_1 \oplus^t \mathcal{Q}_2 = \mathcal{Q}_2 \oplus^t \mathcal{Q}_1\).
3. \(\rho_{t_{c}}\mathcal{Q}_1 \oplus^t \rho_{t_{c}}\mathcal{Q}_2 = \rho_{t_{c}}(\mathcal{Q}_1 \oplus^t \mathcal{Q}_2)\).
4. \((\mathcal{Q}_1)^{\rho_{s_{c-1}}} \oplus^t (\mathcal{Q}_2)^{\rho_{s_{c-2}}} = (\mathcal{Q}_1 \oplus^t \mathcal{Q}_2)^{\rho_{s_{c}}})\).
5. \(\rho_{s_{c-1}}\mathcal{Q}_1 \oplus^t \rho_{s_{c-2}}\mathcal{Q}_1 = (\rho_{s_{c-1}} + \rho_{s_{c-2}})(\mathcal{Q}_1)\).
6. \((\mathcal{Q}_1)^{\rho_{s_{c-1}}} \oplus^t (\mathcal{Q}_1)^{\rho_{s_{c-2}}} = (\mathcal{Q}_1)^{\rho_{s_{c-1}} + \rho_{s_{c-2}}}\)).

**Proof.**

1. Based on Definition 7, we obtain:
\[ Q_1 \oplus^\dagger Q_2 = \left( 1 - \left( 1 + \left( \frac{\mathcal{M}_1^q}{1 - \mathcal{M}_1^q} \right)^{\sigma_{SC}} + \left( \frac{\mathcal{M}_2^q}{1 - \mathcal{M}_2^q} \right)^{\sigma_{SC}} \right) \right) \left( \frac{1}{\mathcal{Q}_1} \right), \]

\[ = \left( 1 - \left( 1 + \left( \frac{\mathcal{M}_2^q}{1 - \mathcal{M}_2^q} \right)^{\sigma_{SC}} + \left( \frac{\mathcal{M}_1^q}{1 - \mathcal{M}_1^q} \right)^{\sigma_{SC}} \right) \right) \left( \frac{1}{\mathcal{Q}_2} \right), \]

(2) Omitted because it is similar to step 1.

(3) Based on Definition 7, we obtain:
\[
\varphi_{SC} Q_1 \Theta' \varphi_{SC} Q_2
\]

\[
= \left( 1 - \left( 1 + \left( \varphi_{SC} \left( \frac{\mathcal{M}^q_{Q_1}}{1 - \mathcal{M}^q_{Q_1}} \right) \varphi_{SC} \right) \frac{1}{\varphi_{SC}} \right) \right) \Theta'
\]

\[
= \varphi_{SC} (Q_1 \Theta' Q_2).
\]

(4) Omitted because it is similar to step 3.

(5) Based on Definition 7, we obtain:
\[ (1 - \frac{1}{1 + \left( \frac{M_{q,1}^{q_{sc}}}{1 - M_{q,1}^{q_{sc}}} \right)^{\frac{1}{q_{sc}}}}) \left( 1 - \frac{1}{1 + \left( \frac{M_{q,1}^{q_{sc}}}{1 - M_{q,1}^{q_{sc}}} \right)^{\frac{1}{q_{sc}}}} \right) \]

\[ = \left( \frac{1}{1 + \left( \frac{M_{q,1}^{q_{sc}}}{1 - M_{q,1}^{q_{sc}}} \right)^{\frac{1}{q_{sc}}}} \right)^{\frac{1}{q_{sc}}} \]

\[ = \left( \frac{1}{1 + \left( \frac{M_{q,1}^{q_{sc}}}{1 - M_{q,1}^{q_{sc}}} \right)^{\frac{1}{q_{sc}}}} \right)^{\frac{1}{q_{sc}}} \]

(6) Omitted because it is similar to step 5. □

4. Cirq-ROF Domki Averaging/Geometric Aggregation Operators

This section introduces the Cirq-ROFDWA operator, Cirq-ROFDOWG operator, Cirq-ROFDWG operator, and Cirq-ROFDOGW operator. Further, we evaluate their properties of idempotency, monotonicity, and boundedness.

Definition 8. Let \( Q_j = \left( M_{q,j}, N_{q,j}, R_{q,j} \right), j = 1, 2, ..., n \) be a family of Cirq-ROFNs. Then

\[ Cirq - ROFDWA^{t_{c}}(Q_1, Q_2, ..., Q_n) = e_1 Q_1^{t_{c}} \oplus^{t_{c}} e_2 Q_2^{t_{c}} \oplus^{t_{c}} ... \oplus^{t_{c}} e_n Q_n^{t_{c}} = \oplus_{j=1}^{n} \left( e_j Q_j^{t_{c}} \right) \]

\[ Cirq - ROFDWA^{t}(Q_1, Q_2, ..., Q_n) = e_1 Q_1^{t} \oplus^{t} e_2 Q_2^{t} \oplus^{t} ... \oplus^{t} e_n Q_n^{t} = \oplus_{j=1}^{n} \left( e_j Q_j^{t} \right) \]

These are signified as a Cirq-ROFDWA operator for both t-conorm and t-norm, where the weight vector is represented by \( e_j \in [0,1] \) with \( \oplus_{j=1}^{n} e_j = 1 \).

Theorem 3. Let \( Q_j = \left( M_{q,j}, N_{q,j}, R_{q,j} \right), j = 1, 2, ..., n \) be a family of Cirq-ROFNs. Then, we show that the final result of the Cirq-ROFDWA operator for t-conorm and t-norm is again a Cirq-ROFN:
\[
Cirq - ROFDW A^{tc}(Q_1, Q_2, \ldots, Q_n) = \left(1 - \left\{ 1 + \left( \sum_{j=1}^{n} q_j \left( \frac{\sigma_{sc} \rho_{Q_j}}{1 - \rho_{Q_j}} \right) \right) \right\} \right) \frac{1}{\sigma_{sc}}
\]

\[
Cirq - ROFDWA^{tc}(Q_1, Q_2, \ldots, Q_n) = \left(1 + \left( \sum_{j=1}^{n} q_j \left( \frac{\sigma_{sc} \rho_{Q_j}}{1 - \rho_{Q_j}} \right) \right) \right) \frac{1}{\sigma_{sc}}
\]

**Proof.** In the presence of the induction of mathematics, we evaluate our results. To do so, first, we derive it for \( n = 2 \).

\[
\theta_1^{tc} = \left(1 - \left( 1 + \left( \theta_1 \left( \frac{\rho_{Q_1} \sigma_{sc}}{1 - \rho_{Q_1}} \right) \right) \right) \right) \frac{1}{\sigma_{sc}}
\]

\[
\theta_1^{tc} = \left(1 + \left( \theta_1 \left( \frac{\rho_{Q_1} \sigma_{sc}}{1 - \rho_{Q_1}} \right) \right) \right) \frac{1}{\sigma_{sc}}
\]

and
\[
q_{2}Q_{2}^{tc} = \left( 1 - \frac{1}{1 + \left( q_{2} \frac{M_{q_{2}} q_{2}^{\sigma_{q_{2}}}}{1 - M_{q_{2}} q_{2}^{\sigma_{q_{2}}}} \right)} \frac{1}{q_{SC}} \right) \left( 1 - \frac{1}{1 + \left( q_{2} \frac{M_{q_{2}} q_{2}^{\sigma_{q_{2}}}}{1 - M_{q_{2}} q_{2}^{\sigma_{q_{2}}}} \right)} \frac{1}{q_{SC}} \right).
\]

Thus,

\[
Cirq \rightarrow ROFDWA \left( \mathcal{Q}_{1}, \mathcal{Q}_{2} \right) = q_{1}Q_{1}^{tc} \oplus^{tc} q_{1}Q_{1}^{tc}
\]

\[
= \left( 1 - \frac{1}{1 + \left( q_{1} \frac{M_{q_{1}} q_{1}^{\sigma_{q_{1}}}}{1 - M_{q_{1}} q_{1}^{\sigma_{q_{1}}}} \right)} \frac{1}{q_{SC}} \right) \left( 1 - \frac{1}{1 + \left( q_{1} \frac{M_{q_{1}} q_{1}^{\sigma_{q_{1}}}}{1 - M_{q_{1}} q_{1}^{\sigma_{q_{1}}}} \right)} \frac{1}{q_{SC}} \right) \oplus^{tc} \left( 1 - \frac{1}{1 + \left( q_{2} \frac{M_{q_{2}} q_{2}^{\sigma_{q_{2}}}}{1 - M_{q_{2}} q_{2}^{\sigma_{q_{2}}}} \right)} \frac{1}{q_{SC}} \right) \left( 1 - \frac{1}{1 + \left( q_{2} \frac{M_{q_{2}} q_{2}^{\sigma_{q_{2}}}}{1 - M_{q_{2}} q_{2}^{\sigma_{q_{2}}}} \right)} \frac{1}{q_{SC}} \right).
\]

\[
= \left( 1 - \frac{1}{1 + \left( \sum_{j=1}^{2} \frac{M_{q_{j}} q_{j}^{\sigma_{q_{j}}}}{1 - M_{q_{j}} q_{j}^{\sigma_{q_{j}}}} \right)} \frac{1}{q_{SC}} \right) \left( 1 - \frac{1}{1 + \left( \sum_{j=1}^{2} \frac{M_{q_{j}} q_{j}^{\sigma_{q_{j}}}}{1 - M_{q_{j}} q_{j}^{\sigma_{q_{j}}}} \right)} \frac{1}{q_{SC}} \right)
\]

For our considered value, it has been done. For \( n = m \), we have
Then, we derive our target for $n = m + 1$. 

$$Cirq - ROFDWA^{fc}(Q_1, Q_2, ..., Q_m) =$$

$$\left(1 - \left(\frac{1}{1 + \left(\sum_{j=1}^{m} \varrho_j \left(\frac{M_{q_j}^{\text{gsc}}}{1 - M_{q_j}^{\text{gsc}}} \right)^{\sigma_{q_j}^{\text{gsc}}} \frac{1}{\sigma_{q_j}^{\text{gsc}}} \right)^{\frac{1}{\sigma_{q_j}^{\text{gsc}}}} \right)\right)$$

$$\left(1 - \left(\frac{1}{1 + \left(\sum_{j=1}^{m} \varrho_j \left(\frac{1 - N_{q_j}^{\text{gsc}}}{N_{q_j}^{\text{gsc}}} \right)^{\sigma_{q_j}^{\text{gsc}}} \frac{1}{\sigma_{q_j}^{\text{gsc}}} \right)^{\frac{1}{\sigma_{q_j}^{\text{gsc}}}} \right)\right)$$

$$\left(1 - \left(\frac{1}{1 + \left(\sum_{j=1}^{m} \varrho_j \left(\frac{R_{q_j}^{\text{gsc}}}{1 - R_{q_j}^{\text{gsc}}} \right)^{\sigma_{q_j}^{\text{gsc}}} \frac{1}{\sigma_{q_j}^{\text{gsc}}} \right)^{\frac{1}{\sigma_{q_j}^{\text{gsc}}}} \right)\right)$$
\[
\text{Cirq} - ROFDWA^\text{tc}(Q_1, Q_2, ..., Q_{m+1}) = \text{Cirq} - ROFDWA^\text{tc}(Q_1, Q_2, ..., Q_m) \oplus^\text{tc} \theta_{m+1}Q_{m+1}^\text{tc} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m} \theta_j \left( \frac{M_j^{qsc}}{1 - M_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \oplus^\text{tc} \theta_{m+1}Q_{m+1}^\text{tc} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m} \theta_j \left( \frac{1 - \theta_j^{qsc}}{1 - \theta_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \oplus^\text{tc} \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m} \theta_j \left( \frac{R_j^{qsc}}{1 - R_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m+1} \theta_j \left( \frac{M_j^{qsc}}{1 - M_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m+1} \theta_j \left( \frac{1 - \theta_j^{qsc}}{1 - \theta_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m+1} \theta_j \left( \frac{R_j^{qsc}}{1 - R_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \\
= \left(1 - \left( \frac{1}{m} \left( \sum_{j=1}^{m+1} \theta_j \left( \frac{M_j^{qsc}}{1 - M_j^{qsc}} \frac{1}{\sigma_{sc}} \right) \right) \right) \right)^{\frac{1}{q_{sc}}} \\
\]

Hence, our results are proven for all non-negative integers. Similarly, we follow the same procedure to obtain our major target: \(\text{Cirq} - ROFDWA^\text{tc}(Q_1, Q_2, ..., Q_n)\). The proof is completed. \(\square\)

**Property 1.** Let \(Q_j = (M_j, N_j, R_j)\), \(j = 1, 2, ..., n\) be a family of Cirq-ROFNs. Then

1. **(Idempotency)** If \(Q_j = Q = (M_j, N_j, R_j)\), \(j = 1, 2, ..., n\), then
ROFN that the final result of the Cirq

\[ Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) = Q; \]
\[ Cirq - ROFDA^{t}(Q_1, Q_2, ..., Q_n) = Q. \]

(2) (Monotonicity) If \( Q_j \leq Q_j^\oplus, j = 1, 2, ..., n \), then

\[ Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) \leq Cirq - ROFDA^{tc}(Q_1^\oplus, Q_2^\oplus, ..., Q_n^\oplus); \]
\[ Cirq - ROFDA^{t}(Q_1, Q_2, ..., Q_n) \leq Cirq - ROFDA^{t}(Q_1^\oplus, Q_2^\oplus, ..., Q_n^\oplus). \]

(3) (Boundedness) If \( Q_j^{te} = \left( \min \left( M_{Q_j}, \max \left( N_{Q_j} \right) \right), \max \left( R_{Q_j} \right) \right) \), then \( Q_j^{te} \leq Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) \leq Q_j^{tc} \);
\[ Q_j^{te} \leq Cirq - ROFDA^{t}(Q_1, Q_2, ..., Q_n) \leq Q_j^{tc}. \]

**Proof.** The proofs are straightforward. \( \square \)

**Definition 9.** Let \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}) \), \( j = 1, 2, ..., n \) be a family of Cirq-ROFNs. Then,

\[ Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) = e_1 Q_{o(1)}^{te} \oplus e_2 Q_{o(2)}^{te} \oplus ... \oplus e_n Q_{o(n)}^{te} = \bigoplus_{j=1}^{n} (e_j Q_{o(j)}^{te}); \]
\[ Cirq - ROFDA^{t}(Q_1, Q_2, ..., Q_n) = e_1 Q_{o(1)}^{t} \oplus e_2 Q_{o(2)}^{t} \oplus ... \oplus e_n Q_{o(n)}^{t} = \bigoplus_{j=1}^{n} (e_j Q_{o(j)}^{t}) \]

These are signified as a Cirq-ROFDA operator for both \( t \)-conorm and \( t \)-norm, where the weight vector is represented by \( \varrho_j \in [0,1] \) with \( \bigoplus_{j=1}^{n} \varrho_j = 1 \) with an order \( o(j) \leq o(j-1) \).

**Theorem 4.** Let \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}) \), \( j = 1, 2, ..., n \) be a family of Cirq-ROFNs. Then, we show that the final result of the Cirq-ROFDA operator for \( t \)-conorm and \( t \)-norm is again a Cirq-ROFN:

\[ Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) = \left( 1 - \left( 1 + \sum_{j=1}^{n} \varrho_j \left( \frac{M_{Q_{o(j)}}^{t\text{sc}}}{1 - M_{Q_{o(j)}}^{t\text{sc}}} \right)^{\frac{1}{t\text{sc}}} \right) \right)^{\frac{1}{t\text{sc}}}, \]
\[ Cirq - ROFDA^{t}(Q_1, Q_2, ..., Q_n) = \left( 1 + \sum_{j=1}^{n} \varrho_j \left( \frac{1 - M_{Q_{o(j)}}^{t\text{sc}}}{N_{Q_{o(j)}}^{t\text{sc}}} \right)^{\frac{1}{t\text{sc}}} \right)^{\frac{1}{t\text{sc}}}, \]
\[ Cirq - ROFDA^{tc}(Q_1, Q_2, ..., Q_n) = \left( 1 - \left( 1 + \sum_{j=1}^{n} \varrho_j \left( \frac{R_{Q_{o(j)}}^{t\text{sc}}}{1 - R_{Q_{o(j)}}^{t\text{sc}}} \right)^{\frac{1}{t\text{sc}}} \right) \right)^{\frac{1}{t\text{sc}}}. \]
that the final result of the Cirq-ROFDWG operator for both t-conorm and t-norm is again a Cirq-ROFN:  

\[ \text{Cirq - ROFDWG}^t(Q_1, Q_2, ..., Q_n) = \left( \left( \begin{array}{cc} 1 - \left( 1 / \left( 1 + \left( \sum_{j=1}^{n} q_j \left( \frac{\mathcal{M}^{\text{sc}}_{\mathcal{Q}(j)} - 1}{\mathcal{M}^{\text{sc}}_{\mathcal{Q}(j)}} \right) \right) \right) \right) \right) \right) \]

\[ \left( \begin{array}{cc} 1 - \left( 1 / \left( 1 + \left( \sum_{j=1}^{n} q_j \left( \frac{1 - \mathcal{N}^{\text{sc}}_{\mathcal{Q}(j)}}{\mathcal{N}^{\text{sc}}_{\mathcal{Q}(j)}} \right) \right) \right) \right) \right) \]

\[ \left( \begin{array}{cc} 1 - \left( 1 / \left( 1 + \left( \sum_{j=1}^{n} q_j \left( \frac{1 - \mathcal{R}^{\text{sc}}_{\mathcal{Q}(j)}}{\mathcal{R}^{\text{sc}}_{\mathcal{Q}(j)}} \right) \right) \right) \right) \right) \]

**Proof.** The proof is similar to Theorem 3. □

**Property 2.** Let \( Q_j = (\mathcal{M}_j, \mathcal{N}_j, \mathcal{R}_j), j = 1, 2, ..., n \) be family of Cirq-ROFNs. Then

1. (Idempotency) If \( Q_j = Q = (\mathcal{M}_j, \mathcal{N}_j, \mathcal{R}_j), j = 1, 2, ..., n, \) then 
   \[ \text{Cirq - ROFDWA}^t(Q_1, Q_2, ..., Q_n) = Q; \]

2. (Monotonicity) If \( Q_j \leq Q'_j, j = 1, 2, ..., n, \) then 
   \[ \text{Cirq - ROFDWA}^t(Q_1, Q_2, ..., Q_n) \leq \text{Cirq - ROFDWA}^t(Q'_1, Q'_2, ..., Q'_n); \]

   \[ \text{Cirq - ROFDWA}^t(Q_1, Q_2, ..., Q_n) \leq \text{Cirq - ROFDWA}^t(Q'_1, Q'_2, ..., Q'_n). \]

3. (Boundedness) If \( Q_j^{t^{\text{tc}}} = \left( \min \left( \mathcal{M}_j \right), \max \left( \mathcal{N}_j \right) \right), Q_j^{t^{\text{tc}}} = \left( \min \left( \mathcal{M}_j \right), \max \left( \mathcal{N}_j \right) \right) \) and \( Q_j^{t^{\text{tc}}} = \left( \max \left( \mathcal{M}_j \right), \min \left( \mathcal{N}_j \right) \right), Q_j^{t^{\text{tc}}} = \left( \max \left( \mathcal{M}_j \right), \min \left( \mathcal{N}_j \right) \right), \) then 
   \[ Q_j^{t^{\text{tc}}} \leq \text{Cirq - ROFDWA}^t(Q_1, Q_2, ..., Q_n) \leq Q_j^{t^{\text{tc}}}; \]

   \[ Q_j^{t^{\text{tc}}} \leq \text{Cirq - ROFDWA}^t(Q_1, Q_2, ..., Q_n) \leq Q_j^{t^{\text{tc}}}. \]

**Proof.** The proofs are straightforward. □

**Definition 10.** Let \( Q_j = (\mathcal{M}_j, \mathcal{N}_j, \mathcal{R}_j), j = 1, 2, ..., n \) be family of Cirq-ROFNs. Then 

\[ \text{Cirq - ROFDWG}^t(Q_1, Q_2, ..., Q_n) = (Q_1^{t^{\text{tc}}})^{e_1} \otimes (Q_2^{t^{\text{tc}}})^{e_2} \otimes ... \otimes (Q_n^{t^{\text{tc}}})^{e_n} = \otimes_{j=1}^{e_n} (Q_j^{t^{\text{tc}}})^{e_j}; \]

\[ \text{Cirq - ROFDWG}^t(Q_1, Q_2, ..., Q_n) = (Q_1^{t^{\text{tc}}})^{e_1} \otimes (Q_2^{t^{\text{tc}}})^{e_2} \otimes ... \otimes (Q_n^{t^{\text{tc}}})^{e_n} = \otimes_{j=1}^{e_n} (Q_j^{t^{\text{tc}}})^{e_j}. \]

These are signified as a Cirq-ROFDWG operator for both t-conorm and t-norm, where the weight vector is represented by \( q_j \in [0,1] \) with \( \sum_{j=1}^{n} q_j = 1. \)

**Theorem 5.** Let \( Q_j = (\mathcal{M}_j, \mathcal{N}_j, \mathcal{R}_j), j = 1, 2, ..., n \) be family of Cirq-ROFNs. Then, we show that the final result of the Cirq-ROFDWG operator for t-conorm and t-norm is again a Cirq-ROFN:
Proof. The proof is similar to Theorem 3. □

Property 3. Let \( Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}), j = 1, 2, ..., n \) be family of Cirq-ROFNs. Then

1. (Idempotency) If \( Q_j = Q = (M_Q, N_Q, R_Q), j = 1, 2, ..., n \), then
   \[ \text{Cirq} - \text{ROFDWG}^c(Q_1, Q_2, ..., Q_n) = Q; \]
   \[ \text{Cirq} - \text{ROFDWG}^d(Q_1, Q_2, ..., Q_n) = Q. \]

2. (Monotonicity) If \( Q_j \leq Q_j^*, j = 1, 2, ..., n \), then
   \[ \text{Cirq} - \text{ROFDWG}^c(Q_1, Q_2, ..., Q_n) \leq \text{Cirq} - \text{ROFDWG}^c(Q_1^*, Q_2^*, ..., Q_n^*); \]
   \[ \text{Cirq} - \text{ROFDWG}^d(Q_1, Q_2, ..., Q_n) \leq \text{Cirq} - \text{ROFDWG}^d(Q_1^*, Q_2^*, ..., Q_n^*). \]

3. (Boundedness) If \( Q_j^c = \left( \min\left(M_{Q_j}\right), \max\left(N_{Q_j}\right) \right), \]
   \( Q_j^d = \left( \min\left(M_{Q_j}\right), \max\left(R_{Q_j}\right) \right) \)
   and \( Q_j^* = \left( \max\left(M_{Q_j}\right), \min\left(N_{Q_j}\right) \right), \]
   \( Q_j^* = \left( \max\left(M_{Q_j}\right), \min\left(R_{Q_j}\right) \right) \),
   then
   \[ Q_j^c \leq \text{Cirq} - \text{ROFDWG}^c(Q_1, Q_2, ..., Q_n) \leq Q_j^*; \]
   \[ Q_j^d \leq \text{Cirq} - \text{ROFDWG}^d(Q_1, Q_2, ..., Q_n) \leq Q_j^*. \]

Proof. The proofs are straightforward. □
Definition 11. Let \( Q_j = \left( M_{Q_j}, N_{Q_j}, R_{Q_j} \right), j = 1, 2, \ldots, n \) be family of Cirq-ROFNs. Then
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = (Q_{o(1)}^{tc})^1 \otimes (Q_{o(2)}^{tc})^2 \otimes \cdots \otimes (Q_{o(n)}^{tc})^n = \otimes_{j=1}^n (Q_{o(j)}^{tc})^j;
\]
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = (Q_{o(1)}^{tc})^1 \otimes (Q_{o(2)}^{tc})^2 \otimes \cdots \otimes (Q_{o(n)}^{tc})^n = \otimes_{j=1}^n (Q_{o(j)}^{tc})^j.
\]
These are signified as a Cirq-ROFDOWG operator for both t-conorm and t-norm, where the weight vector is represented by \( q_j \in [0, 1] \) with \( \sum_{j=1}^n q_j = 1 \) with order \( o(j) \leq o(j - 1) \).

Theorem 6. Let \( Q_j = \left( M_{Q_j}, N_{Q_j}, R_{Q_j} \right), j = 1, 2, \ldots, n \) be family of Cirq-ROFNs. Then, we show that the final result of the Cirq-ROFDOWG operator for t-conorm and t-norm is again a Cirq-ROFN:
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = \left( \begin{array}{c}
1/1 + \left( \sum_{j=1}^n q_j \left( \frac{1 - M_{Q_j}^{sc}}{M_{Q_j}^{sc}} \right) \frac{1}{q_{sc}} \right) \frac{1}{q_{sc}}
\end{array} \right)
\]
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = \left( \begin{array}{c}
1 - \left( \frac{1}{1 + \left( \sum_{j=1}^n q_j \left( \frac{N_{Q_j}^{sc}}{1 - N_{Q_j}^{sc}} \right) \frac{1}{q_{sc}} \right) \frac{1}{q_{sc}}} \right)
\end{array} \right)
\]
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = \left( \begin{array}{c}
1/1 + \left( \sum_{j=1}^n q_j \left( \frac{1 - R_{Q_j}^{sc}}{R_{Q_j}^{sc}} \right) \frac{1}{q_{sc}} \right) \frac{1}{q_{sc}}
\end{array} \right)
\]

Proof. The proof is similar to Theorem 3. □

Property 4. Let \( Q_j = \left( M_{Q_j}, N_{Q_j}, R_{Q_j} \right), j = 1, 2, \ldots, n \) be family of Cirq-ROFNs. Then
\( \text{(Idempotency)} \) if \( Q_j = Q = (M_Q, N_Q, R_Q), j = 1, 2, \ldots, n, \) then
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = Q;
\]
\[
\text{Cirq} - \text{ROFDOWG}^t(Q_1, Q_2, \ldots, Q_n) = Q.
\]
Step 2: To consider the normalization information, we evaluate the aggregated values $T_i$.

Step 1: To compute the matrix, we use Cirq-

\[ Cirq \rightarrow ROFDW G^{tc}(Q_1, Q_2, ..., Q_n) \leq Cirq \rightarrow ROFDW G^{tc}(Q_1^*, Q_2^*, ..., Q_n^*); \]

\[ Cirq \rightarrow ROFDW G^{t}(Q_1, Q_2, ..., Q_n) \leq Cirq \rightarrow ROFDW G^{t}(Q_1^*, Q_2^*, ..., Q_n^*). \]

(3) (Boundedness) If $Q_i^{tc} = \left( \min \left( M_{q_i}, N_{q_i} \right), \max \left( N_{q_i}, M_{q_i} \right) \right)$, then $Q_i^{tc} \leq Cirq \rightarrow ROFDW G^{tc}(Q_1, Q_2, ..., Q_n) \leq Q_i^{tc}$; $Q_i^{tc} \geq Cirq \rightarrow ROFDW G^{t}(Q_1, Q_2, ..., Q_n) \leq Q_i^{tc}$.

**Proof.** The proofs are straightforward. □

The proposed techniques in this section can be extensions of the existing techniques based on FSs, IFs, PFSs, q-ROFs, Cir-Fs, Cir-IFSs, and Cir-PFSSs. We next used these proposed operators to create a Cirq-ROF multi-attribute decision-making (MADM) technique.

5. The MADM Method Based on the Initiated Operators

In this section, we present the MADM technique based on the proposed operators, such as the Cirq-ROF and Cirq-ROFDWG operators for both norms, called DTN and DTCN, to improve the validity of the initiated techniques. Consider $Q_1, Q_2, ..., Q_n$ to be the collection of alternative information with the family of attributes for each alternative, such as $Q_1^A, Q_2^A, ..., Q_m^A$ with weight vectors such as $q_j \in [0,1]$, where the condition of the weight vector is $\sum_{j=1}^{n} q_j = 1$. Further, we assign Cirq-ROFNs to each attribute in every alternative. We know that the truth grade is represented by $M_{q_{0c}}$ and the falsity grade is shown by $N_{q_{0c}}$ with $0 \leq M_{q_{0c}} + N_{q_{0c}} \leq 1$, and a radius for the truth grade and falsity grade is $R_{q_{0c}} = [0,1]$. The refusal grade is $R_{q_{0c}}(x) = \left( 1 - \left( M_{q_{0c}}^{qsc} + N_{q_{0c}}^{qsc} \right) \right)^{1/q_{sc}}$, where the Cirq-ROFN is presented by $Q_j = \left( M_{q_j}, N_{q_j}, R_{q_j} \right), j = 1,2, ..., n$. We present the major steps of the MADM technique for evaluating any kind of real-life problems as follows:

Step 1: To compute the matrix, we use Cirq-ROFNs $Q_j = \left( M_{q_j}, N_{q_j}, R_{q_j} \right), j = 1,2, ..., n$.

Further, we evaluate the normalization of the matrix by using the theory below:

\[ N = \left\{ \begin{array}{ll}
(M_{q_j}, N_{q_j}, R_{q_j}) & \text{for benefit} \\
(N_{q_j}, M_{q_j}, R_{q_j}) & \text{for cost}
\end{array} \right. \]

The normalization in the matrix is not carried out if we have beneficial types of information.

Step 2: To consider the normalization information, we evaluate the aggregated values with the help of the Cirq-ROFSDWA operators of $Cirq \rightarrow ROFDWA^{tc}(Q_1, Q_2, ..., Q_n)$ and $Cirq \rightarrow ROFDWA^{t}(Q_1, Q_2, ..., Q_n)$, and the Cirq-ROFDWG operators of $Cirq \rightarrow ROFDWG^{tc}(Q_1, Q_2, ..., Q_n)$ and $Cirq \rightarrow ROFDWG^{t}(Q_1, Q_2, ..., Q_n)$ for both the DTN and DTCN norms, which are presented in Theorems 3 and 5 in Section 4.

Step 3: Calculate the score values based on the information of the score function.

\[ SCQ(Q_j) = R_{q_{0j}}^{qsc} \ast \left( M_{q_j}^{qsc} - N_{q_j}^{qsc} \right) \in [-1,1]; \]
ACC(Q_j) = R_{Q_j}^{q_{SC}} \cdot \left( (M_{Q_j}^{q_{SC}}) + (N_{Q_j}^{q_{SC}}) \right) \in [0,1].

Step 4: Calculate the ranking of alternatives by using the score values and evaluate the best one.

We next used the above-created Cirq-ROF-MADM technique for the application of symmetry analysis in artificial intelligence.

6. Symmetry Analysis in Artificial Intelligence Based on the Proposed Cirq-ROF-MADM

In this section, we evaluate the supremacy and validity of the derived operators by resolving the problem of symmetry analysis in artificial intelligence under the presence of the initiated Cirq-ROF-MADM technique. Symmetry analysis plays a significant and efficient role in various fields, including machine learning, artificial intelligence, neural networks, and data mining, to cope with vague and complicated problems. In the context of artificial intelligence, symmetry analysis can refer to many aspects. In general, the following six aspects are important: Q₁: algorithmic symmetry (optimization algorithm and machine learning models), Q₂: data symmetry (data augmentation), Q₃: knowledge representation (symbolic artificial intelligence), Q₄: natural language processing (semantic symmetry), Q₅: fairness and bias (algorithm fairness), and Q₆: explainability (interpretability of models).

To evaluate the best decision among the above six, we select the following weight vectors: (0.2,0.2,0.2,0.4)ᵀ. For these weight vectors, we have the following attributes: growth analysis, social impact, political impact, and environmental impact. Finally, we present the major steps of the Cirq-ROF-MADM technique for evaluating the real-life problem as follows:

Step 1: To compute the matrix as shown in Table 3, we use the Cirq-ROFNs Q_j = (M_{Q_j}, N_{Q_j}, R_{Q_j}), j = 1, 2, ..., n. Further, we evaluate the normalization of the matrix by using the theory below:

\[ N = \begin{cases} 
(M_{Q_j}, N_{Q_j}, R_{Q_j}) & \text{for benefit} \\
(N_{Q_j}, M_{Q_j}, R_{Q_j}) & \text{for cost} 
\end{cases} \]

Further, the normalization in the matrix is not carried out if we have beneficial types of information, so the data in Table 3 do not need to be normalized.

Step 2: To consider the normalization information, we evaluate the aggregated values with the help of the Cirq-ROFDWA and Cirq-ROFDWG operators, as shown in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>Q₁⁴ᵀ</th>
<th>Q₂⁴ᵀ</th>
<th>Q₃⁴ᵀ</th>
<th>Q₄⁴ᵀ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q₁</td>
<td>(0.9,0.5,0.7)</td>
<td>(0.9,0.5,0.7)</td>
<td>(0.9,0.5,0.7)</td>
<td>(0.9,0.5,0.7)</td>
</tr>
<tr>
<td>Q₂</td>
<td>(0.8,0.7,0.3)</td>
<td>(0.8,0.7,0.3)</td>
<td>(0.8,0.7,0.3)</td>
<td>(0.8,0.7,0.3)</td>
</tr>
<tr>
<td>Q₃</td>
<td>(0.7,0.5,0.1)</td>
<td>(0.7,0.5,0.1)</td>
<td>(0.7,0.5,0.1)</td>
<td>(0.7,0.5,0.1)</td>
</tr>
<tr>
<td>Q₄</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.6,0.3,0.5)</td>
</tr>
<tr>
<td>Q₅</td>
<td>(0.5,0.3,0.4)</td>
<td>(0.5,0.3,0.4)</td>
<td>(0.5,0.3,0.4)</td>
<td>(0.5,0.3,0.4)</td>
</tr>
<tr>
<td>Q₆</td>
<td>(0.9,0.7,0.3)</td>
<td>(0.9,0.7,0.3)</td>
<td>(0.9,0.7,0.3)</td>
<td>(0.9,0.7,0.3)</td>
</tr>
</tbody>
</table>
Table 4. Cirq-ROF aggregated matrix.

<table>
<thead>
<tr>
<th></th>
<th>Cirq – ROFDA$^{tc}$</th>
<th>Cirq – ROFDA$^{t}$</th>
<th>Cirq – ROFDWG$^{tc}$</th>
<th>Cirq – ROFDWG$^{t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>(0.9236,0.510,9943)</td>
<td>(0.9236,0.510,5137)</td>
<td>(0.91,0.9994,0.874)</td>
<td>(0.91,0.9994,0.9994)</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>(0.9743,0.7118,9785)</td>
<td>(0.9743,0.7118,7062)</td>
<td>(0.8116,0.9915,0.7951)</td>
<td>(0.8116,0.9915,0.992)</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>(0.9915,0.510,9992)</td>
<td>(0.9915,0.510,5097)</td>
<td>(0.7116,0.9994,0.7062)</td>
<td>(0.7116,0.9994,0.9994)</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>(0.9976,0.4087,9976)</td>
<td>(0.9976,0.4087,4088)</td>
<td>(0.6109,0.9999,0.6099)</td>
<td>(0.6109,0.9999,0.9999)</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>(0.9976,0.6109,9976)</td>
<td>(0.9976,0.6109,6096)</td>
<td>(0.6109,0.0076,0.6096)</td>
<td>(0.6109,0.0076,0.9976)</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>(0.9236,0.7116,9519)</td>
<td>(0.9236,0.7116,7062)</td>
<td>(0.91,0.9915,0.8692)</td>
<td>(0.91,0.9915,0.9992)</td>
</tr>
</tbody>
</table>

Step 3: Calculate the score values based on the information in the score function, as shown in Table 5.

Table 5. Representation of the score terms.

<table>
<thead>
<tr>
<th></th>
<th>Cirq – ROFDA$^{tc}$</th>
<th>Cirq – ROFDA$^{t}$</th>
<th>Cirq – ROFDWG$^{tc}$</th>
<th>Cirq – ROFDWG$^{t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>0.4985</td>
<td>0.2697</td>
<td>−0.459</td>
<td>−0.525</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>0.73</td>
<td>0.5269</td>
<td>−0.593</td>
<td>−0.74</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.9223</td>
<td>0.4739</td>
<td>−0.657</td>
<td>−0.929</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>0.9775</td>
<td>0.4006</td>
<td>−0.598</td>
<td>−0.98</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>0.9589</td>
<td>0.586</td>
<td>−0.586</td>
<td>−0.959</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>0.4416</td>
<td>0.3276</td>
<td>−0.403</td>
<td>−0.46</td>
</tr>
</tbody>
</table>

Step 4: The score values are used to obtain the ranking results of all alternatives and evaluate the best one, as shown in Table 6.

Table 6. Representation of ranking values.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Ranking Values</th>
<th>Most Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cirq – ROFDA$^{tc}$</td>
<td>$Q_4 &gt; Q_5 &gt; Q_6 &gt; Q_7 &gt; Q_8 &gt; Q_1$</td>
<td>$Q_4$</td>
</tr>
<tr>
<td>Cirq – ROFDA$^{t}$</td>
<td>$Q_4 &gt; Q_5 &gt; Q_6 &gt; Q_7 &gt; Q_8 &gt; Q_1$</td>
<td>$Q_5$</td>
</tr>
<tr>
<td>Cirq – ROFDWG$^{tc}$</td>
<td>$Q_1 &gt; Q_6 &gt; Q_7 &gt; Q_8 &gt; Q_9 &gt; Q_2$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>Cirq – ROFDWG$^{t}$</td>
<td>$Q_6 &gt; Q_7 &gt; Q_8 &gt; Q_9 &gt; Q_1$</td>
<td>$Q_6$</td>
</tr>
</tbody>
</table>

According to the results in Table 6, the best choice is as follows: Cirq – ROFDA$^{tc}$ chooses $Q_4$, Cirq – ROFDA$^{t}$ chooses $Q_5$, Cirq – ROFDWG$^{tc}$ chooses $Q_1$, and Cirq – ROFDWG$^{t}$ chooses $Q_6$.

7. Comparative Analysis

For the evaluation comparisons, in this section, we consider the data in Table 3 and try to evaluate them by using some prevailing techniques. To do so, we collected some existing techniques and tried to compare their ranking results with our initial ranking results to show the supremacy and validity of the derived theory. For comparison, we use the following existing techniques, such as that of Liu and Wang [17], who presented aggregation operators for q-ROFSs. The Dombi operators based on the PFSs were presented by Khan et al. [28]. Jana et al. [29] initiated the Dombi operators for q-ROFSs and their applications. Du [30] evaluated more about Dombi aggregation operators for q-ROFSs, and Seikh and Mandal [31] examined the Dombi operators for IFSs. Further, Akram et al. [32] proposed the fuzzy N-soft sets, Yang et al. [33] exposed the partitioned Bonferroni mean operators for complex q-rung orthopair uncertain linguistic sets, and Suri et al. [34] exposed the entropy measures for q-ROFSs. Finally, based on the data in Table 3, the comparative analysis is listed in Table 7.
Table 7. Representation of comparative analysis.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Ranking Values</th>
<th>Best Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liu and Wang [17]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Khan et al. [28]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Jana et al. [29]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Du [30]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Seikh and Mandal [31]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Akram et al. [32]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Yang et al. [33]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>Suri et al. [34]</td>
<td>Limited features</td>
<td>Limited features</td>
</tr>
<tr>
<td>(Cirq \sim ROFDW A^{1c})</td>
<td>(Q_4 &gt; Q_5 &gt; Q_3 &gt; Q_2 &gt; Q_1 &gt; Q_6)</td>
<td>(Q_4)</td>
</tr>
<tr>
<td>(Cirq \sim ROFDW A^{1})</td>
<td>(Q_4 &gt; Q_5 &gt; Q_3 &gt; Q_2 &gt; Q_6 &gt; Q_1)</td>
<td>(Q_5)</td>
</tr>
<tr>
<td>(Cirq \sim ROFDWG^{1c})</td>
<td>(Q_4 &gt; Q_5 &gt; Q_3 &gt; Q_2 &gt; Q_4 &gt; Q_2)</td>
<td>(Q_1)</td>
</tr>
<tr>
<td>(Cirq \sim ROFDWG^{1})</td>
<td>(Q_4 &gt; Q_5 &gt; Q_3 &gt; Q_2 &gt; Q_5 &gt; Q_4)</td>
<td>(Q_6)</td>
</tr>
</tbody>
</table>

Based on the initiated technique, we obtained the following best optimal features: \(Q_4, Q_5, Q_1,\) and \(Q_6\). From the results in Table 7, we can see that these existing techniques have a problem with limited features, and so they failed to evaluate the data in Table 3. The weakness of the existing techniques by Liu and Wang [17], Khan et al. [28], Jana et al. [29], Du [30], Seikh and Mandal [31], Akram et al. [32], Yang et al. [33], and Suri et al. [34] are that they are operators, methods, and measures based on existing techniques of q-ROFs without circular behavior. However, the proposed methods are computed based on a new way of Cirq-ROFs that provides circular behavior. The advantages and disadvantages of the proposed methods and existing techniques are discussed as follows:

1. Seikh and Mandal [31] proposed the AOs for intuitionistic FSs (IFSs), and Khan et al. [28] proposed the AOs for Pythagorean FSs (PFSs), where the IFSs and PFSs contain the truth grade and the falsity grade but without circular behavior. However, our proposed AOs are based on Cirq-ROFs, which are a circular extended version of IFSs and PFSs. This means that the methods of Seikh and Mandal [31] and Khan et al. [28] cannot handle the data in Table 3.

2. Akram et al. [32] proposed the AOs for fuzzy N-soft sets, which only contain the truth grade, so they fail to process the data in Table 3 under the Cirq-RÖF environment.

3. Liu and Wang [17], Jana et al. [29], Du [30], Yang et al. [33], and Suri et al. [34] proposed the AOs for q-ROFs, which are special cases of Cirq-ROFs without circular behavior. This means that these existing methods cannot deal with the Cirq-ROF data shown in Table 3.

8. Conclusions

Since circular q-rung orthopair fuzzy sets (Cirq-ROFs) were recently proposed as an extension of FSs, IFSs, q-ROFs, Cir-IFSs, and Cir-PFSs by Yusoff et al. [26] with no other research in the literature, we conduct an advanced study of Cirq-ROFs with more properties. Cirq-ROFs are good for depicting vagueness and fuzziness for life problems. In this paper, we continue working on Cirq-ROFs by proposing Dombi aggregation operators (AOs) for Cirq-ROFs with more operational laws. These include the Cirq-ROFDWA, Cirq-ROFDOWA, Cirq-ROFDWG, and Cirq-ROFDOWG operators. We also provide more properties, such as idempotency, monotonicity, and boundedness, for the proposed operators. We then construct the Cirq-ROF multi-attribute decision-making technique and apply it to the context of artificial intelligence in which symmetry analysis can compute the major aspect based on the proposed operators for Cirq-ROFs. We use some existing techniques to compare our results to show the validity and supremacy of the proposed method. Up to now, no one had proposed the distances, similarity measures, or entropy based on Cirq-ROFs. In the future, we will conduct these advanced studies...
on Cirq-ROFSs. We should also consider new extensions of Cirq-ROFSs to ball-extended T-spherical FSs and complex versions of them by utilizing some phase terms. Furthermore, we will discuss their applications in the field of artificial intelligence, machine learning, neural networks, green supply chain management, and decision-making to enhance the worth of the proposed theory.

**Author Contributions:** Conceptualization, Z.A. and M.-S.Y.; methodology, Z.A. and M.-S.Y.; validation, Z.A.; formal analysis, Z.A.; investigation, Z.A. and M.-S.Y.; writing—original draft preparation, Z.A.; writing—review and editing, M.-S.Y.; supervision, M.-S.Y.; funding acquisition, M.-S.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was supported in part by the National Science and Technology Council, Taiwan, under grant NSTC 112-2118-M-033-004.

**Data Availability Statement:** Data are contained within the article.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

27. Dombi, J. A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets Syst.* 1982, 8, 149–163.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.