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Generalized Dynamic Inequalities of Copson Type on Time Scales

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Abstract: This paper introduces novel generalizations of dynamic inequalities of Copson type within the framework of time scales delta calculus. The proposed generalizations leverage mathematical tools such as Hölder's inequality, Minkowski's inequality, the chain rule on time scales, and the properties of power rules on time scales. As special cases of our results, particularly when the time scale $\mathbb{T}$ equals the real line ($\mathbb{T} = \mathbb{R}$), we derive some classical continuous analogs of previous inequalities. Furthermore, when $\mathbb{T}$ corresponds to the set of natural numbers including zero ($\mathbb{T} = \mathbb{N}_0$), the obtained results, to the best of the authors' knowledge, represent innovative contributions to the field.

Keywords: Copson's inequality; Hölder's inequality; Minkowski's inequality; time scales calculus

MSC: 26D10; 34N05; 47B38; 39A12



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1. Introduction

Hardy, known primarily as a problem-solver, contributed significantly to the development of inequalities. His initial systematic treatment of inequalities laid the groundwork for subsequent work, much of which was influenced by his collaboration with Littlewood and Pólya on the book *Inequalities*, published in 1934 [1]. This seminal work spurred widespread study in the field, with notable contributions from Copson [2,3] and Knopp [4].

One of Hardy's key contributions is the classical discrete inequality established in 1920 [5], expressed as follows:

$$\sum_{\kappa=1}^{\infty} \left(\frac{1}{\kappa} \sum_{i=1}^{\kappa} a(i) \right)^{\ell} \leq \left(\frac{\ell}{\ell-1} \right)^{\ell} \sum_{\kappa=1}^{\infty} [a(\kappa)]^{\ell}, \quad \ell > 1, \quad (1)$$

where $a(\kappa) \geq 0$ for $\kappa \geq 1$ and $\sum_{\kappa=1}^{\infty} [a(\kappa)]^{\ell} < \infty$. Hardy later extended this to an integral inequality [6], providing conditions under which a function $\Xi \geq 0$ is integrable over $(0, \lambda)$, $\lambda \in (0, \infty)$, given by:

$$\int_0^{\infty} \left(\frac{1}{\lambda} \int_0^{\lambda} \Xi(\tau) d\tau \right)^{\ell} d\lambda \leq \left(\frac{\ell}{\ell-1} \right)^{\ell} \int_0^{\infty} [\Xi(\lambda)]^{\ell} d\lambda, \quad \ell > 1, \quad (2)$$

where $(\ell/(\ell-1))^\ell$ in (1) and (2) is the optimal constant. The authors of [7] showed that if $\Xi(\lambda) \geq 0$ and $\int_0^\infty [\Xi(\lambda)]^\ell d\lambda < \infty$, then

$$\int_0^\infty \left(\frac{1}{\lambda} \int_\lambda^\infty \Xi(\tau) d\tau \right)^\ell d\lambda \geq \left(\frac{\ell}{1-\ell} \right)^\ell \int_0^\infty [\Xi(\lambda)]^\ell d\lambda, \quad 0 < \ell < 1.$$

Also, the constant $(\ell/(1-\ell))^\ell$ is the best possible.

Building upon Hardy's work, Copson [8] provided a generalized form of the integral inequality (2), introducing parameters $\ell \geq 1$ and $\alpha > 1$, expressed as follows:

$$\int_0^\varepsilon \xi(\vartheta) \frac{(Y(\vartheta))^\ell}{(\Lambda(\vartheta))^\alpha} d\vartheta \leq \left(\frac{\ell}{\alpha-1} \right)^\ell \int_0^\varepsilon \xi(\vartheta) (\Lambda(\vartheta))^{\ell-\alpha} (\Xi(\vartheta))^\ell d\vartheta, \quad (3)$$

where

$$Y(\vartheta) = \int_0^\vartheta \xi(x) \Xi(x) dx \quad \text{and} \quad \Lambda(\vartheta) = \int_0^\vartheta \xi(x) dx.$$

Mohapatra and Vajravelu [9] showed that if $\xi(\vartheta) \in C[0, \infty)$ is a positive function, $\Xi(\vartheta) \geq 0$, $\ell > 1$ and $\exists G, H > 0$ such that

$$\vartheta |\xi'(\vartheta)| \leq G \xi(\vartheta) \quad \text{and} \quad \vartheta \xi(\vartheta) \leq H \Lambda(\vartheta), \quad \forall \vartheta > 0, \quad (4)$$

then

$$\int_0^\infty \xi(\vartheta) \left(\frac{Y(\vartheta)}{\Lambda(\vartheta)} \right)^\ell d\vartheta \leq (G+H)^\ell \int_0^\infty \left(\frac{1}{\vartheta} \int_0^\vartheta (\xi(y))^\frac{1}{\ell} \Xi(y) dy \right)^\ell d\vartheta, \quad (5)$$

and

$$\int_0^\infty \xi(\vartheta) \left(\int_\vartheta^\infty \frac{\xi(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} d\kappa \right)^\ell d\vartheta \leq (H + \ell C)^\ell \int_0^\infty \left(\int_\vartheta^\infty \frac{(\xi(y))^\frac{1}{\ell} \Xi(y)}{y} dy \right)^\ell d\vartheta, \quad (6)$$

where $Y(\vartheta) = \int_0^\vartheta \xi(\kappa) \Xi(\kappa) d\kappa$, $\Lambda(\vartheta) = \int_0^\vartheta \xi(\kappa) d\kappa$ and $C = G + H + 1$.

Also, some authors presented the discrete Formulas (5) and (6) (see [10]).

The purpose of this study is to establish some new generalizations of (5) and (6) on time scales by leveraging Hölder's inequality, Minkowski's inequality and chain rule formula. The paper's exploration of novel inequalities within the framework of time scales delta calculus hints at potential applications of symmetry, suggesting a deeper understanding of the inherent balance and invariance present in the mathematical structures under consideration. The concept of symmetry in this context may be connected to exploring patterns, invariances or relationships within the established dynamic inequalities on time scales. For more results about the dynamic inequalities, see the book [11] and the papers [12,13].

In [14], El-Deeb et al. proved that if $0 \leq x \in \mathbb{T}$, $\delta \geq a$, then Ω and ϕ are non-negative (γ, a) -nabla fractional differentiable and locally integrable functions on $[x, \infty)_{\mathbb{T}}$. Let $\exists \Lambda > 0$ such that $\frac{\omega^\rho(\delta)}{\omega(\delta)} \leq \Lambda$ for $\delta \in [x, \infty)_{\mathbb{T}}$. If $k \geq 0$, $\ell > 1$, $k + \theta > 1$ and $0 < \gamma < 1$ are real constants, then

$$\int_x^\infty \frac{\phi(\delta) [\Psi^\rho(\delta)]^{k+\ell-\gamma+1}}{[\omega^\rho(\delta)]^{k+\theta-\gamma+1}} \nabla_a^\gamma \delta \leq \frac{k+\ell-\gamma+1}{k+\theta-\gamma} \int_x^\infty \frac{\phi(\delta) \Omega(\delta) [\Psi^\rho(\delta)]^{k+\ell-\gamma}}{[\omega(\delta)]^{k+\theta-\gamma}} \nabla_a^\gamma \delta,$$

where $\omega(\delta) = \int_\delta^\infty \phi(t) \nabla_a^\gamma t$, $\Psi(\delta) = \int_\delta^\infty \phi(t) \Omega(t) \nabla_a^\gamma t$, $\Psi(x) < \infty$ and $\int_x^\infty \frac{\phi(\delta)}{\omega^\rho(\delta)} \nabla_a^\gamma \delta < \infty$.

In [15], Saker et al. proved that if $\lambda, \Omega \geq 0$ functions, $\Lambda(\vartheta) = \int_\varepsilon^\vartheta \lambda(x) \Delta x$, $\Lambda(\infty) = \infty$ and $\Psi(\vartheta) = \int_\varepsilon^\vartheta \lambda(x) \Omega(x) \Delta x$ for any $\vartheta \in [\varepsilon, \infty)_{\mathbb{T}}$, then

$$\int_{\varepsilon}^{\infty} \frac{\lambda(\vartheta)[\Psi^{\sigma}(\vartheta)]^{\ell}}{[\Lambda^{\sigma}(\vartheta)]^{\ell}} \Delta\vartheta \leq \frac{\ell}{\ell-1} \int_{\varepsilon}^{\infty} \frac{\lambda(\vartheta)\Omega(\vartheta)\Psi^{\ell-1}(\vartheta)}{\Lambda^{\ell-1}(\vartheta)} \Delta\vartheta, \quad \ell > 1.$$

In [16], Saker et al. showed that if $\lambda, \Omega \geq 0$ functions, $\Lambda(\vartheta) = \int_{\varepsilon}^{\vartheta} \lambda(x)\Delta x$, $\Phi(\vartheta) = \int_{\varepsilon}^{\vartheta} \lambda(x)\Omega(x)\Delta x$ for any $\vartheta \in [\varepsilon, \infty)_{\mathbb{T}}$, $1 < c \leq \ell$, then

$$\int_{\varepsilon}^{\infty} \frac{\lambda(\vartheta)[\Phi^{\sigma}(\vartheta)]^{\ell}}{[\Lambda^{\sigma}(\vartheta)]^c} \Delta\vartheta \leq \frac{\ell}{c-1} \int_{\varepsilon}^{\infty} \Lambda^{1-c}(\vartheta)\lambda(\vartheta)\Omega(\vartheta)\Psi^{\ell-1}(\vartheta)\Delta\vartheta,$$

and

$$\int_{\varepsilon}^{\infty} \frac{\lambda(\vartheta)[\Phi^{\sigma}(\vartheta)]^{\ell}}{[\Lambda^{\sigma}(\vartheta)]^c} \Delta\vartheta \leq \left(\frac{\ell}{c-1}\right)^{\ell} \int_{\varepsilon}^{\infty} \frac{(\Lambda^{\sigma}(\vartheta))^{(\ell-1)c}}{(\Lambda(\vartheta))^{\ell(c-1)}} \lambda(\vartheta)\Omega^{\ell}(\vartheta)\Delta\vartheta.$$

In [17], Ashraf et al. proved that $\mathbb{T}_1, \dots, \mathbb{T}_n$ denote time scales. For $\ell > 1, j = 1, \dots, n$, consider $b_j \in [0, \infty)_{\mathbb{T}}$, $\Omega : \mathbb{T}_1 \times \dots \times \mathbb{T}_n \rightarrow \mathbb{R}^+$, $\xi_j : \mathbb{T}_j \rightarrow \mathbb{R}^+$, $\Lambda_j(y_j) = \int_{b_j}^{y_j} \xi_i(\vartheta_i)\Delta\vartheta_i$ and $\Lambda_j(\infty) = 0$. If

$$\phi(y_1, \dots, y_n) = \int_{y_1}^{\infty} \dots \int_{y_n}^{\infty} \prod_{j=1}^n \frac{\xi_j(\vartheta_j)}{\Lambda_j^{\sigma_j}(\vartheta_j)} \Omega(\vartheta_1, \dots, \vartheta_n) \Delta\vartheta_1 \dots \Delta\vartheta_n,$$

then

$$\begin{aligned} & \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \prod_{j=1}^n \xi_j(y_j) \phi^{\ell}(y_1, \dots, y_n) \Delta y_1 \dots \Delta y_n \\ & \leq \ell^{n\ell} \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \prod_{j=1}^n \xi_j(y_j) \Omega^{\ell}(y_1, \dots, y_n) \Delta y_1 \dots \Delta y_n. \end{aligned}$$

In [18], Kayar and Kaymakçalan observed that ω, Ξ are non-negative, along with ld-continuous, ∇ -differentiable and locally nabla integrable functions. In addition, if $G(\vartheta) = \int_{\varepsilon}^{\vartheta} \omega(y)\nabla y$ and $H(\vartheta) = \int_{\varepsilon}^{\vartheta} \omega(y)\Xi(y)\nabla y$ with $H(\infty) < \infty$, then $\int_{\varepsilon}^{\infty} \frac{\omega(\vartheta)}{G^{l+r}(\vartheta)} \nabla\vartheta < \infty$. If there exists $L > 0$ such that $\frac{G(\vartheta)}{G^{\rho}(\vartheta)} \leq L$ for $\vartheta \in (\varepsilon, \infty)_{\mathbb{T}}$, $l \geq 0, \varsigma > 1, l+r > 1$ are real constants, then

$$\int_{\varepsilon}^{\infty} \frac{\omega(\vartheta)H^{l+\varsigma}(\vartheta)}{[G(\vartheta)]^{l+r}} \nabla\vartheta \leq \frac{l+\varsigma}{l+r-1} \int_{\varepsilon}^{\infty} \frac{\omega(\vartheta)\Xi(\vartheta)H^{l+\varsigma-1}(\vartheta)}{[G^{\rho}(\vartheta)]^{l+r-1}} \nabla\vartheta.$$

In [19], Saker et al. proved that if u, h are positive rd-continuous functions on $[\varepsilon, b]_{\mathbb{T}}$, $U(\kappa) = \int_{\varepsilon}^{\kappa} u(\vartheta)\Delta\vartheta$ and $1 < \ell < \infty$, then

$$\int_{\varepsilon}^b u(\kappa) \left(\int_{\kappa}^b h(\vartheta)\Delta\vartheta \right)^{\ell} \Delta\kappa \leq \ell^{\ell} \int_{\varepsilon}^b u(\kappa) \left(\frac{h(\kappa)U^{\sigma}(\kappa)}{u(\kappa)} \right)^{\ell} \Delta\kappa. \quad (7)$$

The inequality (7) is reversed for $0 < \ell < 1$, while $\ell = 1$ holds equality.

The structure of this article unfolds as follows: In Section 2, we introduce several lemmas on time scales, which are essential for the proofs presented in Section 3. The outcomes we establish in Section 3 encompass the inequalities (5) and (6), previously demonstrated by Mohapatra and Vajravelu [9] in the special cases where $\mathbb{T}$ equals the real numbers ($\mathbb{T} = \mathbb{R}$). Additionally, when $\mathbb{T}$ is equivalent to the set of natural numbers ($\mathbb{T} = \mathbb{N}$), our results, to the best of the authors' knowledge, represent novel findings.

2. Preliminaries

Bohner and Peterson [20] defined the time scale $\mathbb{T}$ as an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. Thus

$$\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0,$$

i.e., the real numbers, non-negative real numbers, the integers, the natural numbers, and the non-negative integers are examples of time scales. Also, they defined the forward jump operator by $\sigma(\varepsilon) := \inf\{w \in \mathbb{T} : w > \varepsilon\}$ and the graininess function μ by $\mu(\varepsilon) := \sigma(\varepsilon) - \varepsilon \geq 0$. For any function $\Xi : \mathbb{T} \rightarrow \mathbb{R}$, $\Xi^\sigma(\varepsilon)$ denotes $\Xi(\sigma(\varepsilon))$.

The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. In summary,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1 ([20]). $\Xi : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left dense points in $\mathbb{T}$. The set of *rd-continuous* functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2 ([20]). Assume that $\Xi : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $\xi \in \mathbb{T}$. We define $\Xi^\Delta(\xi)$ to be the number, if it exists, as follows: for any $\varepsilon > 0$, there is a neighborhood $U = (\xi - \delta, \xi + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|\Xi(\sigma(\xi)) - \Xi(y) - \Xi^\Delta(\xi)(\sigma(\xi) - y)| \leq \varepsilon |\sigma(\xi) - y| \quad \forall y \in U, y \neq \sigma(\xi).$$

In this case, we say that $\Xi^\Delta(\xi)$ is the *delta derivative* of Ξ at ξ .

Theorem 1 ([20]). Assume $\Xi, \Omega : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $\xi \in \mathbb{T}$. Then

1. $\Xi + \Omega : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at ξ , and

$$(\Xi + \Omega)^\Delta(\xi) = \Xi^\Delta(\xi) + \Omega^\Delta(\xi).$$

2. $\Xi\Omega : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at ξ and the “product rule,” which is defined by

$$(\Xi\Omega)^\Delta(\xi) = \Xi^\Delta(\xi)\Omega(\xi) + \Xi(\sigma(\xi))\Omega^\Delta(\xi) = \Xi(\xi)\Omega^\Delta(\xi) + \Xi^\Delta(\xi)\Omega(\sigma(\xi)),$$

holds.

3. If $\Omega(\xi)\Omega(\sigma(\xi)) \neq 0$, then we have the quotient $\Xi/\Omega : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at ξ , and the “quotient rule”

$$\left(\frac{\Xi}{\Omega}\right)^\Delta(\xi) = \frac{\Xi^\Delta(\xi)\Omega(\xi) - \Xi(\xi)\Omega^\Delta(\xi)}{\Omega(\xi)\Omega(\sigma(\xi))},$$

holds.

Theorem 2 ([20]). Suppose there is a continuous function $\Omega : \mathbb{R} \rightarrow \mathbb{R}$ whose restriction $\Omega : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable on $\mathbb{T}$ and $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then

$$(\Xi \circ \Omega)^\Delta(\xi) = \Xi'(\Omega(c))\Omega^\Delta(\xi); \quad c \in [\xi, \sigma(\xi)]. \quad (8)$$

Definition 3 ([20]). $Y : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $\Xi : \mathbb{T} \rightarrow \mathbb{R}$ if

$$Y^\Delta(\xi) = \Xi(\xi), \quad \forall \xi \in \mathbb{T}^\kappa.$$

In this case, the *Cauchy integral* of Ξ is defined by

$$\int_r^y \Xi(\xi) \Delta \xi = Y(y) - Y(r), \quad \forall r, y \in \mathbb{T}.$$

Theorem 3 ([20]). Every rd-continuous function $\Xi : \mathbb{T} \rightarrow \mathbb{R}$ has an antiderivative. In particular, if $\xi_0 \in \mathbb{T}$, then

$$\left(\int_{\xi_0}^{\xi} \Xi(\tau) \Delta \tau \right)^{\Delta} = \Xi(\xi), \quad \text{for } \xi \in \mathbb{T}.$$

Lemma 1 ([21]). If $r, s \in \mathbb{T}$ and $u, v \in C_{rd}([r, s]_{\mathbb{T}}, \mathbb{R})$, then

$$\int_r^s u(\xi) v^{\Delta}(\xi) \Delta \xi = [u(\xi) v(\xi)]_r^s - \int_r^s u^{\Delta}(\xi) v^{\sigma}(\xi) \Delta \xi. \quad (9)$$

Lemma 2 ([21]). If $r, s \in \mathbb{T}$ and $\lambda, \omega \in C_{rd}([r, s]_{\mathbb{T}}, \mathbb{R}^+)$, then

$$\int_r^s \lambda(\xi) \omega(\xi) \Delta \xi \leq \left[\int_r^s [\lambda(\xi)]^{\gamma} \Delta \xi \right]^{\frac{1}{\gamma}} \left[\int_r^s [\omega(\xi)]^{\nu} \Delta \xi \right]^{\frac{1}{\nu}}, \quad (10)$$

where $\gamma > 1$ and $1/\gamma + 1/\nu = 1$.

Lemma 3 (Weighted Minkowski's Inequality [21]). Let $\chi, \lambda, \omega \in C_{rd}([r, s]_{\mathbb{T}}, \mathbb{R}^+)$ and $\ell > 1$. Then

$$\begin{aligned} & \left[\int_r^s \chi(\xi) [\lambda(\xi) + \omega(\xi)]^{\ell} \Delta \xi \right]^{\frac{1}{\ell}} \\ & \leq \left[\int_r^s \chi(\xi) [\lambda(\xi)]^{\ell} \Delta \xi \right]^{\frac{1}{\ell}} + \left[\int_r^s \chi(\xi) [\omega(\xi)]^{\ell} \Delta \xi \right]^{\frac{1}{\ell}}. \end{aligned} \quad (11)$$

Lemma 4 ([22]). Assume $r, \xi \in \mathbb{T}$ and $\Omega \in C_{rd}([r, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. If $\ell \geq 1$, then

$$\left(\int_r^{\sigma(\xi)} \Omega(\kappa) \Delta \kappa \right)^{\ell} \leq \ell \int_r^{\sigma(\xi)} \Omega(\kappa) \left(\int_r^{\sigma(\kappa)} \Omega(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa.$$

Lemma 5 ([22]). Let $r \in \mathbb{T}$ and $\varphi, \eta \in C_{rd}([r, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. Then

$$\int_r^{\infty} \varphi(\xi) \left(\int_{\xi}^{\infty} \eta(\kappa) \Delta \kappa \right) \Delta \xi = \int_{\varepsilon}^{\infty} \eta(\xi) \left(\int_{\varepsilon}^{\sigma(\xi)} \varphi(\kappa) \Delta \kappa \right) \Delta \xi.$$

Lemma 6 ([23]). Let $\Omega \in C_{rd}([r, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\ell \geq 1$. Then

$$\left(\int_{\xi}^{\infty} \Omega(\kappa) \Delta \kappa \right)^{\ell} \leq \ell \int_{\xi}^{\infty} \Omega(\kappa) \left(\int_{\kappa}^{\infty} \Omega(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa.$$

Lemma 7. Assume that $\varepsilon \in \mathbb{T}$, $\ell > 1$ with $1/\ell + 1/\ell^* = 1$ and $\Xi : [\varepsilon, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}^+$. If $Z : [\varepsilon, \infty)_{\mathbb{T}} \rightarrow \mathbb{R}^+$ is a positive, differentiable and nonincreasing function with

$$\varphi(\xi) = \int_{\xi}^{\infty} \frac{[Z(\delta)]^{\frac{1}{\ell}} \Xi(\delta)}{\delta - \varepsilon} \Delta \delta, \quad \Lambda(\xi) = \int_{\varepsilon}^{\xi} Z(\delta) \Delta \delta,$$

and there exists a positive constant H such that $(\xi - \varepsilon)Z(\xi) \leq H\Lambda(\xi)$, then

$$\lim_{\kappa \rightarrow \infty} \frac{(\kappa - \varepsilon)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) = 0.$$

Proof. Since $Z(\delta)$ is a positive and non-increasing function, we have for $\delta \geq \kappa$, that $Z(\delta) \leq Z(\kappa)$ and then

$$\int_{\kappa}^{\infty} \frac{[Z(\delta)]^{\frac{1}{\ell}} \Xi(\delta)}{\delta - \varepsilon} \Delta \delta \leq [Z(\kappa)]^{\frac{1}{\ell}} \int_{\kappa}^{\infty} \frac{\Xi(\delta)}{\delta - \varepsilon} \Delta \delta. \quad (12)$$

Since

$$\frac{(\kappa - \varepsilon)Z(\kappa)}{\Lambda(\kappa)} \leq H,$$

we have from (12) that

$$\begin{aligned} 0 &\leq \frac{(\kappa - \varepsilon)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) = \frac{(\kappa - \varepsilon)Z(\kappa)}{\Lambda(\kappa)} [Z(\kappa)]^{\frac{-1}{\ell}} \int_{\kappa}^{\infty} \frac{[Z(\delta)]^{\frac{1}{\ell}} \Xi(\delta)}{\delta - \varepsilon} \Delta \delta \\ &\leq H[Z(\kappa)]^{\frac{-1}{\ell}} \int_{\kappa}^{\infty} \frac{[Z(\delta)]^{\frac{1}{\ell}} \Xi(\delta)}{\delta - \varepsilon} \Delta \delta \leq H \int_{\kappa}^{\infty} \frac{\Xi(\delta)}{\delta - \varepsilon} \Delta \delta, \end{aligned}$$

then

$$0 \leq \lim_{\kappa \rightarrow \infty} \frac{(\kappa - \varepsilon)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) \leq H \lim_{\kappa \rightarrow \infty} \int_{\kappa}^{\infty} \frac{\Xi(\delta)}{\delta - \varepsilon} \Delta \delta = 0,$$

and then by using the Sandwich theorem, we obtain

$$\lim_{\kappa \rightarrow \infty} \frac{(\kappa - \varepsilon)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) = 0.$$

□

3. Main Results

Throughout the following results, we will assume that the functions are rd-continuous functions on $[\varepsilon, \infty)_{\mathbb{T}}$ and the integrals considered are assumed to exist.

Theorem 4. Assume $\varepsilon \in \mathbb{T}$ and $\ell > 1$ with $1/\ell + 1/\ell^* = 1$. Let $Z(\xi) > 0$ be a differentiable and non-increasing function on $[\varepsilon, \infty)_{\mathbb{T}}$ and there exists positive constants G, H such that

$$(\sigma(\xi) - \varepsilon) \left| Z^{\Delta}(\xi) \right| \leq G Z^{\sigma}(\xi) \quad \text{and} \quad (\sigma(\xi) - \varepsilon) Z(\xi) \leq H \Lambda^{\sigma}(\xi). \quad (13)$$

Then

$$\begin{aligned} &\left(\int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{Y^{\sigma}(\xi)}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ &\leq G \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} + H \left(\int_{\varepsilon}^{\infty} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}, \end{aligned} \quad (14)$$

where

$$Y(\xi) = \int_{\varepsilon}^{\xi} Z(\kappa) \Xi(\kappa) \Delta \kappa, \quad \psi(\xi) = \int_{\varepsilon}^{\xi} [Z(y)]^{\frac{1}{\ell}} \Xi(y) \Delta y \quad \text{and} \quad \Lambda(\xi) = \int_{\varepsilon}^{\xi} Z(\kappa) \Delta \kappa. \quad (15)$$

Proof. Denote

$$h(\kappa) = (Z^{\sigma}(\kappa))^{-\frac{1}{\ell}} \left| Z^{\Delta}(\kappa) \right| \psi^{\sigma}(\kappa). \quad (16)$$

Since $1/\ell + 1/\ell^* = 1$, we obtain

$$\begin{aligned}
 & \int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{Y^{\sigma}(\xi)}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \\
 &= \int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{\int_{\varepsilon}^{\sigma(\xi)} Z(\kappa) \Xi(\kappa) \Delta \kappa}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \\
 &= \int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{\int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} [Z(\kappa)]^{\frac{1}{\ell}} \Xi(\kappa) \Delta \kappa}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \\
 &= \int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{\int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} \psi^{\Delta}(\kappa) \Delta \kappa}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi. \tag{17}
 \end{aligned}$$

Applying (9) on $\int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} \psi^{\Delta}(\kappa) \Delta \kappa$, with $u(\kappa) = [Z(\kappa)]^{\frac{1}{\ell^*}}$ and $v(\kappa) = \psi(\kappa)$, we see that

$$\begin{aligned}
 & \int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} \psi^{\Delta}(\kappa) \Delta \kappa \\
 &= [Z(\kappa)]^{\frac{1}{\ell^*}} \psi(\kappa) \Big|_{\varepsilon}^{\sigma(\xi)} - \int_{\varepsilon}^{\sigma(\xi)} \left[[Z(\kappa)]^{\frac{1}{\ell^*}} \right]^{\Delta} \psi^{\sigma}(\kappa) \Delta \kappa. \tag{18}
 \end{aligned}$$

Since $\psi(\varepsilon) = 0$, we have that

$$\begin{aligned}
 & \int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} \psi^{\Delta}(\kappa) \Delta \kappa \\
 &= (Z^{\sigma}(\xi))^{\frac{1}{\ell^*}} \psi^{\sigma}(\xi) + \int_{\varepsilon}^{\sigma(\xi)} \left[-[Z(\kappa)]^{\frac{1}{\ell^*}} \right]^{\Delta} \psi^{\sigma}(\kappa) \Delta \kappa. \tag{19}
 \end{aligned}$$

Applying (8) on $\left[-[Z(\kappa)]^{\frac{1}{\ell^*}} \right]^{\Delta}$, we observe that

$$\left[-[Z(\kappa)]^{\frac{1}{\ell^*}} \right]^{\Delta} = \frac{-1}{\ell^*} Z^{-\frac{1}{\ell}}(\xi) Z^{\Delta}(\kappa), \quad \xi \in [\kappa, \sigma(\kappa)]. \tag{20}$$

Since $Z(\kappa)$ is a non-increasing function, then $Z^{\Delta}(\kappa) \leq 0$ and then

$$\left| Z^{\Delta}(\kappa) \right| = -Z^{\Delta}(\kappa). \tag{21}$$

Since $\xi \leq \sigma(\kappa)$ and $Z(\kappa)$ is a non-increasing function, we observe that

$$Z^{-\frac{1}{\ell}}(\xi) \leq [Z^{\sigma}(\kappa)]^{-\frac{1}{\ell}}. \tag{22}$$

Substituting (21) and (22) into (20), we obtain

$$\left[-[Z(\kappa)]^{\frac{1}{\ell^*}} \right]^{\Delta} \leq \frac{1}{\ell^*} [Z^{\sigma}(\kappa)]^{-\frac{1}{\ell}} \left| Z^{\Delta}(\kappa) \right|,$$

and then we have from (19) that

$$\begin{aligned}
 & \int_{\varepsilon}^{\sigma(\xi)} [Z(\kappa)]^{\frac{1}{\ell^*}} \psi^{\Delta}(\kappa) \Delta \kappa \\
 &\leq (Z^{\sigma}(\xi))^{\frac{1}{\ell^*}} \psi^{\sigma}(\xi) + \frac{1}{\ell^*} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{-\frac{1}{\ell}} \left| Z^{\Delta}(\kappa) \right| \psi^{\sigma}(\kappa) \Delta \kappa. \tag{23}
 \end{aligned}$$

Substituting (23) into (17), we observe that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{Y^{\sigma}(\xi)}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \\ & \leq \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \\ & \quad \times \left((Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi) + \frac{1}{\ell^{*}} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi, \end{aligned}$$

and then

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{Y^{\sigma}(\xi)}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right. \\ & \quad \times \left. \left((Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi) + \frac{1}{\ell^{*}} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (24)$$

Applying (11) to

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right. \\ & \quad \times \left. \left((Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi) + \frac{1}{\ell^{*}} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}, \end{aligned}$$

with

$$\chi(\xi) = \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}}, \quad \lambda(\xi) = (Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi),$$

and

$$\omega(\xi) = \frac{1}{\ell^{*}} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa,$$

we see that

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right. \\ & \quad \times \left. \left((Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi) + \frac{1}{\ell^{*}} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[(Z^{\sigma}(\xi))^{\frac{1}{\ell^{*}}} \psi^{\sigma}(\xi) \right]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \quad + \frac{1}{\ell^{*}} \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (25)$$

Since $Z(\xi)$ is a non-increasing function and $\sigma(\xi) \geq \xi$, we observe that $Z^\sigma(\xi) \leq Z(\xi)$. Moreover, we then have from (13) that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left[(Z^\sigma(\xi))^{\frac{1}{\ell^*}} \psi^\sigma(\xi) \right]^\ell \Delta \xi \\ & \leq \int_{\varepsilon}^{\infty} \frac{[Z(\xi)]^\ell}{(\Lambda^\sigma(\xi))^\ell} [\psi^\sigma(\xi)]^\ell \Delta \xi \\ & \leq H^\ell \int_{\varepsilon}^{\infty} \left[\frac{\psi^\sigma(\xi)}{\sigma(\xi) - \varepsilon} \right]^\ell \Delta \xi. \end{aligned} \quad (26)$$

From (16), we observe that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^\sigma(\kappa))^{\frac{-1}{\ell}} |Z^\Delta(\kappa)| \psi^\sigma(\kappa) \Delta \kappa \right]^\ell \Delta \xi \\ & = \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left[\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \Delta \kappa \right]^\ell \Delta \xi. \end{aligned} \quad (27)$$

Applying Lemma 4, we see that

$$\begin{aligned} & \left[\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \Delta \kappa \right]^\ell \\ & \leq \ell \int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \left(\int_{\varepsilon}^{\sigma(\kappa)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa, \end{aligned} \quad (28)$$

and then we have from (27) that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^\sigma(\kappa))^{\frac{-1}{\ell}} |Z^\Delta(\kappa)| \psi^\sigma(\kappa) \Delta \kappa \right]^\ell \Delta \xi \\ & \leq \ell \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left(\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \left(\int_{\varepsilon}^{\sigma(\kappa)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa \right) \Delta \xi. \end{aligned} \quad (29)$$

Applying Lemma 5, we observe that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^\sigma(\xi))^\ell} \left(\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \left(\int_{\varepsilon}^{\sigma(\kappa)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa \right) \Delta \xi \\ & = \int_{\varepsilon}^{\infty} h(\xi) \left(\int_{\varepsilon}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \left(\int_{\xi}^{\infty} \frac{Z(\kappa)}{(\Lambda^\sigma(\kappa))^\ell} \Delta \kappa \right) \Delta \xi. \end{aligned} \quad (30)$$

Applying (8) on $\Lambda^{1-\ell}(\kappa)$, we see that

$$\frac{1}{1-\ell} \left[(\Lambda(\kappa))^{1-\ell} \right]^\Delta = [\Lambda(\xi)]^{-\ell} Z(\kappa),$$

where $\xi \in [\kappa, \sigma(\kappa)]$. Since $\Lambda^\Delta(\kappa) = Z(\kappa) > 0$, Λ is an increasing function on $[\varepsilon, \infty)_{\mathbb{T}}$; and then we have for $\xi \leq \sigma(\kappa)$ that

$$[\Lambda(\xi)]^{-\ell} \geq [\Lambda^\sigma(\kappa)]^{-\ell},$$

thus

$$\frac{1}{1-\ell} [(\Lambda(\kappa))^{1-\ell}]^{\Delta} \geq \frac{Z(\kappa)}{(\Lambda^{\sigma}(\kappa))^{\ell}}, \quad (31)$$

Integrating (31) over κ from ξ to ∞ , we have

$$\begin{aligned} \int_{\xi}^{\infty} \frac{Z(\kappa)}{(\Lambda^{\sigma}(\kappa))^{\ell}} \Delta \kappa &\leq \frac{1}{1-\ell} \int_{\xi}^{\infty} [(\Lambda(\kappa))^{1-\ell}]^{\Delta} \Delta \kappa \\ &= \frac{1}{\ell-1} \left[[\Lambda(\xi)]^{1-\ell} - \left(\int_{\xi}^{\infty} Z(\kappa) \Delta \kappa \right)^{1-\ell} \right] \\ &\leq \frac{1}{\ell-1} [\Lambda(\xi)]^{1-\ell}. \end{aligned} \quad (32)$$

Substituting (32) into (30), we observe that

$$\begin{aligned} &\int_{\xi}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left(\int_{\xi}^{\sigma(\xi)} h(\kappa) \left(\int_{\xi}^{\sigma(\kappa)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa \right) \Delta \xi \\ &\leq \frac{1}{\ell-1} \int_{\xi}^{\infty} h(\xi) [\Lambda(\xi)]^{1-\ell} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi. \end{aligned} \quad (33)$$

From (33) and (29), we have that

$$\begin{aligned} &\int_{\xi}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\xi}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \\ &\leq \frac{\ell}{\ell-1} \int_{\xi}^{\infty} h(\xi) [\Lambda(\xi)]^{1-\ell} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi \\ &= \frac{\ell}{\ell-1} \int_{\xi}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\frac{\ell-1}{\ell}} h(\xi) [\Lambda(\xi)]^{1-\ell} \\ &\quad \times \left[\frac{Z^{\sigma}(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right]^{\frac{\ell-1}{\ell}} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi. \end{aligned} \quad (34)$$

Applying (10) to

$$\int_{\xi}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\frac{\ell-1}{\ell}} h(\xi) [\Lambda(\xi)]^{1-\ell} \left[\frac{Z^{\sigma}(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right]^{\frac{\ell-1}{\ell}} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi,$$

with indices $\ell > 1$ and $\ell/(\ell-1)$, we obtain

$$\begin{aligned} &\int_{\xi}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\frac{\ell-1}{\ell}} h(\xi) [\Lambda(\xi)]^{1-\ell} \\ &\quad \times \left[\frac{Z^{\sigma}(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right]^{\frac{\ell-1}{\ell}} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi \\ &\leq \left(\int_{\xi}^{\infty} \frac{Z^{\sigma}(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left(\int_{\xi}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell} \Delta \xi \right)^{\frac{\ell-1}{\ell}} \\ &\quad \times \left(\int_{\xi}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\ell-1} h^{\ell}(\xi) [\Lambda(\xi)]^{\ell(1-\ell)} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (35)$$

Substituting (35) into (34), we observe that

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \\ & \leq \frac{\ell}{\ell-1} \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left(\int_{\varepsilon}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell} \Delta \xi \right)^{\frac{\ell-1}{\ell}} \\ & \quad \times \left(\int_{\varepsilon}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\ell-1} h^{\ell}(\xi) [\Lambda(\xi)]^{\ell(1-\ell)} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (36)$$

Since Z is a non-increasing function, then $Z^{\sigma}(\xi) \leq Z(\xi)$, and the inequality (36) becomes

$$\begin{aligned} & \int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \\ & \leq \frac{\ell}{\ell-1} \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left(\int_{\varepsilon}^{\sigma(\xi)} h(\tau) \Delta \tau \right)^{\ell} \Delta \xi \right)^{\frac{\ell-1}{\ell}} \\ & \quad \times \left(\int_{\varepsilon}^{\infty} \left[\frac{(\Lambda^{\sigma}(\xi))^{\ell}}{Z^{\sigma}(\xi)} \right]^{\ell-1} h^{\ell}(\xi) [\Lambda(\xi)]^{\ell(1-\ell)} \Delta \xi \right)^{\frac{1}{\ell}}, \end{aligned}$$

and from (16), we have that

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq \frac{\ell}{\ell-1} \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} h^{\ell}(\xi) [Z^{\sigma}(\xi)]^{1-\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & = \frac{\ell}{\ell-1} \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \frac{1}{(Z^{\sigma}(\xi))^{\ell}} |Z^{\Delta}(\xi)|^{\ell} (\psi^{\sigma}(\xi))^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (37)$$

Substituting (13) into (37), we observe that

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} h(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq G^{\ell*} \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \frac{(\psi^{\sigma}(\xi))^{\ell}}{(\sigma(\xi) - \varepsilon)^{\ell}} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (38)$$

From (27) and (38), we see that

$$\begin{aligned} & \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \left[\int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq G^{\ell*} \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \frac{(\psi^{\sigma}(\xi))^{\ell}}{(\sigma(\xi) - \varepsilon)^{\ell}} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (39)$$

From (25), (26) and (39), we have that

$$\begin{aligned}
& \left(\int_{\varepsilon}^{\infty} \frac{Z(\xi)}{(\Lambda^{\sigma}(\xi))^{\ell}} \right. \\
& \times \left. \left((Z^{\sigma}(\xi))^{\frac{1}{\ell^*}} \psi^{\sigma}(\xi) + \frac{1}{\ell^*} \int_{\varepsilon}^{\sigma(\xi)} (Z^{\sigma}(\kappa))^{\frac{-1}{\ell}} |Z^{\Delta}(\kappa)| \psi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\
& \leq G \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} + H \left(\int_{\varepsilon}^{\infty} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \quad (40)
\end{aligned}$$

Substituting (40) into (24), we observe that

$$\begin{aligned}
& \left(\int_{\varepsilon}^{\infty} Z(\xi) \left(\frac{Y^{\sigma}(\xi)}{\Lambda^{\sigma}(\xi)} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\
& \leq G \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} + H \left(\int_{\varepsilon}^{\infty} \left(\frac{\psi^{\sigma}(\xi)}{\sigma(\xi) - \varepsilon} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}},
\end{aligned}$$

which is (14). $\square$

Remark 1. If $\mathbb{T} = \mathbb{R}$ and $\varepsilon = 0$, then $\sigma(\xi) = \xi$, $Z^{\sigma}(\xi) = Z(\xi)$, $Z^{\Delta}(\xi) = Z'(\xi)$, $\Lambda^{\sigma}(\xi) = \Lambda(\xi)$ and $\frac{\Lambda^{\sigma}(\xi)}{\Lambda(\xi)} = 1$; thus, we obtain (5).

Corollary 1. If $\mathbb{T} = \mathbb{N}_0$, $\varepsilon = 0$, $\ell > 1$, then $Z(\xi) > 0$ is a non-increasing function and there exists G, H , which are a positive constants, such that

$$(\xi + 1)|\Delta Z(\xi)| \leq GZ(\xi + 1), \text{ and } (\xi + 1)Z(\xi) \leq H\Lambda(\xi + 1),$$

then $\sigma(\xi) = \xi + 1$, and the inequality

$$\begin{aligned}
& \left[\sum_{\xi=0}^{\infty} Z(\xi) \left(\frac{Y(\xi + 1)}{\Lambda(\xi + 1)} \right)^{\ell} \right]^{\frac{1}{\ell}} \\
& \leq (2^{\ell-1}G + H) \left(\sum_{\xi=0}^{\infty} \left(\frac{\psi(\xi + 1)}{\xi + 1} \right)^{\ell} \right)^{\frac{1}{\ell}}, \quad (41)
\end{aligned}$$

is satisfied, where

$$Y(\xi) = \sum_{\kappa=0}^{\xi-1} Z(\kappa) \Xi(\kappa), \quad \psi(\xi) = \sum_{j=0}^{\xi-1} [Z(j)]^{\frac{1}{\ell}} \Xi(j),$$

and

$$\Lambda(\xi) = \sum_{\kappa=0}^{\xi-1} Z(\kappa).$$

Proof. For $\mathbb{T} = \mathbb{N}_0$, $\varepsilon = 0$, we have $\sigma(\xi) = \xi + 1$ and (14) becomes

$$\begin{aligned}
& \left[\sum_{\xi=0}^{\infty} Z(\xi) \left(\frac{Y(\xi + 1)}{\Lambda(\xi + 1)} \right)^{\ell} \right]^{\frac{1}{\ell}} \\
& \leq G \left(\int_{\varepsilon}^{\infty} \left[\frac{\Lambda(\xi + 1)}{\Lambda(\xi)} \right]^{\ell(\ell-1)} \left(\frac{\psi(\xi + 1)}{\xi + 1} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\
& \quad + H \left(\int_{\varepsilon}^{\infty} \left(\frac{\psi(\xi + 1)}{\xi + 1} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \quad (42)
\end{aligned}$$

Since $Z(\xi)$ is a non-increasing function, we obtain $Z(\xi) \leq Z(\xi - 1)$, and so

$$\begin{aligned}\Lambda(\xi + 1) &= \sum_{\kappa=0}^{\xi} Z(\kappa) = \sum_{\kappa=0}^{\xi-1} Z(\kappa) + Z(\xi) \\ &\leq \sum_{\kappa=0}^{\xi-1} Z(\kappa) + Z(\xi - 1) \leq 2 \sum_{\kappa=0}^{\xi-1} Z(\kappa) = 2\Lambda(\xi).\end{aligned}\quad (43)$$

Substituting (43) into (42), we observe that

$$\left[\sum_{\xi=0}^{\infty} Z(\xi) \left(\frac{Y(\xi+1)}{\Lambda(\xi+1)} \right)^{\ell} \right]^{\frac{1}{\ell'}} \leq (2^{\ell-1}G + H) \left(\int_{\varepsilon}^{\infty} \left(\frac{\psi(\xi+1)}{\xi+1} \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}},$$

which is (41). $\square$

Theorem 5. Assume that $\varepsilon \in \mathbb{T}$, $\ell > 1$ with $1/\ell + 1/\ell^* = 1$ and $\Xi(\xi)$ is a non-negative function on $[\varepsilon, \infty)_{\mathbb{T}}$. If $Z(\xi) > 0$ is a differentiable and non-increasing function on $[\varepsilon, \infty)_{\mathbb{T}}$ and there exists G, H , which are positive constants, such that

$$(\xi - b) \left| Z^{\Delta}(\xi) \right| \leq GZ^{\sigma}(\xi) \text{ and } (\xi - b)Z(\xi) \leq B\Lambda(\xi), \quad (44)$$

then

$$\left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \frac{Z(\kappa)\Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \leq (H + \ell C) \left(\int_b^{\infty} [\varphi(\xi)]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}, \quad (45)$$

where $C = 1 + G + H$, $\varphi(\xi) = \int_{\xi}^{\infty} \frac{[Z(y)]^{\frac{1}{\ell}} \Xi(y)}{y-b} \Delta y$ and $\Lambda(\xi) = \int_b^{\xi} Z(y) \Delta y$.

Proof. Denote

$$\lambda(\kappa) = \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} \varphi^{\sigma}(\kappa). \quad (46)$$

Since

$$\varphi^{\Delta}(\xi) = \frac{-[Z(\xi)]^{\frac{1}{\ell}} \Xi(\xi)}{\xi - b} \leq 0 \text{ for } \xi > b,$$

then φ is a non-increasing function and thus

$$\begin{aligned}& \int_{\xi}^{\infty} \frac{Z(\kappa)\Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \\ &= - \int_{\xi}^{\infty} \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi^{\Delta}(\kappa) \Delta \kappa.\end{aligned}\quad (47)$$

Applying (9) to $\int_{\xi}^{\infty} \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi^{\Delta}(\kappa) \Delta \kappa$, with

$$u(\kappa) = \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \text{ and } v(\kappa) = \varphi(\kappa),$$

we see that

$$\begin{aligned}
& \int_{\xi}^{\infty} \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi^{\Delta}(\kappa) \Delta \kappa \\
&= \left. \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) \right|_{\xi}^{\infty} \\
&\quad - \int_{\xi}^{\infty} \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \varphi^{\sigma}(\kappa) \Delta \kappa.
\end{aligned} \tag{48}$$

From Lemma 7, we observe that

$$\lim_{\kappa \rightarrow \infty} \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi(\kappa) = 0,$$

and then we have from (48) that

$$\begin{aligned}
& \int_{\xi}^{\infty} \frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \varphi^{\Delta}(\kappa) \Delta \kappa \\
&= - \frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \\
&\quad - \int_{\xi}^{\infty} \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \varphi^{\sigma}(\kappa) \Delta \kappa.
\end{aligned} \tag{49}$$

From (47) and (49), we observe that

$$\begin{aligned}
& \int_{\xi}^{\infty} \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \\
&= \frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) + \int_{\xi}^{\infty} \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \varphi^{\sigma}(\kappa) \Delta \kappa \\
&\leq \frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \\
&\quad + \int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa.
\end{aligned} \tag{50}$$

From (50), we see that

$$\begin{aligned}
& \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\
&\leq \left(\int_b^{\infty} Z(\xi) \left(\frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \right. \right. \\
&\quad \left. \left. + \int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}.
\end{aligned} \tag{51}$$

Applying (11) onto the right-hand side of (51) with

$$\chi(\xi) = Z(\xi), \quad \lambda(\xi) = \frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi),$$

and

$$\omega(\xi) = \int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa,$$

we see that

$$\begin{aligned} & \left(\int_b^{\infty} Z(\xi) \left(\frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \right. \right. \\ & \quad \left. \left. + \int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq \left(\int_b^{\infty} Z(\xi) \left(\frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \quad + \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (52)$$

Substituting (52) into (51), we obtain that

$$\begin{aligned} & \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq \left(\int_b^{\infty} Z(\xi) \left(\frac{(\xi - b)[Z(\xi)]^{\frac{1}{\ell^*}}}{\Lambda(\xi)} \varphi(\xi) \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \quad + \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (53)$$

From (44) and (53), we have that

$$\begin{aligned} & \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq H \left(\int_b^{\infty} [\varphi(\xi)]^{\ell} \Delta \xi \right)^{\frac{1}{\ell}} \\ & \quad + \left(\int_b^{\infty} Z(\xi) \left(\int_{\xi}^{\infty} \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^{\Delta} \right| \varphi^{\sigma}(\kappa) \Delta \kappa \right)^{\ell} \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (54)$$

Applying the quotient rule of differentiation to $\left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)$, we observe that

$$\left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)^{\Delta} = \frac{\Lambda(\kappa)\left[(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}\right]^{\Delta} - (\kappa-b)[Z(\kappa)]^{1+\frac{1}{\ell^*}}}{\Lambda(\kappa)\Lambda^{\sigma}(\kappa)}. \quad (55)$$

Applying the multiple rule of differentiation to $(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}$, we have that

$$\left[(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}\right]^{\Delta} = (\kappa-b)\left[Z(\kappa)]^{\frac{1}{\ell^*}}\right]^{\Delta} + [Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}. \quad (56)$$

Applying (8) to $[Z(\kappa)]^{\frac{1}{\ell^*}}$, $\ell^* > 1$, we see that

$$\left[Z(\kappa)]^{\frac{1}{\ell^*}}\right]^{\Delta} = \frac{1}{\ell^*}[Z(\xi)]^{\frac{1}{\ell^*}-1}Z^{\Delta}(\kappa), \quad \xi \in [\kappa, \sigma(\kappa)]. \quad (57)$$

From (55) to (57), we obtain

$$\begin{aligned} & \left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)^{\Delta} \\ &= \frac{\frac{1}{\ell^*}(\kappa-b)[Z(\xi)]^{\frac{1}{\ell^*}-1}Z^{\Delta}(\kappa)}{\Lambda^{\sigma}(\kappa)} + \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} - \frac{(\kappa-b)[Z(\kappa)]^{1+\frac{1}{\ell^*}}}{\Lambda(\kappa)\Lambda^{\sigma}(\kappa)}, \end{aligned}$$

and then

$$\begin{aligned} & \left|\left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)^{\Delta}\right| \\ &\leq \frac{\frac{1}{\ell^*}(\kappa-b)[Z(\xi)]^{\frac{1}{\ell^*}-1}|Z^{\Delta}(\kappa)|}{\Lambda^{\sigma}(\kappa)} + \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} + \frac{(\kappa-b)[Z(\kappa)]^{1+\frac{1}{\ell^*}}}{\Lambda(\kappa)\Lambda^{\sigma}(\kappa)}. \end{aligned} \quad (58)$$

Since $\xi \leq \sigma(\kappa)$ and $Z(\kappa)$ is a non-increasing function, we have that $Z(\xi) \geq Z^{\sigma}(\kappa)$, and then

$$[Z(\xi)]^{\frac{1}{\ell^*}-1} \leq [Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}-1}. \quad (59)$$

Substituting (59) into (58), we observe that

$$\begin{aligned} & \left|\left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)^{\Delta}\right| \\ &\leq \frac{\frac{1}{\ell^*}(\kappa-b)[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}-1}|Z^{\Delta}(\kappa)|}{\Lambda^{\sigma}(\kappa)} + \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} + \frac{(\kappa-b)[Z(\kappa)]^{1+\frac{1}{\ell^*}}}{\Lambda(\kappa)\Lambda^{\sigma}(\kappa)} \\ &= \frac{\frac{1}{\ell^*}(\kappa-b)[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}-1}|Z^{\Delta}(\kappa)|}{Z^{\sigma}(\kappa)\Lambda^{\sigma}(\kappa)} + \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} + \frac{(\kappa-b)[Z(\kappa)]^{1+\frac{1}{\ell^*}}}{\Lambda(\kappa)\Lambda^{\sigma}(\kappa)}. \end{aligned} \quad (60)$$

Using (44), the inequality (60) becomes

$$\begin{aligned} & \left|\left(\frac{(\kappa-b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)}\right)^{\Delta}\right| \\ &\leq \frac{b}{\ell^*} \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} + \frac{[Z^{\sigma}(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)} + H \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^{\sigma}(\kappa)}. \end{aligned} \quad (61)$$

Since $Z(\kappa)$ is non-increasing and $\sigma(\kappa) \geq \kappa$, we have that $Z^\sigma(\kappa) \leq Z(\kappa)$, and then the inequality (61) is

$$\begin{aligned} & \left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^\Delta \right| \\ & \leq \frac{b}{\ell^*} \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} + \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} + H \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \\ & = \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \left(\frac{G}{\ell^*} + 1 + H \right). \end{aligned} \quad (62)$$

Since $1/\ell^* < 1$, then the inequality (62) becomes

$$\left| \left(\frac{(\kappa - b)[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda(\kappa)} \right)^\Delta \right| \leq \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} [1 + G + H] = C \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)}. \quad (63)$$

Substituting (63) into (54), we have that

$$\begin{aligned} & \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^\ell \Delta \xi \right)^{\frac{1}{\ell}} \\ & \leq H \left(\int_b^\infty [\varphi(\xi)]^\ell \Delta \xi \right)^{\frac{1}{\ell}} \\ & + C \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta \kappa \right)^\ell \Delta \xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (64)$$

Using (46), and by applying Lemma 6, we see that

$$\begin{aligned} & \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta \kappa \right)^\ell \\ & = \left(\int_\xi^\infty \lambda(\kappa) \Delta \kappa \right)^\ell \\ & \leq \ell \int_\xi^\infty \lambda(\kappa) \left(\int_\kappa^\infty \lambda(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa, \end{aligned}$$

and then

$$\begin{aligned} & \int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta \kappa \right)^\ell \Delta \xi \\ & \leq \ell \int_b^\infty Z(\xi) \left[\int_\xi^\infty \lambda(\kappa) \left(\int_\kappa^\infty \lambda(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa \right] \Delta \xi. \end{aligned} \quad (65)$$

Applying Lemma 5, we observe that

$$\begin{aligned} & \int_b^\infty Z(\xi) \left[\int_\xi^\infty \lambda(\kappa) \left(\int_\kappa^\infty \lambda(\tau) \Delta \tau \right)^{\ell-1} \Delta \kappa \right] \Delta \xi \\ & = \int_b^\infty \lambda(\xi) \left(\int_\xi^\infty \lambda(\tau) \Delta \tau \right)^{\ell-1} \left(\int_b^{\sigma(\xi)} Z(\kappa) \Delta \kappa \right) \Delta \xi \\ & = \int_b^\infty \lambda(\xi) \Lambda^\sigma(\xi) \left(\int_\xi^\infty \lambda(\tau) \Delta \tau \right)^{\ell-1} \Delta \xi. \end{aligned} \quad (66)$$

Substituting (66) into (65), we have that

$$\begin{aligned} & \int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta\kappa \right)^\ell \Delta\xi \\ & \leq \ell \int_b^\infty \lambda(\xi) \Lambda^\sigma(\xi) \left(\int_\xi^\infty \lambda(\tau) \Delta\tau \right)^{\ell-1} \Delta\xi \\ & = \ell \int_b^\infty \frac{\lambda(\xi) \Lambda^\sigma(\xi)}{[Z(\xi)]^{\frac{1}{\ell^*}}} [Z(\xi)]^{\frac{1}{\ell^*}} \left(\int_\xi^\infty \lambda(\tau) \Delta\tau \right)^{\ell-1} \Delta\xi. \end{aligned} \quad (67)$$

Applying (10), we see that

$$\begin{aligned} & \int_b^\infty \frac{\lambda(\xi) \Lambda^\sigma(\xi)}{[Z(\xi)]^{\frac{1}{\ell^*}}} \left[[Z(\xi)]^{\frac{1}{\ell^*}} \left(\int_\xi^\infty \lambda(\tau) \Delta\tau \right)^{\ell-1} \right] \Delta\xi \\ & \leq \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \lambda(\tau) \Delta\tau \right)^\ell \Delta\xi \right)^{\frac{\ell-1}{\ell}} \left(\int_b^\infty \frac{(\lambda(\xi) \Lambda^\sigma(\xi))^\ell}{[Z(\xi)]^{\ell-1}} \Delta\xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (68)$$

Substituting (68) into (67), we see that

$$\begin{aligned} & \int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta\kappa \right)^\ell \Delta\xi \\ & \leq \ell \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \lambda(\tau) \Delta\tau \right)^\ell \Delta\xi \right)^{1-\frac{1}{\ell}} \left(\int_b^\infty \frac{(\lambda(\xi) \Lambda^\sigma(\xi))^\ell}{Z^{\ell-1}(\xi)} \Delta\xi \right)^{\frac{1}{\ell}}. \end{aligned}$$

From (46), the last inequality becomes

$$\begin{aligned} & \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{[Z(\kappa)]^{\frac{1}{\ell^*}}}{\Lambda^\sigma(\kappa)} \varphi^\sigma(\kappa) \Delta\kappa \right)^\ell \Delta\xi \right)^{\frac{1}{\ell}} \\ & \leq \ell \left(\int_b^\infty \frac{(\lambda(\xi) \Lambda^\sigma(\xi))^\ell}{[Z(\xi)]^{\ell-1}} \Delta\xi \right)^{\frac{1}{\ell}} \\ & = \ell \left(\int_b^\infty (\varphi^\sigma(\xi))^\ell \Delta\xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (69)$$

Substituting (69) into (64), we see that

$$\begin{aligned} & \left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta\kappa \right)^\ell \Delta\xi \right)^{\frac{1}{\ell}} \\ & \leq H \left(\int_b^\infty [\varphi(\xi)]^\ell \Delta\xi \right)^{\frac{1}{\ell}} \\ & \quad + \ell C \left(\int_b^\infty (\varphi^\sigma(\xi))^\ell \Delta\xi \right)^{\frac{1}{\ell}}. \end{aligned} \quad (70)$$

Since φ is a non-increasing function and $\sigma(\xi) \geq \xi$, then $\varphi^\sigma(\xi) \leq \varphi(\xi)$, and then the inequality (70) becomes

$$\left(\int_b^\infty Z(\xi) \left(\int_\xi^\infty \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \Delta \kappa \right)^\ell \Delta \xi \right)^{\frac{1}{\ell}} \\ \leq (H + \ell C) \left(\int_b^\infty [\varphi(\xi)]^\ell \Delta \xi \right)^{\frac{1}{\ell}},$$

which is (45). $\square$

Remark 2. If $\mathbb{T} = \mathbb{R}$, $b = 0$, then $\sigma(\xi) = \xi$, $Z^\sigma(\xi) = Z(\xi)$ and $Z^\Delta(\xi) = Z'(\xi)$, and so we obtain (6).

Corollary 2. Assume that $\mathbb{T} = \mathbb{N}_0$, $b = 1$, $\ell > 1$ and $\Xi(\xi)$ is a non-negative sequence. Let $Z(\xi) > 0$ be a non-increasing sequence and there exists G, H as positive constants, such that

$$(\xi - 1)|\Delta Z(\xi)| \leq GZ(\xi + 1) \text{ and } (\xi - 1)Z(\xi) \leq H\Lambda(\xi).$$

Then

$$\sum_{\xi=1}^{\infty} Z(\xi) \left(\sum_{\kappa=\xi}^{\infty} \frac{Z(\kappa) \Xi(\kappa)}{\Lambda(\kappa)} \right)^\ell \leq (H + \ell C)^\ell \sum_{\xi=1}^{\infty} [\varphi(\xi)]^\ell,$$

where $C = 1 + G + H$,

$$\varphi(\xi) = \sum_{\delta=\xi}^{\infty} \frac{[Z(\delta)]^{\frac{1}{\ell}} \Xi(\delta)}{\delta} \text{ and } \Lambda(\xi) = \sum_{\kappa=1}^{\xi-1} Z(\kappa).$$

4. Conclusions

In this manuscript, we have demonstrated novel Copson-type inequalities within the framework of time scales delta calculus for the unification of inequalities in the two (the continuous and the discrete) calculi, focusing on a positive, differentiable, and non-increasing function. Looking ahead, we plan to extend our findings by establishing fresh dynamic inequalities of Copson type using time scales nabla calculus. Furthermore, we aim to generalize these inequalities to diamond alpha calculus, a linear combination of delta calculus and nabla calculus. Additionally, our future work includes the exploration of new dynamic inequalities through conformable delta and nabla fractional calculus.

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