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Best Proximity Point Results via Simulation Function with Application to Fuzzy Fractional Differential Equations

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Abstract: In this study, we prove the existence and uniqueness of a best proximity point in the setting of non-Archimedean modular metric spaces via the concept of simulation functions. A non-Archimedean metric modular is shaped as a parameterized family of classical metrics; therefore, for each value of the parameter, the positivity, the symmetry, the triangle inequality, or the continuity is ensured. Also, we demonstrate how analogous theorems in modular metric spaces may be used to generate the best proximity point results in triangular fuzzy metric spaces. The utility of our findings is further demonstrated by certain examples, illustrated consequences, and an application to fuzzy fractional differential equations.

Keywords: metric space; simulation functions; triangular fuzzy metric space; approximately compact

1. Introduction

In 2010, Chistyakov introduced a novel concept known as the modular metric space, which fundamentally involves a metric function denoted as $d : \chi \to \chi$ operating on a nonempty set χ to provide a finite non-negative measurement of distance between any two elements p and q within χ. Modular metric spaces represent a compelling and intuitive extension of traditional modulars defined over linear spaces, such as Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, and Calderon–Lozanovskii spaces, among others. This broader conceptual framework offers a rich and diverse landscape for exploring mathematical structures and phenomena beyond the confines of linear spaces. Specifically, this metric function delineates the spatial separation between points p and q. Furthermore, within the framework of modular metric spaces, a modular metric function $\varpi_\lambda : \chi \times \chi \to [0, \infty]$ is introduced, wherein $\lambda > 0$ signifies a designated time interval. This modular metric function characterizes the absolute value of an average velocity, potentially accommodating infinite values, thereby quantifying the distance traversed between points p and q over the specified time duration λ. A non-Archimedean modular metric is shaped as a parameterized family of classical metrics; therefore, for each value of the parameter, the positivity, the symmetry, the triangle inequality, or the continuity is ensured. Additionally, in 2010, Basha [1] initiated the concept of the best proximity point for non-self mappings.

Simulation functions, as highlighted by Jleli et al. [2], serve as integral components in fixed point theory, contributing additional tools to establish both the existence and uniqueness of fixed points across various metrical settings. Simulation functions epitomize a notable unifying capability, consolidating diverse established results within a coherent framework. They streamline the proof process by introducing auxiliary functions, thereby enhancing the manageability of analysis and facilitating more elegant and concise proofs. This approach is exemplified in works such as those by [2–8] and references therein.

The exploration and resolution of intricate differential equations and variational problems that permeate various branches of applied sciences constitute formidable challenges

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that continually drive mathematicians and researchers to delve into the intricacies of fixed point problems within modular metric spaces. Numerous studies, including [9–20], have been dedicated to this pursuit. These endeavors predominantly aim to derive general theorems, often leveraging the concept of simulation functions, as discussed in works: [2,21,22].

In this study, we introduce some new results for the existence and uniqueness of the best proximity point in modular metric spaces via simulation functions, and we obtain some results in fuzzy metric spaces as a consequence of those given for a modular metric [23–27]. Consequently, we get some fixed points as corollaries in both modular and fuzzy metrics influenced by simulation functions.

2. Preliminaries

In this section, we endeavor to expound upon pivotal concepts to guarantee the self-sufficiency of our study.

Definition 1 ([2,21])**.** *A simulation function, denoted as* $\varsigma : [0,\infty) \times [0,\infty) \to \mathbb{R}$*, is characterized by the following criteria:*

(ς1) *The function* $\varsigma(r,j) < j - r$ *holds true for all* $r, j > 0$*.*
(ς2) *For sequences* r_n *and* j_n *in* $(0,\infty)$ *converging to* τ *as* $n \to \infty$*, where* $\tau > 0$*, the superior limit as* $n \to \infty$ *of* $\varsigma(r_n, j_n)$ *is less than* 0*.*

We represent the collection of all simulation functions as $\mathfrak{Z}$*.*

Example 1. *Let us introduce the following list of simulation functions; see also [2,21]:*

(1) $\varsigma(r,j) = j - \frac{r+2}{r+1} r$ *for all* $r, j > 0$*.*
(2) $\varsigma(r,j) = j - \phi(j) - r$*, where* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(r) = 0 \Longleftrightarrow r = 0$ *for all* $r, j > 0$*.*
(3) $\varsigma(r,j) = j - \phi(r)$*, where* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(r) > r$ *for all* $r, j > 0$*.*
(4) $\varsigma(r,j) = [\phi(j)]^k - \phi(r)$*, where* $\phi : [0,\infty) \to [1,\infty)$ *is a continuous function such that* $\phi(j) - \phi(t) < j - r$ *for all* $r, j > 0$ *and* $k \in [0,1)$*.*

Let χ *be a nonempty set and* $\varpi : (0,+\infty) \times \chi \times \chi \to [0,+\infty]$ *be a function; for simplicity, we will write*

$$\varpi_\lambda(p,q) = \varpi(\lambda, p, q),$$

for all $\lambda > 0$ *and* $p, q \in \chi$*.*

Definition 2 ([14,28])**.** *A mapping* $\varpi : (0,+\infty) \times \chi \times \chi \to [0,+\infty]$ *is termed a modular metric on* χ *if it satisfies the following conditions for all* $\lambda_1, \lambda_2 > 0$*, and* $p, q, c \in \chi$*: (i)* $p = q$ *if and only if* $\varpi_{\lambda_1}(p,q) = 0$*; (ii)* $\varpi_{\lambda_1}(p,q) = \varpi_{\lambda_1}(q,p)$*; (iii)* $\varpi_{\lambda_1+\lambda_2}(p,q) \leq \varpi_{\lambda_1}(p,c) + \varpi_{\lambda_2}(c,q)$*.*

Note that the pseudomodular metric ϖ *satisfies the following condition instead of (i) in Definition 2:*

(i') $\varpi_{\lambda_1}(p,p) = 0$ *for all* $\lambda_1 > 0$ *and* $p \in \chi$*.*
Moreover, ϖ *is referred to as regular if condition (i) is replaced with:*
$p = q$ *if and only if* $\varpi_{\lambda_1}(q,p) = 0$ *for some* $\lambda_1 > 0$*.*
Furthermore, if for $\lambda_1, \lambda_2 > 0$*, and* $p, q, c \in \chi$*, the inequality:*

$$\varpi_{\lambda_1+\lambda_2}(p,q) \leq \frac{\lambda_1}{\lambda_1+\lambda_2}\varpi_{\lambda_1}(p,c) + \frac{\lambda_2}{\lambda_1+\lambda_2}\varpi_{\lambda_2}(c,q),$$

holds, then ϖ *is called convex.*

Remark 1. *The function* ϖ_λ *is termed non-Archimedean if it satisfies conditions (i) and (ii) of Definition 2 and replaces condition (iii) with: (iii')*$\varpi_{\max\{\lambda_1,\lambda_2\}}(p,q) \leq \varpi_{\lambda_1}(p,c) + \varpi_{\lambda_2}(c,q)$ *for all* $\lambda_1, \lambda_2 > 0$*;* $p, q, c \in \chi$*.*

It is worth noting that condition (iii′) entails condition (iii), thus indicating that a non-Archimedean modular metric satisfies the properties of a modular metric.

Remark 2. *Indeed, if* $0 < \lambda_1 < \lambda_2$*, then*

$$\varpi_{\lambda_2}(p,q) \leq \varpi_{\lambda_2-\lambda_1}(p,p) + \varpi_{\lambda_1}(p,q) = \varpi_{\lambda_1}(p,q).$$

Definition 3 ([14,28])**.** *Let* ϖ *denote a pseudomodular metric on* χ*, and let* $p_0 \in \chi$ *be a fixed element. Consider the two sets*

$$\chi_\varpi = \chi_\omega(p_0) = \{p \in \chi : \varpi_\lambda(p,p_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\},$$

and

$$\chi^*_\varpi = \chi^*_\varpi(p_0) = \{p \in \chi : \exists \lambda = \lambda(p) > 0 \text{ such that } \varpi_\lambda(p,p_0) < +\infty\}.$$

The sets χ_ϖ *and* χ^*_ϖ *are termed modular spaces (around* p_0*).*

It is evident that $\chi_\varpi \subset \chi^*_\varpi$*. It is noteworthy that the set* χ_ω *can be equipped with a metric defined as follows:*

$$d_\varpi(p,q) = \inf\{\lambda > 0 : \varpi_\lambda(p,q) \leq \lambda\} \quad \text{for all} \quad p,q \in \chi_\varpi.$$

If ϖ *is convex, then* $\chi_\varpi = \chi_\varpi$*, and we can introduce the metric* d_ϖ *defined as follows:*

$$d^*_\varpi(p,q) = \inf\{\lambda > 0 : \varpi_\lambda(p,q) \leq 1\} \quad \text{for all} \quad p,q \in \chi_\varpi.$$

See [14,28].

Definition 4. *Let* χ_ϖ *denote a modular metric space, and let M be a subset of* χ_ϖ*. Then:*

(1) A sequence $p_n \in \chi_\varpi$ *is defined as* ϖ*-convergent to some* $p \in \chi_\varpi$ *if* $\varpi_\lambda(p_n,p) \to 0$ *as* $n \to +\infty$*. Here, p is termed the* ϖ*-limit of* p_n*.*

(2) p_n *is referred to as* ϖ*-Cauchy if* $\varpi_\lambda(p_m,p_n) \to 0$ *as* $m,n \to +\infty$*.*

(3) Regarding a ϖ*-convergent* $p_n \in M$ *that converges to some* $p \in \chi_\varpi$*, if* $p \in M$*, then M is termed* ϖ*-closed.*

(4) For a ϖ*-Cauchy sequence* $p_n \in M$*, if* p_n *converges to some* $p \in M$*, then M is termed* ϖ*-complete.*

3. Best Proximity Point Results

Consider two non-empty subsets P_1 and P_2 within a modular metric space χ_ω. We denote the sets $(P_1)0^\lambda$ and $(P2)^\lambda_0$ as follows:

$$(P_1)^\lambda_0 = \{p \in P_1 : \varpi_\lambda(p,q) = \varpi_\lambda(P_1,P_2), \text{ for some } q \in P_2\}$$
$$(P_2)^\lambda_0 = \{q \in P_2 : \varpi_\lambda(p,q) = \varpi_\lambda(P_1,P_2), \text{ for some } p \in P_1\},$$

where $\varpi_\lambda(P_1,P_2) = \inf\{\varpi_\lambda(p,q) : a \in P_1 \text{ and } q \in P_2\}$.

Definition 5 ([29])**.** *A subset* P_2 *is characterized as approximately compact with respect to* P_1 *if, for every sequence* q_n *in* P_2 *and some* $p \in P_1$*, the condition* $\varpi_\lambda(p,q_n) \to \varpi_\lambda(p,P_2)$ *implies* $p \in (P_1)^\lambda_0$*.*

In all subsequent results, please note that

- ϖ *is assumed to be of regular nature.*
- *The symbol* ϕ *represents a lower semi-continuous function, defined as* $\phi : \chi_\varpi \to [0,\infty)$*, while* ς *denotes a simulation function belonging to* $\mathfrak{Z}$*.*
- *For a non-self mapping* $g : P_1 \to P_2$*, a point* $p^* \in P_1$ *is termed the best proximity point of the mapping g if*

$$\varpi(p^*,gp^*) = \varpi(P_1,P_2).$$

Theorem 1. *Consider a complete non-Archimedean modular metric space denoted by χ_{ϖ}. Let P_1 and P_2 be two non-empty subsets of χ_{ϖ}, where P_1 is assumed to be closed. Suppose there exists a mapping $g : P_1 \to P_2$ such that $g\Big((P_1)_0^\lambda\Big) \subseteq (P_2)_0^\lambda$. Additionally, assume the existence of p0 and p_1 in $(P_1)_0^\lambda$ such that $\varpi\lambda(p_1, gp_0) = \varpi_\lambda(P_1, P_2)$.*

For $p, q, u, v \in A_1$ with $\varpi_\lambda(u, gp) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(v, gq)$, we have:

$$\varsigma(\varpi_\lambda(u,v) + \phi(u) + \phi(v), \varpi_\lambda(p,q) + \phi(p) + \phi(q)) \geq 0. \tag{1}$$

Assuming g is ϖ-continuous, it possesses a unique best proximity point $p \in P_1$ satisfying $\phi(p) = 0$.

Proof. From the assumptions $\varpi_\lambda(p_1, gp_0) = \varpi_\lambda(P_1, P_2)$ and $g\Big((P_1)_0^\lambda\Big) \subseteq (P_2)_0^\lambda$, there exists $p_2 \in (P_1)_0^\lambda$ such that $\varpi_\lambda(p_2, gp_1) = \varpi_\lambda(P_1, P_2)$. Again, for $p_2 \in (P_1)_0^\lambda$ and $g\Big((P_1)_0^\lambda\Big) \subseteq (P_2)_0^\lambda$, there exists $p_3 \in (P_1)_0^\lambda$ such that $\varpi_\lambda(p_3, gp_2) = \varpi_\lambda(P_1, P_2)$.

Continuing this process we get,

$$\varpi_\lambda(p_{n+1}, gp_n) = \varpi_\lambda(P_1, P_2)$$

for all $n \in \mathbb{N} \cup \{0\}$. Applying (1), we get

$$\varsigma(\varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n), \varpi_\lambda(p_n, p_{n-1}) + \phi(p_{n-1}) + \phi(p_n)) \geq 0.$$

For every $n \in \mathbb{N} \cup 0$, suppose there exists $j \in \mathbb{N}$ such that $\varpi_\lambda(p_{j+1}, p_j) = 0$. By virtue of the regularity property of ϖ_λ, we deduce that p_j serves as the best proximity point of g.

Consequently, let us consider the scenario where $\varpi_\lambda(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N} \cup 0$. Therefore, according to ($\varsigma 1$), we obtain:

$$\begin{aligned} 0 &\leq \varsigma(\varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n), \varpi_\lambda(p_n, p_{n-1}) + \phi(p_{n-1}) + \phi(p_n) \\ &< \varpi_\lambda(p_n, p_{n-1}) + \phi(p_{n-1}) + \phi(p_n)) - [\varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n)]. \end{aligned}$$

This implies that

$$\varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n) < \varpi_\lambda(p_n, p_{n-1}) + \phi(p_{n-1}) + \phi(p_n).$$

Consider the sequence $\tau_n = \varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n)$, which forms a decreasing sequence of positive real numbers. Consequently, there exists a non-negative $\tau \geq 0$ such that $\lim n \to \infty \tau n = \tau$.

Suppose $\tau > 0$, then according to condition ($\varsigma 2$), we have:

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \varsigma(\varpi_\lambda(p_{n+1}, p_n) + \phi(p_{n+1}) + \phi(p_n), \varpi_\lambda(p_n, p_{n-1}) + \phi(p_{n-1}) + \phi(p_n)) \\ &< 0, \end{aligned}$$

which is a contradiction. We conclude that $\tau = 0$, that is,

$$\lim_{n\to\infty} \varpi_\lambda(p_{n+1}, p_n) + \phi(a_{n+1}) + \phi(p_n) = 0.$$

Since ϕ takes only non-negative values, we get

$$\lim_{n\to\infty} \phi(p_n) \to 0, \text{ and } \lim_{n\to\infty} \varpi_\lambda(p_{n+1}, p_n) \to 0. \tag{2}$$

To establish the ϖ-Cauchy property of the sequence p_n within P_1, let us assume the contrary, i.e., we can suppose that $\limsup m, n \to \infty \varpi\lambda(p_m, p_n) > 0$. Consequently, there exist $\epsilon > 0$ along with two subsequences p_{n_k} and p_{m_k} of p_n

$$\text{for } n_k > m_k \geq k, \varpi_\lambda(p_{m_k}, p_{n_k}) \geq \epsilon \text{ and } \varpi_\lambda(p_{m_k}, p_{n_{k-1}}) < \epsilon.$$

Thus,

$$\begin{aligned}\epsilon &\leq \varpi_\lambda\left(p_{m_k}, p_{n_k}\right) \\ &= \varpi_{\max\{\lambda,\lambda\}}\left(p_{m_k}, p_{n_k}\right) \\ &\leq \varpi_\lambda\left(p_{m_k}, p_{n_{k-1}}\right) + \varpi_\lambda\left(p_{n_{k-1}}, p_{n_k}\right) \\ &< \epsilon + \varpi_\lambda\left(p_{n_{k-1}}, p_{n_k}\right).\end{aligned}$$

Taking the limits as $k \to \infty$, we get,

$$\lim_{k\to\infty} \varpi_\lambda\left(p_{m_k}, p_{n_k}\right) = \epsilon. \tag{3}$$

Similarly,

$$\begin{aligned}\epsilon &\leq \varpi_\lambda\left(p_{m_k}, p_{n_k}\right) \\ &= \varpi_{\max\{\lambda,\lambda\}}\left(p_{m_k}, p_{n_k}\right) \\ &\leq \varpi_\lambda\left(p_{m_k}, p_{m_{k-1}}\right) + \varpi_\lambda\left(p_{m_{k-1}}, p_{n_{k-1}}\right) + \varpi_\lambda\left(p_{n_{k-1}}, p_{n_k}\right).\end{aligned}$$

and

$$\begin{aligned}\varpi_\lambda\left(p_{m_{k-1}}, p_{n_{k-1}}\right) &\leq \varpi_\lambda\left(p_{m_{k-1}}, p_{m_k}\right) + \varpi_\lambda\left(p_{m_k}, p_{n_{k-1}}\right) \\ &\leq \varpi_\lambda\left(p_{m_{k-1}}, p_{m_k}\right) + \varpi_\lambda\left(p_{m_k}, p_{n_k}\right) + \varpi_\lambda\left(p_{n_k}, p_{n_{k-1}}\right).\end{aligned}$$

Taking the limits in the above inequalities as $k \to \infty$, we get

$$\lim_{k\to\infty} \varpi_\lambda\left(p_{m_{k-1}}, p_{n_{k-1}}\right) = \epsilon. \tag{4}$$

Combining the equations in (2), (3), and (4), we get

$$\lim_{k\to\infty} \varpi_\lambda\left(p_{m_k}, p_{n_k}\right) + \phi\left(p_{m_k}\right) + \phi\left(p_{n_k}\right) = \epsilon$$

and

$$\lim_{k\to\infty} \varpi_\lambda\left(p_{m_{k-1}}, p_{n_{k-1}}\right) + \phi\left(p_{m_{k-1}}\right) + \phi\left(p_{n_{k-1}}\right) = \epsilon.$$

We get

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \varsigma\left(\varpi_\lambda\left(p_{m_k}, p_{n_k}\right) + \phi\left(p_{m_k}\right) + \phi\left(p_{n_k}\right), \varpi_\lambda\left(p_{m_{k-1}}, p_{n_{k-1}}\right)\right. \\ &\quad \left. + \phi\left(p_{m_{k-1}}\right) + \phi\left(p_{n_{k-1}}\right)\right) \\ &< 0.\end{aligned}$$

This assumption leads to a contradiction. Thus, p_n forms a ϖ-Cauchy sequence in P_1. Since P_1 is a closed subset of a complete modular metric space χ_ϖ, it follows that P_1 is also complete. Consequently, there exists a point $p \in P_1$ such that $\varpi\lambda(p_n, p) = 0$ as $n \to \infty$.

Recalling the limit expression in (2) and utilizing the lower semi-continuity of the function ϕ, we obtain:

$$0 \leq \phi(p^*) \leq \liminf_{n\to\infty} \phi(p_n) = 0 \Longrightarrow \phi(p^*) = 0.$$

Since g is ϖ-continuous, $\Longrightarrow \varpi_\lambda(gp_n, gp^*) \to 0$ as $n \to \infty$.

Now

$$\begin{aligned}\varpi_\lambda(p^*, gp^*) &\leq \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(p_{n+1}, gp_n) + \varpi_\lambda(gp_n, gp^*) \\ &= \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(P_1, P_2) + \varpi_\lambda(gp_n, gp^*).\end{aligned} \tag{5}$$

Taking the limit as $n \to \infty$ in (5) , we get $\varpi_\lambda(p^*, gp^*) = \varpi_\lambda(P_1, P_2)$, and hence, p^* is the best proximity point of g.

In order to establish the uniqueness of p as the best proximity point of g, suppose otherwise. That is, assume the existence of another best proximity point $p \in P_1$ such that

$\varpi_\lambda(p, p^{**}) > 0$. That is, $\varpi_\lambda(p^*, gp^*) = \varpi_\lambda(P_1, P_2)$ and $\varpi_\lambda(p^{**}, gp^{**}) = \varpi_\lambda(P_1, P_2)$. So from (2.1) together with $(\varsigma 1)$,

$$\begin{aligned}0 &\leq \varsigma(\varpi_\lambda(p^*, p^{**}) + \phi(p^*) + \phi(p^{**}), \varpi_\lambda(p^*, p^{**}) + \phi(p^*) + \phi(p^{**}) \\ &< \varpi_\lambda(p^*, p^{**}) + \phi(p^*) + \phi(p^{**}) - [\varpi_\lambda(p^*, p^{**}) + \phi(p^*) + \phi(p^{**})] \\ &= 0.\end{aligned}$$

This is a contradiction, and hence, g has a unique best proximity point. □

Theorem 2. *Instead of the continuity condition of g as stated in Theorem 1, let us consider the assumption that P_2 is approximately compact with respect to P_1. Under this assumption, g possesses a unique best proximity point $p^* \in P_1$.*

Proof. Applying analogous steps as in Theorem 1, it can be concluded that p_n constitutes a ϖ-Cauchy sequence in P_1 and converges to a certain p satisfying $\phi(p) = 0$.

$$\begin{aligned}\omega_1(P_1, P_2) &\leq \varpi_\lambda(p_{n+1}, gp_n) \\ &\leq \varpi_\lambda(p_{n+1}, p^*) + \varpi_\lambda(p^*, gp_n) \\ &\leq \varpi_\lambda(p_{n+1}, p^*) + \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(p_{n+1}, gp_n) \\ &= \varpi_\lambda(p_{n+1}, p^*) + \varpi_\lambda(p^*, p_{n+1}) + \omega_1(P_1, P_2).\end{aligned}$$

Take the limit as $n \to \infty \implies \varpi_\lambda(p^*, gp_n) \to \varpi_\lambda(P_1, P_2)$, and so by the approximate compactness of P_2, $p^* \in (P_1)_0^\lambda$. But $g\Big((P_1)_0^\lambda\Big) \subseteq (P_2)_0^\lambda \implies gp^* \in (P_2)_0^\lambda$. Therefore, there exists $q^* \in P_1$ such that

$$\varpi_\lambda(q^*, gp^*) = \varpi_\lambda(P_1, P_2).$$

So we have

$$\varpi_\lambda(q^*, gp^*) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(p_{n+1}, gp_n). \tag{6}$$

Without loss of generality, we may assume that $q^* \neq p_n$ and $p^* \neq p_n$ for all $n \in \mathbb{N}$. Thus, by (1), (6) and $(\varsigma 1)$, we have

$$\begin{aligned}0 &\leq \varsigma(\varpi_\lambda(p_{n+1}, q^*) + \phi(p_{n+1}) + \phi(q^*), \varpi_\lambda(p^*, p_n) + \phi(p^*) + \phi(p_n)) \\ &< \varpi_\lambda(p^*, p_n) + \phi(p^*) + \phi(p_n) - [\varpi_\lambda(p_{n+1}, q^*) + \phi(p_{n+1}) + \phi(q^*)] \\ &\implies \varpi_\lambda(p_{n+1}, q^*) + \phi(a_{n+1}) + \phi(q^*) < \varpi_\lambda(p^*, p_n) + \phi(p^*) + \phi(p_n).\end{aligned}$$

Now

$$\begin{aligned}0 &\leq \varpi_\lambda(p^*, q^*) \\ &\leq \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(p_{n+1}, q^*) \\ &\leq \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(p^*n+1, q^*) + \phi(p_{n+1}) + \phi(q^*) \\ &< \varpi_\lambda(p^*, p_{n+1}) + \varpi_\lambda(p^*, p_n) + \phi(p^*) + \phi(p_n).\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\varpi_\lambda(p^*, q^*) = 0 \implies p^* = q^*.$$

By substituting in (6), we get,

$$\varpi_\lambda(p^*, gp^*) = \varpi_\lambda(P_1, P_2),$$

Thus, p^* emerges as the best proximity point of g. The uniqueness aspect remains consistent with Theorem 1. □

In the subsequent corollaries, we derive various outcomes in best proximity point theory using alternative simulation functions.

Corollary 1. *Consider χ_ϖ to be a complete non-Archimedean modular metric space, where P_1 and P_2 are two non-empty subsets of χ_ϖ, with P_1 being closed. Let $g : P_1 \to P_2$ be a mapping such that $g\big((P_1)0^\lambda\big) \subseteq (P2)_0^\lambda$. Suppose there exist $p_0, p_1 \in (P_1)_0^\lambda$ such that $\varpi_\lambda(p_1, gP_0) = \varpi_\lambda(P_1, P_2)$.*

For $p, q, u, v \in A_1$ with $\varpi_\lambda(u, gp) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(v, gq)$, then

$$\varpi_\lambda(u, v) + \phi(u) + \phi(v) \leq r(\varpi_\lambda(p, q) + \phi(p) + \phi(q)), \tag{7}$$

where $r \in [0, 1)$. If g is either ϖ-continuous or P_2 is approximately compact with respect to P_1, then it has a unique best proximity point $p^* \in P_1$, with $\phi(p^*) = 0$.

Proof. Define the simulation function $\varsigma \in \mathfrak{Z}$ by

$$\varsigma(r, j) = rj - r \text{ for all } r, j \in [0, \infty).$$

□

Corollary 2. *Consider χ_ϖ to be a complete non-Archimedean modular metric space. Let P_1 and P_2 be two non-empty subsets of χ_ϖ, with P_1 being closed. Suppose $g : P_1 \to P_2$ is a mapping such that $g\big((P_1)0^\lambda\big) \subseteq (P2)_0^\lambda$. Suppose there exist $p_0, p_1 \in (P_1)_0^\lambda$ such that $\varpi_\lambda(p_1, gp_0) = \varpi_\lambda(P_1, P_2)$. For $p, q, u, v \in P_1$ with $\varpi_\lambda(u, gp) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(v, gq)$, then*

$$\varpi_\lambda(u, v) + \phi(u) + \phi(v) \leq (\varpi_\lambda(p, q) + \phi(p) + \phi(q)) \cdot \eta(\varpi_\lambda(p, q) + \phi(p) + \phi(q)), \tag{8}$$

where $\eta : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t \to r} \eta(t) < 1$ for all $r > 0$. If g is either ϖ-continuous or P_2 is approximately compact with respect to p_1, then it has a unique best proximity point $p^* \in p_1$, with $\phi(p^*) = 0$.

Proof. Define the simulation function $\varsigma \in \mathfrak{Z}$ by

$$\varsigma(t, s) = s\eta(s) - t \text{ for all } t, s \in [0, \infty).$$

□

Corollary 3. *Consider χ_ϖ as a complete non-Archimedean modular metric space. Suppose P_1 and P_2 are non-empty subsets of χ_ϖ, with P_1 being closed. Let $g : P_1 \to P_2$ be a mapping such that $g\big((P_1)0^\lambda\big) \subseteq (P2)_0^\lambda$ Suppose there exist $p_0, p_1 \in (P_1)_0^\lambda$ such that $\varpi_\lambda(p_1, gp_0) = \varpi_\lambda(P_1, P_2)$. For $p, q, u, v \in P_1$ with $\varpi_\lambda(u, gp) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(v, gq)$, then*

$$\psi(\varpi_\lambda(u, v) + \phi(u) + \phi(v)) \leq \varpi_\lambda(p, q) + \phi(p) + \phi(q), \tag{9}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(t) > t$ for all $t > 0$. If g is either ϖ-continuous or P_2 is approximately compact with respect to $P_0 01$, then it has a unique best proximity point $p^* \in P_1$, with $\phi(p^*) = 0$.

Proof. Define the simulation function $\varsigma \in \mathfrak{Z}$ by

$$\varsigma(t, s) = s - \psi(t) \text{ for all } t, s \in [0, \infty).$$

□

Example 2. *Let $\mathbb{R}^2$ be a complete non-Archimedean modular metric space with modular ϖ_λ given by $\varpi_\lambda((p_1, q_1), (p_2, q_2)) = \frac{1}{\lambda}(|p_1 - p_2| + |q_1 - q_2|)$ for all $\lambda > 0$. Define the sets*

$P_1 = \left\{(0,p) : 0 \leq p \leq \frac{1}{2}\right\}$ and $P_2 = \left\{\left(\frac{1}{2}, b\right) : 0 \leq q \leq \frac{1}{2}\right\}$. Clearly, $\varpi_\lambda(P_1, P_2) = \frac{1}{2\lambda}$, and $(P_1)_0^\lambda = P_1$ and $(P_2)_0^\lambda = P_2$. Also, define $g : P_1 \to P_2$ by

$$g(p,q) = \left(\frac{1}{2}, \frac{q^2}{2}\right).$$

Notice that $g\left(P_{10}^\lambda\right) \subseteq P_2^\lambda$. We claim that all conditions of Theorem 1 hold true with respect to the simulation function $\varsigma \in \mathfrak{Z}$ defined by $\varsigma(t,s) = \frac{1}{2}s - t$ if $t < s$ and $\varsigma(t,s) = 0$ if $t \geq s$. Define the lower semi-continuous function $\phi : \chi_\varpi \to [0,\infty)$ given by $\phi((p,q)) = 2b$ for all $(p,q) \in \chi_\varpi$. For all $(0,p), (0,q), (0,u)$ and $(0,v) \in \chi_\varpi$, with $\varpi_\lambda(u, gp) = \varpi_\lambda(P_1, P_2) = \varpi_\lambda(v, gq)$, we have

$$u = \frac{p^2}{2}; v = \frac{q^2}{2}.$$

Now,

$$\varpi_\lambda((0,u),(0,v)) + \phi((0,u)) + \phi((0,v)) = \frac{1}{\lambda}\frac{\left|p^2 - q^2\right|}{2} + p^2 + q^2. \tag{10}$$

$$\varpi_\lambda((0,p),(0,q)) + \phi((0,p)) + \phi((0,q)) = \frac{1}{\lambda}|p-q| + 2a + 2q. \tag{11}$$

$$\begin{aligned}\varsigma(t,s) = \varsigma(&\varpi_\lambda((0,u),(0,v)) + \phi((0,u)) + \phi((0,v)), \omega_\lambda((0,p),(0,q))\\ &+\phi((0,p)) + \phi((0,q)))\end{aligned}$$

which implies

$$\begin{aligned}\varsigma(t,s) &= \frac{1}{2}[|p-q| + 2p + 2q] - \left[\frac{\left|p^2 - q^2\right|}{2} + p^2 + q^2\right],\\ &= (p+b) - \left(p^2 + q^2\right) + \frac{|p-q|}{2} - \frac{\left|p^2 - q^2\right|}{2}.\end{aligned}$$

It is clear that

$$(p+b) - \left(p^2 + q^2\right) \geq 0,$$

and so

$$\frac{|p-q|}{2} - \frac{\left|p^2 - q^2\right|}{2} \geq 0.$$

Therefore,

$$\begin{aligned}\varsigma(\varpi_\lambda((0,u),(0,v)) + \phi((0,u)) + \phi((0,v)), \varpi_\lambda((0,x),(0,y))\\ +\phi((0,p)) + \phi((0,q))) \geq 0,\end{aligned}$$

so by Theorem 1, we deduce that g has a unique best proximity point $(0,0) \in P_1$.

If $P_1 = P_2 = \chi_\varpi$, then we get the fixed point theorem as a corollary as following.

Corollary 4 ([30])**.** *Suppose χ_ϖ is a complete non-Archimedean modular metric space and g represents a self-mapping on χ_ϖ. Given the existence of $\varsigma \in \mathfrak{Z}$ and a lower semi-continuous function ϕ satisfying*

$$\varsigma(\varpi_\lambda(ga, gq) + \phi(gp) + \phi(gq), \varpi_\lambda(p,q) + \phi(p) + \phi(q)) \geq 0$$

for all $p, q \in \chi_\varpi$. Then g has a unique fixed point $p^* \in P_1$ and $\phi(p^*) = 0$.

The subsequent corollaries present various outcomes in fixed point theory using alternative simulation functions.

Corollary 5. *Consider χ_ϖ as a complete non-Archimedean modular metric space. Let g denote a self-mapping on χ_ϖ. Suppose that*

$$\varpi_\lambda(gp, gq) + \phi(gp) + \phi(gq) \leq r\varpi_\lambda(p, q) + \phi(p) + \phi(q)$$

for all $p, q \in \chi_\varpi$. Then g has a unique fixed point $p^* \in P_1$ and $\phi(p^*) = 0$.

Proof. Define the simulation function $\varsigma \in \mathfrak{Z}$ by

$$\varsigma(t, s) = rs - t \text{ for all } t, s \in [0, \infty).$$

□

Corollary 6. *Consider χ_ϖ as a complete non-Archimedean modular metric space. Let g denote a self-mapping on χ_ϖ. Suppose that*

$$\psi(\varpi_\lambda(gp, gq) + \phi(gp) + \phi(gq)) \leq [\psi(\varpi_\lambda(p, q) + \phi(p) + \phi(q))]^k$$

for all $p, q \in \chi_\varpi$, where $\psi : [0, \infty) \to (1, \infty)$ is a continuous function such that $\psi(s) - \psi(t) < s - t$ for all $s, t > 0$ and $k \in [0, 1)$. Then g has a unique fixed point $p^* \in P_1$ and $\phi(p^*) = 0$.

Proof. Define the simulation function $\varsigma \in \mathfrak{Z}$ by

$$\varsigma(r, j) = [\psi(j)]^k - \psi(r) \text{ for all } r, j \in [0, \infty).$$

□

4. Modular Metric Spaces to Fuzzy Metric Spaces

In this section, we illustrate how similar theorems established in modular metric spaces can be employed to derive best proximity point results in triangular fuzzy metric spaces.

Definition 6. *A continuous t-norm, denoted by $* : [0, 1] \times [0, 1] \to [0, 1]$, is characterized by the following properties:* (CTN1)$*$ *is commutative and associative;*

*(CTN1) * is continuous;*

(CTN1) $x * 1 = x$ for all $x \in [0, 1]$;

(CTN1) $x_1 * y_1 \leq x' * y'$ when $x_1 \leq x'$ and $y_1 \leq y'$ and $x_1, y_1, x', y' \in [0, 1]$.

Examples of the t-norm are $x * y = \min\{x, y\}$, $x * y = xy$ and $x * y = \max\{0, x + y - 1\}$.

Definition 7 ([23]). *For a nonempty set χ and a continuous t-norm $*$, along with a fuzzy set $\mu : \chi \times \chi \times (0, +\infty)$, the following conditions hold for all $p, q, c \in \chi$ and $t_1, t_2 > 0$:*

(FM1)$\mu(p, q, t_1) > 0$;

(FM2) $\mu(p, q, t_1) = 1$ iff $p = q$;

(FM3)$\mu(p, q, t_1) = \mu(q, p, t_1)$;

(FM4) $\mu(p, q, t_1) * \mu(q, c, t_2) \leq \mu(p, c, t_1 + t_2)$;

(FM5) $\mu(p, q, \cdot) : (0, +\infty) \to (0, 1]$ is left continuous.

Therefore, the triplet $(\chi, \mu, *)$ defines a fuzzy metric space.

When condition $(FM2)$ is substituted with:

$$\mu(p, q, t_1) = 1 \textit{ if and only if } p = q, \textit{ for some } t_1 > 0,$$

then μ is said to be regular.

$$\textit{If } \mu(p, q, t_1) * \mu(q, c, t_2) \leq \mu(p, c, \max\{t_1, t_2\}).$$

If condition $(FM4)$ is replaced, μ is termed non-Archimedean fuzzy.

It is worth noting that if μ is non-Archimedean, it also qualifies as a fuzzy metric space.

Definition 8 ([31])**.** *Let $(\chi, \mu, *)$ be a fuzzy metric space. The fuzzy metric μ is called triangular whenever*

$$\frac{1}{\mu(p,q,t)} - 1 \leq \frac{1}{\mu(p,c,t)} - 1 + \frac{1}{\mu(c,q,t)} - 1$$

for all $p, q, c \in \chi$ and all $t > 0$.

Definition 9. *Suppose $(\chi, \mu, *)$ constitutes a fuzzy metric space, and let $g : \chi \to \chi$ be a mapping. Then:*

(i) The sequence p_n is considered a μ-Cauchy sequence if, for all $0 < \epsilon < 1$, $\lim m, n \to \infty \mu(pn, p_m, t) = 1$ for all $m > n$ and $t > 0$.
(ii) The sequence p_n is considered to be μ-convergent to some $p \in \chi$ if

$$\lim_{n\to\infty} \mu(p_n, p, t) = 1 \quad \text{for all } t > 0.$$

(iii) The fuzzy metric space $(\chi, \mu, *)$ is deemed μ-complete if every μ-Cauchy sequence p_n in χ converges to some $p \in \chi$.
(iv) g is called a μ-continuous mapping if $\lim_{n\to+\infty} \mu(p_n, p, t) = 1$ implies $\lim_{n\to+\infty} \mu(gp_n, gp, t) = 1$.

Let P_1 and P_2 be two nonempty subsets of the fuzzy metric space $(\chi, \mu, *)$. The following definitions are introduced:

Definition 10.

$$(P_1)_0(t) = \{a \in P_1 : \mu(p,q,t) = \mu(P_1, P_2, t) \text{ for some } q \in P_2\}$$
$$(P_2)_0(t) = \{q \in P_2 : \mu(p,q,t) = \mu(P_1, P_2, t) \text{ for some } p \in P_1\},$$

where $\mu(P_1, P_2, t) = \sup\{\mu(p,q,t) : p \in P_1, q \in P_2\}$. Let $g : P_1 \to P_2$ be a mapping, then a point $a^* \in A_1$ is called a best proximity point in $(\chi, \mu, *)$ if

$$\mu(p^*, gp^*, t) = \mu(P_1, P_2, t), \text{ for all } t > 0.$$

In a recent study by Hussain and Salimi [32], a valuable lemma was presented that highlights a connection between fuzzy metrics and modular metrics.

Lemma 1 ([32])**.** *Let $(\chi, \mu, *)$ be a triangular fuzzy metric space. Define*

$$\varpi_\lambda(p,q) = \frac{1}{\mu(p,q,\lambda)} - 1,$$

for all $p, q \in \chi$ and all $\lambda > 0$. Then ϖ_λ is a modular metric on χ.

By combining Lemma 1 with our earlier theorems, we derive novel findings in triangular non-Archimedean fuzzy metric spaces.

Note that in all subsequent results, the fuzzy metric μ is assumed to be both triangular and regular.

Theorem 3. *Consider $(\chi, \mu, *)$ to be a complete non-Archimedean fuzzy metric space. Suppose P_1 and P_2 are two nonempty subsets of χ, with P_1 being closed. Let $g : P_1 \to P_2$ be a mapping satisfying the condition $g((P_1)_0(t)) \subseteq (P_2)_0(t)$ for all $t > 0$. Suppose that there exist elements p_0*

and p_1 *in* $(P_1)_0(t)$ *such that* $\mu(p_1, gp_0, t) = \mu(P_1, P_2, t)$*. For* $p, q, u, v \in P_1$ *with* $\mu(u, gp, t) = \mu(v, gq, t) = \mu(P_1, P_2, t)$*, then*

$$\varsigma\left(\frac{1}{\mu(u,v,t)} - 1 + \phi(u) + \phi(v), \frac{1}{\mu(u,v,t)} - 1 + \phi(u) + \phi(v)\right) \geq 0.$$

If either g *exhibits* μ*-continuity or* P_2 *is a fuzzy approximately compact set with respect to* P_1*, then* g *possesses a sole optimal proximity point* $p \in P_1$*, where* $\phi(p) = 0$*.*

Proof. Let χ_ϖ denote the modular metric space centered at p_0 and constructed from the modular metric ϖ_λ as outlined in Lemma 1, that is,

$$\chi_\varpi = \left\{p \in \chi : \lim_{t\to\infty} \varpi_t(p, p_0) = 0\right\},$$

or equivalently,

$$X_\varpi = \left\{p \in \chi : \lim_{t\to\infty} \mu(p, p_0, t) = 1\right\}.$$

Trivially, $\chi_\varpi \neq \emptyset$ since $p_0 \in \chi_\varpi$. Now, we demonstrate that χ_ϖ is a closed subset of $(\chi, \mu, *)$. Suppose p_n is a sequence in χ_ϖ converging to some $p \in \chi$. For any $\epsilon \in (0,1)$ and $t_0 > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(p_{n_0}, p, t_0) > 1 - \epsilon$. According to condition $(FM4)$ in Definition 7, we have:

$$\begin{aligned}
\mu(p_0, p, t) &= \mu(p_0, p, (t - t_0 + t_0)) \\
&\geq \mu(p_0, p_{n_0}, t - t_0) * \mu(p_{n_0}, p, t_0), \\
&> \mu(p_0, p_{n_0}, t - t_0) * (1 - \epsilon).
\end{aligned}$$

Taking the limits as $t \to \infty$ in the above inequalities, we get

$$\lim_{n\to\infty} \mu(p_0, p, t) \geq 1 - \epsilon \text{ for all } \epsilon > 0,$$

and hence, $a \in \chi_\varpi$, that is, χ_ϖ is closed in $(\chi, \mu, *)$ and so is complete. All hypotheses of Theorem 1 hold true, so we get the conclusion. □

If $P_1 = P_2 = \chi$, we get a fixed point theorem as a corollary as following.

Corollary 7 ([30])**.** *Consider a complete non-Archimedean fuzzy metric space* $(\chi, \mu, *)$ *and a self-mapping* g *on* χ*. Assume the existence of* $\varsigma \in \mathfrak{Z}$ *and a lower semi-continuous function* ϕ *such that*

$$\varsigma\left(\frac{1}{\mu(gp,gq,t)} - 1 + \phi(gp) + \phi(gq), \frac{1}{\mu(p,q,t)} - 1 + \phi(p) + \phi(q)\right) \geq 0$$

for all $p, q \in \chi$*.*
Then g *has a unique fixed point* $p^* \in \chi$*, with* $\phi(p^*) = 0$*.*

Corollary 8. *Consider a complete non-Archimedean fuzzy metric space* $(\chi, \mu, *)$ *and a self-mapping* g *on* χ*. Suppose there exists a lower semi-continuous function* ϕ *such that*

$$r.\left(\frac{1}{\mu(p,q,t)} - 1 + \phi(p) + \phi(q)\right) \geq \left(\frac{1}{\mu(gp,gq,t)} - 1 + \phi(gp) + \phi(gq)\right)$$

for all $p, q \in \chi$*.*
Then g *has a unique fixed point* $p^* \in \chi$*, with* $\phi(p^*) = 0$*.*

Example 3. *Consider the space χ as the set of real numbers $\mathbb{R}$. Define a fuzzy metric function μ on $\mathbb{R}$ that measures the similarity between two real numbers. This function can be defined as $\mu(p,q,t) = e^{-t|p-q|}$, where t is a parameter controlling the sensitivity of the metric.*

Let $g : \mathbb{R} \to \mathbb{R}$ be a self-mapping: for example, $g(x) = \frac{x}{2}$.

Next, define the function $\phi : \mathbb{R} \to \mathbb{R}$ as $\phi(x) = |x|$. This function is clearly continuous.

Now, let us check if the corollary applies. The inequality condition of the corollary becomes:

$$r \cdot \left(\frac{1}{\mu(p,q,t)} - 1 + \phi(p) + \phi(q) \right) \geq \left(\frac{1}{\mu(gp,gq,t)} - 1 + \phi(gp) + \phi(gq) \right).$$

Using the definitions of g and ϕ, this becomes:

$$r \cdot \left(e^{t|p-q|} - 1 + |p| + |q| \right) \geq e^{t\left|\frac{p}{2} - \frac{q}{2}\right|} - 1 + \left|\frac{p}{2}\right| + \left|\frac{q}{2}\right|.$$

Hence, all the stipulations of the preceding corollary substantiate the derived conclusion.

5. Application to Fuzzy Fractional Differential Equations

In this section, we consider the following initial value problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha,\gamma}\Upsilon(\rho) = \psi(\rho, \Upsilon(\rho)), \quad \rho \in [0,1] \\ \qquad \Upsilon(0) = 0. \end{cases} \tag{12}$$

where ${}^{C}\mathfrak{D}^{\alpha,\gamma}$ is the tempered Caputo fractional derivative of order $\alpha \in]0,1[$, with $\gamma > 0$, and ψ is a continuous function satisfying

$$|\psi(\rho, a\ ) - \psi(\rho, b)| \leqslant (\ln(r) + 1)\Gamma(\alpha + 1)|a - b| \tag{13}$$

for all $\rho \in [0,1]$, $a, b \in \chi$ and $r \in (0,1]$.

The set μ is defined by

$$\mu(a,b,t) = e^{-t\|a-b\|_\infty},$$

and $\phi : \chi \longrightarrow \chi$ is defined by

$$\phi(x) = \|x\|_\infty.$$

Now, for a continuous function $\Upsilon : \mathbb{R}^+ \longrightarrow \mathbb{R}$, the tempered Caputo fractional derivative and integral of order $\alpha \in]0,1[$ and $\gamma \geqslant 0$ are defined by

$${}^{C}\mathfrak{D}^{\alpha,\gamma}\Upsilon(\rho) = \frac{e^{-\gamma\rho}}{\Gamma(1-\alpha)} \int_a^\rho (\rho - s)^{-\alpha} \frac{d}{ds}(e^{\gamma s}\Upsilon(s)), ds$$

and

$$\mathfrak{I}^{\alpha,\gamma}\Upsilon(\rho) = \frac{1}{\Gamma(\alpha)} \int_a^\rho (\rho - s)^{\alpha-1} e^{-\gamma(\rho - s)}\Upsilon(s) ds,$$

which verifies the following properties:

$$\begin{aligned} {}^{C}\mathfrak{D}^{\alpha,\gamma}\mathfrak{I}^{\alpha,\gamma}\Upsilon(\rho) &= \Upsilon(\rho) \\ \mathfrak{I}^{\alpha,\gamma}\,{}^{C}\mathfrak{D}^{\alpha,\gamma}\Upsilon(\rho) &= \Upsilon(\rho) - \Upsilon(a). \end{aligned}$$

In the following, consider the set

$$\chi'_\mu = \chi'_\mu(a) = \left\{ b \in \chi : \exists \lambda = \lambda(b) > 0 \text{ such that } \frac{1}{\mu(a,b,t)} - 1 < +\infty \right\}.$$

Let us denote a set of real-valued functions defined on $[0,1]$ by

$$\chi = \{a| \quad a : [0,1] \to \mathbb{R}\},$$

and

$$\chi' = \{a| \quad a : [0,1] \to \mathbb{R}; \quad a(0) = 0\} \subset \chi.$$

Thus, for problem (12), we take the following triangular fuzzy metric space:

$$\chi_\mu^* = \chi_\mu' \cap \chi' = \left\{a \in \chi_\mu' : \quad a(0) = 0\right\}.$$

We will now provide certain lemmas that can be employed to establish the existence and uniqueness of the solution to problem (12).

Lemma 2. *Consider* $\Upsilon(\rho)$ *as the solution to Equation (12) if and only if*

$$\Upsilon(\rho) = \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} e^{-\gamma(\rho-s)} \psi(s, \Upsilon(s)) ds.$$

Proof. Composing problem (13) by $\mathfrak{I}^{\alpha,\gamma}$ on two sides, we obtain

$$\Upsilon(\rho) = \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} e^{-\gamma(\rho-s)} \psi(s, \Upsilon(s)) ds.$$

As follows, define the function

$$\mathcal{G}(\Upsilon(\rho)) = \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} e^{-\gamma(\rho-s)} \Upsilon(s) ds, \tag{14}$$

where $\rho \in [0,1]$, $\Upsilon \in \chi_\mu^*$. □

Lemma 3. *If the function* $\psi(\rho, \Upsilon(\rho))$ *yields conditions (13), then* $\mathcal{G}$ *maps* χ_μ^* *into itself, i.e.,* $\mathcal{G} : \chi_\mu^* \longrightarrow \chi_\mu^*$.

Proof. Let us start by selecting any $\Upsilon \in \chi_\mu^*$, with $\Upsilon(0) = 0$. To establish that $\mathcal{G}\Upsilon \in \chi_\mu^*$, we need to demonstrate that $\mathcal{G}$ is a mapping from χ_μ^* to χ_μ^*. Initially, as $\Upsilon(0) = 0$, it follows that $\mathcal{G}\Upsilon(0) = 0$, implying $\mathcal{G}\Upsilon \in \chi'$. For a function $\psi(\rho, \Upsilon(\rho))$ that verifies (13), we have for all $\Upsilon \in \chi_\mu^*$:

$$\begin{aligned}
|\mathcal{G}\Upsilon(\rho) - \mathcal{G}\Upsilon(0)| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} \left| e^{-\gamma(\rho-s)} \psi(s, \Upsilon(s)) \right| d\, s \\
&\leqslant \frac{\Gamma(\alpha+1)\ln(r)\|\Upsilon\|_\infty}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} \left| e^{-\gamma(\rho-s)} \right| d\, s \\
&\leqslant \frac{\Gamma(\alpha+1)(\ln(r)+1)\|\Upsilon\|_\infty}{\Gamma(\alpha)} \left[\frac{\rho^\alpha}{\alpha}\right] \\
&\leqslant (\ln(1)+1)\|\Upsilon\|_\infty \leqslant \|\Upsilon\|_\infty.
\end{aligned}$$

Thus, for all $\rho \in [0,1]$,

$$\begin{gathered}
\|\mathcal{G}\Upsilon\|_\infty \leqslant \|\Upsilon\|_\infty \\
e^{t\|\mathcal{G}\Upsilon\|_\infty} \leqslant e^{t\|\Upsilon\|_\infty}, \\
\frac{1}{e^{-t\|\mathcal{G}\Upsilon\|_\infty}} \leqslant \frac{1}{e^{-t\|\Upsilon\|_\infty}} \\
\frac{1}{e^{-t\|\mathcal{G}\Upsilon\|_\infty}} - 1 \leqslant \frac{1}{e^{-t\|\Upsilon\|_\infty}} - 1 \\
\frac{1}{\mu(\mathcal{G}\Upsilon, 0, t)} - 1 \leqslant \frac{1}{\mu(\Upsilon, 0, t)} - 1 \leqslant \infty.
\end{gathered}$$

Hence, $\mathcal{G}\Upsilon \in \chi_\mu'$, implying $\mathcal{G}\Upsilon \in \chi_\mu^*$.
□

Theorem 4. *Examine the mapping $\mathcal{G} : \chi_\mu^* \longrightarrow \chi_\mu^*$ defined by (14). Assuming that the function $\psi(\rho, \Upsilon(\rho))$ satisfies (13), then problem (12) possesses a unique solution.*

Proof. To begin, let us demonstrate that $\mathcal{G}$ satisfies corollary 7. For any $\Upsilon_1, \Upsilon_2 \in \chi_\mu^*$, and $\rho \in [0,1]$, we observe

$$\begin{aligned}
&|\mathcal{G}\Upsilon_1(\rho) - \mathcal{G}\Upsilon_2(\rho)| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} \Big| e^{-\gamma(\rho-s)} [\psi(s, \Upsilon_1(s)) - \psi(s, \Upsilon_2(s))] \Big| d\,s \\
&\leqslant \frac{\Gamma(\alpha+1)(\ln(r)+1)\|\Upsilon_1 - \Upsilon_2\|_\infty}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} \Big| e^{-\gamma(\rho-s)} \Big| d\,s \\
&\leqslant \frac{\Gamma(\alpha+1)(\ln(r)+1)\|\Upsilon_1 - \Upsilon_1\|_\infty}{\Gamma(\alpha)} \left[\frac{\rho^\alpha}{\alpha} \right] \\
&\leqslant (\ln(r)+1)\|\Upsilon_1 - \Upsilon_2\|_\infty.
\end{aligned}$$

Thus, for all $r \in (0,1]$ and $t > 0$, we have

$$\begin{aligned}
\|\mathcal{G}\Upsilon_1 - \mathcal{G}\Upsilon_2\|_\infty &\leqslant (\ln(r)+1)\|\Upsilon_1 - \Upsilon_2\|_\infty \\
e^{t\|\mathcal{G}\Upsilon_1 - \mathcal{G}\Upsilon_2\|_\infty} &\leqslant r^{t\|\Upsilon_1 - \Upsilon_2\|_\infty} e^{t\|\Upsilon_1 - \Upsilon_2\|_\infty} \\
e^{t\|\mathcal{G}\Upsilon_1 - \mathcal{G}\Upsilon_2\|_\infty} &\leqslant r e^{t\|\Upsilon_1 - \Upsilon_2\|_\infty} \\
\frac{1}{e^{-t\|\mathcal{G}\Upsilon_1 - \mathcal{G}\Upsilon_2\|_\infty}} &\leqslant \frac{r}{e^{-t\|\Upsilon_1 - \Upsilon_2\|_\infty}} \\
\frac{1}{e^{-t\|\mathcal{G}\Upsilon_1 - \mathcal{G}\Upsilon_2\|_\infty}} - 1 &\leqslant \frac{r}{e^{-t\|\Upsilon_1 - \Upsilon_2\|_\infty}} - 1
\end{aligned}$$

$$\frac{1}{\mu(\mathcal{G}\Upsilon_1, \mathcal{G}\Upsilon_2, t)} - 1 \leqslant r\left(\frac{1}{\mu(\Upsilon_1, \Upsilon_2, t)} - 1 \right). \tag{15}$$

On the other, we have for all $\rho \in [0,1]$ and $r \in (0,1]$:

$$\begin{aligned}
\phi(\mathcal{G}\Upsilon) = \sup_{\rho \in [0,1]} &\left| \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - s)^{\alpha-1} e^{-\gamma(\rho-s)} \Upsilon(s) ds \right| \\
&\leqslant (\ln(r)+1)\|\Upsilon\|_\infty \leqslant r\|\Upsilon\|_\infty
\end{aligned}$$

which means that

$$\phi(\mathcal{G}\Upsilon_1) + \phi(\mathcal{G}\Upsilon_2) \leqslant r(\|\Upsilon_1\|_\infty + \|\Upsilon_2\|_\infty) = r(\phi(\Upsilon_1) + \phi(\Upsilon_2)). \tag{16}$$

Consequently, from (15) and (16) for all $\Upsilon_1, \Upsilon_2 \in \chi_\mu^*$, $\rho \in [0,1]$, and $r \in (0,1]$, we get

$$\begin{aligned}
&r \cdot \left(\frac{1}{\mu(\Upsilon_1, \Upsilon_2, t)} - 1 + \phi(\Upsilon_1) + \phi(\Upsilon_2) \right) \\
&\quad \geqslant \left(\frac{1}{\mu(\mathcal{G}\Upsilon_1, \mathcal{G}\Upsilon_2, t)} - 1 + \phi(\mathcal{G}\Upsilon_1) + \phi(\mathcal{G}\Upsilon_2) \right).
\end{aligned}$$

Consequently, we conclude from corollary 3.2 that $\mathcal{G}$ has a unique fixed point in χ_μ^*, i.e., problem (12) possesses a unique solution. □

6. Conclusions

In this study, we established the existence and uniqueness of the best proximity point within the domain of non-Archimedean modular metric spaces through the employment of simulation functions. The non-Archimedean metric modular, structured as a parameterized collection of classical metrics, ensures the fulfillment of some essential properties.

Furthermore, we showcased the transferability of analogous theorems from modular metric spaces to the derivation of best proximity point outcomes in triangular fuzzy metric spaces. The practical significance of our findings was further exemplified through specific illustrative examples, the elucidation of consequences, and an insightful application to fuzzy fractional differential equations. Through rigorous analysis and demonstration, our research offers valuable insights into the theoretical underpinnings and practical implications of proximity point theory across diverse metric space frameworks.

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