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On Variance and Average Moduli of Zeros and Critical Points of Polynomials

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Abstract: This paper investigates various aspects of the distribution of roots and critical points of a complex polynomial, including their variance and the relationships between their moduli using an inequality due to de Bruijn. Making use of two other inequalities-again due to de Bruijn-we derive two probabilistic results concerning upper bounds for the average moduli of the imaginary parts of zeros and those of critical points, assuming uniform distribution of the zeros over a unit disc and employing the Markov inequality. The paper also provides an explicit formula for the variance of the roots of a complex polynomial for the case when all the zeros are real. In addition, for polynomials with uniform distribution of roots over the unit disc, the expected variance of the zeros is computed. Furthermore, a bound on the variance of the critical points in terms of the variance of the zeros of a general polynomial is derived, whereby it is established that the variance of the critical points of a polynomial cannot exceed the variance of its roots. Finally, we conjecture a relation between the real parts of the zeros and the critical points of a polynomial.

Keywords: polynomials; mean; variance; de Bruijn inequality; expected value; mean of moduli; mean deviation; critical points

MSC: 12D10; 26D; 62H12; 30C15

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1. Introduction

Polynomials are fundamental and ubiquitous across many mathematical fields. They assist in determining solutions to equations, identifying how many solutions exist, understanding the influence of critical points on the geometric distribution of values, characterizing the traits of geometric curves, and more. The role of polynomials in the progression of mathematics has been profound since ancient times. From the Babylonian study of quadratic equations to the Greek geometric approach to cubics and quartics, the journey of understanding polynomials has been long and intricate [1]. With the emergence of complex numbers during the Renaissance, polynomials began to reveal even more fascinating properties. The 18th and 19th centuries bore witness to a series of ground breaking discoveries in the analytical theory of polynomials, especially in the context of their complex roots. This period saw luminaries like Gauss, Cauchy, Riemann and others whose contributions expanded our understanding of polynomials and proved pivotal in establishing foundational theorems that connected topology, algebra, and complex analysis. This confluence of ideas laid the groundwork for the myriad applications and deeper investigations of polynomials in the modern era. One of the most captivating sub-fields within the larger field of complex analysis is what now goes by the name of the geometric and analytical exploration of complex polynomials. Historically, its foundations were solidified in the early 20th century, setting a distinctive trajectory that diverged from a purely algebraic analysis. Novel generalizations of polynomials with applications to diverse fields of science continue to be developed [2].

When one delves into the realm of theory of polynomials from a complex function viewpoint, it transforms into a geometric spectacle, often characterized as the study of the "landscape" of zeros within the complex plane. For this reason, the field is often referred to as the geometry of zeros of polynomials in the complex plane.

Pioneers like Morris Marden, with his seminal work *Geometry of Zeros of Polynomials* [3], played a pivotal role in establishing a distinct niche area within complex analysis. While his contributions are monumental, the field has further been enriched by works like *Topics in Polynomials: Extremal Problems, Inequalities, and Zeros* by Milovanovic et al. [4] and the comprehensive survey *Analytic Theory of Polynomials* by Rahman and Schmeisser [5].

A recurring theme in the study of polynomials is the dynamic nature of zeros. As one alters some or all coefficients, the zeros dance across the complex plane. This dance, the shifting locations of zeros in response to changes in the coefficients, is captured by the continuity theorem. The study of the roots and zeros of polynomials is an important topic in mathematics with applications in a variety of fields, including physics, engineering, and computer science [4–7]. The study of things like the distribution of zeros and critical points, geometry of polynomials, different norms of polynomials, bounds of roots, extremal problems, and such other things have received tremendous research focus over the years both in polynomial theory and the theory of random polynomials [4,5,8–11]. One important measure of the distribution of roots is the variance of the roots, which gives information about how spread out the roots are. The notion of variance provides a natural and potential tool for analyzing the distribution of roots and exploring their behavior. However, despite its potential importance, the variance of the roots of a polynomial is a less commonly studied topic in the theory of polynomials. The variance of the number of real roots of random polynomials has been extensively [7,12] studied in the context of random polynomials, but little to no studies exist on the variance of the zeros themselves. An extensive search across various databases showed us only the following result about the bounds of variance for monic polynomials with only real zeros [13]:

Theorem 1. *Let the roots of the nth degree monic polynomial equation*

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

be all real and positive. Let x_1 and x_n, respectively, denote the smallest and largest root of the polynomial. Then,

$$x_n \geq \alpha + \sqrt{\frac{\alpha^2 + \beta}{2\gamma}},$$

$$x_1 \leq \alpha - \sqrt{\frac{\alpha^2 + \beta}{2\gamma}},$$

$$x_n - x_1 \geq \sqrt{\frac{\alpha^2 + \beta}{\gamma}},$$

where

$$\alpha = \frac{a_1^2 - 2a_2}{n} - \frac{a_1 a_n}{a_{n-1}},$$
$$\beta = \frac{4a_n}{n}\left(a_1 a_{n-1} - n^2 a_n\right)\left(2na_2 - (n-1)a_1^2\right) - (na_{n-1})^2,$$
$$\gamma = n\frac{a_n}{a_{n-1}} - a_1^n.$$

In this paper, compared to [13], we prove a more general result about the exact variance of the zeros of a general polynomial with only real roots. We also explore the relationship between the variance of the zeros and critical points of a polynomial as well as their average moduli. By measuring the distance of the zeros from their barycenter, the notion of variance captures the circular symmetry of the zeros distribution or the deviation thereof. We explore the behavior of the variance and mean deviation for zeros on the unit circle and discuss their implications. We further use an inequality due to de Bruijn and Vieta's formula for the product of zeros to derive bounds for the sum of the moduli of zeros and the average of the squares of the moduli of zero for a polynomial. Finally, we propose a conjecture regarding the relationship between the real parts of the zeros and the real parts of the critical points. Our results have implications for the study of complex analysis and the distribution of roots of polynomials.

Because of the absence of the any recent results on the variance of roots of a polynomial, we will first perform a brief survey of some classical results that have a bearing on the results we develop. We, however, emphasize the variance of the roots themselves should not be confused with the variance of the number of real roots, for which there are a plethora of results in the literature. It is important to note that the expression for variance involves the moduli of roots. For polynomials of degree four or lower, there are formulas—like the quadratic formula for degree two, Cardano's formula for degree three, and Ferrari's formula for degree four—that allow us to find the roots explicitly and then compute their moduli. For polynomials of a degree greater than 4, it is not possible to calculate the sum of the moduli of the zeros of general complex polynomials of degrees higher than 4 in terms of the coefficients in a closed form. This makes the study of the variance for general polynomials a very significant problem. The theorems developed in this paper can be effectively utilized for Olympiad training, particularly for enhancing problem-solving skills in algebra and inequalities. As demonstrated in [14], a solid understanding of polynomial behavior is crucial for success in mathematics competitions, and our results provide a foundational tool-set for such applications.

Survey of Classical Results on Average Distances of Critical Points and Zeros

Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$ and critical points $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. By Vieta's formulae, it immediately follows:

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \zeta_i = \frac{1}{n} \sum_{i=1}^{n} z_i. \tag{1}$$

We also have the following result [5]:

Theorem 2. *The moduli of imaginary parts of the zeros and the critical points satisfy the following:*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} |Im(\zeta_i)| \leq \frac{1}{n} \sum_{i=1}^{n} |Im(z_i)|. \tag{2}$$

Remark 1. *The geometry of the complex plane and the fundamental properties of entire functions play a crucial role in the relationship between zeros and critical points, and this relationship does not extend in the same way to the real parts of these values. Another reason is that complex roots occur in conjugate pairs.*

We have the following theorem by de Bruijn [15].

Theorem 3. *Let $P(z)$ be a polynomial of degree $n > 1$ with zeros $z_1, \ldots, z_n$ and critical points $\zeta_1, \ldots, \zeta_{n-1}$. Then for $p \geq 1, c \in \mathbb{C}$,*

$$\frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i - c|^p \leq \frac{1}{n}\sum_{i=1}^{n}|z_i - c|^p. \tag{3}$$

Therefore, for $p = 1, c = 0$, we have the following result:

Theorem 4. *Let $P(z)$ be a polynomial of degree $n > 1$ with zeros $z_1, \ldots, z_n$ and critical points $\zeta_1, \ldots, \zeta_{n-1}$. Then*

$$\frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i| \leq \frac{1}{n}\sum_{i=1}^{n}|z_i|. \tag{4}$$

The remainder of the paper is organized as follows. In Section 2, we present some results concerning the average moduli of critical points for the uniform distribution of zeros. In Section 3, we discuss some results regarding the variance of zeros, while Section 4 explores the expected variance for the uniform distribution of zeros. In Section 5, we present a theorem about the variance of the zeros of polynomials with real zeros. In Section 6, we calculate the bounds for the average of squares of the moduli of zeros using a de Bruijn's inequality and Vieta's formula for the product of zeros. In Section 7, we present a relationship between the variance of zeros and critical points. Finally, we present a conjecture regarding a possible relation between the moduli of zeros and critical points.

2. Average of Moduli of Critical Points for Uniform Distribution of Roots

Theorem 5. *Let*

$$\zeta_{ip} = \frac{1}{n-1}\sum_{i=1}^{n}|\zeta_i| \tag{5}$$

be the average of the moduli of the imaginary parts of the critical point of the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ and the roots $z_1, \cdots, z_n$ be independently, uniformly, and randomly chosen over the unit disc $|z| \leq 1$, then for $c > 0$,

$$\mathbb{P}[\zeta_{ip} > c] \leq \frac{4}{3\pi c}. \tag{6}$$

Proof. Since $z_1, \ldots, z_n$ are identically, uniformly, and randomly distributed over the unit circle, the joint distribution of their x and y coordinates is given by

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & (x,y) \in D \\ 0, & \text{else.} \end{cases} \tag{7}$$

The distribution of the random variable Y corresponding to the imaginary parts of zeros is given by:

$$\begin{aligned} f_Y(y) &= \int_{1-y^2}^{\sqrt{1-y^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-y^2}, \quad -1 \leqslant y \leqslant 1. \end{aligned} \tag{8}$$

Hence, the expected value of $|Y|$ can be written as

$$\begin{aligned} E(|Y|) &= \int_{-1}^{1} |y| \cdot \frac{2}{\pi}\sqrt{1-y^2}dy \\ &= \frac{4}{\pi}\int_{0}^{1} y \cdot \sqrt{1-y^2}dy \\ &= \frac{4}{3\pi}. \end{aligned} \tag{9}$$

Now, we have from Theorem 2:

$$\frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i| \leq \sum_{i=1}^{n}|z_i|. \tag{10}$$

Taking expectations on both sides of Inequality (10) and using Equation (9), we have:

$$\mathbb{E}(\zeta_{ip}) \leqslant \frac{4}{3\pi}. \tag{11}$$

Employing Markov inequality [16,17], we have:

$$\mathbb{P}[\zeta_{ip} > c] \leq \frac{4}{3\pi c}. \tag{12}$$

which establishes the claim of the theorem. □

Theorem 6. *Let*

$$\zeta_{av} = \frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i|, \tag{13}$$

be the average of the moduli of the critical points of a polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ *of degree n with zeros* $z_1, \ldots, z_n$ *chosen independently, uniformly, and randomly over the unit disc* $|z| \leq 1$. *Then,*

$$\mathbb{P}[\zeta_{av} > c] \leq \frac{2}{3c}, \tag{14}$$

for $c > 0$.

Proof. Let W be the random variable corresponding to the moduli of the zeros of $P(z)$. Since z_i, $i = 1, 2, \ldots n$, are independently, identically, and randomly chosen over the unit disc $|z| \leq 1$, the *pdf* of $W = |z_i|$ can be written as:

$$f_W(w) = 2w, 0 \leq w \leq 1. \tag{15}$$

Hence, upon taking expectations, we have:

$$\begin{aligned} \mathbb{E}(|z_i|) &= \int_{0}^{1} w \cdot 2w\, dw \\ &= \frac{2}{3}. \end{aligned} \tag{16}$$

Using Theorem 4, we have:

$$\frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i| \leq \frac{1}{n}\sum_{i=1}^{n}|z_i| \leq \frac{2}{3}, \tag{17}$$

where the last inequality follows from the fact that $|z_i| \leq 1$ for all i. Taking expectations yields

$$\mathbb{E}(\zeta_{av}) \leqslant \frac{2}{3}. \tag{18}$$

Finally, employing Markov's inequality, we have:

$$\mathbb{P}[\zeta_{av} > c] \leq \frac{2}{3c}, \tag{19}$$

for $c > 0$. This establishes the Theorem 6. □

3. Variance of Zeros of a Polynomial

3.1. Variance and Mean Deviation Bounds

Let us consider a complex polynomial of degree n:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{20}$$

where $a_0, a_1, \ldots, a_n$ are complex coefficients. Let the roots of this polynomial be denoted by $z_1, z_2, \ldots, z_n$. Define the variance of the roots to be the quantity

$$\mathrm{Var}(z_i) = \frac{1}{n} \sum_{i=1}^{n} |z_i - \mu|^2. \tag{21}$$

We have the following theorem.

Theorem 7. *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial of degree n, where $a_0, a_1, \ldots, a_n$ are complex coefficients with $a_n \neq 0$. Denote the roots of $P(z)$ as $z_1, z_2, \ldots, z_n$. Define the mean of the roots as $\mu = \frac{1}{n} \sum_{i=1}^{n} z_i$. Then, we have the following results:*

(a). The variance of the roots is given by:

$$\mathrm{Var}(z_i) = \frac{1}{n} \sum_{i=1}^{n} |z_i|^2 - \frac{1}{n^2} \left| \frac{a_{n-1}}{a_n} \right|^2, \tag{22}$$

(b). The mean deviation of the roots is bounded as

$$\mathrm{Md}(z_i) \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} \rho^2 - \frac{1}{n^2} \left| \frac{a_{n-1}}{a_n} \right|^2}, \tag{23}$$

where $\rho = 1 + \max_{0 \leq i \leq n-1} |a_i|$.

Proof. We have the mean of the zeros of $P(z)$ is given by:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} z_i, \tag{24}$$

and the variance of the zeros is

$$\mathrm{Var}(z_i) = \frac{1}{n} \sum_{i=1}^{n} |z_i - \mu|^2. \tag{25}$$

We observe that the square of the deviation can be written as:

$$|z_i - \mu|^2 = |z_i|^2 + |\mu|^2 - 2\,\mathrm{Re}(\bar{z}_i \mu), \tag{26}$$

where $\mathrm{Re}(z)$ denotes the real part of z and $\bar{z}$ denotes the complex conjugate of z.

Using Expressions (26) in (25), the expression for variance becomes:

$$\mathrm{Var}(z_i) = \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 + \frac{1}{n}\sum_{i=1}^{n}|\mu|^2 - \frac{2}{n}\sum_{i=1}^{n}\mathrm{Re}(\bar{z}_i\mu) \tag{27}$$

Since $\sum_{i=1}^{n} z_i = -\frac{a_{n-1}}{a_n}$, we have $\mu = -\frac{a_{n-1}}{na_n}$.
Hence, the expression for the variance simplifies to:

$$\begin{aligned}\mathrm{Var}(z_i) &= \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 + \frac{1}{n}|\mu|^2 - \frac{2}{n}\mathrm{Re}(\bar{\mu}\mu) \\ &= \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2.\end{aligned} \tag{28}$$

We define another quantity similar to mean deviation as:

$$\mathrm{Md}(z_i) = \frac{1}{n}\sum_{i=1}^{n}|z_i - \mu|, \tag{29}$$

where as before, μ is the complex mean of the zeros of the polynomial $P(z)$.

Employing the fact that for any real-valued random variable X:

$$\mathbb{E}(X) \leq \sqrt{\mathbb{E}(X^2)}, \tag{30}$$

we immediately obtain:

$$\begin{aligned}\mathrm{Md}(z_i) &\leq \sqrt{\frac{1}{n}\sum_{0}^{n}|z_i - \mu|^2} \\ &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}|z_i|^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2}.\end{aligned} \tag{31}$$

We recall that the Cauchy bound $\rho(P)$ given by:

$$\rho = 1 + \max_{0\leq i\leq n-1}|a_i|, \tag{32}$$

then from Equations (28) and (32), we have:

$$\mathrm{Var}(z_i) \leq \frac{1}{n}\sum_{i=1}^{n}\rho^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2, \tag{33}$$

and

$$\mathrm{Md}(z_i) \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n}\rho^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2}. \tag{34}$$

From the Equation (28) and Inequality (31), we note that if the roots are on the unit circle, then we have the following:

$$\mathrm{Var}(z_i) = 1 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2, \tag{35}$$

and

$$\mathrm{Md}(z_i) \leq \sqrt{1 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2}. \tag{36}$$

This completes the proof of Theorem 7. □

Remark 2. *Based on the value of variance for zeros on the unit circle, we make the following observation:*

(i). From (35), we observe the minimum variance of 0 occurs when all the roots are the same, or in other words, when all the roots coincide at a single point on the unit circle.

(ii). If we assume the roots are all on the unit circle, then $|z_i|^2 = 1$ for all i. Therefore, $\frac{1}{n}\sum_{i=1}^{n}|z_i|^2 = 1$. So, the maximum variance occurs when the second term in the formula for $Var(z_i)$ is as small as possible. From (35), we see that the second term in $\mathrm{Var}(z_i)$ *is*

$$\frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|^2. \tag{37}$$

This term is based on the coefficients a_{n-1} and a_n of the polynomial, which determines the location of the roots. If all roots are on the unit circle, a_{n-1}/a_n would be the negative sum of the roots. The maximum variance would occur when this term is zero or close to zero, i.e., when the sum of the roots is zero. In this case, for the roots to sum to zero, they should be symmetrically distributed around the origin in the complex plane.

3.2. Variance for Quadratic Polynomials

Consider a quadratic polynomial $P(z) = az^2 + bz + c$ with roots z_1 and z_2. The variance (σ^2) of the roots can be defined as the average of the sum of squares of the deviations of each root from the mean of the roots. In this case, the mean of the roots is $\frac{z_1+z_2}{2}$; hence, the variance is given by:

$$\sigma^2 = \frac{1}{2}\left[\left(z_1 - \frac{z_1+z_2}{2}\right)^2 + \left(z_2 - \frac{z_1+z_2}{2}\right)^2\right]. \tag{38}$$

Simplifying Equation (38), we get:

$$\sigma^2 = \frac{1}{2}\left[\frac{|(z_1-z_2)|^2}{4}\right] = \frac{|(z_1-z_2)|^2}{8}. \tag{39}$$

Using the identity $|u-v|^2 = \left|(u+v)^2 - 4uv\right|$ and apply it to $u = z_1$ and $v = z_2$, we obtain:

$$(z_1-z_2)^2 = (z_1+z_2)^2 - 4z_1z_2. \tag{40}$$

Using Vieta's formulae, which state that $z_1 + z_2 = -\frac{b}{a}$ and $z_1z_2 = \frac{c}{a}$ for a quadratic equation $ax^2 + bx + c = 0$, we can substitute into the above expression to obtain:

$$|(z_1-z_2)|^2 = \left|\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}\right| = \left|\frac{b^2-4ac}{a^2}\right|, \tag{41}$$

where $b^2 - 4ac$ is the discriminant of the quadratic equation, often denoted as D. Substituting this into the equation for σ^2 gives

$$\sigma^2 = \left|\frac{\frac{b^2-4ac}{a^2}}{8}\right| = \left|\frac{D}{8a^2}\right|. \tag{42}$$

4. Expected Variance for Uniform Distribution of Roots

Theorem 8. *Let the zeros $z_1, z_2, \ldots, z_n$ of the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, be uniformly distributed over the unit disc $|z| \leq 1$. Then, we have:*

$$\mathbb{E}(Var(z_i)) = \frac{1}{2}, \quad \textit{whenever } n \to \infty. \tag{43}$$

Proof. We observe that the squared modulus of the center of mass of zeros can be written as:

$$|\mu|^2 = \left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right)\overline{\left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right)}$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} |z_i|^2 + 2\mathrm{Re}(z_i\overline{z_j}). \tag{44}$$

Taking expectation on both side of Equation (44), we have:

$$\mathbb{E}\left[|\mu|^2\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[|z_i|^2\right] + \mathbb{E}\left(\sum_{1\leq i\leq j\leq n}\mathrm{Re}(z_i\overline{z_j})\right). \tag{45}$$

As for the the quantity $Re(z_i\bar{z}_j)$ where $i < j$, we can use the fact that the roots are uniformly distributed over the unit disc to calculate its expected value. We can write $z_i = r_i e^{i\theta_i}$ and $z_j = r_j e^{i\theta_j}$, where r_i and r_j are the magnitudes of the roots, and θ_i and θ_j are their arguments. Then, we have:

$$Re(z_i\bar{z}_j) = r_i r_j \cos(\theta_i - \theta_j). \tag{46}$$

Since the roots are uniformly distributed over the unit disc, the magnitudes r_i and r_j are independent and uniformly distributed over the interval $[0,1]$. The difference $\theta_i - \theta_j$ is also uniformly distributed over the interval $[-\pi, \pi)$. Therefore, the expected value of $Re(z_i\bar{z}_j)$ is given by:

$$\mathbb{E}[Re(z_i\bar{z}_j)] = \frac{1}{4\pi^2}\int_0^1\int_0^1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} r_i r_j \cos(\theta_i - \theta_j) dr_i dr_j d\theta_i d\theta_j \tag{47}$$
$$= \frac{1}{4\pi^2}\int_0^1\int_0^1 r_i r_j \left(\int_{-\pi}^{\pi}\cos(\theta_i - \theta_j) d\theta_i d\theta_j\right) dr_i dr_j \tag{48}$$
$$= 0. \tag{49}$$

Therefore, from the Equation (45), we have:

$$\mathbb{E}\left[|\mu|^2\right] = \frac{1}{n^2} \times n \times \int_0^1 2r \cdot dr + 0 = \frac{1}{2n}. \tag{50}$$

It follows from Equation (50),

$$\mathbb{E}\left[|\mu|\right] \leq \sqrt{\frac{1}{2n}}. \tag{51}$$

Since $|\mu|$ is non-negative, we conclude that as n goes to infinity, $\mathbb{E}\left[|\mu|\right]$, and hence $|\mu|$, both approach 0. We now observe from Equation (22) that the expression for variance can also be recast as:

$$\mathrm{Var}(z_i) = \frac{1}{n}\sum_{i=1}^{n} |z_i|^2 - |\mu|^2. \tag{52}$$

Thus, for a uniform distribution on the zeros over a unit disc, we have the variance is given as follows:

$$\mathbb{E}(\mathrm{Var}(z_i)) = \frac{1}{n}\mathbb{E}\left(\sum_1^n |z_i|^2\right) - \frac{1}{2n}$$
$$= \frac{1}{2} - \frac{1}{2n}. \tag{53}$$

This completes the proof. □

Remark 3. *It is worth-noting from Equation (53), that $\mathbb{E}(\mathrm{Var}(z_i))$ approaches $\frac{1}{2}$ as $n \to \infty$, under the assumptions of the Theorem (8).*

5. Variance of Polynomials with Real Roots

Theorem 9. *For a polynomial $P(z) = \sum_0^n a_i z^i$ with only real roots, the variance of the roots is given by:*

$$\sigma^2 = \frac{1}{n}\left[-\frac{a_{n-1}^2}{a_n^2} + \frac{2(n-1)a_{n-1}^2}{a_n^2} - \frac{2(n-2)a_{n-2}}{a_n}\right]. \tag{54}$$

Proof. Given a polynomial in the form

$$P(z) = \sum_{i=0}^n a_i z^i,$$

we can apply the substitution $z = w - \frac{a_{n-1}}{na_n}$ to obtain a new polynomial

$$Q(w) = \sum_{i=0}^n b_i w^i.$$

The coefficients b_i in the polynomial $Q(w)$ can be expressed in terms of the a_i coefficients by substituting $w = z + \frac{a_{n-1}}{na_n}$ into the polynomial $P(z)$, yielding

$$Q(w) = P\left(w - \frac{a_{n-1}}{na_n}\right) = \sum_{i=0}^n a_i \left(w - \frac{a_{n-1}}{na_n}\right)^i. \tag{55}$$

Expanding the w^i terms and collecting terms on the R.H.S of Equation (55), we find expressions for b_n, b_{n-1}, and b_{n-2}:

For b_n, we have:

$$b_n = a_n.$$

For b_{n-1}, we find

$$b_{n-1} = -\frac{a_{n-1}a_n}{a_n} + (n-1)a_{n-1} = 0.$$

For b_{n-2}, we obtain:

$$b_{n-2} = \frac{a_{n-1}^2 a_n}{2a_n^2} - \frac{(n-1)a_{n-1}^2}{a_n} + (n-2)a_{n-2}$$
$$= \frac{a_{n-1}^2}{2a_n} - \frac{(n-1)a_{n-1}^2}{a_n} + (n-2)a_{n-2}.$$

The variance σ^2 of the roots of a polynomial with only real roots can be expressed as:

$$\sigma^2 = -\frac{2b_{n-2}}{nb_n},$$

where b_{n-2} and b_n are coefficients obtained by translating the polynomial so that its sum of roots is zero.

Using the identity

$$(w_1 + \cdots + w_n)^2 = w_1^2 + \cdots + w_n^2 + 2(w_1 w_2 + w_1 w_3 \cdots + w_{n-1} w_n)$$

and the Vièta's formula for the sum of roots taken two at a time, we have:

$$0 = n\sigma^2 + 2\frac{b_{n-2}}{b_n},$$

which yields the expression for σ^2.

$$\sigma^2 = -\frac{2\left(\frac{a_{n-1}^2}{2a_n} - \frac{(n-1)a_{n-1}^2}{a_n} + (n-2)a_{n-2}\right)}{na_n} \tag{56}$$

Simplifying the expression in (56) we get:

$$\sigma^2 = \frac{1}{n}\left[-\frac{a_{n-1}^2}{a_n^2} + \frac{2(n-1)a_{n-1}^2}{a_n^2} - \frac{2(n-2)a_{n-2}}{a_n}\right]. \tag{57}$$

□

6. Calculating Bounds Using de Bruijn Inequality and the Product of Roots

6.1. Bounds Using de Bruijn Inequality

Here, we use the following inequality due to de Bruijn [18] to derive some bounds for the sum of the moduli of zeros of a polynomial.

Theorem 10. *Let $z_1, z_2, \ldots, z_n$ be the roots of a polynomial $\sum_{k=0}^{n} a_k z^k$. Then, a bound for $\frac{1}{n}\sum_{k=1}^{n}|z_k|^2$ is given by:*

$$\frac{1}{n}\sum_{k=1}^{n}|z_k|^2 \geqslant \frac{2}{n^2|a_n|}\left[|a_{n-1}|^2 - |a_{n-1} - 2a_{n-2}|\right]. \tag{58}$$

Proof. In the proof of the above bound, we shall be using the following result by de Bruijn as a lemma. □

Lemma 1 (de Bruijn's Inequality). *If $a = (a_1, \ldots, a_n)$ is an n-tuple of real numbers and $z = (z_1, \ldots, z_n)$ is an n-tuple of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]. \tag{59}$$

Equality holds if and only if for $k \in \{1, \ldots, n\}, a_k = \mathrm{Re}(\lambda z_k)$, where λ is a complex number such that $\lambda^2 \sum_{k=1}^{n} z_k^2$ is a non-negative real number.

As a straightforward corollary, if all a_k for $k = 1, 2, 3, \ldots, n$ are taken to be 1, then the given inequality would simplify to:

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \frac{n}{2}\left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]. \tag{60}$$

which shows that

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sqrt{\frac{n}{2}\left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]}. \tag{61}$$

With $a_k = 1$ for all $k = 1, 2, \ldots, n$, the de Bruijn inequality becomes:

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leq \frac{n}{2}\left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right] \tag{62}$$

We can now find a bound for $\mathrm{Re}(z_i\bar{z}_j)$ for the roots of the polynomial as follows. Expanding the square in the left-hand side, we obtain:

$$\sum_{k=1}^{n}|z_k|^2 + 2\sum_{i<j}\mathrm{Re}(z_i\bar{z}_j) \leq \frac{n}{2}\left[\sum_{k=1}^{n}|z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right] \tag{63}$$

$$\sum_{i<j}\mathrm{Re}(z_i\bar{z}_j) \leq \frac{1}{2}\left[\frac{(n-2)}{2}\left(\sum_{k=1}^{n}|z_k|^2\right) + \left|\sum_{k=1}^{n} z_k^2\right|\right] \tag{64}$$

In order to find a bound for the mean of the moduli of the roots, consider a polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$. We have from Vieta's formulae:

$$\sum z_i = (-1)^{n-1}\frac{a_{n-1}}{a_n} \tag{65}$$

and

$$\left|\sum_{k=1}^{n} z_k^2\right| = \left|\left(\sum z_k\right)^2 - 2\sum z_1 \cdot z_2\right|, \tag{66}$$

Simplifying (66),we get:

$$\left|\sum_{k=1}^{n} z_k^2\right| = \left|(-1)^{n-1}\frac{a_{n-1}}{a_n} - (-1)^{n-2}2\frac{a_{n-2}}{a_n}\right| \tag{67}$$

$$= \left|\frac{a_{n-1}}{a_n} - 2\frac{a_{n-2}}{a_n}\right| \tag{68}$$

$$= \left|\frac{a_{n-1} - 2a_{n-2}}{a_n}\right| \tag{69}$$

From inequality (60), we have:

$$\sum_{k=1}^{n}|zk|^2 \geqslant \frac{2}{n}\left[\left|\sum_{k=1}^{n} z_k\right|^2 - \left|\sum_{k=1}^{n} z_k^2\right|^2\right] \tag{70}$$

Hence, a bound for $\frac{1}{n}\sum_{k=1}^{n}|z_k|^2$ is given by

$$\frac{1}{n}\sum_{k=1}^{n}|z_k|^2 \geqslant \frac{2}{n^2|a_n|}\left[|a_{n-1}|^2 - |a_{n-1} - 2a_{n-2}|\right]. \tag{71}$$

That establishes Lemma (1)

6.2. Bound Using the Products of Roots

Theorem 11. *For any polynomial* $\sum_{i=0}^{n} a_i z^i$*, the lower bound of the average moduli of roots is given by:*

$$\frac{1}{n}\sum_{1}^{n}|z_i| \geq \frac{1}{n}\left(\left|\frac{a_0}{a_n}\right|\right)^{1/n}. \tag{72}$$

Proof. For any polynomial $\sum_{0}^{n} a_i z^i$, we may note that

$$\left|\text{Product of roots}\right| = \left|\frac{a_0}{a_n}\right|.$$

So, if z_{min} is the root min minimum modulus, then

$$|z_{min}| \geq \left(\left|\frac{a_0}{a_n}\right|\right)^{1/n}, \tag{73}$$

so that

$$\frac{1}{n}\sum_{1}^{n}|z_i| \geq \frac{1}{n}\left(\left|\frac{a_0}{a_n}\right|\right)^{1/n}. \tag{74}$$

□

7. Relation between the Variances of Zeros and Critical Points

Theorem 12. *For any complex polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$*, we have the following:*

(a). The variance of the critical points ζ_i *is related to the variance of the zeros* z_i *and the coefficients through the inequality:*

$$\text{Var}(\zeta_i) \leq \text{Var}(z_i) - \frac{1}{n^2(n-1)}\left|\frac{a_{n-1}}{a_n}\right|. \tag{75}$$

(b). The variance $\text{Var}(\zeta_i)$ *of the critical points can never exceed the variance* $\text{Var}(z_i)$ *of the zeros.*

Proof. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$ and critical points $\zeta_1, \ldots, \zeta_{n-1}$. We define the variance of the critical points as:

$$\text{Var}(\zeta_i) = \frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i - \mu_c|^2, \tag{76}$$

where $\mu_c = \frac{1}{n-1}\sum_{i=1}^{n-1}\zeta_i$ is the mean of the critical points.

Also,

$$\text{Var}(z_i) = \frac{1}{n}\sum_{i=1}^{n}|z_i - \mu_z|, \tag{77}$$

where $\mu_z = \frac{1}{n}\sum_{i=1}^{n} z_i$.

Now, $P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$, and

$$\text{Var}(z_i) = \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|. \tag{78}$$

Therefore,

$$\mathrm{Var}(\zeta_i) = \frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i|^2 - \frac{1}{(n-1)^2}\left|\frac{(n-1)a_{n-1}}{na_n}\right| \tag{79}$$

$$= \frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i|^2 - \frac{1}{n(n-1)}\left|\frac{a_{n-1}}{a_n}\right|. \tag{80}$$

Since $\frac{1}{n-1}\sum_{i=1}^{n-1}|\zeta_i|^2 \leq \frac{1}{n}\sum_{i=1}^{n}|z_i|^2$, we have

$$\mathrm{Var}(\zeta_i) \leq \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 - \frac{1}{n(n-1)}\left|\frac{a_{n-1}}{a_n}\right|. \tag{81}$$

From the expression

$$\mathrm{Var}(z_i) = \frac{1}{n}\sum_{i=1}^{n}|z_i|^2 - \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right|, \tag{82}$$

we have:

$$\frac{1}{n}\sum_{i=1}^{n}|z_i|^2 = \mathrm{Var}(z_i) + \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right| \tag{83}$$

Therefore,

$$\mathrm{Var}(\zeta_i) \leq \mathrm{Var}(z_i) + \frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right| - \frac{1}{n(n-1)}\left|\frac{a_{n-1}}{a_n}\right| \tag{84}$$

Since

$$\frac{1}{n^2}\left|\frac{a_{n-1}}{a_n}\right| \leq \frac{1}{n(n-1)}\left|\frac{a_{n-1}}{a_n}\right|, \tag{85}$$

we conclude:

$$\mathrm{Var}(\zeta_i) \leq \mathrm{Var}(z_i), \tag{86}$$

which establishes the theorem. □

8. Real Part Conjecture

Here is a possible relation between the moduli of zeros and critical points:

$$\frac{1}{n-1}\sum_{i=0}^{n-1}|Re(\zeta_i)| \leq \frac{1}{n}\sum_{i=1}^{n}|Re(z_i)| + C_n \tag{87}$$

for some constant C_n that depends on n, the degree of the polynomial.

The intuition here is that the real parts of the zeros provide an upper bound on the real parts of the critical points. The reason we need to add a constant C_n is that for higher-degree polynomials, the zeros can be quite spread out, so we need to account for the possibility of critical points appearing in the gaps between zeros.

9. Conclusions

In this paper, we have presented some novel results and new techniques for the relatively underinvestigated problems concerning the variance of the zeros of polynomials. Significantly, we have derived exact results for the variance of the roots of polynomials with only real roots. The research fills a gap in the existing literature by exploring the variance of the zeros themselves, not just the number of real roots, which has been extensively studied. The paper also investigates the relationship between the variance of the zeros and critical points of a polynomial, as well as their average moduli, utilizing tools like de Bruijn's inequality and some other lesser known classical inequalities while extending the results to some random polynomials using the Markov inequality from theory of probability. Among the things investigated in this paper, the chief things to mention are the average of the moduli of the critical points for the uniform distribution of the roots; the variance and mean deviation bounds and the expected variance for the uniform distribution of the roots and variance of polynomials with real roots; and the relation between the variances of the zeros and critical points. The findings provide valuable insights into the distribution of zeros and the structural characteristics of polynomials, besides providing novel directions for new explorations in the distribution of the zeros of polynomials.

10. Future Research

Building upon the novel findings in this paper, future research holds exciting possibilities to push the boundaries of understanding zero variance and its diverse implications. While the current work focuses on polynomials with real roots, venturing into the realm of complex roots would paint a broader picture, potentially uncovering connections with root location theorems and demanding specialized techniques from complex analysis. Furthermore, extending the analysis beyond univariate polynomials to encompass the intricate world of multivariate settings could involve harnessing tensor decompositions, exploring geometric nuances of higher-dimensional zero sets, and adapting tools from algebraic geometry.

Beyond uniform root distributions, delving into diverse scenarios with weighted distributions, specific polynomial families with known root behavior, or random models with controlled coefficient dependencies could reveal fascinating insights. Additionally, deepening our understanding of the interplay between zero variance and other structural characteristics like degree, coefficient ratios, and root clustering patterns holds immense potential.

From a computational standpoint, developing efficient algorithms for calculating zero variance or related quantities for general polynomials could pave the way for applications in signal processing, optimization, and approximation theory, where root distribution plays a pivotal role. Integrating tools from probability theory and random matrix theory into the analysis of zero variance in random polynomial ensembles could offer invaluable insights into typical behavior and asymptotic properties for large degrees or specific coefficient distributions.

Furthermore, exploring the potential connection between zero variance and the stability of polynomial roots under perturbations could have significant implications for control theory, numerical analysis, and understanding the sensitivity of solutions to data changes.

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References

1. van der Waerden, B.L. *A History of Algebra: From al-Khwārizmī to Emmy Noether*; Springer: Berlin, Germany, 2013.
2. Duran, U.; Acikgoz, M.; Esi, A.; Aracı, S. A note on the (p,q)-Hermite polynomials. *Appl. Mat. Inf. Sci.* **2018**, *12*, 227–231. [CrossRef]
3. Marden, M. *Geometry of Polynomials*; American Mathematical Society: Providence, RI, USA, 1949.
4. Milovanovic, G.V.; Mitrinovic, D.S.; Rassias, T. *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*; World Scientific Publishing Co.: Singapore, 1994.
5. Rahman, Q.I.; Schmeisser, G. *Analytic Theory of Polynomials: Critical Points, Zeros and Extremal Properties*; Oxford University Press: Oxford, UK, 2002.
6. Bharucha-Reid, A.T.; Sambandham, M. *Random Polynomials*; Academic Press: Orlando, FL, USA, 1986.

7. Thangraj, V.; Sambandham, M. On Random Polynomials-II: A Survey. *Neural Parallel Sci. Comput.* **2021**, *29*, 230–251. [CrossRef]
8. Govil, N.K. Inequalities for the derivative of a polynomial. *J. Approx. Theory* **1991**, *66*, 29–35. [CrossRef]
9. Frappier, C.; Rahman, Q.I.; Ruscheweyh, S. New Inequalities for polynomials. *Trans. Am. Math. Soc.* **1958**, *28*, 69–99. [CrossRef]
10. Ahanger, U.M.; Shah, W.M. Inequalities for the derivative of a polynomial with restricted zeros. *J. Anal.* **2021**, *29*, 1367–1374. [CrossRef]
11. Borwein, P.; Erdelyi, T. *Polynomials and Polynomial Inequalities*; Springer: Berlin/Heidelberg, Germany, 2012.
12. Thangaraj, V.; Renganathan, N. A lower bound for the variance of real zeros of random polynomials. *Stoch. Anal. Appl.* **1990**, *8*, 463–481. [CrossRef]
13. Sharma, R.; Gupta, M.; Kapoor, G. Some Better Bounds on the Variance with Applications. *J. Math. Inequalities* **2010**, *4*, 355–363. [CrossRef]
14. Andreescu, T.; Safaei, N.; Ventullo, A. *Awesome Polynomials for Mathematics Competitions*; XYZ Press: Hicksville, NY, USA, 2021.
15. Bruijn, d.N.G. On the zeros of a polynomial and of its derivative. *Proc. Sect. Sci. K. Ned. Akad. Van Wet. Amst.* **1946**, *49*, 1037–1044.
16. Ross, S.M. *Introduction to Probability Models*; Academic Press: Orlando, FL, USA, 2014.
17. Feller, W. *An Introduction to Probability Theory and Its Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2008.
18. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1992; pp. 89–90.