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A Singular Tempered Sub-Diffusion Fractional Model Involving a Non-Symmetrically Quasi-Homogeneous Operator

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Abstract: In this paper, we focus on the existence of positive solutions for a singular tempered sub-diffusion fractional model involving a quasi-homogeneous nonlinear operator. By using the spectrum theory and computing the fixed point index, some new sufficient conditions for the existence of positive solutions are derived. It is worth pointing out that the nonlinearity of the equation contains a tempered fractional sub-diffusion term, and is allowed to possess strong singularities in time and space variables. In particular, the quasi-homogeneous operator is a nonlinear and non-symmetrical operator.

Keywords: singular tempered fractional equation; sub-diffusion; spectrum theory; quasi-homogeneous nonlinear operator

1. Introduction

In this paper, we consider the following singular tempered sub-diffusion fractional model involving a quasi-homogeneous nonlinear operator:

$$\begin{cases} \mathfrak{L}(-\mathbb{D}_t^{\alpha,\mu} z(t)) = f(t, e^{\mu t} z(t), \mathbb{D}_t^{\beta,\mu} z(t)), & t \in (0, 1), \\ \mathbb{D}_t^{\beta,\mu} z(0) = 0, \quad \mathbb{D}_t^{\beta,\mu} z(1) = 0. \end{cases} \quad (1)$$

Here, $1 < \alpha \leq 2$, $0 < \beta < 1$ and $\alpha - \beta > 1$, $f(t, x, y) : (0, 1) \times (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ is continuous, and can be singular at time variables $t = 0, 1$ and/or at space variables $x = 0, y = 0$, and $\mathfrak{L}(x) = x\varphi(x)$ is a nonlinear operator, where φ is a quasi-homogeneous function with degree $\gamma > 0$ as in Definition 1.

The tempered fractional derivative ${}^R_0\mathbb{D}_t^{\alpha,\lambda}$ is a new variant of the Riemann–Liouville fractional derivative ${}^R_0\mathcal{D}_t^\alpha$ for describing the connection between random walks and Brownian motion with semi-heavy tails that feature in anomalous diffusion; this is called tempered fractional Brownian motion in the literature [1]. In mathematics, the tempered fractional derivative can be derived by multiplying the Riemann–Liouville fractional derivatives by an exponential factor, namely

$${}^R_0\mathbb{D}_t^{\alpha,\mu} z(t) = e^{-\lambda t} {}^R_0\mathcal{D}_t^\alpha (e^{\mu t} z(t)). \quad (2)$$

For the definitions and properties of Riemann–Liouville fractional derivatives and integrals, see [2].

In anomalous diffusion, if a plume of particles spreads faster than the anticipation of traditional diffusion, the fractional space derivatives maybe used to model an anomalous super-diffusion, while the fractional time derivative may simulate an anomalous sub-diffusion. Thus, the super-diffusion and sub-diffusion in tempered fractional Brownian motion possess the characteristics of semi-long range dependence, i.e., at moderate time



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scales, the decay of the particle depends on a power law, but at long time scales, it follows an exponential rule. This extension also led to the establishment of a time-domain stochastic process model used in electric power generation facilities for the famous Davenport spectrum of wind speed [3,4]. The tempered diffusion model can also be used to study tempered Lévy flights without sharp cutoffs [5], geophysics problems [6,7] and finance problems [8].

From Relation (2), it is easy to see that the tempered fractional derivative becomes the Riemann–Liouville fractional derivative, provided that $\mu = 0$. This suggests that the tempered fractional derivative extends beyond classical fractional calculus, encompassing special cases like the Riemann–Liouville derivatives, Caputo fractional derivatives, and others [9–15]. Recently, by employing Ricceri’s variational principle, Ledesma et al. [16] studied a tempered fractional sub-diffusion model with an oscillating term:

$$\begin{cases} \mathbb{D}_{d-}^{\alpha,\mu} \left({}^C\mathbb{D}_{c+}^{\alpha,\mu} z(t) \right) = \lambda p(t) f(z(t)), & t \in (c, d), \\ z(c) = z(d) = 0, \end{cases} \quad (3)$$

where $\alpha \in (\frac{1}{2}, 1)$, $\mu > 0$, $\lambda \in \mathbb{R}$, $\mathbb{R}$ is real number set, and $p \in L^\infty(c, d)$, ${}^C\mathbb{D}_{c+}^{\alpha,\lambda}$ and $\mathbb{D}_{d-}^{\alpha,\lambda}$ denote the tempered right Caputo and left Riemann–Liouville fractional derivative, respectively. It has been proven that the tempered fractional sub-diffusion model (3) has infinitely many weak solutions if the nonlinear term f satisfies certain suitable oscillating conditions either at the origin or at infinity. Zhang et al. [17] obtained the existence and asymptotic properties of positive solutions

$$k_1 e^{-\lambda t} t^{\beta-1} \leq w(t) \leq k_2 e^{-\lambda t} t^{\beta-1}, k_1, k_2 > 0,$$

for the following tempered fractional turbulent flow model:

$$\begin{cases} \mathbb{D}_t^{\alpha,\lambda} \left(\varphi_p \left(\mathbb{D}_t^{\beta,\lambda} z(t) \right) \right) = f(t, z(t)), & t \in (0, 1), \\ z(0) = 0, \quad \mathbb{D}_t^{\beta,\lambda} z(0) = 0, \quad z(1) = \int_0^1 e^{-\lambda(1-t)} z(t) dt, \end{cases} \quad (4)$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$. In the past twenty years, there has been a growing interest in the profit derived from advancing space theories [18–20], regular theories [21–25], operator methods [26–30], iterative techniques [31–33], the moving sphere method [34], critical point theories [35–38], and tempered fractional calculus. This surge in attention has not only propelled the rapid progress of these disciplines, but has also spurred corresponding contributions across various fields.

In this paper, we are interested in handling the singularity in space variables for a tempered sub-diffusion fractional model involving a quasi-homogeneous nonlinear operator. Since a small change in the space variable near singular points will give rise to sharp changes in the objective function, it is difficult to deal with the singularity of the space variables. To overcome this difficulty, in this paper, the main tools we use are the spectrum analysis of linear operators and the calculation of the fixed-point index. The main contributions of this paper include the following aspects:

- The nonlinearity of the equation contains a tempered fractional sub-diffusion term, and the reducing order technique of fractional derivatives and integrals is used;
- The equation involves a quasi-homogeneous nonlinear operator, which gives the model a wider range of applications;
- The nonlinear term may include the strong singularities in time and space variables.

The paper’s structure is laid out as follows: In Section 2, we initially outline the properties of the quasi-homogeneous nonlinear operator. Subsequently, we revisit certain lemmas from the spectrum theory for linear operators, and provide some essential preliminaries for the subsequent discussion. The main results are presented and proven in Section 3.

2. Preliminaries and Lemmas

Let $\mathbb{R}^+$ be a nonnegative real number set, I be an interval on $\mathbb{R}^+$ including 0, i.e., I may be interval $[0, L]$, $[0, L)$ or $[0, +\infty)$. We first give the following definition.

Definition 1. Let $\gamma > 0$. A function $\varphi : I \rightarrow (0, +\infty)$ is said to be a quasi-homogeneous function with the degree γ if the following condition holds:

$$\varphi(ax) \leq a^\gamma \varphi(x) \text{ for all } a \in [0, 1] \text{ and } x \in I. \quad (5)$$

Lemma 1. Assume that φ is a quasi-homogeneous function. Let $\mathfrak{L}(x) = x\varphi(x)$, $x \in I$, then, for any $0 < a < 1$, there is a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(x)$ satisfying

$$\mathfrak{L}^{-1}(ax) \geq a^{\frac{1}{1+\gamma}} \mathfrak{L}^{-1}(x), \quad x \in I. \quad (6)$$

Proof. For any $0 \leq y < x \in I$, if $y = 0$, clearly, $\mathfrak{L}(y) = 0 < \mathfrak{L}(x)$. Otherwise, we have $0 < y < x$; let $a = \frac{y}{x}$, then $0 < a < 1$. It follows from (5) that

$$\mathfrak{L}(y) = ax\varphi(ax) \leq a^{\gamma+1}x\varphi(x) < x\varphi(x) = \mathfrak{L}(x).$$

Thus, $\mathfrak{L}$ is a strictly increasing function; consequently, $\mathfrak{L}$ is a bijection from I onto $\mathfrak{L}(I)$, and then it has a nonnegative increasing inverse, mapping $\mathfrak{L}^{-1}(x)$.

In the following, we shall prove (6). In fact, for any $0 < a < 1$ and $y \in I$, from (5), one has

$$\mathfrak{L}(ay) = ay\varphi(ay) \leq a^{1+\gamma}y\varphi(y) = a^{1+\gamma}\mathfrak{L}(y), \text{ for } y \in I.$$

and then

$$ay \leq \mathfrak{L}^{-1}(a^{1+\gamma}\mathfrak{L}(y)), \text{ for } y \in I. \quad (7)$$

Let $x = \mathfrak{L}(y)$; it follows from (7) that

$$\mathfrak{L}^{-1}(ax) = \mathfrak{L}^{-1}(a\mathfrak{L}(y)) \geq a^{\frac{1}{1+\gamma}}y = a^{\frac{1}{1+\gamma}}\mathfrak{L}^{-1}(x), \quad x \in I,$$

consequently, (6) holds. $\square$

Remark 1. The definition of quasi-homogeneous function appeared for the first time in [39], and was developed to study the iterative processes in metric spaces combining with the concept of gauge function [40,41].

Remark 2. By Lemma 1, we have

$$\mathfrak{L}^{-1}(ax) \leq a^{\frac{1}{1+\gamma}} \mathfrak{L}^{-1}(x), \quad a \geq 1. \quad (8)$$

Remark 3. $\mathfrak{L}$ is a general nonlinear operator including many operators as special case. In particular, if $\varphi(x) = |x|^{p-1}$, $p > 1$, then $\mathfrak{L}(x)$ reduces to a p -Laplacian operator.

It follows from Lemma 1 that Equation (1) can be converted to the following form:

$$\begin{cases} -\mathbb{D}_t^{\alpha, \mu} z(t) = \mathfrak{L}^{-1}\left(f\left(t, e^{\mu t} z(t), \mathbb{D}_t^{\beta, \mu} z(t)\right)\right), & t \in (0, 1), \\ \mathbb{D}_t^{\beta, \mu} z(0) = 0, \quad \mathbb{D}_t^{\beta, \mu} z(1) = 0. \end{cases} \quad (9)$$

Now, we recall some useful properties of Riemann–Liouville fractional derivatives and integrals.

Lemma 2 ([2]). Let $\alpha > \beta \in (0, +\infty)$ and $n = [\alpha] + 1$, where $[\alpha]$ is the greatest integer less than or equal to α . If $x(t) \in C[0, 1]$, then

(i)

$$I_0^{\alpha R} \mathcal{D}_t^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, i = 1, 2, 3, \dots, n. \quad (10)$$

(ii)

$$I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t), \quad {}^R_0 \mathcal{D}_t^\beta I^\alpha x(t) = I^{\alpha-\beta} x(t), \quad {}^R_0 \mathcal{D}_t^\beta I^\beta x(t) = x(t).$$

By making the integral transformation

$$z(t) = e^{-\mu t} I^\beta (e^{\mu t} u(t)), \quad t \in [0, 1], \quad (11)$$

and using Lemma 2 and the strategy of [42], Equation (9) can be rewritten by the following reducing order fractional equation:

$$\begin{cases} -{}_0^R \mathbb{D}_t^{\alpha-\beta, \mu} u(t) = \mathfrak{L}^{-1} \left(f(t, I^\beta (e^{\mu t} u(t)), u(t)) \right), & t \in (0, 1), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \quad (12)$$

and the following lemma is valid.

Lemma 3 ([42]). *Under the conditions of $0 < \alpha \leq 2, 0 < \beta < 1$ and $\alpha - \beta > 1$, the reducing order fractional Equation (12) and the singular tempered sub-diffusion fractional Equation (1) are equivalent. Moreover, let u be a positive solution of Equation (12), then $z(t) = e^{-\mu t} I^\beta (e^{\mu t} u(t))$ is a positive solution of the singular Equation (1).*

Lemma 4 ([42]). *Assume that $h \in C[0, 1]$ and $1 < \alpha - \beta \leq 2$. Then, the following linear equation*

$$\begin{cases} -{}_0^R \mathbb{D}_t^{\alpha-\beta, \mu} u(t) = h(t), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \quad (13)$$

has a unique solution

$$u(t) = \int_0^1 H(t, s) h(s) ds, \quad (14)$$

where

$$H(t, s) = \begin{cases} \frac{t^{\alpha-\beta-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} e^{-\mu t} e^{\mu s}, & 0 \leq s \leq t \leq 1; \\ \frac{t^{\alpha-\beta-1} (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} e^{-\mu t} e^{\mu s}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (15)$$

is the Green function of (12), which is nonnegative continuous and possesses the following properties:

$$\begin{aligned} \frac{t^{\alpha-\beta-1} (1-t) e^{-\mu t}}{\Gamma(\alpha-\beta)} (1-s)^{\alpha-\beta-1} s e^{\mu s} \leq H(t, s) \leq \frac{t^{\alpha-\beta-1} (1-t) e^{-\mu t}}{\Gamma(\alpha-\beta)} \\ \text{or } \left(\frac{(1-s)^{\alpha-\beta-1} s e^{\mu s}}{\Gamma(\alpha-\beta)} \right), \quad (t, s) \in [0, 1] \times [0, 1]. \end{aligned} \quad (16)$$

Suppose that X is a Banach space; let $P \subset X$ be a cone. For any $K > k \in (0, \infty)$, let

$$P_K = \{u \in P : \|u\| < K\}, \quad \partial P_K = \{u \in P : \|u\| = K\}, \quad \bar{P}_K \setminus P_K = \{u \in P : k \leq \|u\| \leq K\}.$$

Now, we recall some useful lemmas about spectrum theories and fixed point index used in the rest of this paper.

Lemma 5 ([43]). *(Krein–Rutman). Assume that $L : X \rightarrow X$ is a continuous linear operator, and P is a total cone satisfying $L(P) \subset P$. If there exist $u_0 \in X \setminus (-P)$ and a positive constant c such*

that $cL(u_0) \geq u_0$, then the spectral radius $d(L) \neq 0$, and there exists a positive eigenfunction for the first eigenvalue $\lambda = d(L)^{-1}$.

Lemma 6 ([43]). (Gelfand's formula) For a bounded linear operator L and the operator norm $\|\cdot\|$, the spectral radius of L^n satisfies

$$d(L) = \lim_{n \rightarrow +\infty} \|L^n\|^{\frac{1}{n}}.$$

Lemma 7 ([43]). Assume that $T : \bar{P}_k \rightarrow P$ is a completely continuous operator.

(i) If there exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Tu \neq \lambda u_0, \quad u \in \partial P_k, \quad \lambda \geq 0,$$

then the fixed point index $i(T, P_k, P) = 0$.

(ii) If

$$Tu \neq \lambda u, \quad u \in \partial P_k, \quad \lambda \geq 1,$$

then the fixed point index $i(T, P_k, P) = 1$.

In this paper, we use the following condition:

(F) $f : (0, 1) \times (0, +\infty) \times (0, +\infty) \rightarrow [0, +\infty)$ is continuous and, for any $k < K \in (0, +\infty)$,

$$\lim_{n \rightarrow +\infty} \sup_{u \in \bar{P}_K \setminus P_k} \int_{\Omega_n} t^{\alpha-\beta-1} (1-t) e^{-\mu t} \mathfrak{L}^{-1} \left(f(t, I^\beta(e^{\mu t} u(t)), u(t)) \right) dt = 0,$$

where $\Omega_n = [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$.

Let $X = C([0, 1], \mathbb{R})$ and

$$\|u\| = \max_{t \in [0, 1]} |u(t)|,$$

then X is a Banach space. Now, define the following cone:

$$P = \{u \in X : u(t) \geq t^{\alpha-\beta-1} (1-t) e^{-\mu t} \|u\|\}.$$

Obviously, $\bar{P}_K \setminus P_k \subset P \subset X$. Next define a nonlinear operator $T : \bar{P}_K \setminus P_k \rightarrow P$, as well as a linear operator $L : X \rightarrow X$ below

$$(Tu)(t) = \int_0^1 H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds, \quad (17)$$

$$(Lu)(t) = \int_0^1 H(t, s) u(s) ds. \quad (18)$$

It follows from Lemmas 3 and 4 that the solution of the singular tempered sub-diffusion fractional Equation (1) is equivalent to the fixed point of the operator equation $u = Tu$. In order to find the fixed point of T , the following lemmas are necessary.

Lemma 8. L is a completely continuous operator with $L(P) \subset P$. Moreover, its spectral radius $d(L) \neq 0$, and there is a positive eigenfunction u^* corresponding to the first eigenvalue $\lambda_1 = \frac{1}{d(L)}$ such that $\lambda_1 L u^* = u^*$.

Proof. In fact, for any $u \in P$, by Lemma 4, we have

$$\|Lu\| = \max_{t \in [0, 1]} \int_0^1 H(t, s) u(s) ds \leq \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} u(s) ds, \quad (19)$$

and

$$(Lu)(t) \geq \frac{t^{\alpha-\beta-1}(1-t)e^{-\mu t}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} u(s) ds, \quad (20)$$

where Gamma function

$$\Gamma(\alpha-\beta) = \int_0^{+\infty} t^{\alpha-\beta-1} e^{-t} dt.$$

It follows from (19) and (20) that

$$(Lu)(t) \geq t^{\alpha-\beta-1}(1-t)e^{-\mu t} \|Lu\|.$$

Thus, one has $L(P) \subset P$. Next, noticing that $H(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$, thus $L : P \rightarrow P$ is a completely continuous operator.

On the other hand, since $H(t, s)$ is nonnegative continuous in $[0, 1] \times [0, 1]$, there exists $t_0 \in (0, 1)$ such that $H(t_0, t_0) > 0$. Let us choose $0 < a < t_0 < b < 1$ such that $H(t, s) > 0$ for all $t, s \in [a, b] \subset (0, 1)$, and take $u \in P$, satisfying $u(t_0) > 0$ and $u(s) = 0$, $s \notin [a, b]$. Thus, for any $t \in [a, b]$, we obtain

$$(Lu)(t) = \int_0^1 H(t, s) u(s) ds \geq \int_a^b H(t, s) u(s) ds > 0.$$

Thus there exists $\lambda > 0$ such that $\lambda(Lu)(t) \geq u(t)$ for $t \in [0, 1]$. By employing the Krein–Rutman theorem, the spectral radius $d(L) \neq 0$; moreover, L has the first eigenvalue $\lambda_1 = (d(L))^{-1}$ and a positive eigenfunction u^* satisfying $\lambda_1 Lu^* = u^*$. $\square$

Lemma 9. Assume that the condition (F) is satisfied. Then, the operator $T : \bar{P}_K \setminus P_k \rightarrow P$ is completely continuous.

Proof. We first prove that $T(P) \subset P$ is well defined. By Lemma 4, for any $u \in P$, $t \in [0, 1]$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds \\ &\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds, \end{aligned}$$

and, consequently,

$$\|Tu\| \leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds.$$

By using Lemma 4 again, we also obtain

$$\begin{aligned} (Tu)(t) &= \int_0^1 H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds \\ &\geq \frac{t^{\alpha-\beta-1}(1-t)e^{-\mu t}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds \\ &\geq t^{\alpha-\beta-1}(1-t)e^{-\mu t} \|Tu\|. \end{aligned}$$

Thus, $T(P) \subset P$, and then $T(\bar{P}_K \setminus P_k) \subset P$.

On the other hand, it follows from condition (F) that there exists a natural number n , such that

$$\sup_{u \in \bar{P}_K \setminus P_k} \int_{\Omega_n} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds < \frac{1}{4}. \quad (21)$$

So, for any $u \in \overline{P}_K \setminus P_k$ and $\frac{1}{n} \leq t \leq 1 - \frac{1}{n}$, one has

$$\left(\frac{1}{n}\right)^{\alpha-\beta-1} \left(1 - \frac{1}{n}\right) e^{-\frac{\mu}{n}} k \leq u(t) \leq \|u\| = K, \quad (22)$$

and

$$\begin{aligned} \frac{\left(\frac{1}{n}\right)^{\beta} k}{\Gamma(\beta+1)} &\leq k \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \leq I^{\beta}(e^{\mu s} u(s)) \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} e^{\mu s} u(s) ds \\ &\leq \frac{e^{\mu} K t^{\beta}}{\Gamma(\beta+1)} \leq \frac{e^{\mu} \left(1 - \frac{1}{n}\right)^{\beta} K}{\Gamma(\beta+1)}. \end{aligned} \quad (23)$$

Let

$$Q_n^* = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times \left[\frac{\left(\frac{1}{n}\right)^{\beta} k}{\Gamma(\beta+1)}, \frac{e^{\mu} \left(1 - \frac{1}{n}\right)^{\beta} K}{\Gamma(\beta+1)}\right] \times \left[\left(\frac{1}{n}\right)^{\alpha-\beta-1} \left(1 - \frac{1}{n}\right) e^{-\frac{\mu}{n}} k, K\right],$$

and

$$L_1 = \max_{(t,u,v) \in Q_n^*} \{f(t, u, v)\}.$$

By (21)–(23), we have

$$\begin{aligned} &\sup_{u \in \overline{P}_K \setminus P_k} \int_0^1 H(t, s) \mathfrak{L}^{-1} \left(f(s, I^{\beta}(e^{\mu s} u(s)), u(s)) \right) ds \\ &\leq \sup_{u \in \overline{P}_K \setminus P_k} \int_{\Omega_n} H(t, s) \mathfrak{L}^{-1} \left(f(s, I^{\beta}(e^{\mu s} u(s)), u(s)) \right) ds \\ &\quad + \sup_{u \in \overline{P}_K \setminus P_k} \int_{\frac{1}{n}}^{1-\frac{1}{n}} H(t, s) \mathfrak{L}^{-1} \left(f(s, I^{\beta}(e^{\mu s} u(s)), u(s)) \right) ds \\ &\leq \frac{1}{4} + \frac{1}{\Gamma(\alpha-\beta)} \sup_{u \in \overline{P}_K \setminus P_k} \int_{\frac{1}{n}}^{1-\frac{1}{n}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^{\beta}(e^{\mu s} u(s)), u(s)) \right) ds \\ &\leq \frac{1}{4} + \frac{\mathfrak{L}^{-1}(L_1)}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} ds < +\infty. \end{aligned} \quad (24)$$

Thus, T is well defined and uniformly bounded in any bounded set.

Next, we assert that $T : \overline{P}_K \setminus P_k \rightarrow P$ is continuous. In fact, for any $\varepsilon > 0$ and $u \in P$, in view of (F), there exists a natural number $n_0 > 0$ such that

$$\sup_{u \in \overline{P}_K \setminus P_k} \int_{\Omega_{n_0}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^{\beta}(e^{\mu s} u(s)), u(s)) \right) ds < \frac{\varepsilon}{4}. \quad (25)$$

Now, suppose $u_n, u_0 \in \overline{P}_K \setminus P_k$, satisfying $\|u_n - u_0\| \rightarrow 0$ ($n \rightarrow \infty$). Since $\mathfrak{L}^{-1}(f(t, u, v))$ is uniformly continuous on the close interval

$$Q_{n_0}^* = \left[\frac{1}{n_0}, 1 - \frac{1}{n_0}\right] \times \left[\frac{\left(\frac{1}{n_0}\right)^{\beta} k}{\Gamma(\beta+1)}, \frac{e^{\mu} \left(1 - \frac{1}{n_0}\right)^{\beta} K}{\Gamma(\beta+1)}\right] \times \left[\left(\frac{1}{n_0}\right)^{\alpha-\beta-1} \left(1 - \frac{1}{n_0}\right) e^{-\frac{\mu}{n_0}} k, K\right],$$

one has

$$\lim_{n \rightarrow +\infty} \left| \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_n(s)), u_n(s)) \right) - \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_0(s)), u_0(s)) \right) \right| = 0$$

which uniformly holds for s on $\left[\frac{1}{n_0}, 1 - \frac{1}{n_0}\right]$. It follows from the Lebesgue control convergence theorem that

$$\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \left| \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_n(s)), u_n(s)) \right) - \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_0(s)), u_0(s)) \right) \right| ds \rightarrow 0, \quad n \rightarrow \infty.$$

Thus for the above $\varepsilon > 0$, there exists a natural number N such that for any $n > N$, one has

$$\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \left| \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_n(s)), u_n(s)) \right) - \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_0(s)), u_0(s)) \right) \right| ds < \frac{\varepsilon}{2}. \quad (26)$$

Hence, for any $n > N$, (25) and (26) imply that

$$\begin{aligned} \|Tu_n - Tu_0\| &\leq 2 \sup_{u_0 \in \overline{P}_K \setminus P_k} \int_{\Omega_{n_0}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_0(s)), u_0(s)) \right) ds \\ &+ \int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \left| \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_n(s)), u_n(s)) \right) - \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_0(s)), u_0(s)) \right) \right| ds \\ &< 2 \times \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So, $T : \overline{P}_K \setminus P_k \rightarrow P$ is continuous.

Finally, we show that T is equicontinuous. In view of (F), for any $\varepsilon > 0$, there exists a positive integer n_1 such that

$$\sup_{u \in \overline{P}_K \setminus P_k} \int_{\Omega_{n_1}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds < \frac{\varepsilon}{4}.$$

Let

$$Q_{n_1}^* = \left[\frac{1}{n_1}, 1 - \frac{1}{n_1} \right] \times \left[\frac{\left(\frac{1}{n_1}\right)^\beta k}{\Gamma(\beta+1)}, \frac{e^\mu \left(1 - \frac{1}{n_1}\right)^\beta K}{\Gamma(\beta+1)} \right] \times \left[\left(\frac{1}{n_1}\right)^{\alpha-\beta-1} \left(1 - \frac{1}{n_1}\right) e^{-\frac{\mu}{n_1}} k, K \right],$$

and take

$$L_2 = \max_{(t,x,y) \in Q_{n_1}^*} \{f(t, x, y)\}.$$

Since $H(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$, for the above $\varepsilon > 0$ and any fixed $s \in \left[\frac{1}{n_1}, 1 - \frac{1}{n_1}\right]$, there exists $\delta > 0$ such that, for any $|t_1 - t_2| < \delta$, $t_1, t_2 \in [0, 1]$, one has

$$|H(t_1, s) - H(t_2, s)| \leq \frac{1}{2\mathfrak{L}^{-1}(L_2)} \varepsilon.$$

Thus, for any $|t_1 - t_2| < \delta$, $t_1, t_2 \in [0, 1]$, we obtain

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &\leq 2 \sup_{u \in \overline{P}_K \setminus P_k} \int_{\Omega_{n_1}} (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds \\ &+ \sup_{u \in \overline{P}_K \setminus P_k} \int_{\frac{1}{n_1}}^{1-\frac{1}{n_1}} |H(t_1, s) - H(t_2, s)| \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u(s)), u(s)) \right) ds \\ &< 2 \times \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e., T is equicontinuous. It follows from the Arzelà–Ascoli theorem that $T : \overline{P}_K \setminus P_k \rightarrow P$ is completely continuous. $\square$

3. Main Results

We state the main results of this paper as follows:

Theorem 1. Suppose that (F) holds. Let λ_1 be the first eigenvalue of the linear operator L defined in (18). If

$$\limsup_{\substack{\tau+\xi \rightarrow +\infty \\ \xi \rightarrow +\infty}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\xi} < \lambda_1 < \liminf_{\substack{\tau \rightarrow 0^+ \\ \xi \rightarrow 0^+}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\tau + \xi}, \quad (27)$$

uniformly holds on $t \in [0, 1]$, then the singular tempered sub-diffusion fractional model (1) has at least one positive solution.

Proof. Firstly, since $T : \overline{P}_K \setminus P_k \rightarrow P$ is a completely continuous operator, by using the extension theorem of the completely continuous operator (see [43]), for any $K > 0$, we can find an extension operator T^* , which is still completely continuous, and $T^* : \overline{P}_K \rightarrow P$. For convenience, we write this expansion operator as T .

Now, by using (27), we know that there exists $k > 0$ such that

$$\mathfrak{L}^{-1}(f(t, \tau, \xi)) \geq \lambda_1(\tau + \xi), \quad |\tau| \leq \frac{e^\mu k}{\Gamma(\beta + 1)}, \quad |\xi| \leq k, \quad t \in [0, 1]. \quad (28)$$

Similarly to (23), for any $u \in \partial P_k$ and $s \in [0, 1]$, one has

$$\left| I^\beta(e^{\mu s} u(s)) \right| \leq \frac{e^\mu k}{\Gamma(\beta + 1)}, \quad |u(s)| \leq k. \quad (29)$$

Consequently, for any $u \in \partial P_k$, (28) and (29) imply that

$$\begin{aligned} (Tu)(t) &= \int_0^1 H(t, s) \mathfrak{L}^{-1}\left(f(s, I^\beta(e^{\mu s} u(s)), u(s))\right) ds \\ &\geq \lambda_1 \int_0^1 H(t, s) \left| I^\beta(e^{\mu s} u(s)) + u(s) \right| ds \\ &\geq \lambda_1 (Lu)(t), \quad t \in [0, 1]. \end{aligned} \quad (30)$$

Suppose that u^* is the positive eigenfunction corresponding to λ_1 , that is, $u^* = \lambda_1 Lu^*$. Now, we prove that

$$u - Tu \neq \lambda u^*, \quad u \in \partial P_k, \quad \lambda \geq 0. \quad (31)$$

If not, there exist $u_0 \in \partial P_k$ and $\lambda_0 \geq 0$ such that $u_0 - Tu_0 = \lambda_0 u^*$. Since T has no fixed points on ∂P_k (if not, the theorem holds), we have $\lambda_0 > 0$ and

$$u_0 = Tu_0 + \lambda_0 u^* \geq \lambda_0 u^*.$$

Let $\bar{\lambda} = \sup\{\lambda \mid u_0 \geq \lambda u^*\}$. Obviously, $\bar{\lambda} \geq \lambda_0$, $u_0 \geq \bar{\lambda} u^*$. Thus, we have

$$\lambda_1 Lu_0 \geq \lambda_1 \bar{\lambda} Lu^* = \bar{\lambda} u^*. \quad (32)$$

Therefore, from (30) and (32), we have

$$u_0 = Tu_0 + \lambda_0 u^* \geq \lambda_1 Lu_0 + \lambda_0 u^* \geq \bar{\lambda} u^* + \lambda_0 u^* = (\bar{\lambda} + \lambda_0) u^*.$$

This is a contradiction with the definition of $\bar{\lambda}$, which implies that (31) holds. It follows from Lemma 7 that

$$i(T, P_k, P) = 0. \quad (33)$$

Next, by (27), there exist $K_1 > k$ and $0 < \vartheta < 1$ such that

$$\mathfrak{L}^{-1}(f(t, \tau, \xi)) \leq \vartheta \lambda_1 |\xi|, \text{ for } |\tau + \xi| \geq K_1, |\xi| \geq K_1. \quad (34)$$

Let $\tilde{L}u = \vartheta \lambda_1 Lu$, then $\tilde{L} : X \rightarrow X$ is still a bounded linear operator with $\tilde{L}(P) \subset P$. Thus, we have

$$d^{-1}(\tilde{L}) = (\vartheta \lambda_1 d(L))^{-1} = \vartheta^{-1} > 1.$$

According to Gelfand's formula, one has

$$\vartheta = \lim_{n \rightarrow +\infty} \|\tilde{L}^n\|^{\frac{1}{n}}. \quad (35)$$

Let $\varepsilon_0 = \frac{1}{2}(1 - \vartheta)$; then, from (35), there is a large natural number N such that, for any $n > N$,

$$\|\tilde{L}^n\| \leq [\vartheta + \varepsilon_0]^n. \quad (36)$$

Now, define

$$\|u\|^* = \sum_{i=1}^N [\vartheta + \varepsilon_0]^{N-i} \|\tilde{L}^{i-1}u\|, \quad u \in X, \quad (37)$$

where $\tilde{L}^0 = I$ is the identity operator. Clearly, $\|\cdot\|^*$ is also the norm of X . Let

$$M = \sup_{u \in \partial P_{K_1}} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s}u(s)), u(s)) \right) ds,$$

then it follows from (24) that $M < +\infty$. Let $M^* = \|M\|^*$, and choose

$$K_2 > \max \left\{ K_1, \frac{2}{\varepsilon_0} M^* \right\}.$$

From (37), we have $\|u\|^* > [\vartheta + \varepsilon_0]^{N-1} \|u\|$, and thus take $K > \frac{K_2}{[\vartheta + \varepsilon_0]^{N-1}}$; then, if $\|u\| \geq K$ holds, one has $\|u\|^* > K_2$.

In what follows, we prove

$$Tu \neq \lambda u, \quad u \in \partial P_K, \quad \lambda \geq 1. \quad (38)$$

If not, there exist $u_1 \in \partial P_K$ and $\lambda^* \geq 1$ such that

$$Tu_1 = \lambda^* u_1. \quad (39)$$

Let $\tilde{u}(t) = \min\{u_1(t), K_1\}$ and

$$D(u_1) = \{t \in [0, 1] : u_1(t) > K_1\}.$$

As $u_1 \in C[0, 1]$ and $u_1(t) \leq \|u_1\| = K$, there exists $0 < t_0 \leq 1$ such that

$$u_1(t_0) = K.$$

Thus, for any $t \in [0, 1]$, one has $\tilde{u}(t) = \min\{u_1(t), K_1\} \leq \min\{K, K_1\} = K_1$ and $\tilde{u}(t_0) = \min\{u_1(t_0), K_1\} = \min\{K, K_1\} = K_1$, which yields $\|\tilde{u}(t)\| = K_1$; that is, $\tilde{u} \in \partial P_{K_1}$. Hence, it follows from Lemma 4 and (39) that

$$\begin{aligned} \lambda^* u_1 &= (Tu_1)(t) = \int_0^1 H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_1(s)), u_1(s)) \right) ds \\ &\leq \int_{D(u_1)} H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_1(s)), u_1(s)) \right) ds \\ &\quad + \int_{[0,1] \setminus D(u_1)} H(t, s) \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} u_1(s)), u_1(s)) \right) ds \\ &\leq \vartheta \lambda_1 \int_0^1 H(t, s) u_1(s) ds + \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} \mathfrak{L}^{-1} \left(f(s, I^\beta(e^{\mu s} \tilde{u}(s)), \tilde{u}(s)) \right) ds \\ &\leq (\tilde{L}u_1)(t) + M, \quad t \in [0, 1]. \end{aligned} \quad (40)$$

Thus, by $\tilde{L}(P) \subset P$, (39) and (40), one has

$$0 \leq (\tilde{L}^j(Tu_1))(t) = (\tilde{L}^j(\lambda^* u_1))(t) \leq (\tilde{L}^j(\tilde{L}u_1 + M))(t), \quad j = 0, 1, 2, \dots, N-1. \quad (41)$$

It follows from the fact P is a normal cone with normality constant 1 and (41) that

$$\|\tilde{L}^j(Tu_1)\| \leq \|\tilde{L}^j(\tilde{L}u_1 + M)\|, \quad j = 0, 1, 2, \dots, N-1.$$

Thus,

$$\begin{aligned} \|Tu_1\|^* &= \sum_{i=1}^N [\vartheta + \varepsilon_0]^{N-i} \|\tilde{L}^{i-1}(Tu_1)\| \\ &\leq \sum_{i=1}^N [\vartheta + \varepsilon_0]^{N-i} \|\tilde{L}^{i-1}(\tilde{L}u_1 + M)\| = \|\tilde{L}u_1 + M\|^*. \end{aligned} \quad (42)$$

It follows from $M^* < \frac{\varepsilon_0}{2} K_2$, (36), (37), (42) and $u_1 \in \partial P_K$ that

$$\begin{aligned} \lambda^* \|u_1\|^* &= \|Tu_1\|^* \leq \|\tilde{L}u_1\|^* + M^* = \sum_{i=1}^N [\vartheta + \varepsilon_0]^{N-i} \|\tilde{L}^i u_1\| + M^* \\ &= [\vartheta + \varepsilon_0] \sum_{i=1}^{N-1} [\vartheta + \varepsilon_0]^{N-i-1} \|\tilde{L}^i u_1\| + \|\tilde{L}^N u_1\| + M^* \\ &\leq [\vartheta + \varepsilon_0] \sum_{i=1}^{N-1} [\vartheta + \varepsilon_0]^{N-i-1} \|\tilde{L}^i u_1\| + [\vartheta + \varepsilon_0]^N \|u_1\| + M^* \\ &= [\vartheta + \varepsilon_0] \sum_{i=1}^N [\vartheta + \varepsilon_0]^{N-i} \|\tilde{L}^{i-1} u_1\| + M^* \leq [\vartheta + \varepsilon_0] \|u_1\|^* + \frac{\varepsilon_0}{2} K_2 \\ &\leq [\vartheta + \varepsilon_0] \|u_1\|^* + \frac{\varepsilon_0}{2} \|u_1\|^* = \left[\frac{1}{4} \vartheta + \frac{3}{4} \right] \|u_1\|^*, \end{aligned}$$

which implies that

$$\frac{1}{4} \vartheta + \frac{3}{4} \geq \lambda^* \geq 1,$$

that is, $\vartheta \geq 1$. This contradicts $0 < \vartheta < 1$. Consequently (38) holds, and then, from Lemma 7, we have

$$i(T, P_K, P) = 1. \quad (43)$$

It follows from (33) and (43) that

$$i(T, P_K \setminus \bar{P}_K, P) = i(T, P_K, P) - i(T, P_K, P) = 1.$$

Thus, T has at least one fixed point u on $P_K \setminus \bar{P}_k$. By using Lemma 3, $z(t) = e^{-\mu t} I^\beta(e^{\mu t} u(t))$ is a positive solution of the singular sub-diffusion tempered fractional model (1). $\square$

Before we give the second main result of this paper, for a sufficiently small $0 < \sigma < 1$, let us introduce the following linear operator L_σ :

$$(L_\sigma u)(t) = \int_\sigma^{1-\sigma} H(t, s) u(s) ds, \quad t \in [0, 1]. \quad (44)$$

It follows from Lemma 8 that $L_\sigma(P) \subset P$ is completely continuous with the first eigenvalue $\lambda_\sigma = (d(L_\sigma))^{-1}$, where $d(L_\sigma) \neq 0$ is the spectral radius of L_σ . Moreover, there exists a positive eigenfunction u_σ corresponding to the first eigenvalue, such that

$$\lambda_\sigma (L_\sigma u_\sigma)(t) = u_\sigma.$$

Lemma 10. *The linear operator L has an eigenvalue λ^* such that*

$$\lim_{\sigma \rightarrow 0^+} \lambda_\sigma = \lambda^*.$$

Proof. Firstly, construct a sequence $\{\sigma_m\}$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq \dots$ and

$$\lim_{m \rightarrow \infty} \sigma_m = 0.$$

For any $u \in P$ and $k > m$, it follows from (44) that

$$(L_{\sigma_m} u)(t) \leq (L_{\sigma_k} u)(t) \leq (Lu)(t), \quad t \in [0, 1]. \quad (45)$$

Let $L_{\sigma_m}^n = L(L_{\sigma_m}^{n-1})$, $n = 2, 3, \dots$; by (45), one has

$$(L_{\sigma_m}^n u)(t) \leq (L_{\sigma_k}^n u)(t) \leq (L^n u)(t), \quad t \in [0, 1], \quad n = 2, 3, \dots$$

Since P is a normal cone with normality constant 1, one has

$$\|L_{\sigma_m}^n\| \leq \|L_{\sigma_k}^n\| \leq \|L^n\|, \quad n = 1, 2, \dots$$

It follows from Gelfand's formula that

$$d(L_{\sigma_m}) \leq d(L_{\sigma_k}) \leq d(L),$$

where $d(L)$ is the spectral radius of L . Thus $\{d(L_{\sigma_m})\}$ is a monotonically increasing sequence with upper bound $d(L)$ and, consequently, there exists a λ^* such that

$$\lim_{m \rightarrow +\infty} \lambda_{\sigma_m} = \lambda^*.$$

In what follows, we prove that λ^* is the eigenvalue of L . To do this, suppose that u_{σ_m} are positive eigenfunctions of L_{σ_m} corresponding to λ_{σ_m} and $\|u_{\sigma_m}\| = 1$, $m = 1, 2, \dots$. Since

$$u_{\sigma_m}(t) = \lambda_{\sigma_m} (L_{\sigma_m} u_{\sigma_m})(t) = \lambda_{\sigma_m} \int_{\sigma_m}^{1-\sigma_m} H(t, s) u_{\sigma_m}(s) ds, \quad t \in [0, 1], \quad (46)$$

we have

$$\|L_{\sigma_m}\| = \max_{0 \leq t \leq 1} \int_{\sigma_m}^{1-\sigma_m} H(t, s) u_{\sigma_m}(s) ds \leq \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\mu s} ds < +\infty, \quad (m = 1, 2, \dots),$$

which implies that $L_{\sigma_m}(P) \subset P$ is uniformly bounded.

On the other hand, for any natural number m and $0 \leq \bar{t}, \tilde{t} \leq 1$, one has

$$|L_{\sigma_m} u_{\sigma_m}(\bar{t}) - L_{\sigma_m} u_{\sigma_m}(\tilde{t})| \leq \int_{\sigma_m}^{1-\sigma_m} |H(\bar{t}, s) - H(\tilde{t}, s)| u_{\sigma_m}(s) ds. \quad (47)$$

It follows from the uniform continuity of $H(t, s)$ in $[0, 1] \times [0, 1]$ and (47) that $L_{\sigma_m}(P) \subset P$ is equicontinuous. Noticing that $\lim_{m \rightarrow \infty} \sigma_m = 0$ and $\lambda_{\sigma_m} \geq \lambda_{\sigma_k} \geq \lambda_1$, then $\{u_{\sigma_m}\}$ is a monotonically increasing sequence with an upper bound, and then there exists a $u_0 \in P$ such that

$$u_{\sigma_m} \rightarrow u_0, m \rightarrow \infty, \|u_0\| = 1.$$

Let $m \rightarrow \infty$ in (46), by noticing that $\lim_{m \rightarrow +\infty} \lambda_{\sigma_m} = \lambda^*$, according to the Arzelà–Ascoli theorem, one gets

$$u_0(t) = \lambda^* \int_0^1 H(t, s) u_0(s) ds, \quad t \in [0, 1],$$

i.e.,

$$u_0 = \lambda^* L u_0,$$

which implies that λ^* is an eigenvalue of L . $\square$

Theorem 2. Let λ^* and λ_1 be any eigenvalue and the first eigenvalue of L , respectively. If (F) holds, and

$$\limsup_{\substack{\tau \rightarrow 0^+ \\ \xi \rightarrow 0^+}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\xi} < \lambda_1, \quad \liminf_{\tau + \xi \rightarrow +\infty} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\tau + \xi} > \lambda^* \quad (48)$$

uniformly for t on $[0, 1]$, then the singular tempered sub-diffusion fractional model (1) has at least one positive solution.

Proof. Firstly, according to (48), for any $0 \leq t \leq 1$, there exists $k > 0$ such that

$$\mathfrak{L}^{-1}(f(t, \tau, \xi)) \leq \lambda_1 \xi, \quad |\tau| \leq \frac{e^{\mu k}}{\Gamma(\beta + 1)}, \quad |\xi| \leq k. \quad (49)$$

Noticing that

$$|I^\beta(e^{\mu s} u(s))| = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} e^{\mu s} u(s) ds \leq \frac{e^{\mu k} t^\beta}{\Gamma(\beta + 1)} \leq \frac{e^{\mu k}}{\Gamma(\beta + 1)}, \quad |u(s)| \leq \|u\| = k,$$

for any $u \in \partial P_k$, one has

$$\begin{aligned} (Tu)(t) &= \int_0^1 H(t, s) \mathfrak{L}^{-1}(f(s, I^\beta(e^{\mu s} u(s)), u(s))) ds \\ &\leq \lambda_1 \int_0^1 H(t, s) u(s) ds = \lambda_1 (Lu)(t), \quad t \in [0, 1]. \end{aligned} \quad (50)$$

Now, we prove that

$$Tu \neq \lambda u, \text{ for any } u \in \partial P_k, \lambda \geq 1. \quad (51)$$

Suppose that (51) does not hold, then there exist $u_0 \in \partial P_k$ and $\lambda_0 \geq 1$ such that

$$Tu_0 = \lambda_0 u_0. \quad (52)$$

Clearly, $\lambda_0 > 1$, or otherwise the proof is completed. By (50) and (52), one has

$$\lambda_0 u_0 = Tu_0 \leq \lambda_1 Lu_0. \quad (53)$$

By induction, one gets

$$\lambda_0^n u_0 \leq \lambda_1^n L^n u_0, \quad n = 1, 2, \dots,$$

which yields

$$\|L^n\| \geq \frac{\|L^n u_0\|}{\|u_0\|} \geq \frac{\lambda_0^n \|u_0\|}{\lambda_1^n \|u_0\|} = \left(\frac{\lambda_0}{\lambda_1}\right)^n.$$

So, by Gelfand's formula, we have

$$d(L) = \lim_{n \rightarrow \infty} \sqrt[n]{\|L^n\|} \geq \frac{\lambda_0}{\lambda_1} > \lambda_1^{-1}.$$

This is a contradiction with $d(L) = \lambda_1^{-1}$.

Consequently, (47) holds, and from Lemma 7, we have

$$i(T, P_k, P) = 1. \quad (54)$$

Since $\lim_{\sigma \rightarrow 0^+} \lambda_\sigma = \lambda^*$, it follows from (48) that there exist $K > k$ and a sufficiently small $\sigma > 0$ such that

$$\mathfrak{L}^{-1}(f(t, \tau, \xi)) \geq \lambda_\sigma(\tau + \xi), \quad \tau + \xi \geq \left(\frac{\beta\Gamma(\alpha - \beta)\sigma^{\alpha-1}e^{-\mu\sigma^2}}{\Gamma(\alpha + 1)} + \sigma^{\alpha-\beta}e^{-\mu(1-\sigma)} \right) K, \quad t \in [0, 1], \quad (55)$$

where λ_σ is the first eigenvalue of L_σ .

For any $u \in \partial P_K$, $s \in [\sigma, 1 - \sigma]$, one has

$$\begin{aligned} I^\beta(e^{\mu s} u(s)) + u(s) &\geq \int_0^s \frac{(s - \xi)^{\beta-1}}{\Gamma(\beta)} e^{\mu \xi} u(\xi) d\xi + s^{\alpha-\beta-1}(1-s)e^{-\mu s} \|u\| \\ &\geq \int_0^\sigma \frac{(\sigma - \xi)^{\beta-1}}{\Gamma(\beta)} \xi^{\alpha-\beta-1}(1-\xi)e^{-\mu \xi} d\xi \|u\| + \sigma^{\alpha-\beta}e^{-\mu(1-\sigma)} \|u\| \\ &\geq \frac{\beta\Gamma(\alpha - \beta)\sigma^{\alpha-1}e^{-\mu\sigma^2}}{\Gamma(\alpha + 1)} \|u\| + \sigma^{\alpha-\beta}e^{-\mu(1-\sigma)} \|u\| \\ &\geq \left(\frac{\beta\Gamma(\alpha - \beta)\sigma^{\alpha-1}e^{-\mu\sigma^2}}{\Gamma(\alpha + 1)} + \sigma^{\alpha-\beta}e^{-\mu(1-\sigma)} \right) K. \end{aligned} \quad (56)$$

Let u_σ be the positive eigenfunction corresponding to λ_σ , that is

$$u_\sigma = \lambda_\sigma L_\sigma u_\sigma.$$

By (55) and (56), we have

$$\begin{aligned} (Tu_\sigma)(t) &= \int_0^1 H(t, s) \mathfrak{L}^{-1}\left(f(s, I^\beta(e^{\mu s} u_\sigma(s)), u_\sigma(s))\right) ds \\ &\geq \int_\sigma^{1-\sigma} H(t, s) \mathfrak{L}^{-1}\left(f(s, I^\beta(e^{\mu s} u_\sigma(s)), u_\sigma(s))\right) ds \\ &\geq \lambda_\sigma \int_\sigma^{1-\sigma} H(t, s) \left(I^\beta(e^{\mu s} u_\sigma(s)) + u_\sigma(s)\right) ds \\ &\geq \lambda_\sigma \int_\sigma^{1-\sigma} H(t, s) u_\sigma(s) ds \\ &= \lambda_\sigma (L_\sigma u_\sigma)(t), \quad t \in [0, 1]. \end{aligned}$$

Similarly to the proof of Theorem 1, we have

$$u - Tu \neq \lambda u_\sigma, \quad u \in \partial P_K, \quad \lambda \geq 0.$$

According to Lemma 7, we have

$$i(T, P_K, P) = 0. \quad (57)$$

Thus, by (54) and (57), we obtain

$$i(T, P_K \setminus \bar{P}_r, P) = i(T, P_K, P) - i(T, P_r, P) = -1,$$

which implies that T has at least one fixed point u on $P_K \setminus \bar{P}_k$, and thus the singular tempered sub-diffusion fractional model (1) has at least one positive solution $z(t) = e^{-\mu t} I^\beta(e^{\mu t} u(t))$. $\square$

The particle's random walks of anomalous diffusion in Brownian motion is modelled by the fractional diffusion equation

$$\partial_t^\beta f(z, t) = \partial_z^\alpha f(z, t),$$

where $0 < \alpha < 2, 0 < \beta < 1$, and $f(z, t)$ is the particle jump density function. In what follows, we give a complex singular tempered sub-diffusion fractional model to illustrate how to use our main results in practice.

Let $\varphi(x) = x^2$, then $\varphi(x)$ is a quasi-homogeneous function with degree $\gamma = 2$, and $\mathfrak{L}(x) = x\varphi(x) = x^3$.

Example 1. Consider the following singular tempered sub-diffusion fractional model involving a quasi-homogeneous nonlinear operator:

$$\begin{cases} \mathfrak{L}(-\mathbb{D}_t^{\frac{3}{2}, 2} z(t)) = \left[\left(e^{2t} z(t) + \mathbb{D}_t^{\frac{1}{4}, 2} z(t) \right)^{-\frac{1}{3}} + |\ln \mathbb{D}_t^{\frac{1}{4}, 2} z(t)| \right]^3, & t \in (0, 1), \\ \mathbb{D}_t^{\frac{1}{4}, 2} z(0) = 0, \quad \mathbb{D}_t^{\frac{1}{4}, 2} z(1) = 0. \end{cases} \quad (58)$$

The singular tempered sub-diffusion fractional model (58) has at least one positive solution.

Proof. Let $\alpha = \frac{3}{2}, \beta = \frac{1}{4}, \mu = 2$, and

$$f(t, x, y) = \left[(x + y)^{-\frac{1}{3}} + |\ln y| \right]^3.$$

Thus, f can be singular at $t = 0, 1, x = 0$ and $y = 0$.

Define a cone in X

$$P = \{u \in X : u(t) \geq t^{\frac{1}{4}}(1-t)e^{-2t}||u||\}. \quad (59)$$

Now, let $u(t) = \mathbb{D}_t^{\frac{1}{4}, 2} z(t)$, then from [42], we have $e^{2t} z(t) = I^{\frac{1}{4}}(e^{2t} u(t))$. Thus, for any $0 < k < K < +\infty$ and $u \in \bar{K}_R \setminus K_r$, by (23) and (59), we have

$$0 \leq ke^{-2}t^{\frac{1}{4}}(1-t) \leq kt^{\frac{1}{4}}(1-t)e^{-2t} \leq u(t) \leq K, \quad t \in [0, 1],$$

$$0 \leq \frac{kt^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \leq I^{\frac{1}{4}}(e^{2t} u(t)) \leq \frac{e^2 K}{\Gamma(\frac{5}{4})}, \quad t \in [0, 1]$$

Noticing that the function $w(y) = |\ln y|$ is decreasing on $(0, 1)$ and increasing in $(1, +\infty)$, one has

$$\begin{aligned}
|\ln u(t)| &\leq \left| \ln ke^{-2} t^{\frac{1}{4}}(1-t) \right| + |\ln K| \leq 2 + |\ln k| + |\ln K| + \left| \ln t^{\frac{1}{4}}(1-t) \right|, \\
\left[u(t) + I^{\frac{1}{4}}(e^{2t}u(t)) \right]^{-\frac{1}{3}} &\leq \left[ke^{-2} t^{\frac{1}{4}}(1-t) + \frac{kt^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} \leq \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}}.
\end{aligned} \tag{60}$$

Thus,

$$\begin{aligned}
&\int_0^1 \left| \ln t^{\frac{1}{4}}(1-t) \right| + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}} dt \\
&\leq \frac{1}{4} \int_0^1 |\ln t| dt + \int_0^1 |\ln(1-t)| dt + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} \int_0^1 t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}} dt \\
&\leq \frac{5}{4} + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} \frac{\Gamma(\frac{11}{12})\Gamma(\frac{2}{3})}{\Gamma(\frac{19}{12})} < +\infty.
\end{aligned}$$

According to the absolute continuity of the integral, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} \left| \ln t^{\frac{1}{4}}(1-t) \right| + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}} dt = 0. \tag{61}$$

It follows from (60) and (61) that

$$\begin{aligned}
&\lim_{n \rightarrow +\infty} \sup_{u \in \bar{P}_K \setminus P_k} \int_{\Omega_n} t^{\frac{1}{4}}(1-t)e^{-2t} \mathfrak{L}^{-1} \left(f(t, I^{\frac{1}{4}}(e^{2t}u(t)), u(t)) \right) dt \\
&\leq \lim_{n \rightarrow +\infty} \sup_{u \in \bar{P}_K \setminus P_k} \int_{\Omega_n} \left[\left(u(t) + I^{\frac{1}{4}}(e^{2t}u(t)) \right)^{-\frac{1}{3}} + |\ln u(t)| \right] dt \\
&\leq \lim_{n \rightarrow +\infty} \sup_{u \in \bar{P}_K \setminus P_k} \int_{\Omega_n} \left[2 + |\ln k| + |\ln K| + \left| \ln t^{\frac{1}{4}}(1-t) \right| + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}} \right] dt \\
&= [2 + |\ln k| + |\ln K|] \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \int_{\Omega_n} \left| \ln t^{\frac{1}{4}}(1-t) \right| + \left[ke^{-2} + \frac{k}{\Gamma(\frac{5}{4})} \right]^{-\frac{1}{3}} t^{-\frac{1}{12}}(1-t)^{-\frac{1}{3}} dt \\
&= 0.
\end{aligned}$$

Thus, (F) is satisfied.

In what follows, we verify the condition (27). In fact, we have

$$\begin{aligned}
\liminf_{\substack{\tau \rightarrow 0^+ \\ \xi \rightarrow 0^+}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\tau + \xi} &= \liminf_{\substack{\tau \rightarrow 0^+ \\ \xi \rightarrow 0^+}} \frac{(\xi + \tau)^{-\frac{1}{3}} + |\ln \xi|}{\xi + \tau} = +\infty, \\
\limsup_{\substack{\tau + \xi \rightarrow +\infty \\ \xi \rightarrow +\infty}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\xi} &= \limsup_{\substack{\tau + \xi \rightarrow +\infty \\ \xi \rightarrow +\infty}} \frac{(\xi + \tau)^{-\frac{1}{3}} + |\ln \xi|}{\xi} = 0.
\end{aligned}$$

This yields

$$\limsup_{\substack{\tau + \xi \rightarrow +\infty \\ \xi \rightarrow +\infty}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\xi} < \lambda_1 < \liminf_{\substack{\tau \rightarrow 0^+ \\ \xi \rightarrow 0^+}} \frac{\mathfrak{L}^{-1}(f(t, \tau, \xi))}{\tau + \xi},$$

thus (27) holds.

According to Theorem 1, Equation (58) has at least one positive solution. $\square$

4. Conclusions

Singularity arises from many fields of physics, bioscience, hydrodynamics, mathematics and engineering. Because a small change in the variable near the singular points will give rise to sharp changes in the property of the objective function, it is difficult to deal with the singularity of the space and time variables. Hence, the study of singularity is a very challenging and interesting project. In this paper, we introduce a limit-type control condition to overcome the difficulty of a singularity of a nonlinear term at space variables, which is effective and reasonable for deriving the solution of the model. Moreover, the condition can also be used to deal with other types of singular nonlinear problems. In addition, the noteworthy aspects include the presence of a tempered fractional sub-diffusion term within the equation's nonlinearity, along with the involvement of a quasi-homogeneous nonlinear operator. In addition, we only consider the case of $1 < \alpha \leq 2, 0 < \beta < 1$ and $\alpha - \beta > 1$ for Equation (1), which is a natural condition. For the case $1 < \alpha \leq 2, 0 < \beta < 1$ and $\alpha - \beta > 1$, after suitable reducing order, it is a initial problem, so this is still a interesting question for further study.

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