Article

Directed Path 3-Arc-Connectivity of Cartesian Product Digraphs

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Abstract: Let \( D = (V(D), A(D)) \) be a digraph of order \( n \) and let \( r \in S \subseteq V(D) \) with \( 2 \leq |S| \leq n \). A directed \((S, r)\)-Steiner path (or an \((S, r)\)-path for short) is a directed path \( P \) beginning at \( r \) such that \( S \subseteq V(P) \). Arc-disjoint between two \((S, r)\)-paths is characterized by the absence of common arcs. Let \( \lambda_{S,r}^k(D) \) be the maximum number of arc-disjoint \((S, r)\)-paths in \( D \). The directed path \( k \)-arc-connectivity of \( D \) is defined as \( \lambda_k^D(D) = \min\{\lambda_{S,r}^k(D) \mid S \subseteq V(D), |S| = k, r \in S\} \). In this paper, we shall investigate the directed path \( 3 \)-arc-connectivity of Cartesian product digraph \( \lambda_{S,r}^3(G \boxplus H) \) and prove that if \( G \) and \( H \) are two digraphs such that \( \delta^{	ext{out}}(G) \geq 4, \delta^{	ext{out}}(H) \geq 4, \) and \( \kappa(G) \geq 2, \kappa(H) \geq 2, \) then \( \lambda_{S,r}^3(G \boxplus H) \geq \min\{2\kappa(G), 2\kappa(H)\} \); moreover, this bound is sharp. We also obtain exact values for \( \lambda_k^3(G \boxplus H) \) for some digraph classes \( G \) and \( H \), and most of these digraphs are symmetric.

Keywords: connectivity; directed path \( k \)-connectivity; Cartesian product

1. Introduction

For a detailed explanation of graph theoretical notation and terminology not provided here, readers are directed to reference [1]. It should be noted that all digraphs discussed in this paper do not contain parallel arcs or loops. The set of all natural numbers from 1 to \( n \) is denoted by \([n]\). If a directed graph \( D \) can be obtained from its underlying graph \( G \) by replacing each edge in \( G \) with corresponding arcs in both directions, then \( D \) is said to be symmetric, denoted as \( D = \overset{\text{sym}}{G} \). The notation \( \overset{\text{sym}}{C}_n \) is used for a symmetric digraph whose underlying graph forms a tree of order \( n \). The notation \( \overset{\text{sym}}{C}_n \) is used for a symmetric digraph whose underlying graph forms a cycle of order \( n \). The cycle digraph of order \( n \) is denoted by \( \overset{\text{sym}}{C}_n \). We denote the complete digraph of order \( n \) as \( \overset{\text{sym}}{K}_n \).

The well-known Steiner tree packing problem is characterized as follows. Given a graph \( G \) and a set of terminal vertices \( S \subseteq V(G) \), the goal is to identify as many edge-disjoint \( S \)-Steiner trees (i.e., trees \( T \) in \( G \) with \( S \subseteq V(T) \)) as feasible. This particular problem, along with its associated topics, garners significant interest from researchers due to its extensive applications in VLSI circuit design [2–4] and Internet Domain [5]. In practical applications, the construction of vertex-disjoint or arc-disjoint paths in graphs holds significance, as they play a crucial role in improving transmission reliability and boosting network transmission rates [6]. This paper will specifically delve into a variant of the directed Steiner tree packing problem, termed the directed Steiner path packing problem, closely interconnected with the Steiner path problem and the Steiner path cover problem [7,8].

We now consider two types of directed Steiner path packing problems and related parameters. Let \( D = (V(D), A(D)) \) be a digraph of order \( n \) and let \( r \in S \subseteq V(D) \) with \( 2 \leq |S| \leq n \). A directed \((S, r)\)-Steiner path, or simply an \((S, r)\)-path, refers to a directed path \( P \) originating from \( r \) such that \( S \) is a subset of the vertices in \( P \). Arc-disjoint between two \((S, r)\)-paths implies that they share no common arcs, while two arc-disjoint \((S, r)\)-paths are internally disjoint when their common vertex set is precisely \( S \). Let \( \lambda_{S,r}^p(D) \) and \( \kappa_{S,r}^p(D) \) represent the maximum number of arc-disjoint (and internally disjoint) \((S, r)\)-paths in \( D \), respectively. The Arc-disjoint (or Internally disjoint) Directed Steiner Path Packing problem is formulated as follows. Given a digraph \( D \) and letting \( r \in S \subseteq V(D) \), the objective is

\[ \text{Maximize} \quad \lambda_{S,r}^p(D) \quad \text{subject to} \quad \lambda_{S,r}^p(D) \leq \kappa_{S,r}^p(D). \]
to maximize the count of arc-disjoint (or internally disjoint) \((S,r)\)-paths. The notion of directed path connectivity, which is a derivative of path connectivity in undirected graphs, is intricately linked to the directed Steiner path packing problem and serves as a logical progression from path connectivity in directed graphs (refer to [5] for the initial presentation of path connectivity). The directed path \(k\)-connectivity [9] of \(D\) is defined as

\[
\kappa^p_k(D) = \min\{\kappa^p_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}
\]

Similarly, the directed path \(k\)-arc-connectivity [9] of \(D\) is defined as

\[
\lambda^p_k(D) = \min\{\lambda^p_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}
\]

The concepts of directed path \(k\)-connectivity and directed path \(k\)-arc-connectivity are synonymous with directed path connectivity. In the context of \(k = 2\), \(\kappa^p_2(D)\) equates to \(\kappa(D)\) and \(\lambda^p_2(D)\) equates to \(\lambda(D)\), where \(\kappa(D)\) and \(\lambda(D)\) denote vertex-strong connectivity and arc-strong connectivity of digraphs, respectively. Hence, these parameters can be viewed as extensions of the classical connectivity measures in a digraph. It is pertinent to emphasize the close relationship between strong subgraph connectivity and directed path connectivity; refer to [10–12] for further insights on this interconnected topic.

It is a widely recognized fact that Cartesian products of digraphs are of great interest in graph theory and its applications. For a comprehensive overview of various findings on Cartesian products of digraphs, one may refer to a recent survey chapter by Hammack [13]. In this paper, we continue research on directed path connectivity and focus on the directed path \(3\)-arc-connectivity of Cartesian products of digraphs.

In Section 2, we introduce terminology and notation on Cartesian products of digraphs. In Section 3, we prove that if \(G\) and \(H\) are two digraphs such that \(\delta^1(G) \geq 4\), \(\delta^0(H) \geq 4\), and \(\kappa(G) \geq 2\), \(\kappa(H) \geq 2\), then

\[
\lambda^p_3(G \Box H) \geq \min\{2\kappa(G), 2\kappa(H)\};
\]

moreover, this bound is sharp. Finally, we obtain exact values of \(\lambda^p_G(G \Box H)\) for some digraph classes \(G\) and \(H\) in Section 4.

2. Cartesian Product of Digraphs

Consider two digraphs \(G\) and \(H\) with vertex sets \(V(G) = \{u_i \mid i \in [n]\}\) and \(V(H) = \{v_j \mid j \in [m]\}\). The Cartesian product of \(G\) and \(H\), denoted by \(G \Box H\), is a digraph with vertex set

\[
V(G \Box H) = V(G) \times V(H) = \{(x,x') \mid x \in V(G), x' \in V(H)\}.
\]

The arc set of \(G \Box H\), denoted by \(A(G \Box H)\), is given by \(\{(x,x')(y,y') \mid xy \in A(G), x' = y', \text{or } x = y, x'y' \in A(H)\}\). It is worth noting that Cartesian product is an associative and commutative operation. Furthermore, \(G \Box H\) is strongly connected if and only if both \(G\) and \(H\) are strongly connected, as shown in a recent survey chapter by Hammack [13].

In the rest of the paper, we will use \(u_{ij}\) to denote \((u_i,v_j)\). Additionally, \(G(v_j)\) will refer to the subgraph of \(G \Box H\) induced by the vertex set \(\{u_{ij} \mid i \in [n]\}\) with \(j \in [m]\), while \(H(u_i)\) will denote the subgraph of \(G \Box H\) induced by the vertex set \(\{u_{ij} \mid j \in [m]\}\) with \(i \in [n]\). It is evident that \(G(v_j)\) is isomorphic to \(G\) and \(H(u_i)\) is isomorphic to \(H\). To illustrate this, refer to Figure 1 (this figure comes from [14]), where it can be observed that \(G(v_j)\) is isomorphic to \(G\) for \(1 \leq j \leq 4\), and \(H(u_i)\) is isomorphic to \(H\) for \(1 \leq i \leq 3\).

For distinct indices \(j_1\) and \(j_2\) with \(1 \leq j_1 \neq j_2 \leq m\), the vertices \(u_{ij_1}\) and \(u_{ij_2}\) belong to the same digraph \(H(u_i)\), where \(u_i\) is an element of \(V(G)\). \(u_{ij_1}\) is referred to as the vertex corresponding to \(u_{ij_1}\) in \(G(v_j)\). Similarly, for distinct indices \(i_1\) and \(i_2\) with \(1 \leq i_1 \neq i_2 \leq n\), \(u_{i_1j}\) is the vertex corresponding to \(u_{i_1j}\) in \(H(u_i)\). Analogously, the subgraph corresponding to a given subgraph can also be defined. For instance, in the digraph (c) depicted in Figure 1,
if we label the path 1 as $P_1$ (and the path 2 as $P_2$) in $H(u_1)$ ($H(u_2)$), then $P_2$ is identified as the path that corresponds to $P_1$ in $H(u_2)$.

![Diagram](https://example.com/diagram.png)

**Figure 1.** $G$, $H$ and their Cartesian product [14] (1 denotes arc $u_{1,1}v_{1,2}$, $u_{1,2}v_{1,3}$ and arc $u_{1,3}u_{1,i}$; 2 denotes arc $u_{2,1}u_{2,2}$, $u_{2,2}u_{2,3}$ and arc $u_{2,3}u_{2,4}$).

Sun and Zhang proved some results of directed path connectivity, that is, the following lemma.

**Lemma 1** ([9]). Let $D$ be a digraph of order $n$, and let $k$ be an integer satisfying $2 \leq k \leq n$. The following statements are valid:

1. $\lambda^p_k(D) \leq \lambda^p_k(D)$ when $k \leq n - 1$.
2. $k^p(D) \leq \lambda^p_k(D) \leq \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$.

**Lemma 2** ([15]). $\kappa(\overrightarrow{K_n}) = n - 1$.

3. A General Lower Bound

Now we will provide a lower bound for $\lambda^p_3(G \square H)$.

**Theorem 1.** Let $G$ and $H$ be two digraphs such that $\delta^0(G) \geq 4$, $\delta^0(H) \geq 4$, and $\kappa(G) \geq 2$, $\kappa(H) \geq 2$. We have

$$\lambda^p_3(G \square H) \geq \min\{2\kappa(G), 2\kappa(H)\}.$$

Furthermore, this bound is sharp.

**Proof.** It suffices to show that there are at least $2\kappa(G)$ or $2\kappa(H)$ arc-disjoint $(S, r)$-paths for any $S \subseteq V(G \square H)$ with $|S| = 3$, $r \in S$. Let $S = \{x, y, z\}$ and let $r = x$. Without loss of generality, we may assume $\kappa(G) \leq \kappa(H)$ and consider the following six cases.

**Case 1.** Let $x, y$ and $z$ be in the same $H(u_i)$ or $G(v_j)$ for some $i \in [n], j \in [m]$. Without loss of generality, we may assume that $x = u_{1,1}, y = u_{2,1}, z = u_{3,1}$. In this case, our overall goal is that we will use arc-disjoint paths between $x$ and $y$ in $G(v_1)$, $y$ and $z$ in $G(v_1)$, and its out-neighbors in $H(u_1)$, $y$ and its in-neighbors in $H(u_2)$, $z$ and its in-neighbors in $H(u_3)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 2. The vertices and paths contained in Figure 2 are explained below.

Let $S_1 = \{x, y\}, r_1 = x$. It is known that there are at least $\kappa(G)$ internally disjoint $(S_1, r_1)$-paths in $G(v_1)$, denoted as $P_{1i}$ ($i \in [\kappa(G)]$). Considering $S'_1 = \{y, z\}, r'_1 = y$, there are at least $\kappa(G)$ internally disjoint $(S'_1, r'_1)$-paths in $G(v_1)$, denoted as $P_{j2}$ ($j \in [\kappa(G)]$). For each $j \in [\kappa(G)]$, let $u_{a,1}$ be the out-neighbor of $y$ in $P_{j2}$; clearly these out-neighbors are distinct. Similarly, an in-neighbor $u_{k,1}$ ($j \in [\kappa(G)]$) of $z$ in $P_{j2}$ can be chosen such that these in-neighbors are distinct. In $H(u_1)$, if there is a vertex that is not an out-neighbor of $x$, then choose such a vertex as $u_{1, a}$, where $a \neq 1$. If there is no such vertex, that is, all vertices are out-neighbors of $x$, then choose any vertex as $u_{1, a}$, where $a \neq 1$. In $H(u_1)$, let $S'_2 = \{x, u_{1, a}\}, r'_2 = x$, and it is established that there exist at least $\kappa(G)$ internally disjoint $(S'_2, r'_2)$-paths, say $P_{2j}$ ($j \in [\kappa(G)]$). In $G(v_3)$, let $S'_3 = \{u_{1, a}, u_{2, a}\}, r'_3 = u_{1, a}$,
and it is established that there exist at least \( \kappa(G) \) internally disjoint \( (S_3', r_3') \)-paths, say \( \vec{P}_{2j} \) \( (j \in [\kappa(G)]) \). In \( H(u_2) \), let \( S'_4 = \{y, u_2a\} \), \( r'_4 = u_{2,a} \), and it is established that there exist at least \( \kappa(G) \) internally disjoint \( (S_4', r_4') \)-paths, say \( \vec{P}_{2j} \) \( (j \in [\kappa(G)]) \).

![Figure 2. Depiction of the arc-disjoint paths found in Case 1 of the proof of Theorem 1.](image)

In \( H(u_1) \), if there is a vertex that is not an out-neighbor of \( x \) in \( \vec{P}_{2j} \), then choose such a vertex as \( u_{1,d} \), with \( d \not\in \{1, a\} \). If there is no such vertex, then choose any vertex as \( u_{1,d} \), with \( d \not\in \{1, a\} \). In \( H(u_2) \), with \( S_2 = \{y, u_2a\} \) and \( r_2 = y \), it is known that there are at least \( \kappa(G) \) internally disjoint \( (S_2', r_2') \)-paths, denoted as \( \vec{P}_{2j} \) \( (i \in [\kappa(G)]) \). For each \( i \in [\kappa(G)] \), let \( u_{2,f_i} \) be the out-neighbor of \( y \) in \( \vec{P}_{1i} \); clearly these out-neighbors are distinct. For each \( i \in [\kappa(G)] \), since \( d^3(G) \geq 4 \), an out-neighbor of \( u_{2,f_i} \) in \( G(v_1) \), denoted by \( u_{b,f_i} \) \( (b \in [n]) \), can be chosen, with \( b \not\in \{1, 3\} \). If there exists a vertex \( u_{s_i,1} \not\in \{u_{1,1}, u_{3,1}\} \), let \( b = s_j \). If there is no such vertex, then let \( b \neq k_j \). In \( H(u_b) \), \( \vec{P}_{1i} \) is the \( (S_3, r_3) \)-path corresponding to \( \vec{P}_{1i} \), where \( S_3 = \{u_{b,1}, u_{b,d}\} \), and \( r_3 = u_{b,1} \). In \( \vec{P}_{1i}' \), the path from vertex \( u_{b,f_i} \) to \( u_{b,d} \) is denoted as \( \vec{P}_{1i}' \). Let \( S_4 = \{u_{b,d}, u_{3,d}\} \), \( r_4 = u_{b,d} \), and it is established that there exist at least \( \kappa(G) \) internally disjoint \( (S_4, r_4) \)-paths, say \( \vec{P}_{3i} \) \( (i \in [\kappa(G)]) \). If \( u_{2,f_i} = u_{2,d} \) \( (t \in [\kappa(G)]) \), then let \( u_{2,d} \not\in \vec{P}_{2j} \) in \( \vec{P}_{1i} \). In \( H(u_3) \), let \( S_5 = \{u_{3,d}, z\} \), \( r_5 = u_{3,d} \), and it is established that there exist at least \( \kappa(G) \) internally disjoint \( (S_5, r_5) \)-paths, say \( \vec{P}_{1i} \) \( (i \in [\kappa(G)]) \).

In \( H(u_1) \), if there is an out-neighbor of \( x \) that is not an out-neighbor of \( x \) in \( \vec{P}_{2j} \), then choose such a vertex as \( u_{1,c} \), with \( c \not\in \{a, d\} \). If there is no such vertex, then choose any out-neighbor of \( x \) as \( u_{1,c} \), with \( c \not\in \{a, d\} \). And \( u_{s_i,c} \) is an out-neighbor of \( u_{s_i,1} \) in \( H(u_{s_i}) \). In \( G(v_c) \), \( \vec{P}_{2j} \) is the \( (S_5, r_5) \)-path corresponding to \( \vec{P}_{2j} \), where \( S_5 = \{u_{2,c}, u_{3,c}\} \) and \( r_5 = u_{2,c} \). In \( \vec{P}_{2j} \), the path from vertex \( u_{s_i,c} \) to \( u_{k,s} \) is denoted as \( \vec{P}_{2j} \). If \( u_{s_i,1} = u_{k,1} \) \( (t \in [\kappa(G)]) \), then \( \vec{P}_{2j} = \{y u_{s_i,1}, u_{s_i,1} z\} \). If \( u_{s_i,1} = z \) \( (t \in [\kappa(G)]) \), then \( \vec{P}_{2j} = \{y z\} \). If \( u_{1,c} \not\in \vec{P}_{2j} \) \( (h \in [\kappa(G)]) \), then \( u_{1,c} \not\in \vec{P}_{2h} \). In \( H(u_{s_j}) \), with \( S_{b_j} = \{u_{b,c} \cup u_{s_j,1}\} \), and \( r_{b_j} = u_{b,c} \), it is known that there
exist at least $\kappa(G)$ internally disjoint $(S_6', r_6')$-paths. Then in these paths, one of the paths $\tilde{P}_j$ ($j \in [\kappa(G)]$) is chosen, with $u_{k,1} \notin \tilde{P}_j$.

**Subcase 1.1.** In the set \{u_{j,1}, u_{k,1}\}, there is no vertex such that $u_{j,1} = x$ or $u_{k,1} = x$, and the vertex $z$ is not in path $\tilde{P}_1$. We now construct the arc-disjoint $(S,r)$-paths by letting

- $P_{1l} = \tilde{P}_1 \cup \tilde{P}_1 \cup P_{1l} = \{yv_{u_{j,1}, u_{k,1}, r_{j,1}}\}, j \in [\kappa(G)]$,
- $P_{2l} = \tilde{P}_2 \cup \tilde{P}_2 \cup P_{2l} = \{yv_{u_{j,1}, u_{k,1}, r_{j,1}}\}, j \in [\kappa(G)] \setminus \{1, t\}$,
- $P_{2l} = \tilde{P}_2 \cup \tilde{P}_2 \cup \tilde{P}_{2l} \cup P_{2l} = \{yv_{u_{j,1}, u_{k,1}, r_{j,1}}\}, j \in [\kappa(G)] \setminus \{1, t\}$,
- Then we obtain $2\kappa(G)$ arc-disjoint $(S,r)$-paths.

**Subcase 1.2.** In the set \{u_{j,1}, u_{k,1}\}, there is no vertex such that $u_{j,1} = x$ or $u_{k,1} = x$, and there exist $z \in \tilde{P}_{1h}$ ($h \in [\kappa(G)]$), but there is no arc $u_{k,1}z$ in path $\tilde{P}_{1h}$. Let $P_{1h} = \tilde{P}_{1h}$. The other paths are the same as Subcase 1.1.

**Subcase 1.3.** There is an arc $u_{k,1}z$ in path $\tilde{P}_{1h}$ ($\{r, h \} \subseteq [\kappa(G)]$). In the set \{u_{j,1}, u_{k,1}\} ($j \neq r$), there is no vertex $x$. We can find a path $\tilde{P}_{2r}$ such that $u_{2,r} \notin \tilde{P}_{2r}$. If $u_{b,a} \in \tilde{P}_{1h}$, then let $u_{b,a} \notin \tilde{P}_{2r}$. If $u_{l,d} \in \tilde{P}_{1h}$, then let $u_{l,d} \notin \tilde{P}_{2r}$. In $\tilde{P}_{1h}$ and $P_{1h}$, let $u_{l,d} \notin \tilde{P}_{1h}$ and $u_{b,a} \notin \tilde{P}_{1h}$. Let $P_{2h} = \tilde{P}_{2h} \cup \tilde{P}_{2h} \cup \tilde{P}_{2h} \cup \tilde{P}_{1h} \cup \tilde{P}_{2h} \cup \{l_{u_{j,1}, u_{k,1}}\}$.

The other paths are the same as Subcase 1.1.

**Subcase 1.4.** The set \{u_{j,1}, u_{k,1}\} contains the vertex $u_{j,1} = x$ and $u_{k,1} \neq x$. There is no arc $u_{k,1}z$ in $\tilde{P}_{1h}$. In $\tilde{P}_{2r}$, there is an arc $u_{j,1}z$ ($q \in [\kappa(G)]$, $q \in [h]$). In $\tilde{P}_{2r}$, there exists an out-neighbor $u_{i,1}$ of $x$, where $g_1 \in [\kappa(G)] \setminus \{a, c, d\}$, and this path is denoted by $\tilde{P}_{2r}$.

**Subcase 1.4.1.** There is no arc $xu_{g,1}$ in $\tilde{P}_{1h}$.

In $\tilde{P}_{2r}$, the path from vertex $u_{g,1}$ to $u_{g,1}$ is denoted as $\tilde{P}_{2r}$. In $G(v_{g_1})$, with $S_7' = \{u_{3,3,2}, u_{1,3,2}\}$ and $r_7' = u_{3,3,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S_7', r_7')$-paths. Then in these paths, one of the paths $\tilde{P}_q$ is chosen, with $u_{k,1}g_2 \notin \tilde{P}_q$.

If $u_{3,3,2} \in \tilde{P}_q$, then let $u_{3,3,2} \notin \tilde{P}_q$. In $\tilde{P}_{2r}$, the path from vertex $u_{1,3,2}$ to $u_{1,1}$ is denoted as $\tilde{P}_{2r}$.

Let $P_{2q} = \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \{xu_{g,1}, u_{g,1}, u_{1,1, z}, z_{u_{3,3,2}}\}$.

If $u_{1,1} \notin \tilde{P}_{2q}$, then $P_{2q} = \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \tilde{P}_{2q} \cup \{xu_{g,1}, u_{g,1}, u_{1,1, z}, z_{u_{3,3,2}}\}$. The other paths are the same as Subcases 1.1–1.3.

**Subcase 1.4.2.** If there exists an arc $xu_{g,1}$ in $\tilde{P}_{1g}$ ($g \in [\kappa(G)]$), then in $H(u_{g,1})$, with $S_6 = \{u_{g,1}, u_{g,1}, g_{1,3,2}\}$ and $r_6 = g_{1,3,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S_6, r_6)$-paths. Then in these paths, one of the paths $\tilde{P}_g$ is chosen, with $u_{k,1}g \notin \tilde{P}_g$. In $\tilde{P}_{1g}$, the path from vertex $u_{g,1}$ to $y$ is denoted as $\tilde{P}_{1g}$. Let $P_{2q} = \tilde{P}_{2q}$ be the same as in Subcase 1.4.1.

Let $P_{1g} = \tilde{P}_{1g} \cup \tilde{P}_{1g} \cup \tilde{P}_{1g} \cup \tilde{P}_{1g} \cup \tilde{P}_{1g} \cup \{xu_{g,1}, u_{g,1}, u_{g,1, y_{u_{3,3,2}}}, y_{u_{3,3,2}}\}$.

The other paths are the same as Subcases 1.1–1.3.

**Subcase 1.5.** In the set \{u_{j,1}, u_{k,1}\}, there exists vertex $u_{k,1}z = x$. And there is no arc $u_{k,1}z$ in $\tilde{P}_{1p}$.

In $\tilde{P}_{2r}$, there is an out-neighbor $u_{1,1}$ of $x$ such that $g \in [\kappa(G)] \setminus \{a, c, d\}$, and this path is denoted by $\tilde{P}_{2r}$. In $G(v_{g})$, let $S_8 = \{u_{3,3,2}, u_{1,3,2}\}$, $r_8 = u_{3,3,2}$, and we know there exist at least $\kappa(G)$ internally disjoint $(S_8, r_8)$-paths. Then in these paths, we choose one of the paths $\tilde{P}_p$, and let $u_{3,3,2} \notin \tilde{P}_p$. In $\tilde{P}_{2r}$, we denote the path from vertex $u_{3,3,2}$ to $u_{1,1}$ as $\tilde{P}_{2r}$. Let $P_{2p} = \tilde{P}_p \cup \tilde{P}_p \cup \tilde{P}_p \cup \{x, z_{u_{3,3,2}}\}$. 
The other paths are the same as Subcases 1.1–1.3.

**Case 2.** Let $x$ and $y$ be in the same $G(v_j)$. Let $x$ and $z$ be in the same $H(u_i)$ for some $i \in [n]$, $j \in [m]$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,1}$, and $z = u_{1,2}$. In this case, our overall goal is that we will use arc-disjoint paths between $x$ and $y$ in $G(v_1)$, $y$ and its out-neighbors in $H(u_2)$, $z$ and its in-neighbors in $G(v_2)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 3. The vertices and paths contained in Figure 3 are explained below.

**Figure 3.** Depiction of the arc-disjoint paths found in Case 2 of the proof of Theorem 1.

Considering $S_1 = \{x, y\}$, $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S_1, r_1)$-paths in $G(v_1)$, denoted as $\bar{P}_{1i} (i \in [\kappa(G)])$. Let $S_2 = \{y, u_{2,2}\}$, $r_2 = y$, and there exist at least $\kappa(G)$ internally disjoint $(S_2, r_2)$-paths in $H(u_2)$, denoted as $\bar{P}_{1i} (i \in [\kappa(G)])$. For each $i \in [\kappa(G)]$, let $u_{2,f_i}$ be the out-neighbor of $y$ in $\bar{P}_{1i}$; clearly these out-neighbors are distinct. For each $i \in [\kappa(G)]$, an out-neighbor $u_{b,f_i}$ of $u_{2,f_i}$ in $G(v_f)$ can be chosen, with $b \neq 1$. In $H(u_b)$, with $S_3 = \{u_{b,1}, u_{b,2}\}$ and $r_3 = u_{b,1}$. $\bar{P}_{1i}$ is the $(S_3, r_3)$-path corresponding to $\bar{P}_{1i}$. In $\bar{P}_{1i}$, the path from vertex $u_{b,f_i}$ to $u_{b,2}$ is denoted $\bar{P}_{1i}''$. With $S_4 = \{u_{b,2}, z\}$ and $r_4 = u_{b,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S_4, r_4)$-paths in $G(v_2)$, denoted as $\bar{P}_{1i} (i \in [\kappa(G)])$. If $u_{2,f_i} = u_{2,2}$, then $u_{2,2} \notin \bar{P}_{1i}$. The arc-disjoint $(S, r)$-paths can be constructed as

$$P_{1i} = \bar{P}_{1i} \cup \bar{P}_{1i}' \cup \bar{P}_{1i}'' \cup \{yu_{2,f_i}, u_{2,f_i}, u_{b,f_i}\}, i \in [\kappa(G)].$$

Likewise, we can identify $\kappa(G)$ arc-disjoint $(S, r)$-paths from $x$ to $z$ and subsequently to $y$. Consequently, we can derive $2\kappa(G)$ arc-disjoint $(S, r)$-paths.

**Case 3.** Let $x, y$ and $z$ be in different $H(u_i)$ and $G(v_j)$ for some $i \in [n]$, $j \in [m]$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$, and $z = u_{3,3}$. In this case, our overall goal is that, we will use arc-disjoint paths between $x$ and its out-neighbors in $H(u_1)$, and its out-neighbors in $H(u_2)$, $z$ and its in-neighbors in $G(v_3)$, $x$ and its out-neighbors in $G(v_2)$, and its out-neighbors in $G(v_1)$, $z$ and its in-neighbors in $G(v_3)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 4. The vertices and paths contained in Figure 4 are explained below.

Considering $S_1 = \{x, u_{2,1}\}$, $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S_1, r_1)$-paths in $G(v_1)$, denoted as $\bar{P}_{1i} (i \in [\kappa(G)])$. Let $S_2 = \{u_{2,1}, y\}$, $r_2 = u_{2,1},$
and there exist at least \( \kappa(G) \) internally disjoint \((S_2, r_2)\)-paths in \( H(u_2) \), denoted as \( \hat{P}_{1i} \) (\( i \in \{ \kappa(G) \} \)). Considering \( S'_i = \{ x, u_{11,2} \}, r'_i = x \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_i, r'_i)\)-paths in \( H(u_1) \), denoted as \( \hat{P}_{2j} \) (\( j \in \{ \kappa(G) \} \)). Let \( S_2 = \{ u_{12,1}, y_1 \}, r'_2 = u_{1,2}, \) and there exist at least \( \kappa(G) \) internally disjoint \((S'_2, r'_2)\)-paths in \( G(v_2) \), denoted as \( \hat{P}_{2j} \) (\( j \in \{ \kappa(G) \} \)). In \( H(u_2) \), with \( S'_3 = \{ y, u_{2,3} \}, r'_3 = y \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_3, r'_3)\)-paths, denoted as \( \hat{P}_{2j} \). For each \( j \in \{ \kappa(G) \} \), let \( u_{2, j} \) be the out-neighbor of \( y \) in \( \hat{P}_{2j} \), clearly these out-neighbors are distinct.

Figure 4. Depiction of the arc-disjoint paths found in Case 3 of the proof of Theorem 1.

In \( G(v_2) \), with \( S_3 = \{ y, u_{3,2} \}, r_3 = y \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S_3, r_3)\)-paths in \( G(v_2) \), denoted as \( \hat{P}_{1i} \). For each \( i \in \{ \kappa(G) \} \), let \( u_{3,2} \) be the out-neighbor of \( y \) in \( \hat{P}_{1i} \), clearly these out-neighbors are distinct. For each \( i \in \{ \kappa(G) \} \), an out-neighbor of \( u_{3,2} \) in \( H(u_2) \) can be chosen, denoted by \( u_{3,c} (c \in \{ m \}) \), with \( c \notin \{ 1, 3 \} \). Similarly, an out-neighbor of \( u_{2, j} \) in \( G(v_f) \) can be chosen, denoted by \( u_{b, j} (b \in \{ n \}) \), with \( b \notin \{ 1, 3 \} \).

In \( G(v_3) \), with \( S_4 = \{ u_{2, c}, u_{3, c} \}, r_4 = u_{2, c} \), \( \hat{P}'_{1i} \) is the \((S_4, r_4)\)-path corresponding to \( \hat{P}_{1i} \). In \( \hat{P}'_{1i} \), the path from vertex \( u_{3, c} \) to \( u_{3, c} \) is denoted as \( \hat{P}'_{1i} \). In \( H(u_3) \), with \( S_5 = \{ u_{3, c}, z \}, r_5 = u_{3, c} \) and it is known that there exist at least \( \kappa(G) \) internally disjoint \((S_5, r_5)\)-paths, say \( \hat{P}_{1i} \). In \( H(v_b) \), with \( S'_4 = \{ u_{b,2}, u_{b,3} \}, r'_4 = u_{b,2}, \) \( \hat{P}'_{2j} \) is the \((S'_4, r'_4)\)-path corresponding to \( \hat{P}_{2j} \). In path \( \hat{P}'_{2j} \), the path from vertex \( u_{b, j} \) to \( u_{b,3} \) is denoted as \( \hat{P}'_{2j} \). In \( G(v_3) \), with \( S'_5 = \{ u_{b,3}, z \}, r'_5 = u_{b,3} \), and it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_5, r'_5)\)-paths in \( G(v_3) \), say \( \hat{P}_{2j} \). If \( u_{b,2} = u_{3,2} \), then \( u_{3,2} \notin \hat{P}_{1i} \) (\( k \in \{ \kappa(G) \} \)). If \( u_{b,3} \in \hat{P}_{1i} \), then \( u_{3,3} \notin \hat{P}_{1i} \) (\( l \in \{ \kappa(G) \} \)). Similarly, if \( u_{2, f_2} = u_{2, m} \), then \( u_{2, m} \notin \hat{P}_{2j} \) (\( r \in \{ \kappa(G) \} \)). If \( u_{1,3} \in \hat{P}_{2j} \), then \( u_{1,3} \notin \hat{P}_{2j} \) (\( h \in \{ \kappa(G) \} \)). The arc-disjoint \((S, r)\)-paths can be constructed as \( P_{1i} = \hat{P}_{1i} \cup \hat{P}_{1i} \cup \hat{P}_{1i} \cup \{ y_{u_{b,2}}, u_{3,2}, u_{3, c}, \} \), \( P_{2j} = \hat{P}_{2j} \cup \hat{P}_{2j} \cup \hat{P}_{2j} \cup \{ u_{f_2, j}, u_{f_2, j}, u_{b, j}, \} \).

Then we obtain \( 2x(G) \) arc-disjoint \((S, r)\)-paths.

Case 4. Let \( x \) and \( y \) be in the same \( H(u_i) \). Let \( z, x, y \) be in different \( G(v_j) \) and let \( z, x \) be in different \( H(u_i) \), for some \( i \in \{ n \}, j \in \{ m \} \). Without loss of generality, we can assume
that \( x = u_{2,1}, y = u_{2,2}, z = u_{3,3} \). In this case, our overall goal is that we will use arc-disjoint paths between \( x \) and \( y \) in \( H(u_2) \), \( y \) and its out-neighbors in \( G(v_2) \), \( z \) and its in-neighbors in \( H(u_3) \), \( x \) and its out-neighbors in \( G(v_2) \), \( y \) and its out-neighbors in \( H(u_2) \), \( z \) and its in-neighbors in \( G(v_3) \), and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 5. The vertices and paths contained in Figure 5 are explained below.

![Figure 5](image-url)

**Figure 5.** Depiction of the arc-disjoint paths found in Case 4 of the proof of Theorem 1.

Considering \( S_1 = \{x, y\}, r_1 = x \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S_1, r_1)\)-paths in \( H(u_2) \), denoted as \( \tilde{P}_{1i} \) (i \( \in \) \( \kappa(G) \)). In \( G(v_1) \), with \( S'_1 = \{x, u_{1,1}\} \), and \( r'_1 = x \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_1, r'_1)\)-paths, denoted as \( \tilde{P}_{2j} \). In \( H(u_1) \), with \( S'_2 = \{u_{1,1}, u_{1,2}\} \), and \( r'_2 = u_{1,1} \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_2, r'_2)\)-paths, denoted as \( \tilde{P}_{3j} \). In \( G(v_2) \), with \( S'_3 = \{u_{1,2}, y\} \), and \( r'_3 = u_{1,2} \), it is known that there exist at least \( \kappa(G) \) internally disjoint \((S'_3, r'_3)\)-paths, denoted as \( \tilde{P}_{4j} \). Let \( u_{3,2}, u_{3,c}, u_{2,j}, u_{b,j}, \tilde{P}_{1i}, \tilde{P}_{2j}, \tilde{P}'_{1i}, \tilde{P}'_{2j} \) be the same as in Case 3.

If \( u_{3,2} = u_{3,2}, \) then \( u_{3,2} \notin \tilde{P}_{1i} (k \in \kappa(G)) \). If \( u_{2,j} = u_{2,3} \), then \( u_{2,3} \notin \tilde{P}_{2j} \) \((r \in \kappa(G))\).

If \( u_{1,3} \notin \tilde{P}_{2j}, \) then \( u_{1,3} \notin \tilde{P}_{2h} \) \((h \in \kappa(G))\). If \( u_{b,1} \notin \tilde{P}_{2j}, \) then \( u_{b,1} \notin \tilde{P}_{2l} \) \((t \in \kappa(G))\). If \( u_{1,3} \notin \tilde{P}_{2l}, \) then \( u_{1,3} \notin \tilde{P}_{2l} \) \((t \in \kappa(G))\).

**Subcase 4.1.** If there exists no vertex \( u_{2,j} = x \). Let

\[
\begin{align*}
P_{1i} &= \tilde{P}_{1i} \cup \tilde{P}'_{1i} \cup \tilde{P}_{1i} \cup \{y_{u_{3,2}}, u_{3,2}, u_{3,c}\}, \\
P_{2j} &= \tilde{P}_{2j} \cup \tilde{P}'_{2j} \cup \tilde{P}_{2j} \cup \{u_{2,j}, u_{2,j}u_{b,j}\}.
\end{align*}
\]

**Subcase 4.2.** If there exists a vertex \( u_{2,j} = x \) \((g \in \kappa(G))\), then in \( G(v_1) \), there exists an out-neighbor \( u_{b,1} \) of \( x \). If \( u_{b,1} \notin \tilde{P}_{2j} \), this path is denoted by \( \tilde{P}_{2g} \).

In \( H(u_3) \), there exists an out-neighbor \( u_{3,g1} \) of \( z \) such that \( g_1 \in [m] \setminus \{c, 2, 1\} \). In \( G(v_2) \), there exists an in-neighbor \( u_{g2,2} \) of \( y \) such that \( g_2 \in [n] \setminus \{1, b, 3\} \). If \( u_{g2,2} \notin \tilde{P}_{2j} \), this path is denoted by \( \tilde{P}_{2g} \). Then in \( H(u_{g2}) \), with \( S'_4 = \{u_{g2,1}, u_{g2,2}\} \), and \( r'_4 = u_{g2,1} \), it is known that there are at least \( \kappa(G) \) internally disjoint \((S'_4, r'_4)\)-paths. One such \((S'_4, r'_4)\)-path is chosen, denoted as \( \tilde{P}_{2g} \), with \( u_{g2,2} \notin \tilde{P}_{2g} \). In \( G(v_{g2}) \), with \( S'_5 = \{u_{g2,1}, u_{g2,2}\} \), and \( r'_5 = u_{g2,1} \), it is known that there are at least \( \kappa(G) \) internally disjoint \((S'_5, r'_5)\)-paths. One such \((S'_5, r'_5)\)-path is chosen, denoted as \( \tilde{P}_{2g} \), with \( u_{g2,2} \notin \tilde{P}_{2g} \). Then, \( P_{2g} \) is constructed as

\[
P_{2g} = \tilde{P}'_{2g} \cup \tilde{P}_{2j} \cup \tilde{P}_{2g} \cup \tilde{P}_{2g} \cup \{x_{u_{b,1}}, u_{3,g1}, u_{3,g2}\}.
\]

The other paths are the same as Subcase 4.1. Then we obtain \( 2\kappa(G) \) arc-disjoint \((S, r)\)-paths.
Case 5. Let x and y be in the same $H(u_i)$. Let y and z be in the same $G(v_j)$, for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{i,1}$, $y = u_{i,2}$, $z = u_{i,2}$. In this case, our overall goal is that we will use arc-disjoint paths between x and y in $H(u_i)$, y and z in $G(v_j)$, and x and its out-neighbors in $G(v_j)$, and y and z in $G(v_j)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 6. The vertices and paths contained in Figure 6 are explained below.

![Figure 6](image)

It is known that there exist at least $\kappa(G)$ internally disjoint $(S_1, r_1)$-paths in $H(u_i)$, denoted as $\tilde{P}_{1i}$ ($i \in [\kappa(G)]$), where $S_1 = \{x, y\}$ and $r_1 = x$. In $G(v_j)$, there exist at least $\kappa(G)$ internally disjoint $(S_2, r_2)$-paths, denoted as $\tilde{P}_{1i}$ ($i \in [\kappa(G)]$), where $S_2 = \{y, z\}$ and $r_2 = y$. Similarly, in $G(v_j)$, there exist at least $\kappa(G)$ internally disjoint $(S'_2, r'_2)$-paths, denoted as $\tilde{P}_{2j}$ ($j \in [\kappa(G)]$), where $S'_2 = \{u_{2,1}, z\}$ and $r'_2 = u_{2,1}$. In $G(v_j)$, there exist at least $\kappa(G)$ internally disjoint $(S'_2, r'_2)$-paths, denoted as $\tilde{P}_{2j}$ ($j \in [\kappa(G)]$), where $S'_2 = \{z, y\}$ and $r'_2 = z$. For each $j \in [\kappa(G)]$, let $u_{s,j,2}$ be the in-neighbor of $y$ in $\tilde{P}_{2j}$, and clearly these in-neighbors are distinct. Similarly, let $u_{k,j,2}$ ($j \in [\kappa(G)]$) be the out-neighbor of z in $\tilde{P}_{2j}$. For each $j \in [\kappa(G)]$, an out-neighbor $u_{k,j,2}$ of $u_{k,j,2}$ is chosen in $H(u_{k,j})$, where $b \neq 1$.

In $G(v_j)$, with $S'_2 = \{u_{2,1}, u_{1,2}\}$ and $r'_2 = u_{2,2}$. $\tilde{P}_{2j}$ is the $(S'_2, r'_2)$-path corresponding to $\tilde{P}_{2j}$. In $\tilde{P}_{2j}$, the path from vertex $u_{k,j,2}$ to $u_{s,j,2}$ is denoted as $\tilde{P}_{2j}'$. Then, in $H(u_{k,j})$, with $S'_2 = \{u_{k,j}, u_{s,j,2}\}$ and $r'_2 = u_{k,j,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint $(S'_2, r'_2)$-paths. One such $(S'_2, r'_2)$-path, denoted as $\tilde{P}_{2j}$ ($j \in [\kappa(G)]$), is chosen, with $u_{s,j,2} \notin \tilde{P}_{2j}$. The arc-disjoint $(S, r)$-paths can be constructed as:

$P_{1i} = \tilde{P}_{1i} \cup \tilde{P}_{1i}$,

$P_{2j} = \tilde{P}_{2j} \cup \tilde{P}_{2j} \cup \tilde{P}_{2j}' \cup \tilde{P}_{2j} \cup \{z u_{k,j,2}, u_{s,j,2} y, u_{k,j,2} u_{k,j,2}\}$.

If $u_{s,j,2} = u_{k,j,2}$ ($l \in [\kappa(G)]$), then $P_{2j} = \tilde{P}_{2j} \cup \tilde{P}_{2j} \cup \{z u_{k,j,2}, u_{s,j,2} y\}$. And if $u_{k,j,2} = y$ ($l \in [\kappa(G)]$), then $P_{2j} = \tilde{P}_{2j} \cup \tilde{P}_{2j} \cup \{z y\}$. This results in obtaining $2\kappa(G)$ arc-disjoint $(S, r)$-paths.

Case 6. Let y and z be in the same $G(v_j)$. Let x, y be in different $G(v_j)$ and x, y, z be in different $H(u_i)$, for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{i,1}$, $y = u_{i,2}$, $z = u_{i,2}$. Let $u_{s,j,2}$ ($j \in [\kappa(G)]$), $u_{k,j,2}$, $\tilde{P}_{1i}$, $\tilde{P}_{2j}$, $\tilde{P}_{2j}$ be the same as in Case 5. In $G(v_j)$, with $S'_1 = \{x, u_{i,1}\}$ and $r'_1 = x$, it is known that there exist at least $\kappa(G)$ internally
disjoint \((S'_1, r'_1)\)-paths in \(G(v_1)\), denoted as \(\overline{P}_{2j}\). In this case, our overall goal is that we will use arc-disjoint paths between \(x\) and its out-neighbors in \(H(u_3)\), \(y\) and its in-neighbors in \(H(u_1)\), \(y\) and \(z\) in \(G(v_2)\), \(x\) and its out-neighbors in \(G(v_1)\), \(z\) and its in-neighbors in \(H(u_2)\), \(z\) and \(y\) in \(G(v_2)\), and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 7. The vertices and paths contained in Figure 7 are explained below.

**Figure 7.** Depiction of the arc-disjoint paths found in Case 6 of the proof of Theorem 1.

**Subcase 6.1.** In the set \(\{u_{s_1,2}, u_{k_1,2}\}\), there does not exist \(u_{3,2} \in \{u_{s_1,2}, u_{k_1,2}\}\). Thus, \(u_{s_1,2}, u_{k_1,2}, \overline{P}_{2j}, \overline{P}_{2j}'\) remain the same as in Case 5.

In \(H(u_3)\), with \(S_1 = \{x, u_{s_1,2}\} (c \in \{m \setminus \{1, 2, b\}\) and \(r_1 = x\), it is known that there exist at least \(\kappa(G)\) internally disjoint \((S_1, r_1)\)-paths in \(H(u_3)\), denoted as \(\overline{P}_{1i}\). In \(G(v_i)\), \(S_2 = \{u_{s_1,2}, u_{k_1,2}\} \) and \(r_2 = u_{s_1,2}\), it is known that there exist at least \(\kappa(G)\) internally disjoint \((S_2, r_2)\)-paths in \(G(v_i)\), denoted as \(\overline{P}_{2j}\). In \(H(u_1)\), with \(S_3 = \{u_{s_1,2}, u_{k_1,2}\}\) and \(r_3 = u_{s_1,2}\), it is known that there exist at least \(\kappa(G)\) internally disjoint \((S_3, r_3)\)-paths in \(H(u_1)\), denoted as \(\overline{P}_{1i}\). If \(u_{3,2} \in \overline{P}_{1i}\), then \(u_{3,2} \not\in \overline{P}_{1i}\). Let

\[
\begin{align*}
\overline{P}_{1i} &= P_{2i} \cup P_{2i}' \cup \overline{P}_{2j} \cup \overline{P}_{2j}' , \\
\overline{P}_{2j} &= \overline{P}_{2j} \cup \overline{P}_{2j}' \cup \overline{P}_{2j} \cup \overline{P}_{2j}' \cup \{zu_{k_1,2}, u_{k_1,2}u_{s_1,2}, u_{s_1,2}y\}.
\end{align*}
\]

And if \(u_{k_1,2} = u_{s_1,2} (l \in \kappa(G))\), then \(P_{2i} = P_{2i} \cup \overline{P}_{2j} \cup \{zu_{k_1,2}, u_{s_1,2}y\}\). If \(u_{k_1,2} = y (l \in \kappa(G))\), then \(P_{2i} = P_{2i} \cup \overline{P}_{2j} \cup \{zy\}\). Now we obtain \(2\kappa(G)\) arc-disjoint \((S, r)\)-paths.

**Subcase 6.2.** In the set \(\{u_{s_1,2}, u_{k_1,2}\}\), only one vertex \(u_{k_1,2} = u_{3,2} (r \in \kappa(G))\) exists. Thus, \(u_{s_1,2}, u_{s_1,2}, \overline{P}_{2j}, \overline{P}_{2j}'\) remain the same as in Case 5.

If \(u_{k_1,2}u_{k_1,2} \not\in \overline{P}_{1i}\) in \(\overline{P}_{1i}\), then \(\overline{P}_{1i}, \overline{P}_{2j}\) remain the same as in Subcase 6.1. If an arc \(u_{k_1,2}u_{k_1,2}\) is in \(P_{1i}\), since \(\delta(G) \geq 4\), then an out-neighbor \(u_{k_1,2}\) of \(u_{k_1,2}\) can be found in \(H(u_3)\) such that \(u_{k_1,2}u_{k_1,2} \not\in \overline{P}_{1i}\) and \(a \in \{m \setminus \{c, 1\}\). In \(G(v_i)\), \(P_{2j}'\) is the \((S'_1, r'_1)\)-path corresponding to \(\overline{P}_{2j}\), where \(S'_1 = \{u_{k_1,2}, u_{k_1,2}\}, r'_1 = u_{k_1,2}\). In \(H(u_3)\), with \(S'_1 = \{u_{s_1,2}, u_{s_1,2}\}\) and \(r'_1 = u_{s_1,2}\), it is known that there exist at least \(\kappa(G)\) internally disjoint \((S'_1, r'_1)\)-paths. Then in these paths, one of the paths \(\overline{P}_{2j}\) is chosen, with \(u_{s_1,2} \not\in P_{2j}\) \((j \neq r)\) and \(P_{2i}\) remain the same as in Subcase 6.1. \(P_{2j}\) is constructed as

\[
P_{2j} = \overline{P}_{2j} \cup P_{2j}' \cup \overline{P}_{2j} \cup P_{2j}' \cup \{zu_{k_1,2}, u_{k_1,2}u_{s_1,2}, u_{s_1,2}y\}.
\]
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Subcase 6.3. In the set \{u_{s,2}, u_{k,2}\}, there is only one vertex \(u_{s,2} = u_{3,2} \ (g \in \kappa(G))\).

For each \(j \in [\kappa(G)]\), an in-neighbor of \(u_{s,j,1}\) in \(H(u_{j})\) can be chosen, denoted by \(u_{s,j,d} \ (d \in [m])\), where \(d \neq c, 1\). In \(G(v_{j})\), let \(P_{2j}\) be the \((S_{i,j}', r_{i}')\)-path corresponding to \(P_{2j}\), where \(S_{i,j}' = \{u_{d,2}, u_{d,1}\}, r_{i}' = u_{d,2}\). The path from vertex \(u_{s,j,d}\) to \(u_{s,j,1}\) in path \(P_{2j}\) is denoted as \(P_{2j}'\). In \(H(u_{ij})\), let \(S_{i,j}' = \{u_{k,2}, u_{k,1}\}, r_{i}' = u_{k,2}\), and at least \(\kappa(G)\) internally disjoint \((S_{i,j}', r_{i}')\)-paths are known to exist. Then, one of the paths \(P_{2j}\) \((j \in [\kappa(G)])\) is chosen, where \(u_{s,j,1} \notin P_{2j}\). If \(u_{s,j+1} = u_{x,2} \ (t \in [\kappa(G)])\), \(P_{2t} = \bigcup P_{2i} \cup \bigcup \{z u_{k,2}, u_{x,2}, y\}\). And if \(u_{s,j} = y \ (l \in [\kappa(G)])\), \(P_{2l} = \bigcup P_{2i} \cup \{z y\}\). If \(u_{s,j,d} u_{s,j,2} \notin P_{2l}\) in the path \(P_{2l}\). Let

\[
P_{2j} = P_{2i} \cup P_{2j} \cup P_{2i} \cup \bigcup \{z u_{k,2}, u_{s,j,2}, u_{s,j,2}, y\}.
\]

If an arc \(u_{s,j,d} u_{s,j,2}\) is in path \(P_{2i}\), an in-neighbor \(u_{s,j,f}\) of \(u_{s,j,2}\) can be found in \(H(u_{j})\) such that \(u_{s,j,f} u_{s,j,2} \notin P_{2i}\) and \(f \in [m] \setminus \{c, 1\}\). In \(G(v_{j})\), let \(P_{2j}'\) be the \((S_{j}', r_{j}')\)-path corresponding to \(P_{2j}'\), where \(S_{j}' = \{u_{k,2}, u_{x,2}\}, r_{j}' = u_{k,2}\). In \(H(u_{j})\), let \(S_{j}' = \{u_{k,2}, u_{x,2}\}, r_{j}' = u_{k,2}\), and at least \(\kappa(G)\) internally disjoint \((S_{j}', r_{j}')\)-paths are known to exist. Then, one of the paths \(P_{2j}'\) is chosen, and let \(u_{s,j} \notin P_{2j}'\). Let

\[
P_{2j} = P_{2j} \cup P_{2j}' \cup P_{2j} \cup \bigcup \{z u_{k,2}, u_{s,j,2}, u_{s,j,2}, y\}.
\]

Hence, we obtain \(2 \kappa(G)\) arc-disjoint \((S, r)\)-paths.

Now we prove that this bound is sharp. By Proposition 1, \(\lambda^p_{C}(\overrightarrow{K_n} \square \overrightarrow{K_m}) = n + m - 2\). By Lemma 2, \(\kappa(\overrightarrow{K_n}) = n - 1\). So we have \(\lambda^p_{C}(\overrightarrow{K_n} \square \overrightarrow{K_n}) = 2 \kappa(\overrightarrow{K_n}) = 2n - 2\), with \(n \geq 5\). Therefore, the lower bound holds and is sharp. □

4. Exact Values for Digraph Classes

In this section, we aim to determine precise values for the directed path 3-arc-connectivity of the Cartesian product of two digraphs within specific digraph classes.

Proposition 1. We have \(\lambda^p_{C}(\overrightarrow{K_n} \square \overrightarrow{K_m}) = n + m - 2\).

Proof. Consider \(S = \{x, y, z\}\) and \(r = x\). We will focus solely on scenarios where \(x, y,\) and \(z\) do not all belong to the same \(\overrightarrow{K_m}(u_i)\) or the same \(\overrightarrow{K_n}(v_j)\) for any \(i \in [n], j \in [m]\). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \(x = u_{1,1}, y = u_{2,2}, z = u_{3,3}\). It is feasible to derive \(n + m - 2\) arc-disjoint \((S, r)\)-paths in \(\overrightarrow{K_n} \square \overrightarrow{K_m}\), say \(P_1, P_2, \ldots, P_n \ (a = \min(i + 1, 3, i < l \leq n)), P_{i+1} (4 < i < n), \ldots, P_4 (b = \min(n - j + 3, 3, j \leq n)), P_{n+2} (4 < j \leq m)\) (as shown in Figure 8) such that

\[
P_1 : xu_{1,1}u_{3,2}z, P_2 : xu_{1,2}u_{2,2}z, P_3 : xu_{1,3}z, P_4 : xu_{3,2}z,
\]

\[
P_5 : xu_{1,4}u_{2,1}u_{2,2}z, P_6 : xu_{1,4}u_{3,4}u_{3,1}u_{2,2}u_{1,2}y, P_7 : xu_{1,4}u_{3,4}u_{3,1}u_{2,1}u_{2,2}y,
\]

\[
P_8 : xu_{1,4}u_{4,2}u_{2,2}u_{1,2}y, P_9 : xu_{1,4}u_{2,2}u_{1,2}u_{2,2}u_{1,2}y, P_{10} : xu_{1,4}u_{4,2}u_{2,1}u_{2,1}u_{3,3}z.
\]

Now, we add two cases to prove that the proposition holds, so as to show that the proposition has no constraint conditions.

First, let \(n = m = 4\). We can assume that \(x = u_{1,1}, y = u_{2,2}, z = u_{3,3}\). Let

\[
P_1 : xu_{1,1}u_{3,2}z, P_2 : xu_{1,2}u_{2,2}z, P_3 : xu_{1,3}u_{3,2}z,
\]

\[
P_4 : xu_{1,4}u_{2,1}u_{2,2}z, P_5 : xu_{1,4}u_{2,4}u_{2,2}u_{2,2}u_{1,2}y, P_6 : xu_{1,4}u_{2,2}u_{1,2}u_{2,2}u_{1,2}y,
\]

Furthermore, let \(n = 2, m = 4\). We can assume that \(x = u_{1,1}, y = u_{1,2}, z = u_{1,3}\). Let

\[
P_1 : xy, P_2 : xy, P_3 : xu_{1,4}u_{2,2}u_{2,2}u_{1,2}y, P_4 : xu_{1,2}u_{2,3}u_{2,2}u_{4,4}y.
\]

Then we have \(n + m - 2 = \min \{\delta^+(D), \delta^-(D)\} \geq \lambda^p_{C}(\overrightarrow{K_n} \square \overrightarrow{K_m}) \geq n + m - 2\). This concludes the proof. □
Proposition 2. We have $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{K}_m) = m + 1$, with $n \geq 3$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where $x, y,$ and $z$ are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overrightarrow{K}_m(v_j)$ for any $i \in [n], j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}, y = u_{2,2}, z = u_{3,3}$. We can obtain $m + 1$ arc-disjoint $(S, r)$-paths in $\overrightarrow{C}_n \square \overrightarrow{K}_m$, say $P_1, P_2, \ldots, P_{m+1}$ ($4 < i \leq m$), $P_m, P_1$ (as shown in Figure 9) such that $P_1 : x u_{1,2} y u_{3,2} z$, $P_2 : x u_{1,2} y u_{3,2} z$, $P_3 : x u_{1,2} y u_{3,2} z$, $P_4 : x u_{1,2} y u_{3,2} z$, ..., $P_m : x u_{1,2} y u_{3,2} z$, $P_{m+1} : x u_{1,2} y u_{3,2} z$.

Now, we add two cases to prove that the proposition holds, so as to show that the proposition has no constraint conditions.

First, let $n = 3, m = 4$. We can assume that $x = u_{1,1}, y = u_{2,1}, z = u_{3,1}$. Let $P_1 : x y z, P_2 : x y z, P_3 : x u_{1,2} y u_{3,2} z, P_4 : x u_{1,2} y u_{3,2} z, P_5 : x u_{1,2} y u_{3,2} z, P_6 : x u_{1,2} y u_{3,2} z, P_7 : x u_{1,2} y u_{3,2} z, P_8 : x u_{1,2} y u_{3,2} z, P_9 : x u_{1,2} y u_{3,2} z, P_{10} : x u_{1,2} y u_{3,2} z$.

Furthermore, let $n = 3, m = 2$. We can assume that $x = u_{1,1}, y = u_{1,2}, z = u_{1,3}$. Let $P_1 : x y z, P_2 : x u_{1,2} y u_{3,2} z, P_3 : x u_{1,2} y u_{3,2} z$.

Then we have $m + 1 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{K}_m) \geq m + 1$. This concludes the proof. □

Proposition 3. We have $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{K}_m) = m$. 

Figure 9. $\overrightarrow{C}_n \square \overrightarrow{K}_m$. 

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Proof. Let \( S = \{x, y, z\}, r = x \), and we only examine the case where \( x, y, \) and \( z \) are not all within the same \( C_n(u_i) \) or the same \( C_m(v_j) \) for any \( i \in [n], j \in [m] \). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \( x = u_{1,1}, y = u_{2,2}, z = u_{3,3} \). We can obtain \( m \) arc-disjoint \((S,r)\)-paths in \( C_n \sqcup C_m \).

First assume that \( m \) is even number, let
\[
P_i : xu_{1,1}yu_{3,3}z,  \quad P_2 : xu_{1,2}yu_{2,3}z,  \quad P_3 : xu_{3,1}yu_{1,2}z.
\]
Then we have 3 \( \lambda_3^P(\sqrt{C_n} \sqcup \sqrt{C_m}) \geq m \). This completes the proof. \( \blacksquare \)

Proposition 4. We have \( \lambda_3^P(T_n \sqcup C_m) = m \).

Proof. Let \( S = \{x, y, z\}, r = x \), and we only examine the case where \( x, y, \) and \( z \) are not all within the same \( T_n(u_i) \) or the same \( C_m(v_j) \) for any \( i \in [n], j \in [m] \). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \( x = u_{1,1}, y = u_{2,2}, z = u_{3,3} \). We can obtain \( m \) arc-disjoint \((S,r)\)-paths in \( T_n \sqcup C_m \), say \( P_1, P_2, \ldots, P_i \). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \( x = u_{1,1}, y = u_{2,2}, z = u_{3,3} \). We can obtain \( m \) arc-disjoint \((S,r)\)-paths in \( C_n \sqcup C_m \), say \( P_1, P_2, P_3 \) such that
\[
P_1 : xu_{1,1}yu_{3,3}z,  \quad P_2 : xu_{1,2}yu_{2,3}z,  \quad P_3 : xu_{3,1}yu_{1,2}z.
\]
Then we have \( m = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^P(C_n \sqcup C_m) \geq m \). This completes the proof. \( \blacksquare \)

Proposition 5. We have \( \lambda_3^P(C_n \sqcup C_m) = 3 \).

Proof. Let \( S = \{x, y, z\}, r = x \), and we only examine the case where \( x, y, \) and \( z \) are not all within the same \( C_n(u_i) \) or the same \( C_m(v_j) \) for any \( i \in [n], j \in [m] \). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \( x = u_{1,1}, y = u_{2,2}, z = u_{3,3} \). We can obtain two arc-disjoint \((S,r)\)-paths in \( C_n \sqcup C_m \), say \( P_1, P_2 \) such that
\[
P_1 : xu_{1,1}yu_{3,3}z,  \quad P_2 : xu_{1,2}yu_{2,3}z.
\]
Then we have \( \lambda_3^P(C_n \sqcup C_m) \geq 2 \). This completes the proof. \( \blacksquare \)

Proposition 6. We have \( \lambda_3^P(C_n \sqcup C_m) = 3 \), with \( m \geq 3 \).

Proof. Let \( S = \{x, y, z\}, r = x \), and we only examine the case where \( x, y, \) and \( z \) are not all within the same \( C_n(u_i) \) or the same \( C_m(v_j) \) for any \( i \in [n], j \in [m] \). The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume \( x = u_{1,1}, y = u_{2,2}, z = u_{3,3} \). We can obtain three arc-disjoint \((S,r)\)-paths in \( C_n \sqcup C_m \), say \( P_1, P_2, P_3 \) such that
\[
P_1 : xu_{1,1}yu_{3,3}z,  \quad P_2 : xu_{1,2}yu_{2,3}z,
\]
\[
P_3 : xu_{m,1}yu_{m,2}u_{m-1} \cdots u_{3,2}yu_{1,3}u_{1,2}u_{1,3}u_{m-1} \cdots z.
\]
Then we have \( \lambda_3^P(C_n \sqcup C_m) \geq 3 \). This completes the proof. \( \blacksquare \)

Proposition 7. We have \( \lambda_3^P(C_n \sqcup C_m) = 4 \), with \( n \geq 3, m \geq 3 \).
Proof. Let $S = \{x, y, z\}, r = x$, and we only examine the case where $x, y, z$ and $z$ are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overrightarrow{C}_m(v_j)$ for any $i \in [n], j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}, y = u_{2,2}, z = u_{3,3}$. We can obtain four arc-disjoint $(S, r)$-paths in $\overrightarrow{C}_n \square \overrightarrow{C}_m$, say $P_1, P_2, P_3, P_4$ such that

\begin{align*}
P_1 & : xu_{1,1}yu_{3,3}z, \\
P_2 & : xu_{1,2}yu_{2,2,3}z, \\
P_3 & : xu_{m,1}yu_{m,2}yu_{m,3}, u_{m-1,3} \cdots z u_{3,2}y, \\
P_4 & : xu_{1,1}yu_{2,2,3}zu_{3,2}y.
\end{align*}

Then we have $4 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_4^p(\overrightarrow{C}_n \square \overrightarrow{C}_m) \geq 4$. This completes the proof. □

Proposition 8. We have $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{T}_m) = 2$.

Proof. Let $S = \{x, y, z\}, r = x$, and we only examine the case where $x, y, z$ and $z$ are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overrightarrow{T}_m(v_j)$ for any $i \in [n], j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}, y = u_{2,2}, z = u_{3,3}$. We can obtain three arc-disjoint $(S, r)$-paths in $\overrightarrow{C}_n \square \overrightarrow{T}_m$, say $P_1$ and $P_2$ such that

\begin{align*}
P_1 & : xu_{1,1}yu_{3,3}z, \\
P_2 & : xu_{1,2}yu_{2,2,3}z.
\end{align*}

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{T}_m) \geq 2$. This completes the proof. □

Proposition 9. We have $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{T}_m) = 3$, with $n \geq 3$.

Proof. Let $S = \{x, y, z\}, r = x$, and we only examine the case where $x, y, z$ and $z$ are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overrightarrow{T}_m(v_j)$ for any $i \in [n], j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}, y = u_{2,2}, z = u_{3,3}$. We can obtain three arc-disjoint $(S, r)$-paths in $\overrightarrow{C}_n \square \overrightarrow{T}_m$, say $P_1, P_2, P_3$ such that

\begin{align*}
P_1 & : xu_{1,1}yu_{3,3}z, \\
P_2 & : xu_{1,2}yu_{2,2,3}z, \\
P_3 & : xu_{m,1}yu_{m,2}yu_{m,3}, u_{m-1,3} \cdots z u_{3,2}y.
\end{align*}

Then we have $3 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{T}_m) \geq 3$. This completes the proof. □

Proposition 10. We have $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{T}_m) = 2$.

Proof. Let $S = \{x, y, z\}, r = x$, and we only examine the case where $x, y, z$ and $z$ are not all within the same $\overrightarrow{T}_m(v_j)$ or the same $\overrightarrow{T}_m(v_j)$ for any $i \in [n], j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}, y = u_{2,2}, z = u_{3,3}$. We can obtain three arc-disjoint $(S, r)$-paths in $\overrightarrow{T}_n \square \overrightarrow{T}_m$, say $P_1$ and $P_2$ such that

\begin{align*}
P_1 & : xu_{1,1}yu_{3,3}z, \\
P_2 & : xu_{1,2}yu_{2,2,3}z.
\end{align*}

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{T}_n \square \overrightarrow{T}_m) \geq 2$. This completes the proof. □

According to Propositions 1–9, we find that the directed path 3-arc-connectivity of some Cartesian products of digraphs is equal to the minimum semi-degrees. Based on this discovery, we can consider under what conditions the directed path 3-arc-connectivity of Cartesian products of digraphs can be equal to the minimum semi-degrees, which is a problem we can consider next.

5. Conclusions

In this paper, we prove that if $G$ and $H$ are two digraphs such that $\delta(G) \geq 4, \delta(H) \geq 4,$ and $\kappa(G) \geq 2, \kappa(H) \geq 2,$ then $\lambda_5^3(G \square H) \geq \min\{2\kappa(G), 2\kappa(H)\}$, and moreover, this bound
is sharp. Finally, we obtain exact values of $\lambda^G_p (G \sqcap H)$ for some digraph classes $G$ and $H$. In practical terms, constructing vertex-disjoint or arc-disjoint paths in graphs is crucial. These paths play a significant role in improving transmission reliability and boosting network transmission speeds.

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