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Relation-Preserving Functional Contractions Involving a Triplet of Auxiliary Functions with an Application to Integral Equations

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Abstract: This article addresses certain fixed-point results in a metric space equipped with a locally transitive binary relation under a functional contraction containing three auxiliary functions. The findings proved herein enrich and improve a number of existing results. In order to prove the credibility of our findings, an illustrative example is provided. Making use of our findings, we study the genuineness of the unique solution to a Fredholm integral equation.

Keywords: Fredholm integral equations; finitely transitive relation; fixed point

MSC: 47H10; 45B05; 06A75



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1. Introduction

At present, nonlinear analysis is one of the most rapidly developing areas in mathematics. From a theoretical perspective, there are numerous approaches to handling various problems that emerge in real-world situations. Owing to their potential applications, various disciplines within nonlinear analysis have drawn significant attention in recent years. In this context, we mainly refer the most recent findings of Rad et al. [1], Ou et al. [2], and Cui et al. [3]. The nature of nonlinear analysis allows for generalizing, expanding, and obtaining the most general forms of existing concepts and results.

One of the most attractive disciplines in the domain of nonlinear analysis is fixed-point theory. The motive of this discipline is to study the existence, uniqueness, and characteristics of solutions to equations of the type $\mathcal{S}(p) = p$, where $\mathcal{S}$ is a known self-map. In mathematics, this deceptively easy equation has significant ramifications as it describes a state of equilibrium and stability and solutions to a variety of differential equations, matrix equations, integral equations, approximation theory, variational inequalities, operator equations, and functional equations and the existence of invariant subspaces of linear operators, etc. Fixed-point theory offers data and techniques needed to thoroughly examine and comprehend the existence, features, and initiatives of these special points. On the other hand, fixed-point theory provides essential concepts and approaches for addressing a variety of real-world problems that can be reduced to equivalent fixed-point problems. These problems occur in different branches of the social and natural sciences which include statistics, optimization, computer science, chemistry, biology, physics, engineering, medicine, image recovery, economics, game theory, global analysis, control theory, and fractal theory.

The classical BCP forms serve as the basis of fixed-point theory; it was established by a Polish mathematician, Banach [4], in 1922 and states the following:

Theorem 1. (BCP) Suppose that $(\mathbb{P}, \sigma)$ is a CMS, and $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map. If $\kappa \in [0, 1)$ is a constant such that

$$\sigma(\mathcal{S}p, \mathcal{S}q) \leq \kappa \cdot \sigma(p, q), \quad \forall p, q,$$

then $\mathcal{S}$ owns a unique fixed point.

The map utilized in the BCP is referred to as a *contraction map*. The proof of BCP reveals that the unique fixed point can be determined as the limit of an iteration algorithm formed by successive compositions of the map under the image of an initial point in the CMS. For these reasons, the BCP can be employed to calculate the fixed point numerically as it is a constructive fixed-point theorem. The BCP is critical to the advancement of novel approaches to determining the solutions of a variety of equations, including algebraic equations, boundary value problems, matrix equations, Volterra and Fredholm integral equations, and nonlinear integro-differential equations, and in proving the convergence of algorithms in computational mathematics.

In the past, various authors generalized the BCP by enlarging a class of contraction mappings that use appropriate gauge functions. In this direction, Browder [5] introduced the idea of ϕ -contractions depending on a gauge function, $\phi : [0, \infty) \rightarrow [0, \infty)$, adopted in place of a contraction constant. Later, this outcome was developed by Boyd and Wong [6].

Theorem 2 ([6]). Suppose that $(\mathbb{P}, \sigma)$ is a CMS and $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map. If $\phi : [0, \infty) \rightarrow [0, \infty)$ is an upper-right semiconscious function which verifies $\phi(s) < s$ for all $s > 0$ such that

$$\sigma(\mathcal{S}p, \mathcal{S}q) \leq \phi(\sigma(p, q)), \quad \forall p, q,$$

then $\mathcal{S}$ owns a unique fixed point.

On the other hand, Matkowski [7] improved the fixed point theorem of Browder [5] in the following way:

Theorem 3 ([5]). Suppose that $(\mathbb{P}, \sigma)$ is a CMS, and $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map. If $\phi : [0, \infty) \rightarrow [0, \infty)$ is an increasing function which verifies $\lim_{n \rightarrow \infty} \phi^n(s) = 0$ for all $s > 0$ such that

$$\sigma(\mathcal{S}p, \mathcal{S}q) \leq \phi(\sigma(p, q)), \quad \forall p, q,$$

then $\mathcal{S}$ owns a unique fixed point.

With the limitation $\phi(s) = \kappa s$, $\kappa \in [0, 1)$, a ϕ -contraction reduces to a contraction; consequently, Theorem 2 (Theorem 3 also) reduces to Theorem 1. Dutta and Choudhury [8] initiated the idea of (φ, ψ) -contractions employing a pair of auxiliary functions. Alam et al. [9] improved the class of (φ, ψ) -contractions and used it to prove a variant of the BCP as follows:

Theorem 4 ([9]). Suppose that $(\mathbb{P}, \sigma)$ is a CMS and $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map. If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a right-continuous and increasing function and $\psi : [0, \infty) \rightarrow [0, \infty)$ is a function satisfying $\psi(s) > 0$ for every $s > 0$ and $\liminf_{s \rightarrow r} \psi(s) > 0$ for every $r > 0$ such that

$$\varphi(\sigma(\mathcal{S}p, \mathcal{S}q)) \leq \varphi(\sigma(p, q)) - \psi(\sigma(p, q)), \quad \forall p, q,$$

then $\mathcal{S}$ owns a unique fixed point.

In 2015, Alam and Imdad [10] initiated a relation-theoretic formulation of the BCP which has attracted the attention of numerous researchers, e.g., Refs. [11–19]. In order to ascertain the existence of fixed points for a nonlinear contraction, the transitivity of the underlying relation is also required. Since the transitivity requirement is very restrictive, with a view toward employing an optimal condition of transitivity, several authors (e.g., Alam et al. [13], Sk et al. [19], Turinici [20]) used locally finitely transitive relations. A primary feature of relational contractions is that the contraction inequality is only required to hold for comparative elements, avoiding any element pairing. Consequently, relational

contractions remain weak in comparison to their corresponding ordinary contractions; hence, they are applicable for solving of boundary value problems and integral equations, whereas findings regarding the fixed points of abstract metric space are not employed.

Many variants of the BCP involve more comprehensive contraction circumstances in an (ambient) metric space $(\mathbb{P}, \sigma)$ containing the displacement $\sigma(p, q)$ (where $p, q \in \mathbb{P}$) on the R.H.S. On the other hand, various contraction conditions subsume $\sigma(p, q)$ along with the displacements of $p, q \in \mathbb{P}$ under the map $\mathcal{S}$: $\sigma(p, \mathcal{S}p), \sigma(q, \mathcal{S}q), \sigma(p, \mathcal{S}q), \sigma(q, \mathcal{S}p)$. These types of contractions are referred to as “functional contractions”. We express such a contraction as follows:

$$\sigma(\mathcal{S}p, \mathcal{S}q) \leq \mathcal{F}(\sigma(p, q), \sigma(p, \mathcal{S}p), \sigma(q, \mathcal{S}q), \sigma(p, \mathcal{S}q), \sigma(q, \mathcal{S}p)), \quad \forall p, q \in \mathbb{P},$$

for an adequate selection of the function $\mathcal{F} : [0, \infty)^5 \rightarrow [0, \infty)$. In this context, Ansari et al. [17] investigated the fixed point of a relational functional contraction involving two gauge functions, identifying a few findings, and implemented the same to solve certain nonlinear integral equations.

The focus of this work is to extend the recent results of Ansari et al. [17], employing relatively more generalized contractivity conditions with three auxiliary functions, φ , ψ , and θ . To demonstrate our findings, we deliver an example and a possible application to Fredholm integral equations. Our findings also generalize the corresponding results of Altaweel and Khan [18], Sk et al. [19], and Jleli et al. [21].

2. Preliminaries

As usual, we denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a set $\mathbb{P}$, any subset of $\mathbb{P}^2$ is called a BR. In the following definitions, we presume that $\mathbb{P}$ is a set, $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map, σ is a metric on $\mathbb{P}$, and Λ is a BR on $\mathbb{P}$.

Definition 1 ([10]). Two elements, $p, q \in \mathbb{P}$, are termed Λ -comparative and indicated as $[p, q] \in \Lambda$ if either $(p, q) \in \Lambda$ or $(q, p) \in \Lambda$.

Definition 2 ([22]). $\Lambda^{-1} := \{(p, q) \in \mathbb{P}^2 : (q, p) \in \Lambda\}$ is termed the inverse of Λ .

Definition 3 ([22]). $\Lambda^s := \Lambda \cup \Lambda^{-1}$ is termed the symmetric closure of Λ .

Clearly, $(p, q) \in \Lambda^s$ iff $[p, q] \in \Lambda$.

Definition 4 ([10]). Λ is named $\mathcal{S}$ -closed provided that

$$(p, q) \in \Lambda \Rightarrow (\mathcal{S}p, \mathcal{S}q) \in \Lambda.$$

Proposition 1 ([12]). For every $n \in \mathbb{N}$, Λ is $\mathcal{S}^n$ -closed provided it is $\mathcal{S}$ -closed.

Definition 5 ([10]). Any sequence $\{p_n\} \subset \mathbb{P}$ with the property $(p_n, p_{n+1}) \in \Lambda$ for all $n \in \mathbb{N}_0$ is termed Λ -preserving.

Definition 6 ([23]). The triplet $(\mathbb{P}, \sigma, \Lambda)$ is called a graph metric space.

Definition 7 ([11]). A MS $(\mathbb{P}, \sigma)$ is named Λ -CMS if any Cauchy sequence in $\mathbb{P}$ converges, provided it is Λ -preserving.

Definition 8 ([11]). $\mathcal{S}$ is named Λ -continuous if for any Λ -preserving sequence $\{p_n\} \subset \mathbb{P}$, and for any $p \in \mathbb{P}$ enjoying $p_n \xrightarrow{\sigma} p$, we have

$$\mathcal{S}(p_n) \xrightarrow{\sigma} \mathcal{S}(p).$$

Note that under a universal BR (i.e., $\Lambda = \mathbb{P}^2$), the notions of Λ -completeness and Λ -continuity reduce to completeness and continuity, respectively.

Definition 9 ([10]). Λ is called a σ -self-closed BR if every Λ -preserving convergent sequence in $\mathbb{P}$ has a subsequence in which each term is Λ -comparative with the limit.

Definition 10 ([24]). Any subset $\mathbb{Q} \subset \mathbb{P}$ is referred to as Λ -directed if for all $p, q \in \mathbb{Q}$ there exists $s \in \mathbb{P}$ such that $(p, s) \in \Lambda$ and $(q, s) \in \Lambda$.

Definition 11 ([22]). Given $\mathbb{Q} \subseteq \mathbb{P}$, the set

$$\Lambda|_{\mathbb{Q}} := \Lambda \cap \mathbb{Q}^2.$$

being a BR on $\mathbb{Q}$ is referred to as the restriction of Λ on $\mathbb{Q}$.

Definition 12 ([25]). Let $\tau \in \mathbb{N} - \{1\}$. Λ is named τ -transitive if for every $p_0, p_1, \dots, p_\tau \in \mathbb{P}$, the following holds:

$$(p_{i-1}, p_i) \in \Lambda \text{ for each } i (1 \leq i \leq \tau) \Rightarrow (p_0, p_\tau) \in \Lambda.$$

In the meantime, 2-transitivity equates to transitivity.

Definition 13 ([20]). Λ is named finitely transitive if $\exists \tau \in \mathbb{N} - \{1\}$ for which Λ is τ -transitive.

Definition 14 ([13]). Λ is named locally finitely $\mathcal{S}$ -transitive if for any Λ -preserving sequence $\{p_{n_k}\} \subset \mathcal{S}(\mathbb{P})$ with the range $\mathbb{Q} = \{p_n : n \in \mathbb{N}\}$ the restriction $\Lambda|_{\mathbb{Q}}$ is finitely transitive.

Clearly, transitivity $\implies$ finitely transitivity $\implies$ locally finitely $\mathcal{S}$ -transitivity.

Lemma 1 ([25]). Every non-Cauchy sequence $\{p_n\} \subset \mathbb{P}$ provides the availability of $\epsilon > 0$ and two subsequences $\{p_{n_k}\}$ and $\{p_{m_k}\}$ of $\{p_n\}$, which satisfy

- (i) $\kappa \leq m_k < n_k$, for all $\kappa \in \mathbb{N}$;
- (ii) $\sigma(p_{m_k}, p_{n_k}) \geq \epsilon$, for all $\kappa \in \mathbb{N}$;
- (iii) $\sigma(p_{m_k}, p_{l_k}) < \epsilon$, for all $l_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$.

Moreover, if $\lim_{n \rightarrow \infty} \sigma(p_n, p_{n+1}) = 0$, then $\lim_{\kappa \rightarrow +\infty} \sigma(p_{m_k}, p_{n_k+p}) = \epsilon$, for all $p \in \mathbb{N}_0$.

Lemma 2 ([20]). Let $\mathbb{P}$ be a set endowed with a BR Λ . If $\{p_n\} \subset \mathbb{P}$ is a Λ -preserving sequence and Λ is a τ -transitive on $\mathbb{Q} = \{p_n : n \in \mathbb{N}_0\}$, then

$$(p_n, p_{n+1+k(\tau-1)}) \in \Lambda, \text{ for all } n, k \in \mathbb{N}_0.$$

In what follows, Φ embodies the class of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following

Φ_1 : φ is right-continuous;

Φ_2 : φ is increasing.

Ψ embodies the class of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying

Ψ_1 : $\psi(s) > 0$ for every $s > 0$;

Ψ_2 : $\liminf_{s \rightarrow r} \psi(s) > 0$ for every $r > 0$.

Θ embodies the class of functions $\theta : [0, \infty)^4 \rightarrow [0, \infty)$ satisfying

Θ_1 : θ is continuous;

Θ_2 : $\theta(s_1, s_2, s_3, s_4) = 0 \Leftrightarrow s_1 s_2 s_3 s_4 = 0$.

The first two families (Φ and Ψ) were investigated by Alam et al. [9], while the Θ family was suggested by Jleli et al. [21]. The symmetrical axiom of σ yields the following conclusion:

Proposition 2. Suppose that $(\mathbb{P}, \sigma, \Lambda)$ is a graph MS, and $S : \mathbb{P} \rightarrow \mathbb{P}$ is a map. Then, for any $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$, the following are identical:

$$(I) \quad \varphi(\sigma(Sp, Sq)) \leq \varphi(\sigma(p, q)) - \psi(\sigma(p, q)) + \theta(\sigma(p, Sp), \sigma(q, Sq), \sigma(p, Sq), \sigma(q, Sp)),$$

for all $(p, q) \in \Lambda$,

$$(II) \quad \varphi(\sigma(Sp, Sq)) \leq \varphi(\sigma(p, q)) - \psi(\sigma(p, q)) + \theta(\sigma(p, Sp), \sigma(q, Sq), \sigma(p, Sq), \sigma(q, Sp)),$$

for all $[p, q] \in \Lambda$.

Proposition 3 ([9]). If there exist $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$, where $\phi \in \Phi$ and $\psi \in \Psi$ such that for any $t \in [0, \infty)$ and $s \in (0, \infty)$, $\phi(t) \leq \phi(s) - \psi(s)$, then $t < s$.

The following annotations will be deployed in this text.

- $Fix(S)$:= the set of fixed points of S ;
- $\mathbb{P}(S, \Lambda) := \{p \in \mathbb{P} : (p, Sp) \in \Lambda\}$.

3. Main Results

We now present our fixed-point theorems in relational MS.

Theorem 5. Suppose that $(\mathbb{P}, \sigma, \Lambda)$ is a graph MS, and $S : \mathbb{P} \rightarrow \mathbb{P}$ is a map. Also,

- (i) $(\mathbb{P}, \sigma)$ is Λ -CMS;
- (ii) $\mathbb{P}(S, \Lambda)$ is nonempty;
- (iii) Λ is S -closed and locally finitely S -transitive;
- (iv) S serves as Λ -continuous or Λ serves as σ -self-closed;
- (v) there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying

$$\varphi(\sigma(Sp, Sq)) \leq \varphi(\sigma(p, q)) - \psi(\sigma(p, q)) + \theta(\sigma(p, Sp), \sigma(q, Sq), \sigma(p, Sq), \sigma(q, Sp)),$$

for all $(p, q) \in \Lambda$.

Then S owns at least one fixed point.

Proof. Employing (ii), select $p_0 \in \mathbb{P}(S, \Lambda)$. We have $(p_0, Sp_0) \in \Lambda$. Set

$$p_n := S^n(p_0) = S(p_{n-1}), \quad \forall n \in \mathbb{N}. \quad (1)$$

Using the fact $(p_0, Sp_0) \in \Lambda$ in combination with (ii) and Proposition 1, we obtain

$$(S^n p_0, S^{n+1} p_0) \in \Lambda,$$

which, in view of (1), becomes

$$(p_n, p_{n+1}) \in \Lambda, \quad \forall n \in \mathbb{N}_0. \quad (2)$$

This shows that $\{p_n\}$ is Λ -preserving. If $\sigma(p_{n_0}, p_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then from (1), we obtain $p_{n_0} \in Fix(S)$; hence, this concludes the proof. Otherwise, assume $\sigma_n := \sigma(p_n, p_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. Using (1) and (v), we find

$$\begin{aligned} \varphi(\sigma(p_n, p_{n+1})) &= \varphi(\sigma(Sp_{n-1}, Sp_n)) \\ &\leq \varphi(\sigma(p_{n-1}, p_n)) - \psi(\sigma(p_{n-1}, p_n)) \\ &\quad + \theta(\sigma(Sp_n, p_n), \sigma(Sp_{n-1}, p_{n-1}), \sigma(Sp_{n-1}, p_n), \sigma(Sp_n, p_{n-1})), \end{aligned}$$

i.e.,

$$\varphi(\sigma_n) \leq \varphi(\sigma_{n-1}) - \psi(\sigma_{n-1}) + \theta(\sigma_n, \sigma_{n-1}, 0, \sigma(p_{n-1}, p_{n+1})).$$

By Θ_2 , we obtain

$$\varphi(\sigma_n) \leq \varphi(\sigma_{n-1}) - \psi(\sigma_{n-1}), \quad \forall n \in \mathbb{N} \quad (3)$$

so that

$$\varphi(\sigma_n) \leq \varphi(\sigma_{n-1}), \quad \forall n \in \mathbb{N}.$$

By Proposition 3, we find $\sigma_n < \sigma_{n-1}$ for all $n \in \mathbb{N}$.

Thus, the sequence $\{\sigma_n\}$ continues to decrease. As $\{\sigma_n\}$ is also bounded below, there exists $p \geq 0$ such that

$$\lim_{n \rightarrow \infty} \sigma_n = r. \quad (4)$$

Let $r > 0$. Employing the upper limit in (3), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \varphi(\sigma_n) &\leq \limsup_{n \rightarrow \infty} \varphi(\sigma_{n-1}) + \limsup_{n \rightarrow \infty} [-\psi(\sigma_{n-1})] \\ &\leq \limsup_{n \rightarrow \infty} \varphi(\sigma_{n-1}) - \liminf_{n \rightarrow \infty} \psi(\sigma_{n-1}). \end{aligned}$$

Employing the right continuity of φ , we obtain

$$\varphi(r) \leq \varphi(r) - \liminf_{n \rightarrow \infty} \psi(\sigma_{n-1})$$

so that

$$\liminf_{s \rightarrow r, s > 0} \psi(s) = \liminf_{n \rightarrow \infty} \psi(\sigma_{n-1}) \leq 0$$

which contradicts Ψ_2 . Hence, we conclude that

$$\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} \sigma(p_n, p_{n+1}) = 0. \quad (5)$$

If $\{p_n\}$ is not a Cauchy sequence, then Lemma 1 makes certain the availability of a constant $\epsilon > 0$ and two subsequences, $\{p_{n_k}\}$ and $\{p_{m_k}\}$ of $\{p_n\}$, which verify $\kappa \leq m_k < n_k$, $\sigma(p_{m_k}, p_{n_k}) \geq \epsilon$ and $\sigma(p_{m_k}, p_{l_k}) < \epsilon$, where $l_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$. Again, from (5), we have

$$\lim_{n \rightarrow +\infty} \sigma(p_{m_k}, p_{n_k+p}) = \epsilon \quad \text{for all } p \in \mathbb{N}_0. \quad (6)$$

As $\{p_n\} \subset \mathcal{S}(\mathbb{P})$, the range $\mathbb{Q} = \{p_n : n \in \mathbb{N}_0\}$ is a denumerable subset of $\mathcal{S}(\mathbb{P})$. Via the local finite $\mathcal{S}$ -transitivity of Λ , we can find a natural number $\tau = \tau(\mathbb{Q}) \geq 2$, verifying that $\Lambda|_{\mathbb{Q}}$ is τ -transitive. Since $m_k < n_k$ and $\tau - 1 > 0$, by a division algorithm, we can conclude that

$$\begin{aligned} n_k - m_k &= (\tau - 1)(\alpha_k - 1) + (\tau - \beta_k), \\ \alpha_k - 1 &\geq 0, 0 \leq \tau - \beta_k < \tau - 1 \end{aligned}$$

which is equivalent to

$$\begin{cases} n_k + \beta_k = m_k + 1 + (\tau - 1)\alpha_k, \\ \alpha_k \geq 1, 1 < \beta_k \leq \tau. \end{cases}$$

Naturally, $\beta_k \in (1, \tau]$; so, without a loss of generality, we may consider the subsequences $\{p_{n_k}\}$ and $\{p_{m_k}\}$ of $\{p_n\}$ (enjoying (6)) for which β_k remains constant, such as β . Write

$$m'_k = n_k + \beta = m_k + 1 + (\tau - 1)\alpha_k, \quad (7)$$

By (6) and (7), we obtain

$$\lim_{\kappa \rightarrow +\infty} \sigma(p_{m_\kappa}, p_{m'_\kappa}) = \lim_{\kappa \rightarrow +\infty} \sigma(p_{m_\kappa}, p_{n_\kappa + \beta}) = \epsilon. \quad (8)$$

Making use of the triangular inequality, we obtain

$$\sigma(p_{m_\kappa+1}, p_{m'_\kappa+1}) \leq \sigma(p_{m_\kappa+1}, p_{m_\kappa}) + \sigma(p_{m_\kappa}, p_{m'_\kappa}) + \sigma(p_{m'_\kappa}, p_{m'_\kappa+1}) \quad (9)$$

and

$$\sigma(p_{m_\kappa}, p_{m'_\kappa}) \leq \sigma(p_{m_\kappa}, p_{m_\kappa+1}) + \sigma(p_{m_\kappa+1}, p_{m'_\kappa+1}) + \sigma(p_{m'_\kappa+1}, p_{m'_\kappa})$$

or

$$\sigma(p_{m_\kappa}, p_{m'_\kappa}) - \sigma(p_{m_\kappa}, p_{m_\kappa+1}) - \sigma(p_{m'_\kappa+1}, p_{m'_\kappa}) \leq \sigma(p_{m_\kappa+1}, p_{m'_\kappa+1}). \quad (10)$$

Employing $\kappa \rightarrow \infty$ in (9) and (10) and by (5) and (8), we obtain

$$\lim_{\kappa \rightarrow +\infty} \sigma(p_{m_\kappa+1}, p_{m'_\kappa+1}) = \epsilon. \quad (11)$$

Using (7) and Lemma 2, we have $\sigma(p_{m_\kappa}, p_{m'_\kappa}) \in \Lambda$. Hence, by assumption (v), we find

$$\begin{aligned} \varphi(\sigma(p_{m_\kappa+1}, p_{n_\kappa+1})) &= \varphi(\sigma(\mathcal{S}p_{m_\kappa}, \mathcal{S}p_{n_\kappa})) \\ &\leq \varphi(\sigma(p_{m_\kappa}, p_{n_\kappa})) - \psi(\sigma(p_{m_\kappa}, p_{n_\kappa})) \\ &\quad + \theta(\sigma(p_{m_\kappa}, \mathcal{S}p_{m_\kappa}), \sigma(p_{n_\kappa}, \mathcal{S}p_{n_\kappa}), \sigma(p_{m_\kappa}, \mathcal{S}p_{n_\kappa}), \sigma(p_{n_\kappa}, \mathcal{S}p_{m_\kappa})) \\ &= \varphi(\sigma(p_{m_\kappa}, p_{n_\kappa})) - \psi(\sigma(p_{m_\kappa}, p_{n_\kappa})) \\ &\quad + \theta(\sigma(p_{m_\kappa}, p_{m_\kappa+1}), \sigma(p_{n_\kappa}, p_{n_\kappa+1}), \sigma(p_{m_\kappa}, p_{n_\kappa+1}), \sigma(p_{n_\kappa}, p_{m_\kappa+1})). \end{aligned}$$

Thus, for each $\kappa \in \mathbb{N}_0$, we have

$$\begin{aligned} \varphi(\sigma(p_{m_\kappa+1}, p_{n_\kappa+1})) &\leq \varphi(\sigma(p_{m_\kappa}, p_{n_\kappa})) - \psi(\sigma(p_{m_\kappa}, p_{n_\kappa})) \\ &\quad + \theta(\sigma(p_{m_\kappa}, \mathcal{S}p_{m_\kappa}), \sigma(p_{n_\kappa}, \mathcal{S}p_{n_\kappa}), \sigma(p_{m_\kappa}, \mathcal{S}p_{n_\kappa}), \sigma(p_{n_\kappa}, \mathcal{S}p_{m_\kappa})). \end{aligned}$$

Employing upper limit in the above inequality and using properties of φ , ψ , and θ , we find

$$\begin{aligned} \varphi(\epsilon) &\leq \varphi(\epsilon) - \lim_{q \rightarrow \epsilon^+} \psi(q) + \theta(0, 0, \epsilon, \epsilon) \\ &= \varphi(\epsilon) - \lim_{q \rightarrow \epsilon^+} \psi(q) < \varphi(\epsilon), \end{aligned}$$

thereby yielding

$$\liminf_{\kappa \rightarrow \infty} \psi(\sigma(p_{m_\kappa}, p_{m'_\kappa})) \leq 0,$$

which creates a contradiction. Therefore, $\{p_n\}$ is a Cauchy sequence. As $\{p_n\}$ is also Λ -preserving and $\mathbb{P}$ is a Λ -CMS, there exists $p \in \mathbb{P}$ such that $p_n \xrightarrow{\sigma} p$.

Finally, we will verify that $p \in \mathcal{S}(\mathbb{P})$. If $\mathcal{S}$ is Λ -continuous, then we have $\mathcal{S}(p_n) \xrightarrow{\sigma} \mathcal{S}(p)$ which, employing (1), becomes $p_{n+1} \xrightarrow{\sigma} \mathcal{S}(p)$. Hence, $\mathcal{S}(p) = p$. On the other hand, if Λ remains σ -self closed, then there exists a subsequence $\{p_{n_\kappa}\}$ of $\{p_n\}$ enjoying $[p_{n_\kappa}, p] \in \Lambda$ for all $\kappa \in \mathbb{N}_0$. Upon utilizing (v), Proposition 2, $[p_{n_\kappa}, p] \in \Lambda$ and $p_{n_\kappa} \xrightarrow{\sigma} p$, we find

$$\begin{aligned} \varphi(\sigma(\mathcal{S}p, \mathcal{S}p_{n_\kappa})) &\leq \varphi(\sigma(p, p_{n_\kappa})) - \psi(\sigma(p, p_{n_\kappa})) \\ &\quad + \theta(\sigma(p, \mathcal{S}p), \sigma(p_{n_\kappa}, \mathcal{S}p_{n_\kappa}), \sigma(p, \mathcal{S}p_{n_\kappa}), \sigma(p_{n_\kappa}, \mathcal{S}p)) \\ &\leq \varphi(\sigma(p, p_{n_\kappa})) + \theta(\sigma(p, \mathcal{S}p), \sigma(p_{n_\kappa}, p_{n_\kappa+1}), \sigma(p, p_{n_\kappa+1}), \sigma(p_{n_\kappa}, \mathcal{S}p)), \end{aligned}$$

thereby implying

$$\begin{aligned} \varphi(\sigma(\mathcal{S}p, \mathcal{S}p_{n_k})) &\leq \varphi(\sigma(p, p_{n_k})) + \theta(\sigma(p, \mathcal{S}p), \sigma(p_{n_k}, \mathcal{S}p_{n_k}), \\ &\quad \sigma(p, \mathcal{S}p_{n_k}), \sigma(p_{n_k}, \mathcal{S}p)). \end{aligned} \quad (12)$$

Employing the limit $\kappa \rightarrow \infty$ in (12), we obtain

$$0 \leq \lim_{\kappa \rightarrow \infty} \varphi(\sigma(\mathcal{S}p, \mathcal{S}p_{n_k})) \leq \varphi(0) + \theta(\sigma(p, \mathcal{S}p), 0, 0, \sigma(p, \mathcal{S}p)) = 0$$

thereby implying

$$\lim_{\kappa \rightarrow \infty} \varphi(\sigma(\mathcal{S}p, p_{n_k+1})) = 0. \quad (13)$$

Due to the continuity of φ , we find

$$\lim_{\kappa \rightarrow \infty} \varphi(\sigma(\mathcal{S}p, p_{n_k+1})) = \varphi(\sigma(\mathcal{S}p, p)). \quad (14)$$

Combining (13) and (14), we obtain $\varphi(\sigma(\mathcal{S}p, p)) = 0$ which, upon employing axiom Φ_3 , yields that $\sigma(\mathcal{S}p, p) = 0$ so that $\mathcal{S}(p) = p$. Thus, in each case, $p \in \text{Fix}(\mathcal{S})$. $\square$

The associated uniqueness result is presented as follows:

Theorem 6. *In and alongside Theorem 1, if $\mathcal{S}(\mathbb{P})$ is Λ^s -directed, then $\mathcal{S}$ owns a unique fixed point.*

Proof. From the conclusion of Theorem 1, $\text{Fix}(\mathcal{S}) \neq \emptyset$. Let $p, q \in \text{Fix}(\mathcal{S})$. We conclude that

$$\mathcal{S}^n(p) = p \quad \text{and} \quad \mathcal{S}^n(q) = q, \quad \forall n \in \mathbb{N}_0. \quad (15)$$

As $p, q \in \mathcal{S}(\mathbb{P})$ and $\mathcal{S}(\mathbb{P})$ is Λ^s -directed, there exists $\omega_0 \in \mathbb{P}$ enjoying $[p, \omega_0] \in \Lambda$ and $[q, \omega_0] \in \Lambda$. We define the sequence $\{\omega_n\}$ as follows:

$$\omega_n = \mathcal{S}^n(\omega_0) = \mathcal{S}(\omega_{n-1}), \quad \forall n \in \mathbb{N}_0. \quad (16)$$

By (15), (16), the $\mathcal{S}$ -closedness of Λ , and Proposition 1, we find

$$[p, \omega_n] \in \Lambda \quad \text{and} \quad [q, \omega_n] \in \Lambda, \quad \forall n \in \mathbb{N}_0. \quad (17)$$

Employing (16), (17), and (v), we obtain

$$\begin{aligned} \varphi(\sigma(\omega_{n+1}, p)) &= \varphi(\sigma(\mathcal{S}\omega_n, \mathcal{S}p)) \\ &\leq \varphi(\sigma(\omega_n, p)) - \psi(\sigma(\omega_n, p)) \\ &\quad + \theta(\sigma(\omega_n, Tz_n), \sigma(p, \mathcal{S}p), \sigma(\omega_n, \mathcal{S}p), \sigma(p, Tz_n)) \\ &= \varphi(\sigma(\omega_n, p)) - \psi(\sigma(\omega_n, p)) + \theta(\sigma(\omega_n, \omega_{n+1}), 0, \sigma(\omega_n, \mathcal{S}p), \sigma(p, \omega_{n+1})) \\ &= \varphi(\sigma(\omega_n, p)) - \psi(\sigma(\omega_n, p)) \end{aligned} \quad (18)$$

so that

$$\varphi(\sigma(\omega_{n+1}, p)) \leq \varphi(\sigma(\omega_n, p)), \quad \forall n \in \mathbb{N}_0.$$

Without a loss of generality, it may be assumed that $\omega_n \neq p$. By Proposition 3, we find

$$\sigma(\omega_{n+1}, p) \leq \sigma(\omega_n, p), \quad \forall n \in \mathbb{N}_0.$$

Thus, the sequence $\{\sigma(\omega_n, p)\} \subset (0, \infty)$ is decreasing. Thus, there exists $\delta \geq 0$ verifying

$$\lim_{n \rightarrow \infty} \sigma(\omega_n, p) = \delta^+. \quad (19)$$

Next, we will verify that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. Employing a lower limit in (18), we have

$$\liminf_{n \rightarrow \infty} \varphi(\sigma(\omega_{n+1}, p)) \leq \liminf_{n \rightarrow \infty} \varphi(\sigma(\omega_n, p)) - \liminf_{n \rightarrow \infty} \psi(\sigma(\omega_n, p)).$$

By the right continuity of φ and (19), we find

$$\varphi(\delta) \leq \varphi(\delta) - \lim_{s \rightarrow \delta^+} \psi(s) < \varphi(\delta)$$

which yields a contradiction. Therefore, we conclude $\delta = 0$, i.e.,

$$\lim_{n \rightarrow \infty} \sigma(\omega_n, p) = 0. \quad (20)$$

Similarly, we have

$$\lim_{n \rightarrow \infty} \sigma(\omega_n, q) = 0. \quad (21)$$

By the triangular inequality, (20), and (21), we find

$$\sigma(p, q) \leq \sigma(p, \omega_n) + \sigma(\omega_n, q) \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $p = q$. Thus, $\mathcal{S}$ owns a unique fixed point. $\square$

In particular, for a trivial BR $\Lambda = \mathbb{P}^2$, Theorem 6 reduces to the following result in an abstract MS.

Corollary 1. Assume that $(\mathbb{P}, \sigma)$ is a CMS and $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ is a map. If there exists $\varphi \in \Phi$, $\psi \in \Psi$, and $\theta \in \Theta$ enjoying

$$\varphi(\sigma(\mathcal{S}p, \mathcal{S}q)) \leq \varphi(\sigma(p, q)) - \psi(\sigma(p, q)) + \theta(\sigma(p, \mathcal{S}p), \sigma(q, \mathcal{S}q), \sigma(p, \mathcal{S}q), \sigma(q, \mathcal{S}p)),$$

$$\forall p, q \in \mathbb{P},$$

then $\mathcal{S}$ owns a unique fixed point.

Remark 1. Assume that $\phi : [0, \infty) \rightarrow [0, \infty)$ is an increasing function which verifies $\lim_{n \rightarrow \infty} \phi^n(s) = 0$. Define $\varphi(t) = t$ and $\psi(t) = t - \phi(t)$. Then, by Proposition 3.7 (c.f. [9]), we have $\varphi \in \Phi$ and $\psi \in \Psi$. Under this substitution, Theorems 1 and 6 reduce to Theorems 1 and 2 of Ansari et al. [17]. Thus, our findings improve and extend the outcomes of Ansari et al. [17].

Example 1. Considering $\mathbb{P} = [0, 1] \cup \{2, 3, 4, \dots\}$ with the following metric

$$\sigma(p, q) = \begin{cases} |p - q|, & \text{if } p, q \in [0, 1] \text{ and } p \neq q, \\ p + q, & \text{if at least one of } p \text{ or } q \text{ does not belong to } [0, 1] \text{ and } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

Equip a BR Λ on $\mathbb{P}$ by

$$\Lambda = \{(p, q) \in \mathbb{P}^2 : p > q \text{ and } p \in \{3, 4, 5, \dots\}, q \neq 2\}.$$

Then $\mathbb{P}$ is a Λ -CMS.

Consider the following auxiliary functions:

$$\varphi(s) = \begin{cases} s + 1, & \text{if } 0 \leq s < 1 \\ s^2, & \text{if } s \geq 1, \end{cases} \quad \psi(s) = \begin{cases} \frac{s^2}{4}, & \text{if } 0 \leq s \leq 1, \\ \frac{1}{5}, & \text{if } s > 1, \end{cases} \quad \text{and} \quad \theta(s_1, s_2, s_3, s_4) = 1.$$

Then $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$.

Define a map $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ by

$$\mathcal{S}(p) = \begin{cases} p - \frac{p^3}{4}, & \text{if } 0 \leq p \leq 1, \\ p - 1, & \text{if } p \in \{2, 3, 4, \dots\}. \end{cases}$$

Then Λ is an $\mathcal{S}$ -closed and locally finitely $\mathcal{S}$ -transitive BR.

Now, we check the contractivity condition (v) for $(p, q) \in \Lambda$. If $p \in \{3, 4, \dots\}$, then there are two possibilities for the selection of q . First, we take $q \in [0, 1]$. Then, we have

$$\begin{aligned} \sigma(\mathcal{S}p, \mathcal{S}q) &= \sigma\left(p - 1, q - \frac{1}{4}q^3\right) = p - 1 + q - \frac{q^3}{4} \\ &\leq p + q - 1. \end{aligned}$$

Second, we take $q \in \{3, 4, \dots\}$. Then, we have

$$\begin{aligned} \sigma(\mathcal{S}p, \mathcal{S}q) &= \sigma(p - 1, q - 1) = p + q - 2 \\ &< p + q - 1. \end{aligned}$$

Thus, in both the cases, we have

$$\begin{aligned} \varphi(\sigma(\mathcal{S}p, \mathcal{S}q)) &= (\sigma(\mathcal{S}p, \mathcal{S}q))^2 < (p + q - 1)^2 \\ &< (p + q - 1)(p + q + 1) = (p + q)^2 - 1 \\ &< (p + q)^2 - \frac{1}{5} \\ &= \varphi(\sigma(p, q)) - \psi(\sigma(p, q)). \end{aligned}$$

Therefore, the contraction inequality (v) holds for φ , ψ , and θ . Thus, by virtue of Theorem 1, $\mathcal{S}$ owns a fixed point. Moreover, here, $\mathcal{S}(\mathbb{P})$ is Λ^s -directed; therefore, by Theorem 6, the fixed point of $\mathcal{S}$ remains unique. Note that $\text{Fix}(\mathcal{S}) = \{0\}$.

4. An Application

It is commonly recognised that integral equations are useful tools for modeling a variety of phenomena arising in economics, physical science, and engineering problems. In the recent past, the theory of integral equations drew a significant amount of attention due to its extensive applicability in numerous fields, notably nonlinear analysis and topology. Solutions of integral equations have been studied using fixed-point methods by various researchers, e.g., the authors of Refs. [26–28]. In this section, applying Theorems 1 and 6, we describe the existence and uniqueness of solutions of the following FIE:

$$w(t) = \int_0^1 \mathbf{H}(t, \rho) F(\rho, w(\rho)) d\rho, \quad \forall t \in I, \quad (22)$$

where $\mathbf{H}(t, \rho)$ is kernel defined by

$$\mathbf{H}(t, \rho) = \frac{1}{6} \begin{cases} \rho^2(3t - \rho), & 0 \leq \rho \leq t \leq 1, \\ t^2(3\rho - t), & 0 \leq t \leq \rho \leq 1. \end{cases} \quad (23)$$

We will denote by Γ the family of functions $\eta : [0, \infty) \rightarrow [0, \infty)$ which enjoy

- (i) η is increasing;
- (ii) $\exists \psi \in \Psi$ enjoying $\eta(s) = s - \psi(s)$, $\forall s \in [0, \infty)$.

In what follows, $C(I)$ refers to a family of real, continuous functions on $I = [0, 1]$.

Theorem 7. Alongside (22), let there exist $\eta \in \Gamma$ satisfying

$$0 \leq F(t, a) - F(t, b) \leq \eta(r_1 - r_2), \quad \forall r_1, r_2 \in \mathbb{R} \text{ with } r_1 \geq r_2 \text{ and } \forall t \in I. \quad (24)$$

If there exists $w_0 \in C(I)$ verifying

$$w_0(t) \geq \int_0^1 H(t, \rho) F(\rho, w_0(\rho)) d\rho, \quad \forall t \in I, \quad (25)$$

where H is a kernel delivered by (23), then the FIE (22) encounters a unique solution.

Proof. Keep in mind that

$$0 \leq H(t, \rho) \leq \frac{1}{2} t^2 \rho \quad \forall t, \rho \in I. \quad (26)$$

On $\mathbb{P} := C(I)$, equip a metric σ and a BR Λ given as follows:

$$\sigma(w, x) = \max_{t \in I} |w(t) - x(t)| := \|w - x\|_\infty, \quad \forall w, x \in \mathbb{P},$$

and

$$(w, x) \in \Lambda \Leftrightarrow w(t) \geq x(t), \quad \forall w, x \in \mathbb{P}, \quad \forall t \in I.$$

Clearly, $(\mathbb{P}, \sigma)$ is a Λ -CMS and Λ is σ -self-closed BR.

Define the map $\mathcal{S} : \mathbb{P} \rightarrow \mathbb{P}$ by

$$\mathcal{S}(w)(t) = \int_0^1 H(t, \rho) F(\rho, w(\rho)) d\rho, \quad \forall t \in I, \quad \forall w \in \mathbb{P}.$$

With the help of (24), it can be easily shown that $\mathcal{S}$ is Λ -closed. Also, $\mathcal{S}(\mathbb{P})$ is Λ^s -directed. From (25), we find $w_0(t) \geq \mathcal{S}(w_0)(t)$ so that $w_0 \in \mathbb{P}(\mathcal{S}, \Lambda)$. Using (24), for all $t \in I$ and for all $w, x \in \mathbb{P}$ with $(w, x) \in \Lambda$, we conclude

$$\begin{aligned} |\mathcal{S}(w)(t) - \mathcal{S}(x)(t)| &= \int_0^1 H(t, \rho) (F(\rho, w(\rho)) - F(\rho, x(\rho))) d\rho \\ &\leq \int_0^1 H(t, \rho) \eta(w(\rho) - x(\rho)) d\rho \\ &\leq \left(\int_0^1 H(t, \rho) d\rho \right) (\eta(\|w - x\|_\infty)) \quad (\text{from (i)}) \\ &\leq \frac{\eta(\|w - x\|_\infty)}{4} \quad (\text{using (26)}) \\ &\leq \eta(\|w - x\|_\infty) \\ &= \sigma(w, x) - \psi(\sigma(w, x)) \quad (\text{from (ii)}), \end{aligned}$$

so that

$$\sigma(\mathcal{S}w, \mathcal{S}x) \leq \sigma(w, x) - \psi(\sigma(w, x)).$$

Therefore, all assumptions of Theorems 1 and 6 are verified; therefore, $\exists$ a unique $\bar{w} \in C(I)$ verifying $\mathcal{S}(\bar{w}) = \bar{w}$, which forms the unique solution of (22). $\square$

5. Conclusions

In this article, we presented outcomes on fixed points in a relational MS endowed with a locally finitely $\mathcal{S}$ -transitive BR under certain functional contractions involving three auxiliary functions. Through the implementation of our results, we discussed an existence and uniqueness theorem for certain FIEs prescribed with some additional conditions. The contractivity condition utilized in our results and the class of BR (i.e., locally finitely $\mathcal{S}$ -transitive BR) remain more general than those used in Altaweel and Khan [18]. If we take $\theta = 0$, then our results achieve the corresponding results of Sk et al. [19]. In particular, for $\Lambda = \preceq$, the partial order, our results reduce to the corresponding results of Jleli et al. [21].

In possible future works, Theorems 1 and 6 can be further extended in the following directions:

1. To a variety of metrical structures, such as a dislocated space, quasi-metric space, semi-metric space, b -metric space, etc., equipped with a locally finitely $\mathcal{S}$ -transitive BR;
2. To two or more maps by proving coincidence and common fixed-point theorems;
3. To best-proximity point theorems, following the work of Fallahi et al. [29].

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Abbreviations

This article adopts the following abbreviations:

MS	metric space
CMS	complete metric space
BCP	Banach contraction principle
BR	binary relation
FIE	Fredholm integral equation
iff	if and only if
R.H.S.	right-hand side

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