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Several Symmetric Identities of the Generalized Degenerate Fubini Polynomials by the Fermionic p-Adic Integral on $\mathbb{Z}_p$

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Abstract: After constructions of p-adic q-integrals, in recent years, these integrals with some of their special cases have not only been utilized as integral representations of many special numbers, polynomials, and functions but have also given the chance for deep analysis of many families of special polynomials and numbers, such as Bernoulli, Fubini, Bell, and Changhee polynomials and numbers. One of the main applications of these integrals is to obtain symmetric identities for the special polynomials. In this study, we focus on a novel extension of the degenerate Fubini polynomials and on obtaining some symmetric identities for them. First, we introduce the two-variable degenerate w-torsion Fubini polynomials by means of their exponential generating function. Then, we provide a fermionic p-adic integral representation of these polynomials. By this representation, we derive some new symmetric identities for these polynomials, using some special p-adic integral techniques. Lastly, by using some series manipulation techniques, we obtain more identities of symmetry for the two variable degenerate w-torsion Fubini polynomials.

Keywords: degenerate Fubini polynomials; degenerate w-torsion Fubini polynomials; fermionic p-adic integral on $\mathbb{Z}_p$; symmetric identities

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1. Introduction

With the construction and introduction of the fermionic p-adic integral cf. [1,2], it is utilized for not only integral representations of many special numbers, polynomials, and functions but also for providing a chance for deep analysis of many families of special numbers and polynomials, such as Euler, tangent, Boole, Genocchi, Changhee, Frobenius-Euler, Fubini, polynomials and numbers, cf. [2–17]. One of the most useful aims of the fermionic p-adic integral (abbreviated with "*f.p-a.i.*") is to acquire more formulas and properties of the special numbers and polynomials. In the last ten or more years, by utilizing the fermionic p-adic integrals on $\mathbb{Z}_p$, symmetric identities of some special polynomials, such as w-torsion Fubini polynomials in [11], q-Frobenius–Euler polynomials under S_5, the symmetric group of degree five in [4], q-Genocchi polynomials of higher order under D_3 in [3], (h,q)-Euler polynomials under D_3 in [6], Carlitz's-type twisted (h,q)-tangent polynomials in [7], degenerate q-Euler polynomials in [12], and Fubini polynomials in [15], have been studied and investigated in detail. By means of the p-adic integrals, several special techniques and methods have been used to obtain symmetric identities, where these identities include and generalize many special well-known formulas and properties for the polynomials, such as Raabe formulas, extended recurrence formulas, Miki identity, Carlitz identities, and many other identities for the polynomials. By these motivations, in this study, we focus on a novel extension of the degenerate Fubini polynomials. First, we introduce the two variable degenerate (abbreviated with "*t.w.d.*") w-torsion Fubini polynomials by their exponential generating function. We then provide a *f.p-a.i.* representation of the degenerate w-torsion Fubini polynomials, by which we acquire diverse novel symmetric identities for the degenerate w-torsion Fubini polynomials. Lastly, by using some series manipulation methods, we obtain more identities of symmetry for properties of the *t.w.d.* w-torsion Fubini polynomials.

Along this work, the following notations hold for p be a fixed odd prime number: $\mathbb{Z}_p$ denotes the ring of p-adic integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$. The normalized p-adic norm is provided by $|p|_p = p^{-1}$. For a continuous function $g : \mathbb{Z}_p \to \mathbb{C}_p$, the *f.p-a.i.* of g is provided (cf. [1,8,11]) as follows:

$$\int_{\mathbb{Z}_p} g(\nu) d\mu_{-1}(\nu) = \lim_{N\to\infty} \sum_{\nu=0}^{p^N-1} g(\nu)(-1)^\nu, \tag{1}$$

where $\mu_{-1}(\nu + p^N\mathbb{Z}_p) = (-1)^\nu$.

It is apparent from (1) that

$$\frac{1}{2}\int_{\mathbb{Z}_p} g(\nu+1) d\mu_{-1}(\nu) + \frac{1}{2}\int_{\mathbb{Z}_p} g(\nu) d\mu_{-1}(\nu) = g(0). \tag{2}$$

By invoking (2), we easily obtain (see [11])

$$\frac{1}{2}\int_{\mathbb{Z}_p} (-1)^\nu (\gamma(e^t-1))^\nu d\mu_{-1}(\nu) = \frac{1}{1-\gamma(e^t-1)} = \sum_{m=0}^{\infty} F_m(\gamma)\frac{t^m}{m!}, \tag{3}$$

and

$$\frac{e^{\nu t}}{2}\int_{\mathbb{Z}_p} (-1)^z (\gamma(e^t-1))^z d\mu_{-1}(z) = \frac{1}{1-\gamma(e^t-1)} e^{\nu t} = \sum_{m=0}^{\infty} F_m(\nu;\gamma)\frac{t^m}{m!}, \tag{4}$$

where $F_m(\gamma)$ and $F_m(\nu;\gamma)$ are the Fubini polynomials (also known as the geometric polynomials or the ordered Bell polynomials) and two variable Fubini polynomials, respectively. The convergences of the series (3) and (4) are $|t| < \left|\log\left|1+\gamma^{-1}\right|\right|$ for $\gamma \neq 1$ and $|t| < \log 2$ for $\gamma = 1$. If $\gamma = 1$, $F_m := F_m(0)$ are termed the Fubini numbers, which enumerates the ordered partitions of the set $[n] = 1, 2, \ldots, n$, (cf. [2,5,11,13,15,16,18]). Fubini polynomials and numbers (with several extensions) have been studied and analyzed comprehensively in recent years, cf. [2,5,11,15,18] and see also the references cited therein. In [5], new second-order non-linear recursive polynomials have been defined, and then these recursive polynomials, the properties of the power series, and the combinatorial methods have been used to prove some identities involving the Fubini polynomials, Euler polynomials, and Euler numbers. In [11], the generating function of ω-torsion Fubini polynomials has been considered by means of a fermionic p-adic integral on $\mathbb{Z}_p$ and then, some new symmetric identities for these polynomials have been investigated. Moreover, in [15], the computational problem of one-kind symmetric sums involving Fubini polynomials and Euler numbers has been studied by utilizing elementary methods and the recursive properties of a special sequence, and also, an interesting computational formula has been obtained.

Let $t, \lambda \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. The degenerate exponential function is provided for $\lambda \in \mathbb{R}$ as follows (cf. [2,12,13,16,18,19])

$$e_\lambda^\nu(t) = (1+\lambda t)^{\frac{\nu}{\lambda}} \text{ with } e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{5}$$

The series representations of the function $e_\lambda^\nu(t)$ are presented as follows:

$$e_\lambda^\nu(t) = \sum_{m=0}^{\infty} (\nu)_{m,\lambda} \frac{t^m}{m!}, \tag{6}$$

where $(\nu)_{m,\lambda} := (\nu-(m-1)\lambda)(\nu-(m-2)\lambda)\cdots(\nu-2\lambda)(\nu-\lambda)\nu$ for $m > 0$ and $(\nu)_{0,\lambda} := 1$.

For $k \geq 0$, the degenerate Stirling numbers of the second kind [13,19] are provided by

$$k! \sum_{j=k}^{\infty} S_{2,\lambda}(j,k) \frac{t^j}{j!} = (e_\lambda(t)-1)^k. \tag{7}$$

We compute from (7) that (cf. [19])

$$S_{2,\lambda}(j,k) = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l)_{j,\lambda}. \tag{8}$$

It is obvious that (see [10,13,19])

$$\lim_{\lambda\to 0} S_{2,\lambda}(j,k) = S_2(j,k),$$

which are the usual Stirling numbers of the second kind. In combinatorics, the Stirling numbers of the second kind count the number of ways in which n-distinguishable objects can be partitioned into k indistinguishable subsets when each subset has to contain at least one object, cf. [10,13,19]. Recently, degenerate and probabilistic forms of the Stirling numbers of the second kind have been studied, and many properties and applications have been investigated, cf. [13,19] and see also the references cited therein. In [19], the degenerate Stirling polynomials of the second kind have been considered by their generating function, and some new identities for these polynomials have been analyzed. In [13], the probabilistic degenerate Stirling polynomials of the second kind associated with Y have been introduced, and some properties, explicit expressions, and certain identities for those polynomials have been derived.

It can be observed that

$$F_m(\gamma) = \sum_{k=0}^{m} k! S_2(m,k)\gamma^k \text{ and } F_m = \sum_{k=0}^{m} k! S_2(m,k).$$

The *t.w.d.* Fubini polynomials $F_{m,\lambda}(\nu;\gamma)$ are defined as follows (see [2,18]):

$$\frac{1}{1-\gamma(e_\lambda(t)-1)} e_\lambda^\nu(t) = \sum_{m=0}^{\infty} F_{m,\lambda}(\nu;\gamma) \frac{t^m}{m!}. \tag{9}$$

If $\nu = 0$, $F_{m,\lambda}(\gamma) := F_{m,\lambda}(0;\gamma)$ and $F_{m,\lambda}(1) := F_{m,\lambda}(0;1)$ are termed the degenerate Fubini polynomials and the degenerate Fubini numbers, respectively. In [18], two variable degenerate Fubini polynomials have been first defined and some of their properties, explicit formulas, and recurrence relations have been examined extensively. Also, in [2], two variable higher-order degenerate Fubini polynomials have been considered by utilizing umbral calculus and then, several recurrence relations, explicit formulas, and some correlations, including some families of special functions, have been derived.

Using (2), we give the *f.p-a.i.* representations of the $F_{m,\lambda}(\gamma)$ and $F_{m,\lambda}(\nu;\gamma)$ as follows:

$$\frac{1}{2}\int_{\mathbb{Z}_p} (-1)^\nu (\gamma(e_\lambda(t)-1))^\nu d\mu_{-1}(\nu) = \frac{1}{1-\gamma(e_\lambda(t)-1)} = \sum_{m=0}^{\infty} F_{m,\lambda}(\gamma) \frac{t^m}{m!}, \tag{10}$$

and

$$\frac{e_\lambda^\nu(t)}{2}\int_{\mathbb{Z}_p}(-1)^z(\gamma(e_\lambda(t)-1))^z d\mu_{-1}(z)=\frac{e_\lambda^\nu(t)}{1-\gamma(e_\lambda(t)-1)}=\sum_{m=0}^{\infty}F_{m,\lambda}(\nu;\gamma)\frac{t^m}{m!}. \tag{11}$$

From (6), (10) and (11), we observe that

$$\sum_{l=0}^{m}\binom{m}{l}F_{m-l,\lambda}(\gamma)(\nu)_{l,\lambda}=F_{m,\lambda}(\nu;\gamma) \quad (m\geq 0). \tag{12}$$

The forward difference operator Δ is defined by $\Delta g(v) = g(v+1)-g(v)$ and $\Delta^i=\Delta^{i-1}\Delta$ for $i\in\mathbb{N}$. From this, we see that (cf. [13])

$$\Delta^k g(v)=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}g(v+l),\ (k\geq 0).$$

Therefore, we obtain that

$$\Delta^k(\gamma)_{n,\lambda}=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(\gamma+l)_{n,\lambda},\ (k\geq 0)$$

and

$$\Delta^k(0)_{n,\lambda}=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda},\ (k\geq 0). \tag{13}$$

Hence, it is observed from (8) and (13) that (cf. [13])

$$\Delta^n(0)_{m,\lambda}=\frac{1}{n!}S_{2,\lambda}(m,n). \tag{14}$$

Using (13), it is also examined that

$$\begin{aligned}\frac{1-\gamma^k(e_\lambda(t)-1)^k}{1-\gamma(e_\lambda(t)-1)} &= \sum_{i=0}^{k-1}\gamma^i(e_\lambda(t)-1)^i=\sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{l-i}\gamma^i e_\lambda^l(t)\\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{l-i}\gamma^i(l)_{m,\lambda}\right)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{k-1}\gamma^i\Delta^i(0)_{m,\lambda}\right)\frac{t^m}{m!}.\end{aligned} \tag{15}$$

It can be observed that

$$F_{m,\lambda}(\gamma)=\sum_{k=0}^{m}k!S_{2,\lambda}(m,k)(\gamma)_{k,\lambda} \text{ and } F_{m,\lambda}=\sum_{k=0}^{m}k!S_{2,\lambda}(m,k)(1)_{k,\lambda}.$$

2. Main Results

In this section, we consider *t.w.d.* w-torsion Fubini polynomials in terms of their exponential generating function and a *f.p-a.i.* representation on $\mathbb{Z}_p$ and investigate multifarious formulas and identities of the mentioned polynomials. We begin with our main definition.

Definition 1. *Let $w\in\mathbb{N}$. We define the t.w.d. w-torsion Fubini polynomials as follows:*

$$\frac{1}{1-\gamma^w(e_\lambda(t)-1)^w}e_\lambda^\nu(t)=\sum_{m=0}^{\infty}F_{m,w}(\nu;\gamma|\lambda)\frac{t^m}{m!}. \tag{16}$$

Remark 1. *In some special cases, $F_{m,w}(\gamma|\lambda):=F_{m,w}(0;\gamma|\lambda)$ and $F_{m,w}(1|\lambda):=F_{m,w}(0;1|\lambda)$ are called the degenerate w-torsion Fubini polynomials and numbers, respectively.*

Remark 2. *Upon setting $w = 1$ in (16), the polynomials $F_{m,w}(\nu;\gamma|\lambda)$ become the usual t.w.d. Fubini polynomials $F_m(\nu;\gamma|\lambda)$ in (11).*

Remark 3. *By letting $w = 1$ and $\lambda \to 0$ in (16), the polynomials $F_{m,w}(\nu;\gamma|\lambda)$ become the familiar two-variable Fubini polynomials $F_m(\nu;\gamma)$ in (4).*

Similar to (10) and (11), for $w \in \mathbb{N}$, the *f.p-a.i.* representations of the polynomials $F_{m,w}(\nu;\gamma|\lambda)$ and $F_{m,w}(\nu;\gamma|\lambda)$ are provided by

$$\sum_{m=0}^{\infty} F_{m,w}(\gamma|\lambda)\frac{t^m}{m!} = \frac{1}{1-\gamma^w(e_\lambda(t)-1)^w} = \frac{1}{2}\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^w\gamma^w)^\nu d\mu_{-1}(\nu) \tag{17}$$

and

$$\sum_{m=0}^{\infty} F_{m,w}(\nu;\gamma|\lambda)\frac{t^m}{m!} = \frac{e_\lambda^\nu(t)}{2}\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^w\gamma^w)^\nu d\mu_{-1}(\nu), \tag{18}$$

respectively. Using (14) and (15), it can be derived from (10) and (17) that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)\gamma)^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu)} &= \frac{1-\gamma^{w_1}(e_\lambda(t)-1)_1^w}{1-\gamma(e_\lambda(t)-1)} = \sum_{i=0}^{w_1-1}(e_\lambda(t)-1)^i\gamma^i \\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_1-1}\Delta^i(0)_{m,\lambda}\gamma^i\right)\frac{t^m}{m!}, (w_1 \in \mathbb{N}) \\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_1-1}S_{2,\lambda}(m,i)\gamma^i i!\right)\frac{t^m}{m!}.\end{aligned} \tag{19}$$

Theorem 1. *The following identity*

$$\begin{aligned}&\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_1-1}\gamma^{w_2 i}\Delta^{w_2 i}(0)_{u,\lambda}F_{m-u,w_1}(\gamma|\lambda) \\ =&\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_2-1}\gamma^{w_1 i}\Delta^{w_1 i}(0)_{u,\lambda}F_{m-u,w_2}(\gamma|\lambda)\end{aligned} \tag{20}$$

holds for $m \geq 0$, and w_1, w_2 are two odd numbers.

Proof. For $w_1, w_2 \in \mathbb{N}$, we consider

$$I = \frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^{\nu_1}d\mu_{-1}(\nu_1)\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^{\nu_2}d\mu_{-1}(\nu_2)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)}, \tag{21}$$

which is invariant under the interchange of w_1 and w_2. Then, using (21), we obtain

$$I = \int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu) \times \left(\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)}\right). \tag{22}$$

First, using (14) and (15), we observe that

$$\begin{aligned}
\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)} &= \frac{1-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2}}{1-(e_\lambda(t)-1)^{w_2}\gamma^{w_2}} \\
&= \sum_{i=0}^{w_1-1}(e_\lambda(t)-1)^{w_2i}\gamma^{w_2i} \\
&= \sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2i}\binom{w_2i}{l}(-1)^{w_2i-l}e_\lambda^l(t)\gamma^{w_2i} \\
&= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_1-1}\gamma^{w_2i}\Delta^{w_2i}(0)_{m,\lambda}\right)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_1-1}\gamma^{w_2i}(w_2i)!S_{2,\lambda}(m,w_2i)\right)\frac{t^m}{m!}.
\end{aligned} \tag{23}$$

It can be discovered from (22) and (23) that

$$\begin{aligned}
I &= \int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu)\times\left(\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)}\right) \\
&= \left(\sum_{m=0}^{\infty}2F_{m,w_1}(\gamma|\lambda)\frac{t^m}{m!}\right)\left(\sum_{u=0}^{\infty}\left(\sum_{i=0}^{w_1-1}\gamma^{w_2i}\Delta^{w_2i}(0)_{u,\lambda}\right)\frac{t^u}{u!}\right) \\
&= \sum_{m=0}^{\infty}\left(2\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_1-1}\gamma^{w_2i}\Delta^{w_2i}(0)_{u,\lambda}F_{m-u,w_1}(\gamma|\lambda)\right)\frac{t^m}{m!}.
\end{aligned} \tag{24}$$

Interchanging the roles of w_1 and w_2, using (21), we obtain

$$I=\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^\nu d\mu_{-1}(\nu)\times\left(\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)}\right). \tag{25}$$

Similar to the computations in (23), we obtain that

$$\frac{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p}(-(e_\lambda(t)-1)^{w_1w_2}\gamma^{w_1w_2})^\nu d\mu_{-1}(\nu)} = \sum_{m=0}^{\infty}\left(\sum_{i=0}^{w_2-1}\gamma^{w_1i}(w_1i)!S_{2,\lambda}(m,w_1i)\right)\frac{t^m}{m!}. \tag{26}$$

Utilizing similar computations in (24), we discover from (25) and (26) that

$$I = \sum_{m=0}^{\infty}\left(2\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_2-1}\gamma^{w_1i}\Delta^{w_1i}(0)_{u,\lambda}F_{m-u,w_2}(\gamma|\lambda)\right)\frac{t^m}{m!}. \tag{27}$$

So, the proof is completed as a result of the computations (24) and (27). □

Corollary 1. *By Theorem 1 and Equation (14), the following relation*

$$\begin{aligned}
&\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_1-1}\gamma^{w_2i}(w_2i)!S_{2,\lambda}(m,w_2i)F_{m-u,w_1}(\gamma|\lambda) \\
=&\sum_{u=0}^{m}\binom{m}{u}\sum_{i=0}^{w_2-1}\gamma^{w_1i}(w_1i)!S_{2,\lambda}(m,w_1i)F_{m-u,w_2}(\gamma|\lambda)
\end{aligned} \tag{28}$$

is true for $m \geq 0$, and w_1, w_2 being two odd numbers.

In particular, $w_1 = 1$ in (20) and (28), we obtain the formulas in Remarks 4 and 5.

Remark 4. *The following summation formula holds for $m \geq 0$:*

$$F_m(\gamma|\lambda) = \sum_{u=0}^{m} \binom{m}{u} \sum_{i=0}^{w_2-1} \gamma^i \Delta^i(0)_{u,\lambda} F_{m-u,w_2}(\gamma|\lambda). \tag{29}$$

Remark 5. *The following summation formula holds for $m \geq 0$:*

$$F_m(\gamma|\lambda) = \sum_{u=0}^{m} \binom{m}{u} \sum_{i=0}^{w_2-1} \gamma^i i! S_{2,\lambda}(m,i) F_{m-u,w_2}(\gamma|\lambda). \tag{30}$$

We give the following symmetric relation.

Theorem 2. *The following identity*

$$\begin{aligned} &\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} \gamma^{w_2 i} (-1)^l F_{m,w_1}(w_2 i - l, \gamma|\lambda) \\ = &\sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} \binom{w_1 i}{l} \gamma^{w_1 i} (-1)^l F_{m,w_2}(w_1 i - l, \gamma|\lambda) \end{aligned} \tag{31}$$

holds for $m \geq 0$, and w_1, w_2 being two odd numbers.

Proof. It can be computed from (21) that

$$\begin{aligned} I &= \int_{\mathbb{Z}_p} (-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu) \times \left(\frac{\int_{\mathbb{Z}_p} (-(e_\lambda(t)-1)^{w_2}\gamma^{w_2})^\nu d\mu_{-1}(\nu)}{\int_{\mathbb{Z}_p} (-(e_\lambda(t)-1)^{w_1 w_2}\gamma^{w_1 w_2})^\nu d\mu_{-1}(\nu)} \right) \\ &= \left(\int_{\mathbb{Z}_p} (-(e_\lambda(t)-1)^{w_1}\gamma^{w_1})^\nu d\mu_{-1}(\nu) \right) \times \left(\frac{1-(e_\lambda(t)-1)^{w_1 w_2}\gamma^{w_1 w_2}}{1-(e_\lambda(t)-1)^{w_2}\gamma^{w_2}} \right) \\ &= \left(\sum_{i=0}^{w_1-1} (e_\lambda(t)-1)^{w_2 i}\gamma^{w_2 i} \right) \times \left(\frac{2}{1-(e_\lambda(t)-1)^{w_1}\gamma^{w_1}} \right) \\ &= \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} \gamma^{w_2 i} (-1)^l \frac{2e_\lambda^{w_2 i - l}(t)}{1-(e_\lambda(t)-1)^{w_1}\gamma^{w_1}} \\ &= 2\sum_{m=0}^{\infty} \left(\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} \binom{w_2 i}{l} \gamma^{w_2 i} (-1)^l F_{m,w_1}(w_2 i - l, \gamma|\lambda) \right) \frac{t^m}{m!}. \end{aligned} \tag{32}$$

Similar to the computations in (32), by interchanging the roles of w_1 and w_2, we obtain from (24) that

$$I = 2\sum_{m=0}^{\infty} \left(\sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} \binom{w_1 i}{l} \gamma^{w_1 i} (-1)^l F_{m,w_2}(w_1 i - l, \gamma|\lambda) \right) \frac{t^m}{m!}. \tag{33}$$

So, the proof is completed as a result of the computations (32) and (33). □

On taking $w_1 = 1$ in (31), we obtain the following remark.

Remark 6. *The following summation formula holds for $m \geq 0$:*

$$F_m(\gamma|\lambda) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{i} \binom{i}{l} \gamma^i (-1)^l F_{m,w_2}(i-l, \gamma|\lambda). \tag{34}$$

Now we give the following proposition.

Proposition 1. *We have the following correlation:*

$$F_{n,w}(\nu;\gamma|\lambda)=\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}(kw)!\gamma^{kw}(v)_{n-j,\lambda}S_{2,\lambda}(j,kw). \tag{35}$$

Proof. We observe from (7) and (16) that

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,w}(\nu;\gamma|\lambda)\frac{t^n}{n!} &= \frac{1}{1-\gamma^{w}(e_\lambda(t)-1)^{w}}e_\lambda^{\nu}(t)\\ &= \sum_{k=0}^{\infty}\gamma^{kw}e_\lambda^{\nu}(t)(e_\lambda(t)-1)^{kw}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\gamma^{kw}(v)_{n,\lambda}\frac{t^n}{n!}(kw)!\sum_{j=kw}^{\infty}S_{2,\lambda}(j,kw)\frac{t^j}{j!}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}(kw)!\gamma^{kw}(v)_{n-j,\lambda}S_{2,\lambda}(j,kw)\frac{t^n}{n!},\end{aligned}$$

which completes the proof of the theorem. □

Corollary 2. *By Theorem 2 and Equation (35), the following relation*

$$\begin{aligned}&\sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2 i}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{w_2 i}{l}\binom{m}{j}\gamma^{kw_1+w_2 i}\\ &\times(-1)^l(kw_1)!(w_2 i-l)_{m-j,\lambda}S_{2,\lambda}(j,kw_1)\\ =&\sum_{i=0}^{w_2-1}\sum_{l=0}^{w_1 i}\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{w_1 i}{l}\binom{m}{j}\gamma^{kw_2+w_1 i}\\ &\times(-1)^l(kw_2)!(w_1 i-l)_{m-j,\lambda}S_{2,\lambda}(j,kw_2)\end{aligned}$$

is true for $m\geq 0$, and w_1,w_2 are two odd numbers.

3. Further Remarks

We now aim to derive more symmetric identities for the *t.w.d.* w-torsion Fubini polynomials. Here are some symmetric identities for *t.w.d.* w-torsion Fubini polynomials utilizing some series manipulation methods.

Theorem 3. *The polynomials $F_{m,w}(\nu;\gamma|\lambda)$ fulfill the following identity for $m\in\mathbb{Z}_{\geq 0}$ and $a,b\in\mathbb{R}$ and $m\geq 0$:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}F_{m-k,w}(b\nu;\gamma|b\lambda)\,F_{k,w}(a\nu;\gamma|a\lambda)b^k a^{m-k}\\ =&\sum_{k=0}^{m}\binom{m}{k}F_{m-k,w}(a\nu;\gamma|a\lambda)F_{k,w}(b\nu;\gamma|b\lambda)a^k b^{m-k}.\end{aligned} \tag{36}$$

Proof. We choose that

$$Y=\frac{e_\lambda^{2\nu}(abt)}{(1-(e_{b\lambda}(at)-1)^{w}\gamma^{w})(1-(e_{a\lambda}(bt)-1)^{w}\gamma^{w})},$$

which is symmetric in a and b. We compute from (17) that

$$Y = \left[\sum_{m=0}^{\infty}F_{m,w}(b\nu;\gamma|b\lambda)\frac{(at)^m}{m!}\right]\left[\sum_{m=0}^{\infty}F_{m,w}(a\nu;\gamma|a\lambda)\frac{(bt)^m}{m!}\right]$$

$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}F_{m-k,w}(b\nu;\gamma|b\lambda)\,F_{k,w}(a\nu;\gamma|a\lambda)a^{m-k}b^{k}\right)\frac{t^m}{m!}$$

and in the same way

$$\Upsilon = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}F_{m-k,w}(a\nu;\gamma|a\lambda)F_{k,w}(b\nu;\gamma|b\lambda)a^{k}b^{m-k}\right)\frac{t^m}{m!},$$

which means the assertion (36). □

Here is another symmetric identity for $F_{m,w}(\nu;\gamma|\lambda)$ as follows.

Theorem 4. *The polynomials $F_{m,w}(\nu;\gamma|\lambda)$ fulfill the following identity for $a,b\in\mathbb{R}$ and $m\geq 0$:*

$$\begin{aligned}
&\sum_{k=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\sum_{l=0}^{iw}\sum_{s=0}^{jw}\binom{iw}{l}\binom{m}{k}\binom{jw}{s}\gamma^{w(i+j)}(-1)^{(i+j)w-l-s}\\
&\times F_{m-k,w}(l+s\frac{b}{a}+b\nu_1;\gamma|b\lambda)F_{k,w}(a\nu_2;\gamma|a\lambda)b^{k}a^{m-k}\\
=&\sum_{k=0}^{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{l=0}^{iw}\sum_{s=0}^{jw}\binom{iw}{l}\binom{m}{k}\binom{jw}{s}\gamma^{w(i+j)}(-1)^{(i+j)w-l-s}\\
&F_{k,w}(l+s\frac{a}{b}+a\nu_1;\gamma|a\lambda)F_{m-k,w}(b\nu_2;\gamma|b\lambda)a^{k}b^{m-k}.
\end{aligned}$$

Proof. We consider by (17) that

$$\begin{aligned}
\Psi &= \frac{1-(e_\lambda^{1/b}(abt)-1)^{aw}\gamma^{aw}}{(1-(e_{b\lambda}(at)-1)^{w}\gamma^{w})^2}\frac{1-(e_\lambda^{1/a}(abt)-1)^{bw}\gamma^{bw}}{(1-(e_{a\lambda}(bt)-1)^{w}\gamma^{w})^2}e_\lambda^{\nu_1+\nu_2}(abt)\\
&= \frac{e_\lambda^{\nu_1}(abt)}{1-(e_{b\lambda}(at)-1)^{w}\gamma^{w}}\frac{1-(e_\lambda^{1/b}(abt)-1)^{aw}\gamma^{aw}}{1-(e_{b\lambda}(at)-1)^{w}\gamma^{w}}\\
&\quad\times\frac{e_\lambda^{\nu_2}(abt)}{1-(e_{a\lambda}(bt)-1)^{w}\gamma^{w}}\frac{1-(e_\lambda^{1/a}(abt)-1)^{bw}\gamma^{bw}}{1-(e_{a\lambda}(bt)-1)^{w}\gamma^{w}}.
\end{aligned}$$

Therefore, we compute that

$$\begin{aligned}
\Psi &= \frac{e_\lambda^{\nu_1}(abt)}{1-\gamma^{w}(e_{b\lambda}(at)-1)^{w}}\frac{1-\gamma^{aw}(e_\lambda^{1/b}(abt)-1)^{aw}}{1-\gamma^{w}(e_{b\lambda}(at)-1)^{w}}\\
&\quad\times\frac{e_\lambda^{\nu_2}(abt)}{1-\gamma^{w}(e_{a\lambda}(bt)-1)^{w}}\frac{1-\gamma^{bw}(e_\lambda^{1/a}(abt)-1)^{bw}}{1-\gamma^{w}(e_{a\lambda}(bt)-1)^{w}}\\
&= \frac{e_\lambda^{\nu_1}(abt)}{1-\gamma^{w}(e_{b\lambda}(at)-1)^{w}}\sum_{i=0}^{b-1}\gamma^{iw}(e_{b\lambda}(at)-1)^{iw}\\
&\quad\times\frac{e_\lambda^{\nu_2}(abt)}{1-\gamma^{w}(e_{a\lambda}(bt)-1)^{w}}\sum_{j=0}^{a-1}\gamma^{jw}(e_{a\lambda}(bt)-1)^{jw}\\
&= \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\gamma^{w(i+j)}\frac{1}{1-\gamma^{w}(e_{b\lambda}(at)-1)^{w}}\sum_{l=0}^{iw}\binom{iw}{l}(-1)^{iw-l}e_{b\lambda}^{l+b\nu_1}(at)\\
&\quad\times\frac{1}{1-(e_{a\lambda}(bt)-1)^{w}\gamma^{w}}\sum_{s=0}^{jw}\binom{jw}{s}(-1)^{jw-s}e_{a\lambda}^{s+a\nu_2}(bt)\\
&= \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\sum_{l=0}^{iw}\sum_{s=0}^{jw}\binom{iw}{l}\gamma^{w(i+j)}\binom{jw}{s}(-1)^{(i+j)w-l-s}
\end{aligned}$$

$$\times \frac{e_{b\lambda}^{l+s\frac{b}{a}+b\nu_1}(at)}{1-(e_{b\lambda}(at)-1)^w\gamma^w}\frac{e_{a\lambda}^{a\nu_2}(bt)}{1-(e_{a\lambda}(bt)-1)^w\gamma^w}$$
$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\sum_{l=0}^{iw}\sum_{s=0}^{jw}\binom{m}{k}\binom{iw}{l}\gamma^{w(i+j)}\binom{jw}{s}\right.$$
$$\left.(-1)^{(i+j)w-l-s}F_{m-k,w}(l+s\frac{b}{a}+b\nu_1;\gamma|b\lambda)F_{k,w}(a\nu_2;\gamma|a\lambda)b^k a^{m-k}\right)\frac{t^m}{m!},$$

and similarly,

$$\Psi = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{l=0}^{iw}\sum_{s=0}^{jw}\binom{m}{k}\binom{iw}{l}\gamma^{w(i+j)}\binom{jw}{s}\right.$$
$$\left.(-1)^{(i+j)w-l-s}F_{k,w}(l+s\frac{a}{b}+a\nu_1;\gamma|a\lambda)F_{m-k,w}(b\nu_2;\gamma|b\lambda)a^k b^{m-k}\right)\frac{t^m}{m!},$$

which yields the claimed symmetric identity in the theorem. □

4. Conclusions

After constructions of p-adic q-integrals by Teakyun Kim, a Korean mathematician, in recent years, p-adic q-integrals with some of their special cases have not only been utilized as integral representations of many special polynomials and functions but also have given the chance to deeply analyze many families of special polynomials and numbers, such as Bernoulli, Fubini, Bell and Changhee polynomials and numbers. In the presented work, we focused on a novel extension of the degenerate Fubini polynomials. We first defined the *t.w.d.* w-torsion Fubini polynomials by means of their exponential generating function. We then discovered a *f.p-a.i.* representation of the degenerate w-torsion Fubini polynomials, by which we attained diverse novel symmetric identities for the degenerate w-torsion Fubini and *t.w.d.* w-torsion Fubini polynomials. Finally, by using some series manipulation methods, we acquired more identities of symmetry for properties of the *t.w.d.* w-torsion Fubini polynomials. To the best of our knowledge, the results presented in this paper are new and do not seem to be reported in the literature. In general, these results have the potential to be used in many branches of mathematics, probability, statistics, mathematical physics, and engineering.

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