

Diamond-Type Dirac Dynamic System in Mathematical Physics

Tuba Gulsen ¹, Ayşe Çiğdem Yar ² and Emrah Yılmaz ^{1,*}

¹ Department of Mathematics, Faculty of Science, Firat University, 23119 Elazığ, Türkiye; tyalcin@firat.edu.tr

² Department of Mathematics, Institute of Science, Firat University, 23119 Elazığ, Türkiye; 192121201@firat.edu.tr or ayseyar23@gmail.com

* Correspondence: eyilmaz@firat.edu.tr

Abstract: In order to merge continuous and discrete analyses, a number of dynamic derivative equations have been put out in the process of developing a time-scale calculus. The investigations that incorporated combined dynamic derivatives have led to the proposal of improved approximation expressions for computational application. One such expression is the diamond alpha ($\diamond_\alpha$) derivative, which is defined as a linear combination of delta and nabla derivatives. Several dynamic equations and inequalities, as well as hybrid dynamic behavior—which does not occur in the real line or on discrete time scales—are analyzed using this combined concept. In this study, we consider a $\diamond_\alpha$ Dirac system under boundary conditions on a uniform time scale. We examined some basic spectral properties of the problem we are considering, such as the simplicity, the reality of eigenvalues, orthogonality of eigenfunctions, and self adjointness of the operator. Finally, we construct an expression for the eigenfunction of the $\diamond_\alpha$ Dirac boundary value problem (BVP) on a uniform time scale.

Keywords: time scale; ordinary differential operators; Dirac system



Citation: Gulsen, T.; Yar, A.C.; Yılmaz, E. Diamond-Type Dirac Dynamic System in Mathematical Physics. *Symmetry* **2024**, *16*, 318. <https://doi.org/10.3390/sym16030318>

Academic Editors: Abraham A. Ungar, Branko Dragovich and Sergei D. Odintsov

Received: 21 January 2024

Revised: 22 February 2024

Accepted: 29 February 2024

Published: 7 March 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary, non-empty, closed subset of $\mathbb{R}$. The time-scale theory was first stated by Hilger [1]. Hilger and Aulbach have obtained important results on this subject [2–4]. As a result of the selection of two operators considered on a time scale, σ and ρ operators, respectively, we encounter the concepts of delta and nabla calculations. A detailed theory of delta calculus was discussed by Bohner and Peterson [5,6]. The nabla theory can be found in the relevant references [7–10]. $\diamond_\alpha$ calculus, which is considered as a combination of delta and nabla calculi, was introduced by Sheng et al. [11]. They studied the basic properties of first and second order $\diamond_\alpha$ derivatives, which are linear combinations of delta and nabla dynamic calculus on a time scale. Here, diamond calculus corresponds to delta calculus when $\alpha = 1$ and nabla calculus when $\alpha = 0$. On the other hand, $\alpha = \frac{1}{2}$ represents a central derivative formula on any uniform discrete time scales [11]. There are many studies related to $\diamond_\alpha$ derivative on a time scale. In 2007, Rogers and Sheng studied the case of $\diamond_\alpha$ and the $\diamond_\alpha$ integral as a dynamic derivative on the time scale [12]. In this same year, Sheng studied an approach to the classical second-order derivative via $\diamond_\alpha$ dynamical differentiation under the necessary conditions on a time scale [13]. In 2008, Ammi and Torres gave the basic properties of $\diamond_\alpha$ derivatives and proved a generalized version of Jensen's inequalities on a time scale via $\diamond_\alpha$ integral [14]. In 2009, Mozyrska and Torres worked on a $\diamond_\alpha$ exponential function and homogeneous linear dynamic $\diamond_\alpha$ equations on a time scale [15]. In the same year, Mozyrska and Torres studied $\diamond_\alpha$ integrals on time scales and proved their mean value theorem [16]. In another study, they discussed the results of generalized polynomials covering the definitions and properties of delta and nabla derivatives on a time scale and generalized these results using the concept of the $\diamond_\alpha$ derivative [17]. In 2009, Ozkan and Kaymakçalan introduced double integral calculus for two-variable functions through partial $\diamond_\alpha$ dynamic derivatives and the $\diamond_\alpha$ dynamic

integral [18]. In 2010, Ammi and Torres studied of the Hardy-type dynamic inequalities on a time scale via the $\diamond_\alpha$ integral [19]. In 2015, Cruz, Martins, and Torres gave an improved version of the $\diamond_\alpha$ integral and proved new versions of some important inequalities [20].

To obtain the fundamental results for (5), we need to recall a few basic time-scale ideas. The following are the definitions of the forward and backward jump operators of $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$, for $t \in \mathbb{T}$:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, $\rho(t) < t$, $\rho(t) < t < \sigma(t)$, t is a right-scattered point, a left-scattered point, an isolated (discrete) point, respectively. On the other hand, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$; $t > \inf \mathbb{T}$ and $\rho(t) = t$; and $\rho(t) = t = \sigma(t)$; t is called the right-dense, left-dense, and dense point, respectively. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$. $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ if there is a maximum m point of $\mathbb{T}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. The $f : \mathbb{T} \rightarrow \mathbb{R}$ function is called rd-continuous, provided that $\mathbb{T}$ has a left-sided limit at its right-dense points and at its left-scattered points, and $C_{rd}(\mathbb{T})$ will be used to denote the collection of $f : \mathbb{T} \rightarrow \mathbb{R}$ rd-continuous functions. For all $t \in \mathbb{T}^\kappa$, if $1 + \mu(t)p(t) \neq 0$ holds, we say that function $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive [5].

Let $t \in \mathbb{T}^\kappa$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. For all $\varepsilon > 0$, and for every s in a neighborhood U of point t , if there is a real number $f^\Delta(t)$, such that

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \forall s \in U,$$

$f^\Delta(t)$ is called the Δ (delta) derivative of f at point t . Furthermore, if $f^\Delta(t)$ exists for every $t \in \mathbb{T}^\kappa$, we state that f is Δ differentiable. For all $\varepsilon > 0$, and for every s in a neighborhood U of point $t \in \mathbb{T}_\kappa$, if there is a real number $f^\nabla(t)$, such that

$$\left| [f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s] \right| \leq \varepsilon |\rho(t) - s|, \forall s \in U,$$

$f^\nabla(t)$ is called the ∇ (delta) derivative of f at point t . Moreover, we declare that f^∇ differentiable on $\mathbb{T}_\kappa$ if $f^\nabla(t)$ exists for each $t \in \mathbb{T}_\kappa$ [3,5,6,9].

Now, we will give the important definitions and theorems related to $\diamond_\alpha$ calculus [11–23], which we use as the basis for our study on a time scale.

Assume that $f(t)$ is differentiable in Δ and ∇ senses on a time scale $\mathbb{T}$. For $t \in \mathbb{T}_\kappa^\kappa$, where $\mathbb{T}_\kappa^\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$, we define $\diamond_\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t), \quad \alpha \in [0, 1]. \quad (1)$$

Consequently, if and only if f is both Δ and ∇ differentiable, then f is $\diamond_\alpha$ differentiable. We assume that $t \in \mathbb{T}$ is a dense point. If $f'(t)$ exists, then $f^{\diamond_\alpha}(t) = f^\Delta(t) = f^\nabla(t) = f'(t)$. This shows that $\diamond_\alpha$ derivative is an important generalization.

If $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are $\diamond_\alpha$ differentiable at $t_0 \in \mathbb{T}$, the following properties hold:

$$(\gamma_1 f + \gamma_2 g)^{\diamond_\alpha}(t_0) = (\gamma_1 f^{\diamond_\alpha} + \gamma_2 g^{\diamond_\alpha})(t_0), \text{ for all } \gamma_1, \gamma_2 \in \mathbb{R},$$

$$(fg)^{\diamond_\alpha}(t_0) = (f^{\diamond_\alpha} g + \alpha f^\sigma g^\Delta + (1 - \alpha) f^\rho g^\nabla)(t_0),$$

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t_0) = \frac{(f^{\diamond_\alpha} g^\sigma g^\rho - \alpha f^\sigma g^\rho g^\Delta - (1 - \alpha) f^\rho g^\sigma g^\nabla)(t_0)}{(g g^\sigma g^\rho)(t_0)}, \text{ where } g g^\sigma g^\rho(t_0) \neq 0.$$

Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \rightarrow \mathbb{R}$. The $\diamond_\alpha$ integral of h is defined as

$$\int_a^t h(\zeta) \diamond_\alpha \zeta = \alpha \int_a^t h(\zeta) \Delta \zeta + (1 - \alpha) \int_a^t h(\zeta) \nabla \zeta. \quad (2)$$

The following property is provided for $a, t \in \mathbb{T}, b, c \in \mathbb{R}$.

$$\int_a^t [bf(\zeta) + cg(\zeta)] \diamond_\alpha \zeta = b \int_a^t f(\zeta) \diamond_\alpha \zeta + c \int_a^t g(\zeta) \diamond_\alpha \zeta.$$

Let $t, t_i \in \mathbb{T}$, $p : \mathbb{T} \rightarrow \mathbb{R}$ is a regressive function, $p(t) \equiv p$, and $1 + v^2(t)p^2 \neq 0$ for all $t \in \mathbb{T}$ where $v(t)$ is the backward graininess function. Then, for $t \in \mathbb{T}_k^\kappa$ [24]

$$\sin_p^{\diamond_\alpha}(t, t_i) = \frac{p}{1 + v^2 p^2} ((1 + \alpha v^2 p^2) \cos_p(t, t_i) + (1 - \alpha) v p \sin_p(t, t_i)), \quad (3)$$

and

$$\cos_p^{\diamond_\alpha}(t, t_i) = \frac{-p}{1 + v^2 p^2} ((1 + \alpha v^2 p^2) \sin_p(t, t_i) - (1 - \alpha) v p \cos_p(t, t_i)). \quad (4)$$

The space of all square $\diamond_\alpha$ integrable functions on $[a, b]$, where $a, b \in \mathbb{T}$ and $\alpha \in [0, 1]$, is as follows.

$$L_2^{\diamond_\alpha}[a, b] = \left\{ f : \int_a^b f^2(t) \diamond_\alpha t < \infty \right\}.$$

We define the diamond-type inner product of $y(t, \lambda) = \begin{pmatrix} y_1(t, \lambda) \\ y_2(t, \lambda) \end{pmatrix}$ and $z(t, \lambda) = \begin{pmatrix} z_1(t, \lambda) \\ z_2(t, \lambda) \end{pmatrix}$ on $L_2^{\diamond_\alpha}[a, b]$ by

$$\langle y, z \rangle_{\diamond_\alpha} = \int_a^b [y_1(t, \lambda) z_1(t, \lambda) + y_2(t, \lambda) z_2(t, \lambda)] \diamond_\alpha t.$$

The space $L_2^{\diamond_\alpha}[0, N]$ and $\diamond_\alpha$ inner product will be used in the proof of spectral properties of a $\diamond_\alpha$ Dirac problem in the next section.

In this study, we consider a $\diamond_\alpha$ Dirac eigenvalue problem

$$D_{\diamond_\alpha} y \equiv B y^{\diamond_\alpha}(t) + Q(t) y(t) = \lambda y(t), t \in [0, N], N \in \mathbb{R}, \quad (5)$$

which is handled with boundary conditions

$$y_1(0) - h y_2(0) = 0, \quad (6)$$

$$y_1(N) + H y_2(N) = 0, \quad (7)$$

where

$$Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & r(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$h, H \in \mathbb{R}$; λ is a spectral parameter, and $q, r : [0, N] \cap \mathbb{T} \rightarrow \mathbb{R}$ are continuous functions. If the necessary adjustments are made in Equation (5), the following system is obtained:

$$\begin{aligned} y_2^{\diamond_\alpha}(t) &= (\lambda - q(t)) y_1(t), \\ y_1^{\diamond_\alpha}(t) &= (-\lambda + r(t)) y_2(t). \end{aligned} \quad (8)$$

By setting $\mathbb{T} = \mathbb{R}$ in (8), we obtain following classical Dirac system:

$$\begin{aligned} y_2'(t) &= (\lambda - q(t)) y_1(t), \\ y_1'(t) &= (-\lambda + r(t)) y_2(t). \end{aligned} \quad (9)$$

Equation (9) is called the first canonical form of the Dirac system. In quantum physics, the Dirac operator is the relativistic Schrödinger operator. Dirac proposed the Dirac operator for some difficulties encountered in quantum physics [25]. There are many studies on the Dirac operator and related problems, which is an important subject of study in

this field. Levitan and Sargsjan have discussed the spectral theory of Dirac operators in detail [26]. A study on the eigenfunction expansion for the Dirac operator was made by Joa and Minkin [27]. Bairamov, Aygar, and Olgun found the Jost solutions of self-adjoint Dirac systems and examined their analytical properties and asymptotic behavior. They demonstrated that there is a finite number of simple real eigenvalues in the Dirac system [28]. Keskin and Ozkan studied the properties of eigenvalues and eigenfunctions of the inverse spectral problem for the Dirac operator [29]. The Dirac system has also been studied on the time scale. Gulsen and Yilmaz examined the spectral theory of the Dirac system on the time scale [30]. Hovhannisyan studied the linear Dirac equation on the discrete continuous and quantum time scale [31]. Allahverdiev and Tuna introduced the q analogue of the Dirac equation and examined some spectral properties by investigating the existence and uniqueness of the solution [32]. Gulsen, Yilmaz, and Goktas examine the conformable Dirac system on time scales [33]. Koprubasi studied the impulsive discrete Dirac system with hyperbolic eigenparameters [34]. There are many studied theories regarding different versions of the Dirac system [35–42].

This study is organized as follows: After a reminder of some basic definitions and theorems regarding the calculation of $\diamond_\alpha$ on time scales in the introduction section, in Section 2, we examine some spectral properties of a $\diamond_\alpha$ Dirac problem on a uniform ($\mu(t) = \nu(t)$, for all $t \in \mathbb{T}_k^\kappa$ [12]) time scale, and we obtain an expression for the eigenfunction of the $\diamond_\alpha$ Dirac system.

2. Main Results

In this section, a $\diamond_\alpha$ Dirac BVP,

$$y_2^{\diamond_\alpha}(t) = (\lambda - q(t))y_1(t), \quad y_1^{\diamond_\alpha}(t) = (-\lambda + r(t))y_2(t), \quad (10)$$

$$y_1(0) - hy_2(0) = 0, \quad (11)$$

$$y_1(N) + Hy_2(N) = 0, \quad (12)$$

is discussed on a uniform time scale. The realness of eigenvalues and orthogonality of eigenfunctions on $L_2^{\diamond_\alpha}[0, N]$ are examined. The symmetry, boundedness, and linearity properties of Dirac operator,

$$D_{\diamond_\alpha} : X \rightarrow Y, D_{\diamond_\alpha} y = \begin{pmatrix} y_2^{\diamond_\alpha}(t) - [\lambda - q(t)]y_1(t) \\ y_1^{\diamond_\alpha}(t) - [-\lambda + r(t)]y_2(t) \end{pmatrix},$$

have been proven on $L_2^{\diamond_\alpha}[0, N]$, where X and Y are vector spaces. Later, we obtained some eigenfunction expansions for a $\diamond_\alpha$ Dirac problem. In this section, it is assumed that $\mathbb{T}$ is uniform.

Theorem 1 ([5]). *Let us suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$ on $\mathbb{T}^k$ is Δ differentiable. Then, for any $t \in \mathbb{T}^k$ such that $\sigma(\rho(t)) = t$, f is ∇ differentiable at t and $f^\nabla(t) = f^\Delta(\rho(t))$. Let us assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ on $\mathbb{T}_k$ is ∇ differentiable. Then, for any $t \in \mathbb{T}_k$ such that $\rho(\sigma(t)) = t$, f is Δ differentiable at t and $f^\Delta(t) = f^\nabla(\sigma(t))$.*

Lemma 1. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be $\diamond_\alpha$ integrable for $\forall t \in \mathbb{T}_k^\kappa$ and $\alpha \in (0, 1]$, $d_1, d_2 \in \mathbb{T}$. Then, the following equality holds:*

$$\int_{d_1}^{d_2} f^{\diamond_\alpha}(t) \diamond_\alpha t = (2\alpha^2 - 2\alpha + 1)f(t) \Big|_{d_1}^{d_2} + \alpha(1 - \alpha)(f^\rho + f^\sigma)(t) \Big|_{d_1}^{d_2}.$$

Proof. Using (1) then (2) and Theorem 1, we obtain

$$\begin{aligned}\int_{d_1}^{d_2} f^{\diamond_\alpha}(t) \diamond_\alpha t &= \int_{d_1}^{d_2} (\alpha f^\Delta + (1-\alpha)f^\nabla)(t) \diamond_\alpha t \\ &= \alpha^2 \int_{d_1}^{d_2} f^\Delta(t) \Delta t + \alpha(1-\alpha) \int_{d_1}^{d_2} f^{\rho\Delta}(t) \Delta t \\ &\quad + \alpha(1-\alpha) \int_{d_1}^{d_2} f^{\sigma\nabla}(t) \nabla t + (1-\alpha)^2 \int_{d_1}^{d_2} f^\nabla(t) \nabla t \\ &= (2\alpha^2 - 2\alpha + 1)f(t) \Big|_{d_1}^{d_2} + \alpha(1-\alpha)(f^\rho + f^\sigma)(t) \Big|_{d_1}^{d_2}.\end{aligned}$$

□

Lemma 2. Let $a, b \in \mathbb{T}^* = \mathbb{T} - \{\min \mathbb{T}, \max \mathbb{T}\}$, $h_1, h_2 : \mathbb{T} \rightarrow \mathbb{R}$ be $\diamond_\alpha$ integrable for $\forall \zeta \in \mathbb{T}_\kappa^\kappa$ and $\alpha \in (0, 1]$. Then, the $\diamond_\alpha$ integration by parts formula on $\mathbb{T}$ is constructed by

$$\begin{aligned}\int_a^b (h_1^{\diamond_\alpha} h_2)(\zeta) \diamond_\alpha \zeta &= (1 - 2\alpha + 2\alpha^2)(h_1 h_2)(\zeta) \Big|_a^b + \alpha(1-\alpha)[(h_1 h_2)^\rho + (h_1 h_2)^\sigma](\zeta) \Big|_a^b \\ &\quad + (1-\alpha) \int_a^b h_2^\nabla (h_1^\sigma - h_1^\rho)(\zeta) \diamond_\alpha \zeta - \int_a^b (h_1^\sigma h_2^{\diamond_\alpha})(\zeta) \diamond_\alpha \zeta.\end{aligned}$$

Proof. By applying the $\diamond_\alpha$ integral to the below equality,

$$(h_1 h_2)^{\diamond_\alpha}(\zeta) = h_1^{\diamond_\alpha}(\zeta) h_2(\zeta) + \alpha h_1^\sigma(\zeta) h_2^\Delta(\zeta) + (1-\alpha) h_1^\rho(\zeta) h_2^\nabla(\zeta),$$

on $[a, b]$, we get

$$\int_a^b (h_1 h_2)^{\diamond_\alpha}(\zeta) \diamond_\alpha \zeta = \int_a^b h_1^{\diamond_\alpha}(\zeta) h_2(\zeta) \diamond_\alpha \zeta + \alpha \int_a^b h_1^\sigma(\zeta) h_2^\Delta(\zeta) \diamond_\alpha \zeta + (1-\alpha) \int_a^b h_1^\rho(\zeta) h_2^\nabla(\zeta) \diamond_\alpha \zeta.$$

In the last equality, isolating the term $\int_a^b h_1^{\diamond_\alpha}(\zeta) h_2(\zeta) \diamond_\alpha \zeta$ yields

$$\begin{aligned}\int_a^b h_1^{\diamond_\alpha}(\zeta) h_2(\zeta) \diamond_\alpha \zeta &= (1 - 2\alpha + 2\alpha^2)(h_1 h_2)(\zeta) \Big|_a^b + \alpha(1-\alpha)[(h_1 h_2)^\rho + (h_1 h_2)^\sigma](\zeta) \Big|_a^b \\ &\quad - \alpha \int_a^b (h_1^\sigma h_2^\Delta)(\zeta) \diamond_\alpha \zeta - (1-\alpha) \int_a^b (h_1^\rho h_2^\nabla)(\zeta) \diamond_\alpha \zeta \\ &\quad - (1-\alpha) \int_a^b (h_1^\sigma h_2^\nabla)(\zeta) \diamond_\alpha \zeta + (1-\alpha) \int_a^b (h_1^\rho h_2^\Delta)(\zeta) \diamond_\alpha \zeta \\ &= (1 - 2\alpha + 2\alpha^2)(h_1 h_2)(\zeta) \Big|_a^b + \alpha(1-\alpha)[(h_1 h_2)^\rho + (h_1 h_2)^\sigma](\zeta) \Big|_a^b \\ &\quad - \int_a^b h_1^\sigma [\alpha h_2^\Delta + (1-\alpha) h_2^\nabla](\zeta) \diamond_\alpha \zeta + (1-\alpha) \int_a^b h_2^\nabla [h_1^\sigma - h_1^\rho](\zeta) \diamond_\alpha \zeta.\end{aligned}$$

□

Theorem 2. For $\lambda_1 \neq \lambda_2$, the distinct eigenfunctions of $\diamond_\alpha$ Dirac BVP (10)–(12) are orthogonal on $L_2^{\diamond_\alpha}[0, N]$.

Proof. Let $y(t, \lambda_1) = \begin{pmatrix} y_1(t, \lambda_1) \\ y_2(t, \lambda_1) \end{pmatrix}$ and $z(t, \lambda_2) = \begin{pmatrix} z_1(t, \lambda_2) \\ z_2(t, \lambda_2) \end{pmatrix}$ be two eigenfunctions of (10)–(12). Thus, these functions provide (10). Then, we get

$$\begin{aligned} y_2^{\diamond_\alpha} - [\lambda_1 - q(t)]y_1 &= 0, \\ y_1^{\diamond_\alpha} - [-\lambda_1 + r(t)]y_2 &= 0, \\ z_2^{\diamond_\alpha} - [\lambda_2 - q(t)]z_1 &= 0, \\ z_1^{\diamond_\alpha} - [-\lambda_2 + r(t)]z_2 &= 0. \end{aligned}$$

If the above equations are multiplied by z_1 , $-z_2$, $-y_1$, and y_2 , respectively, and rearranged, we get

$$z_1 y_2^{\diamond_\alpha} - z_2 y_1^{\diamond_\alpha} - y_1 z_2^{\diamond_\alpha} + y_2 z_1^{\diamond_\alpha} + [\lambda_1 - \lambda_2](z_1 y_1 + z_2 y_2) = 0.$$

Since $\mathbb{T}$ is uniform, the following equation can be written as

$$z_1 y_2^{\diamond_\alpha} - z_2 y_1^{\diamond_\alpha} - y_1 z_2^{\diamond_\alpha} + y_2 z_1^{\diamond_\alpha} = \diamond_\alpha [-y_1 z_2 + y_2 z_1].$$

If we use this expression in the previous equation and integrate both sides from 0 to N , we get

$$\int_0^N (-y_1 z_2 + y_2 z_1)^{\diamond_\alpha}(t) \diamond_\alpha t = (\lambda_2 - \lambda_1) \int_0^N (z_1 y_1 + z_2 y_2)(t) \diamond_\alpha t.$$

So,

$$\begin{aligned} (\lambda_2 - \lambda_1) \int_0^N (z_1 y_1 + z_2 y_2)(t) \diamond_\alpha t &= (2\alpha^2 - 2\alpha + 1)(z_1 y_2)(t) \Big|_0^N \\ &\quad + \alpha(1 - \alpha)(z_1 y_2)^\rho(t) + (z_1 y_2)^\sigma(t) \Big|_0^N \\ &\quad - (2\alpha^2 - 2\alpha + 1)(z_2 y_1)(t) \Big|_0^N \\ &\quad - \alpha(1 - \alpha)(z_2 y_1)^\rho(t) + (z_2 y_1)^\sigma(t) \Big|_0^N. \end{aligned}$$

Remembering that $\mathbb{T}$ is uniform and $y(x, \lambda_1)$ and $z(x, \lambda_2)$ fulfill the boundary conditions (11) and (12), we get

$$(\lambda_2 - \lambda_1) \int_0^N (z_1 y_1 + z_2 y_2)(t) \diamond_\alpha t = 0.$$

As a result, we have

$$(\lambda_2 - \lambda_1) \int_0^N (z_1 y_1 + z_2 y_2)(t) \diamond_\alpha t = 0, \lambda_2 \neq \lambda_1,$$

or

$$\int_0^N (z_1 y_1 + z_2 y_2)(t) \diamond_\alpha t = 0.$$

Since $\langle y, z \rangle_{\diamond_\alpha} = 0$, $y(t, \lambda_1)$ and $z(t, \lambda_2)$ are orthogonal on $L_2^{\diamond_\alpha}[0, N]$. $\square$

Theorem 3. The eigenvalues of $\diamond_\alpha$ Dirac BVP (10)–(12) are all real.

Proof. Assume that the complex eigenvalue is λ_1 . So, $\bar{\lambda}_1 = u - iv$ is also an eigenvalue. The following equation is obtained from the orthogonality theorem, where $y(t, \lambda_1)$ is an eigenfunction.

$$(\lambda_1 - \bar{\lambda}_1) \int_0^N (y_1 \bar{y}_1 + y_2 \bar{y}_2)(t, \lambda_1) \diamond_\alpha t = 0$$

and, so

$$(\lambda_1 - \bar{\lambda}_1) \int_0^N (|y_1|^2 + |y_2|^2)(t, \lambda_1) \diamond_\alpha t = 0.$$

Since $\lambda_1 \neq \bar{\lambda}_1$, $y_1(t, \lambda_1) = y_2(t, \lambda_1)$. This is a contradiction since eigenfunctions cannot be zero. So, all eigenvalues are real. $\square$

Theorem 4 ([43]). Let X be a Hilbert space and given the bounded linear operator $A : X \rightarrow X$. A necessary and sufficient condition for an operator A to be self-adjoint is that A is symmetric.

Lemma 3. Minkowski's inequality for $\diamond_\alpha$ integral [44]. If $p > 1$ and $f_1, f_2 : \mathbb{T} \rightarrow \mathbb{R}$ are $\diamond_\alpha$ -integrable on $[a, b]_{\mathbb{T}}$, then

$$\left(\int_a^b |f_1(\zeta) + f_2(\zeta)|^p \diamond_\alpha \zeta \right)^{\frac{1}{p}} \leq \left(\int_a^b |f_1(\zeta)|^p \diamond_\alpha \zeta \right)^{\frac{1}{p}} + \left(\int_a^b |f_2(\zeta)|^p \diamond_\alpha \zeta \right)^{\frac{1}{p}}.$$

Theorem 5. $\diamond_\alpha$ Dirac operator $D_{\diamond_\alpha}$ is bounded on $L_2^{\diamond_\alpha}[0, N]$.

Proof. By the definition of $D_{\diamond_\alpha}$ and norm on $L_2^{\diamond_\alpha}$, and lemma 3, the following inequality can be obtained:

$$\begin{aligned} \|D_{\diamond_\alpha} y\|_{L_2^{\diamond_\alpha}} &= \left(\int_0^N (|y_2^{\diamond_\alpha} - [\lambda - q]y_1|^2 + |y_1^{\diamond_\alpha} - [-\lambda + r]y_2|^2) \diamond_\alpha t \right)^{1/2} \\ &\leq \left(\int_0^N |y_2^{\diamond_\alpha} - [\lambda - q]y_1|^2 \diamond_\alpha t \right)^{1/2} + \left(\int_0^N |y_1^{\diamond_\alpha} - [-\lambda + r]y_2|^2 \diamond_\alpha t \right)^{1/2} \\ &\leq 2\|y\|_{L_2^{\diamond_\alpha}}. \end{aligned}$$

Thus, $D_{\diamond_\alpha}$ is bounded on $L_2^{\diamond_\alpha}[0, N]$. $\square$

Theorem 6. $\diamond_\alpha$ Dirac operator $D_{\diamond_\alpha}$ is linear.

Proof. Let $y^{\diamond_\alpha} = \begin{pmatrix} y_1^{\diamond_\alpha} \\ y_2^{\diamond_\alpha} \end{pmatrix}$, $z^{\diamond_\alpha} = \begin{pmatrix} z_1^{\diamond_\alpha} \\ z_2^{\diamond_\alpha} \end{pmatrix}$ and $\beta_1, \beta_2 \in \mathbb{R}$. From the definition of $D_{\diamond_\alpha}$, we get

$$\begin{aligned} D_{\diamond_\alpha}(\beta_1 y + \beta_2 z) &= \begin{pmatrix} (\beta_1 y_2 + \beta_2 z_2)^{\diamond_\alpha} - (\lambda - q)(\beta_1 y_1 + \beta_2 z_1) \\ (\beta_1 y_1 + \beta_2 z_1)^{\diamond_\alpha} - (-\lambda + r)(\beta_1 y_2 + \beta_2 z_2) \end{pmatrix} \\ &= \beta_1 \begin{pmatrix} y_2^{\diamond_\alpha} - (\lambda - q)y_1 \\ y_1^{\diamond_\alpha} - (-\lambda + r)y_2 \end{pmatrix} + \beta_2 \begin{pmatrix} z_2^{\diamond_\alpha} - (\lambda - q)z_1 \\ z_1^{\diamond_\alpha} - (-\lambda + r)z_2 \end{pmatrix} \\ &= \beta_1 D_{\diamond_\alpha}(y) + \beta_2 D_{\diamond_\alpha}(z). \end{aligned}$$

Therefore, $D_{\diamond_\alpha}$ is linear. $\square$

Theorem 7. $\diamond_\alpha$ Dirac operator is symmetric on $L_2^{\diamond_\alpha}[0, N]$.

Proof. Let eigenfunctions $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be defined for the system (5). It is necessary to demonstrate that $\langle x, D_{\diamond_\alpha} y \rangle_{\diamond_\alpha} = \langle D_{\diamond_\alpha} x, y \rangle_{\diamond_\alpha}$. Considering the system (5), we obtain

$$\begin{aligned} x^T D_{\diamond_\alpha} y - y^T D_{\diamond_\alpha} x &= x^T B y^{\diamond_\alpha} - y^T B x^{\diamond_\alpha} \\ &= (x_1 \ x_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1^{\diamond_\alpha} \\ y_2^{\diamond_\alpha} \end{pmatrix} - (y_1 \ y_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1^{\diamond_\alpha} \\ x_2^{\diamond_\alpha} \end{pmatrix} \\ &= y_2^{\diamond_\alpha} x_1 - y_1^{\diamond_\alpha} x_2 - x_2^{\diamond_\alpha} y_1 + x_1^{\diamond_\alpha} y_2 \\ &= \alpha(x_1 y_2^\Delta + y_2 x_1^\Delta) + (1 - \alpha)(x_1 y_2^\nabla + y_2 x_1^\nabla) \\ &\quad - \alpha(x_2 y_1^\Delta + y_1 x_2^\Delta) - (1 - \alpha)(y_1 x_2^\nabla + x_2 y_1^\nabla). \end{aligned}$$

Since $\mathbb{T}$ is uniform, the following equation is obtained.

$$x^T(t) D_{\diamond_\alpha} y(t) - y^T(t) D_{\diamond_\alpha} x(t) = (x_1 y_2)^{\diamond_\alpha}(t) - (y_1 x_2)^{\diamond_\alpha}(t).$$

Let us apply the $\diamond_\alpha$ integral of both sides from 0 to N in the last equation. Then,

$$\begin{aligned} \langle x, D_{\diamond_\alpha} y \rangle_{\diamond_\alpha} - \langle D_{\diamond_\alpha} x, y \rangle_{\diamond_\alpha} &= \left[(2\alpha^2 - 2\alpha + 1)(x_1 y_2)(t) + \alpha(1 - \alpha)(x_1 y_2^\rho)(t) \right. \\ &\quad \left. - (x_1 y_2^\sigma)(t) - (2\alpha^2 - 2\alpha + 1)(y_1 x_2)(t) \right. \\ &\quad \left. - \alpha(1 - \alpha)(y_1 x_2^\rho)(t) - (x_2 y_1^\sigma)(t) \right] \Big|_0^N. \end{aligned}$$

If boundary conditions (11) and (12) are considered, we obtain

$$\langle x, D_{\diamond_\alpha} y \rangle_{\diamond_\alpha} - \langle D_{\diamond_\alpha} x, y \rangle_{\diamond_\alpha} = 0.$$

From here, the $\diamond_\alpha$ Dirac operator $D_{\diamond_\alpha}$ is symmetric on $L_2^{\diamond_\alpha}[0, N]$. $\square$

Conclusion 1. The Dirac operator $D_{\diamond_\alpha}$ is self-adjoint since it is bounded, linear, and symmetrical on $L_2^{\diamond_\alpha}[0, N]$.

Theorem 8. All eigenvalues of the problem (10)–(12) are simple.

Proof. Let $y(t, \lambda)$ and $u(t, \lambda)$ be eigenfunctions of the problem (10)–(12). From (10), the following equations can be written:

$$\begin{aligned} y_2^{\diamond_\alpha} &= (\lambda - q(t))y_1, \\ y_1^{\diamond_\alpha} &= (-\lambda + r(t))y_2, \\ u_2^{\diamond_\alpha} &= (\lambda - q(t))u_1, \\ u_1^{\diamond_\alpha} &= (-\lambda + r(t))u_2. \end{aligned}$$

If these equations are multiplied by u_2 , u_1 , y_2 , and y_1 , respectively, and the last equation is subtracted from the 2nd and the 3rd equation is subtracted from first, we obtain

$$u_1 y_1^{\diamond_\alpha} - y_1 u_1^{\diamond_\alpha} = 0, \tag{13}$$

and

$$u_2 y_2^{\diamond_\alpha} - y_2 u_2^{\diamond_\alpha} = 0. \tag{14}$$

From (13) and (14), respectively, following is obtained:

$$u_1(\alpha y_1^\Delta + (1 - \alpha)y_1^\nabla) - y_1(\alpha u_1^\Delta + (1 - \alpha)u_1^\nabla) = 0,$$

and

$$u_2(\alpha y_2^\Delta + (1 - \alpha)y_2^\nabla) - y_2(\alpha u_2^\Delta + (1 - \alpha)u_2^\nabla) = 0.$$

Then,

$$\alpha \left(\frac{y_1}{u_1} \right)^\Delta + (1 - \alpha) \left(\frac{y_1}{u_1} \right)^\nabla = 0,$$

and

$$\alpha \left(\frac{y_2}{u_2} \right)^\Delta + (1 - \alpha) \left(\frac{y_2}{u_2} \right)^\nabla = 0.$$

Therefore, the following expressions are obtained:

$$\left(\frac{y_1}{u_1} \right)^{\diamond_\alpha} = 0,$$

and

$$\left(\frac{y_2}{u_2} \right)^{\diamond_\alpha} = 0.$$

As a result, $y_1 = cu_1$ and $y_2 = cu_2$ are obtained. This gives $y = cu, c \in \mathbb{R}$. So, it completes the proof. $\square$

Theorem 9. The asymptotic estimates for eigenfunction $y(t, t_0)$ corresponding to the $\diamond_\alpha$ Dirac problem (5)–(7) are

$$\begin{aligned} y_1(t, t_0) = & h \cos_\lambda(t, t_0) - \frac{h(1 - \alpha)v\lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) + \frac{1 + v^2 \lambda^2}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) \\ & + \frac{\lambda v^2 \lambda^2 (1 - \alpha)}{1 + \alpha v^2 \lambda^2} \int_0^t [y_2(s, t_0) \cos_\lambda(t, t_0) + y_1(s, t_0) \sin_\lambda(t, t_0)] \diamond_\alpha s \\ & + \frac{(1 - \alpha)v\lambda^2}{1 + \alpha v^2 \lambda^2} \int_0^t [y_2(s, t_0) \sin_\lambda(t, t_0) - y_1(s, t_0) \cos_\lambda(t, t_0)] \diamond_\alpha s \\ & + \int_0^t \left(r(s) y_2(s, t_0) cs_\lambda^-(t, s) - \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \sin_\lambda(t, s) \right) \diamond_\alpha s, \end{aligned} \quad (15)$$

and

$$\begin{aligned} y_2(t, t_0) = & -h \sin_\lambda(t, t_0) + \cos_\lambda(t, t_0) + \frac{(1 - \alpha)v\lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) \\ & + \frac{(1 - \alpha)v\lambda^2}{1 + \alpha v^2 \lambda^2} \int_0^t [y_1(s, t_0) \sin_\lambda(t, t_0) + y_2(s, t_0) \cos_\lambda(t, t_0)] \diamond_\alpha s \\ & + \int_0^t [-r(s) y_2(s, t_0) \sin_\lambda(t, s) - q(s) y_1(s, t_0) cs_\lambda^+(t, s)] \diamond_\alpha s, \end{aligned} \quad (16)$$

where

$$\begin{aligned} cs_\lambda^+(t, s) &= \cos_\lambda(t, s) + \frac{(1 - \alpha)v\lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, s), \quad cs_\lambda^-(t, s) = \cos_\lambda(t, s) - \frac{(1 - \alpha)v\lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, s), \\ y(t, t_0) &= \begin{pmatrix} y_1(t, t_0) \\ y_2(t, t_0) \end{pmatrix}, \quad v(t) = v, \text{ and } t_0 = 0. \end{aligned}$$

Proof. First, let us consider homogeneous solution of the system (8) $y_2^{\diamond_\alpha \diamond_\alpha} + \lambda^2 y_2 = 0$, as

$$y_2(t, t_0) = c_1 \cos_\lambda(t, t_0) + c_2 \sin_\lambda(t, t_0). \quad (17)$$

If we use $y_2^{\diamond_\alpha} = \lambda y_1$, the following occurs:

$$c_1 \cos_\lambda^{\diamond_\alpha}(t, t_0) + c_2 \sin_\lambda^{\diamond_\alpha}(t, t_0) = \lambda y_1(t, t_0).$$

If (3) and (4) are used, we obtain

$$y_1(t, t_0) = c_1 \left(\frac{-1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \sin_\lambda(t, t_0) - (1 - \alpha) v \lambda \cos_\lambda(t, t_0)] \right) + c_2 \left(\frac{1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \cos_\lambda(t, t_0) + (1 - \alpha) v \lambda \sin_\lambda(t, t_0)] \right). \quad (18)$$

These two homogeneous solutions can be written as follows by using a variation of parameters method [6].

$$c_1^{\diamond_\alpha} \left(\frac{-1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \sin_\lambda(t, t_0) - (1 - \alpha) v \lambda \cos_\lambda(t, t_0)] \right) + c_2^{\diamond_\alpha} \left(\frac{1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \cos_\lambda(t, t_0) + (1 - \alpha) v \lambda \sin_\lambda(t, t_0)] \right) = -q(t) y_1(t, t_0),$$

$$c_1^{\diamond_\alpha} \cos_\lambda(t, t_0) + c_2^{\diamond_\alpha} \sin_\lambda(t, t_0) = r(t) y_2(t, t_0).$$

If the equations are multiplied by $\cos_\lambda(t, t_0)$ and

$$\frac{1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \sin_\lambda(t, t_0) - (1 - \alpha) v \lambda \cos_\lambda(t, t_0)],$$

respectively, and the $\diamond_\alpha$ integral is applied in the range $[0, t]$, we obtain

$$c_2 = \int_0^t \left(r(s) y_2(s, t_0) s c_\lambda(s, t_0) - \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \cos_\lambda(s, t_0) \right) \diamond_\alpha s,$$

and in a similar way, one can obtain

$$c_1 = \int_0^t \left(r(s) y_2(s, t_0) c s_\lambda(s, t_0)^+ + \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \sin_\lambda(s, t_0) \right) \diamond_\alpha s,$$

where $s c_\lambda(t, s) = \sin_\lambda(t, s) - \frac{(1 - \alpha) v \lambda}{1 + \alpha v^2 \lambda^2} \cos_\lambda(t, s)$. If these results are substituted into the solutions (17) and (18), the following solutions are obtained.

$$y_1(t, t_0) = \frac{-1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \sin_\lambda(t, t_0) - (1 - \alpha) v \lambda \cos_\lambda(t, t_0)] \times \int_0^t \left(r(s) y_2(s, t_0) c s_\lambda(s, t_0)^+ + \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \sin_\lambda(s, t_0) \right) \diamond_\alpha s + \frac{1}{1 + v^2 \lambda^2} [(1 + \alpha v^2 \lambda^2) \cos_\lambda(t, t_0) + (1 - \alpha) v \lambda \sin_\lambda(t, t_0)] \times \int_0^t \left(r(s) y_2(s, t_0) s c_\lambda(s, t_0) - \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \cos_\lambda(s, t_0) \right) \diamond_\alpha s,$$

$$y_2(t, t_0) = \cos_\lambda(t, t_0) \int_0^t \left(r(s) y_2(s, t_0) c s_\lambda(s, t_0)^+ + \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \sin_\lambda(s, t_0) \right) \diamond_\alpha s + \sin_\lambda(t, t_0) \int_0^t \left(r(s) y_2(s, t_0) s c_\lambda(s, t_0) - \frac{(1 + v^2 \lambda^2)}{1 + \alpha v^2 \lambda^2} q(s) y_1(s, t_0) \cos_\lambda(s, t_0) \right) \diamond_\alpha s,$$

and then

$$y_1(t, t_0) = - \int_0^t [r(s) y_2(s, t_0) \sin_\lambda(t, s) + c s_\lambda(t, s)^+ q(s) y_1(s, t_0)] \diamond_\alpha s, \quad (19)$$

$$y_2(t, t_0) = \int_0^t [c s_\lambda(t, s)^- r(s) y_2(s, t_0) - \frac{1 + \alpha v^2 \lambda^2}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, s) q(s) y_1(s, t_0)] \diamond_\alpha s. \quad (20)$$

Taking into consideration the system (5),

$$y_1(t, t_0) = - \int_0^t [(\lambda y_2 + y_1^{\diamond_\alpha}) \sin_\lambda(t, s) + cs_\lambda(t, s)^+ (\lambda y_1 - y_2^{\diamond_\alpha})] \diamond_\alpha s, \quad (21)$$

$$y_2(t, t_0) = \int_0^t [cs_\lambda(t, s)^- (\lambda y_2 + y_1^{\diamond_\alpha}) - \frac{1 + \alpha v^2 \lambda^2}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, s) (\lambda y_1 - y_2^{\diamond_\alpha})] \diamond_\alpha s, \quad (22)$$

is obtained first, and

$$\begin{aligned} y_1(t, t_0) &= h \sin_\lambda(t, t_0) + y_2(t, t_0) - \cos_\lambda(t, t_0) - \frac{(1 - \alpha) v \lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) \\ &\quad - \frac{(1 - \alpha) v \lambda^2}{1 + \alpha v^2 \lambda^2} \int_0^t [y_1(s, t_0) \sin_\lambda(t, s) + y_2(s, t_0) \cos_\lambda(t, s)] \diamond_\alpha s, \end{aligned} \quad (23)$$

$$\begin{aligned} y_2(t, t_0) &= -h \cos_\lambda(t, t_0) + y_1(t, t_0) - \frac{h(1 - \alpha) v \lambda}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) - \frac{1 + v^2 \lambda^2}{1 + \alpha v^2 \lambda^2} \sin_\lambda(t, t_0) \\ &\quad - \frac{(1 - \alpha) v^2 \lambda^3}{1 + \alpha v^2 \lambda^2} \int_0^t [y_2(s, t_0) \cos_\lambda(t, t_0) + y_1(s, t_0) \sin_\lambda(t, t_0)] \sin_\lambda(t, s) \diamond_\alpha s \\ &\quad - \frac{(1 - \alpha) v \lambda^2}{1 + \alpha v^2 \lambda^2} \int_0^t [y_2(s, t_0) \sin_\lambda(t, t_0) - y_1(s, t_0) \cos_\lambda(t, t_0)] \cos_\lambda(t, s) \diamond_\alpha s \end{aligned} \quad (24)$$

is subsequently found by applying the integration by parts approach for (21) and (22) considering Lemma 3. Consequently, by taking into account (22) and (24) and (21) and (22), we have arrived at (15) and (16). $\square$

3. Conclusions

In this study, some basic features of the $\diamond_\alpha$ calculation are examined and the $\diamond_\alpha$ Dirac system is discussed on a uniform time scale. Then, basic spectral features are examined. An eigenfunction expansion has been obtained for the $\diamond_\alpha$ Dirac problem, which is considered with a certain boundary condition. Under special assumptions, these solutions can yield different results.

In mathematics and science, spectral theory is essential, especially when examining linear operators on different function spaces. We want to extend the application of spectral theory to a larger class of mathematical objects beyond the classical continuous and discrete situations by investigating the spectral features of Dirac systems with diamond-alpha derivatives on a uniform time scale. The behavior of quantum systems with non-classical dynamics can be better understood by taking into account the spectrum characteristics of Dirac systems on a uniform time scale. New mathematical approaches and methodologies for the time-scale analysis of differential equations and operators can be developed via the study of the spectral theory of Dirac systems with diamond-alpha derivatives. In the future, the spectral theory could be extended to more intricate systems than the simple Dirac equation with diamond-alpha derivatives. This might entail taking into account extra interactions, boundary constraints, and expansions to higher-dimensional spaces. Theoretical physics and mathematical analysis may be affected in this way.

Author Contributions: Conceptualization, T.G. and E.Y.; methodology, E.Y.; validation, T.G. and A.Ç.Y.; formal analysis, T.G. and A.Ç.Y.; investigation, E.Y.; resources, E.Y.; writing—original draft preparation, T.G. and A.Ç.Y.; writing—review and editing, T.G.; supervision, E.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors are very grateful to the anonymous referee for their thorough review of this work and their comments.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

- Hilger, S. Ein Masskettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg, Würzburg, Germany, 1988.
- Hilger, S. Analysis on measure chains—A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56.
- Aulbach, B.; Hilger, S. Linear dynamic processes with inhomogeneous time scale. In *Nonlinear Dynamics and Quantum Dynamical Systems*; Akademie Verlag: Berlin, Germany, 1990; Volume 59, pp. 9–20.
- Aulbach, B.; Hilger, S. A unified approach to continuous and discrete dynamics, Qualitive theory of differential equaitons (Szeged 1988). In *Colloquia Mathematica Societatis János Bolyai*; North-Holland Pub. Co.: Amsterdam, The Netherlands, 1988; Volume 53, pp. 37–56.
- Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser Boston: Boston, MA, USA, 2001.
- Bohner, M.; Peterson, A.C. *Advances in Dynamic Equations on Time Scales*; Birkhäuser Boston: Boston, MA, USA, 2003.
- Guseinov, G.S.; Kaymakçalan, B. Basics of Riemann Δ and ∇ integration on time scales. *J. Differ. Equ. Appl.* **2002**, *8*, 1001–1017.
- Caputo, M.C. Time scales: From nabla calculus to delta calculus and vice versa via duality. *Int. J. Differ. Equ.* **2010**, *5*, 25–40.
- Atici, F.M.; Guseinov, G.S. On Green's functions and positive solutions for boundary value problems on time scales. *J. Comput. Appl. Math.* **2002**, *141*, 75–99.
- Bohner, M.; Georgiev, S. *Multivariable Dynamic Calculus on Time Scales*; Springer: Berlin/Heidelberg, Germany, 2016.
- Sheng, Q.; Fadag, M.; Henderson, J.; Davis, J.M. An exploration of combined dynamic derivatives on time scales and their applications. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 395–413.
- Rogers, J.W., Jr.; Sheng, Q. Notes on the diamond dynamic derivative on time scales. *J. Math. Anal. Appl.* **2007**, *326*, 228–241.
- Sheng, Q. Hybrid approximations via second order combined dynamic derivatives on time scales. *Electron. J. Qual. Theory Differ. Equ.* **2007**, *17*, 1–13.
- Sidi Ammi, M.R.; Ferreira, R.A.; Torres, D.F. Diamond-Jensen's inequality on time scales. *J. Inequalities Appl.* **2008**, *2008*, 1–13.
- Mozyrska, D.; Torres, D.F.M. A study of diamond alpha dynamic equations on regular time scales. *Afr. Diaspora J. Math.* **2009**, *8*, 35–47.
- Malinowska, A.B.; Torres, D.F.M. On the diamond alpha Riemann integral and mean value theorems on time scales. *Dyn. Syst. Appl.* **2009**, *18*, 469–482.
- Mozyrska, D.; Torres, D.F.M. Diamond-alpha polynomial series on time scales. *Int. J. Math. Stat.* **2009**, *5*, 92–101.
- Ozkan, U.M.; Kaymakçalan, B. Basics of diamond α partial dynamic calculus on time scales. *Math. Comput. Model.* **2009**, *50*, 1253–1261.
- Sidi Ammi, M.R.; Torres, D.F. Holder's and Hardy's two dimensional diamond-alpha inequalities on time scales. *Ann. Univ. Craivo, Math. Comput. Sci. Ser.* **2010**, *37*, 1–11.
- Brito da Cruz, A.M.; Martins, N.; Torres, D.F. The diamond integral on time scales. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 1453–1462.
- El-Deeb, A.A.; Awrejcewicz, J. Diamond- α Hardy-type inequalities on time scales. *Symmetry* **2022**, *14*, 2047.
- Agarwal, R.; O'Regan, D.; Saker, S. *Dynamic Inequalities on Time Scales*; Springer: Berlin/Heidelberg, Germany, 2014; Volume 2014.
- Martynyuk, A.A. *Stability Theory for Dynamic Equations on Time Scales*; Springer International Publishing: Basel, Switzerland, 2016.
- Mozyrska, D.; Torres, D.F. Diamond-alpha polynomial series on time scales. *arXiv* **2008**, arXiv:0805.0274.
- Dirac, P.A.M. The quantum theory of the electron. *Proc. R. Soc. Math. Phys. Eng. Sci.* **1928**, *117*, 610–624.
- Levitan, B.M.; Sargsjan, I.S. *Sturm–Liouville and Dirac Operators*; Nauka: Moscow, Russia, 1991.
- Joa, I.; Minkin, A. Eigenfunction estimate for a Dirac operator. *Acta Math. Hung.* **1997**, *76*, 337–349.
- Bairamov, E.; Aygar, Y.; Olgun, M. Jost solution and the spectrum of the discrete Dirac systems. *Bound. Value Probl.* **2010**, 306571, 1–11.
- Keskin, B.; Ozkan, A.S. Inverse spectral problems for Dirac operator with eigenvalue dependent boundary and jump conditions. *Acta Math. Hung.* **2011**, *130*, 309–320.
- Gulsen, T.; Yilmaz, E. Spectral theory of Dirac system on time scales. *Appl. Anal.* **2017**, *96*, 2684–2694.
- Hovhannisyann, G. On Dirac equation on a time scale. *J. Math. Phys.* **2011**, *52*, 102701.
- Allahverdiev, B.P.; Tuna, H. One-dimensional q -Dirac equation. *Math. Methods Appl. Sci.* **2017**, *40*, 7287–7306.
- Gulsen, T.; Yilmaz, E.; Goktas, S. Conformable fractional Dirac system on time scales. *J. Inequalities Appl.* **2017**, *2017*, 161.
- Koprubasi, T. A study of impulsive discrete Dirac system with hyperbolic eigenparameter. *Turk. J. Math.* **2021**, *45*, 540–548.
- Gasymov, M.G.; Levitan, B.M. The inverse problem for the Dirac system. *Dokl. Akad. Nauk. SSSR* **1996**, *167*, 967970.
- Prats, F.; Toll, J.S. Construction of the Dirac equation central potential from phase shifts and bound states. *Phys. Rev.* **1959**, *113*, 363–370.
- Panakhov, E.S. The defining of Dirac system in two incompletely set collection of eigenvalues. *Dokl. Akad. AzSSR* **1985**, *5*, 8–12.
- Watson, B.A. Inverse spectral problems for weighted Dirac systems. *Inverse Probl.* **1999**, *15*, 793–805.
- Kerimov, N.B. A boundary value problem for the Dirac system with a spectral parameter in the boundary conditions. *Differ. Equ.* **2002**, *38*, 164–174.
- Mamedov, K.R.; Akcay, O. Inverse eigenvalue problem for a class of Dirac operators with discontinuous coefficient. *Bound. Value Probl.* **2014**, *2014*, 1–20.

41. Abdullaev, T.S.; Nabiev, I.M.O. An algorithm for reconstructing the Dirac operator with a spectral parameter in the boundary condition. *Comput. Math. Math. Phys.* **2016**, *56*, 256–262.
42. Gulsen, T.; Yilmaz, E.; Koyunbakan, H. Inverse nodal problem for p-laplacian dirac system. *Math. Methods Appl. Sci.* **2017**, *40*, 2329–2335.
43. Brown, A.L.; Page, A. *Elements of Functional Analysis*; Van Nostrand-Reinhold: London, UK, 1970.
44. Atasever, N. On Diamond-Alpha Dynamic Equations and Inequalities. Master's Thesis, Georgia Southern University, Statesboro, GA, USA, 2011.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.