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# Covering-Based Intuitionistic Hesitant Fuzzy Rough Set Models and Their Application to Decision-Making Problems

**Muhammad Kamraz Khan [1], Kamran [1,*], Muhammad Sajjad Ali Khan [2], Ahmad Aloqaily [3,*] and Nabil Mlaiki [3,*]**

[1] Department of Mathematics, Islamia College Peshawar, Peshawar 25120, Khyber Pakhtunkhwa, Pakistan
[2] Department of Mathematics, Khushal Khan Khattak University, Karak 27200, Khyber Pakhtunkhwa, Pakistan; sajjadalimath@yahoo.com
[3] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
* Correspondence: kamran.maths@icp.edu.pk (K.); maloqaily@psu.edu.sa (A.A.); nmlaiki@psu.edu.sa (N.M.)

**Abstract:** In this paper, we present four categories of covering-based intuitionistic hesitant fuzzy rough set (CIHFRS) models using intuitionistic hesitant fuzzy $\beta$-neighborhoods (IHF $\beta$-neighborhoods) and intuitionistic hesitant fuzzy complementary $\beta$-neighborhoods (IHFC $\beta$-neighborhoods. Through theoretical analysis of covering-based IHFRS models, we propose the intuitionistic hesitant fuzzy TOPSIS (IHF-TOPSIS) technique for order of preference by similarity to an ideal solution, addressing multicriteria decision-making (MCDM) challenges concerning the assessment of IHF data. A compelling example aptly showcases the suggested approach. Furthermore, we address MCDM problems regarding the assessment of IHF information based on CIHFRS models. Through comparison and analysis, it is evident that addressing MCDM problems by assessing IHF data using CIHFRS models proves more effective than utilizing intuitionistic fuzzy data with CIFRS models or hesitant fuzzy information with CHFRS models. IHFS emerges as a unique and superior tool for addressing real-world challenges. Additionally, covering-based rough sets (CRSs) have been successfully applied to decision problems due to their robust capability in handling unclear data. In this study, by combining CRSs with IHFS, four classes of CIFRS versions are established using IHF $\beta$-neighborhoods and IHFC $\beta$-neighborhoods. A corresponding approximation axiomatic system is developed for each. The roughness and precision degrees of CBIHFRS models are specifically talked about. The relationship among these four types of IHFRS versions and existing related versions is presented based on theoretical investigations. A method for MCDM problems through IHF information, namely, IHF-TOPSIS, is introduced to further demonstrate its effectiveness and applicability. By conducting a comparative study, the effectiveness of the suggested approach is evaluated.

**Keywords:** $\beta$-covering rough set; decision making; intuitionistic hesitant fuzzy set; intuitionistic hesitant fuzzy $\beta$-covering rough set



**Citation:** Khan, M.K.; Kamran; Sajjad Ali Khan, M.; Aloqaily, A.; Mlaiki, N. Covering-Based Intuitionistic Hesitant Fuzzy Rough Set Models and Their Application to Decision-Making Problems. *Symmetry* **2024**, *16*, 693. https://doi.org/10.3390/sym16060693

Academic Editor: Jian-Qiang Wang

Received: 28 April 2024
Revised: 29 May 2024
Accepted: 31 May 2024
Published: 4 June 2024



## 1. Introduction

Fuzzy set theory (FST) [1] has emerged as a significant method for dealing with uncertainty and has undergone diverse advancements [2,3]. Formally speaking, FST is among the theories initially conceived to extend classical set theory. Furthermore, the relaxation can go beyond just fuzzy relations to any binary relation. This generalization allows for the use of other types of relations, such as tolerance or similarity relations, which do not essentially partition the universe into disjoint classes but still provide a way to approximate sets. These binary relations can capture more nuanced relationships between elements, offering greater flexibility in various applications. Atanassov [4] proposed the idea of an intuitionistic fuzzy set (IFS) to improve the idea of an FS, which expands upon the traditional notion of a set. Within an IFS, the elements possess not only degrees of membership ($\check{m}$) but also degrees of nonmembership ($\check{n}\check{m}$). The theory of IFSs has found application in diverse domains, including logic programming [5,6], DM problems [7], medical diagnosis [8], and more. It has demonstrated its effectiveness as a valuable tool for addressing the complexities associated with determining the level of $\breve{m}$ of an element within a set, particularly in scenarios involving vagueness. Torra [9] expanded upon the concept of an FS by introducing the idea of an HFS. HFS theory is designed to address scenarios in which there is uncertainty in specifying the inclusion of an element within a set, enabling a range of possible values. In conjunction with the aggregation operators presented in [10–14], The HFS is employed to tackle group decision-making challenges. Ref. [15] suggests the introduction of IHFSs; these arise from the amalgamation of IFSs and HFSs. This concept proves beneficial in addressing situations characterized by uncertainty, where specific values may simultaneously represent both $\breve{m}$ and $\breve{n}\breve{m}$ for a given element. In rough set (RS) theory, symmetry can also influence the approximations.

In classical rough set models, symmetry consideration can lead to different approximations and classifications of objects based on their relationship within the data. Pawlak [16,17] introduced the RS, which has proven to be a valuable tool for handling uncertainty. However, its primary focus is on partitioning the universe to create approximation operators. Recognizing this limitation, many researchers have dedicated their efforts to modifying the stringent conditions of Pawlak's RS. For instance, they have relaxed the equivalence relation to a fuzzy relation and introduced a relation covering the universe. Several studies have explored the extension of Pawlak's RS, including the introduction of the FRS and RFS by Dubois and Prade [18], as well as contributions from others in the field [19–23]. Rough set analysis in a fuzzy setting offers important benefits and insights that classical settings are unable to fully achieve. Classical rough sets approach sets using lower and upper boundaries, which helps them deal with ambiguity and insufficient information. They do, however, make the clear distinction between membership and nonmembership. Conversely, FSs support partial membership, which better captures the ambiguity and vagueness present in many real-world issues. An FRS extends the concept of an RS by approximating an FS within a crisp approximation space. This occurs in scenarios where the values of conditional attributes are exact but the values of decision attributes exhibit fuzziness. FRSs aim to establish the lower approximation (LA) and upper approximation (UA) of a set. This becomes essential when the universe of the FS undergoes roughness, either through an equivalence relation or by converting the equivalence relation into a corresponding fuzzy relation. The amalgamation of FRSs and IFSs led Chinram et al. [24] to introduce the IFRS. Additionally, the introduction of the HFRS allows for the application of RS techniques in a hesitant fuzzy environment, focusing on establishing LA and UA within such a domain. Additionally, the concept of an IHFS is included in the IHFRST, which builds upon RST.

In order to successfully handle ambiguity, hesitation, and uncertainty in activities like data processing, decision making (DM), and knowledge representation, IHFRST combines the concepts of the RS, IHFS, and HFRS. According to the text, DM in a fuzzy environment involves ranking alternatives based on decision information, and this is a common practice in the actual world. MCDM emerges as an invaluable instrument for solving complex problems in this context. Hwang and Yoon [25] were the pioneers in investigating this approach. In the context of multicriteria group DM, total possible alternatives are assessed based on conflicting and interactive criteria. The text also delves into the utilization of the TOPSIS method for group DM in the IHF case. Upon reviewing prior research, it appears that there has been limited exploration of the application of the CIHFRS in evaluating IHF information within MCDM problems. This research gap motivated the present study, which focuses on developing CIHFRS models using the intuitionistic hesitant fuzzy $\beta$-neighborhood (IHF $\beta$-neighborhood) and intuitionistic hesitant fuzzy complementary $\beta$-neighborhood (IHFC $\beta$-neighborhood). Furthermore, this paper investigates the practical applications of these models to MCDM problems related to the assessment of IHF information. The advanced features of intuitionistic hesitant fuzzy rough sets (IHFRSs) addressed the limitations of traditional fuzzy sets theory extensions such as the FS, IFS, and several others. The IHFRS builds on traditional models that utilize the value of lower and upper approximations with membership and nonmembership from the unit interval to represent the uncertainty of real-world challenges. The IHFRS extends the space for information about DM problems. The abovementioned constraints motivated us to develop similarity measures based on the IHFRS, which is a modified version of the IFRS that befits the structure of IFRSs. The development of similarity measures for IHFRSs is of crucial academic significance. Lin et al. [26] presented better fuzzy preference relation rough sets into a multigranulation case. In an instant, exploring rough sets from a multigranulation perspective can better deal with the ambiguous information in complex data, and it has rich research content and value. Due to the uncertainty of IHF information, it is impossible to directly make decisions, so we consider combining multigranulation RSs and IHFSs, which is helpful for DM in IHF environments. Covering-based intuitionistic hesitant fuzzy rough sets (CIHFRSs) offer a more effective approach to handling ambiguities. This is especially helpful when there is uncertainty about numerous choices and the information is not merely incomplete. In decision-making processes, CIHFRSs provide a complete framework that covers the hesitancy and dual nature of intuitionistic fuzzy logic, especially in fields related to risk assessment, financial analysis, and medical diagnosis. Decision making becomes more reliable and accurate as a result. Further, a comprehensive framework for handling complicated and uncertain information can be obtained by the study of covering-based intuitionistic hesitant fuzzy rough sets, which improves the tools available for analysis, modeling, and decision making in a variety of contexts.

The rest of this paper is organized as follows: Section 2 recalls some basic concepts of the CIFS, CHFS, and IHFS. In Section 3, we introduce the four types of CIHFRS models via IHF $\beta$-neighborhoods and IHFC $\beta$-neighborhoods. The corresponding axiomatic systems are investigated. In particular, the rough and precision degrees of CIHFRS models are discussed. In Section 4, based on the theoretical analysis of CIHFRS models, we put forward the IHF-TOPSIS methodology for the MCDM problem with the evaluation of IHF information. An effective example is given to illustrate the proposed methodology. Finally, by comparative analysis, we find that it is more effective to deal with the MCDM problem with the evaluation of IHF information based on CIHFRS models than with the evaluation of intuitionistic fuzzy information based on CIFRS models and the evaluation of hesitant fuzzy information based on CHFRS models in Section 5. In Section 6, we conclude this paper, summarizing our findings and providing plans for further research.

## 2. Preliminaries

The notion of the IFS was first introduced by Atanassov [4] as a generalization of the FS [1]. Further, in order to deal with a set of uncertain information, Torra [9] extended the concept of the FS. Furthermore, these concepts are generalized by Chinram et al. [24], and Zhou and Li [27] developed IFRSs, covering-based HFR sets. In the following subsections, some basic definitions, operational laws, and extended forms of the IFS and HFS are discussed.

### 2.1. Covering-Based Intuitionistic Fuzzy Rough Set

In this subsection, the notion of the IFS, operational laws of the IFS, and the covering-based IFRS are presented. Moreover, score and accuracy functions are also given. The IFS can be defined as follows:

**Definition 1** ([4,28])**.** *Assume that $\dot{X} \neq \phi$ is a universe of discourse; then an IFS $\widehat{\imath}$ in $\dot{X}$ is defined by*

$$\widehat{\imath} = \{\langle z, \tau_{\widehat{\imath}}(z), \varrho_{\widehat{\imath}}(z)\rangle \mid z \in \dot{X}\} \quad (1)$$

*where the functions $\tau_{\widehat{\imath}}(z) : \dot{X} \longrightarrow [0,1]$ and $\varrho_{\widehat{\imath}}(z) : \dot{X} \longrightarrow [0,1]$, denoting the degree of membership and degree of nonmembership of $z$ to $\widehat{\imath}$, respectively, and $\forall\, z \in \dot{X}$, $\tau_{\widehat{\imath}}(z)$, $\varrho_{\widehat{\imath}}(z)$ fulfill $0 \leq \tau_{\widehat{\imath}}(z) + \varrho_{\widehat{\imath}}(z) \leq 1$. An intuitionistic fuzzy number (IFN) is denoted by $\dot{m} = \langle \tau_{\dot{m}}, \eta_{\dot{m}} \rangle$.*

**Definition 2** ([29]). *For an IFN, $\dot{m} = \langle \tau_{\dot{m}}, \eta_{\dot{m}} \rangle$, and the score and accuracy functions are defined, respectively, by*

$$\hat{S}(\dot{m}) = \tau_{\dot{m}} - \eta_{\dot{m}}$$

*and*

$$a(\dot{m}) = \tau_{\dot{m}} + \eta_{\dot{m}}$$

*where $-1 \le \hat{S}(\dot{m}) \le 1, 0 \le a(\dot{m}) \le 1$.*

*Further, for IFNs $\dot{m}_1, \dot{m}_2$, the properties of score and accuracy functions are defined as follows [29]:*

(1) *If $\hat{S}(\dot{m}_1) \succ \hat{S}(\dot{m}_2)$, then $\dot{m}_1 \succ \dot{m}_2$.*
(2) *If $\hat{S}(\dot{m}_1) \prec \hat{S}(\dot{m}_2)$, then $\dot{m}_1 \prec \dot{m}_2$.*
(3) *If $\hat{S}(\dot{m}_1) = \hat{S}(\dot{m}_2)$, then*
    (i) *$a(\dot{m}_1) \succ a(\dot{m}_2)$ implies $\dot{m}_1 \succ \dot{m}_2$.*
    (ii) *$a(\dot{m}_1) \prec a(\dot{m}_2)$ implies $\dot{m}_1 \prec \dot{m}_2$.*
    (iii) *$a(\dot{m}_1) = a(\dot{m}_2)$ implies $\dot{m}_1 \sim \dot{m}_2$.*

**Definition 3** ([30]). *Assume that $X \neq \phi$ is a universe of discourse and $G = \{G_1, G_2, \ldots, G_n\}$, where $G_i (i = 1, 2, 3, \ldots, n) \in IF(X)$. For any IFN, $\beta = (\beta_1, \beta_2)$. Then G is called an IF $\beta$-covering of X if $\left(\bigcup_{i=1}^{n} G_i\right)(\varkappa) \ge \beta, \forall\, \varkappa \in X$. In this context, the pair (X,G) is represented as an IF covering approximation space (IFCAS).*

**Definition 4.** *Assume that $(X, G)$ is an IFCAS and $G = \{G_1, G_2, \ldots, G_n\}$ is an IF-$\beta$ covering of X for some $\beta = (\beta_1, \beta_2)$; then,*

$$\mathbb{N}^{\beta}_{G(\varkappa)} = \bigcap \{G_i \in G | G_i(\varkappa) \ge \beta, i = 1, 2, \ldots, n\} \tag{2}$$

*is referred to as an IF $\beta$-neighborhood of $\varkappa$ in X.*

**Definition 5** ([30]). *Assume that $(X, G)$ is an IFCAS, $G = \{G_1, G_2, \ldots, G_m\}$ is an IF $\beta$-covering of X for some IFN $\beta = (\beta_1, \beta_2)$, and $X = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_n\}$. Suppose that $\mathbb{N}^{\beta}_{G(\varkappa_i)} = \left\{\mathbb{N}^{\beta}_{G(\varkappa_i)} | \varkappa_i \in X\right\}$ is an IF $\beta$-neighborhood system induced by G, where $\mathbb{N}^{\beta}_{G(\varkappa_i)} = \left\{\left\langle \varkappa_j, \tau_{\mathbb{N}^{\beta}_{G(\varkappa_i)}}(\varkappa_i, \varkappa_j), \eta_{\mathbb{N}^{\beta}_{G(\varkappa_i)}}(\varkappa_i, \varkappa_j)\right\rangle \right.$ $\left. | \; j = 1, 2, \ldots n\right\} \forall\; i = 1, 2, \ldots, n$. For any $A \in IF(X)$, the lower approximation (LA) and upper approximation (UA) spaces of A with respect to $\mathbb{N}^{\beta}_{G}$ represented as $\underline{\mathbb{N}^{\beta}_{G}}(A)$ and $\overline{\mathbb{N}^{\beta}_{G}}(A)$, respectively, are two IFSs on X defined by*

$$\underline{\mathbb{N}^{\beta}_{G}}(A) = \left\{\left\langle \varkappa_i, \tau_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa_i), \eta_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa_i)\right\rangle | \; i = 1, 2, \ldots n\right\} \tag{3}$$

*and*

$$\overline{\mathbb{N}^{\beta}_{G}}(A) = \left\{\left\langle \varkappa_i, \tau_{\overline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa_i), \eta_{\overline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa_i)\right\rangle | \; i = 1, 2, \ldots n\right\} \tag{4}$$

*where*

$$\tau_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa) = \bigwedge_{j=1}^{n} \{\eta_{\mathbb{N}^{\beta}_{G(\varkappa_i)}}(\varkappa_i, \varkappa_j) \veebar \tau_A(\varkappa_j)\}$$

$$\eta_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(\varkappa) = \bigvee_{j=1}^{n} \{\tau_{\mathbb{N}^{\beta}_{G(\varkappa_i)}}(\varkappa_i, \varkappa_j) \barwedge \eta_A(\varkappa_j)\}$$

*and*

$$\tau_{\overline{\mathbb{N}_G^\beta}(A)}(x_i) = \bigvee_{j=1}^{n} \{\tau_{\mathbb{N}^\beta_{G(x_i)}}(x_i, x_j) \barwedge \tau_A(x_j)\}$$

$$\eta_{\overline{\mathbb{N}_G^\beta}(A)}(\varkappa) = \bigwedge_{j=1}^{n} \{\eta_{\mathbb{N}^\beta_{G(\varkappa_i)}}(\varkappa_i, \varkappa_j) \veebar \eta_A(\varkappa_j)\}$$

$\underline{\mathbb{N}_G^\beta}(A)$, $\overline{\mathbb{N}_G^\beta}(A) : IF(X) \to IF(X)$ *are called the lower and upper IF rough approximation operators (IFRAOs) with respect to* $\mathbb{N}_G^\beta$*, respectively. The pair* $(\underline{\mathbb{N}_G^\beta}(A), \overline{\mathbb{N}_G^\beta}(A))$ *refers to a covering-based IF rough set (CIFRS).*

*2.2. Covering-Based Hesitant Fuzzy Rough Set*

In this subsection, we examine the HFS, score function, and accuracy function of a hesitant fuzzy element (HFE) and a hesitant fuzzy rough set (HFRS). The HFS can be defined as follows:

**Definition 6** ([9,31])**.** *Assume that* $X \neq \phi$ *universe of discourse; then an HFS denoted as* $\hat{H}$ *in the set X can be defined as*

$$\hat{H} = \{\langle z, \Omega_{\hat{H}}(z)\rangle \mid z \in X\} \tag{5}$$

*where* $\Omega_{\hat{H}}(z)$ *is a collection of possible values belonging to the interval [0, 1] representing the degree of membership of* $z \in X$ *to* $\hat{H}$*. If X is a singleton set, then* $\widehat{h(z)}$ *is a hesitant fuzzy element.*

**Definition 7** ([32])**.** *Assuming that* $\widehat{h}$ *is an HF element, the score function of* $\widehat{h}$ *is defined as follows:*

$$s(\widehat{h}) = \frac{1}{l(\widehat{h})} \sum_{s=1}^{l(\overleftrightarrow{h})} \widehat{h}^{\sigma(s)}.$$

*Furthermore [32], for any two hesitant elements, say,* $\dot{h}_1, \dot{h}_2$*, see as follows:*

*(i)* *If* $s(\widehat{h}_1) \succ s(\widehat{h}_2)$*, then* $\widehat{h}_1 \succ \widehat{h}_2$*.*
*(ii)* *If* $s(\widehat{h}_1) = s(\widehat{h}_2)$*, then* $\widehat{h}_1 \sim \widehat{h}_2$*.*

**Definition 8** ([27])**.** *Assume that* $X \neq \phi$ *is a universe of discourse,* $G = \{G_1, G_2, \ldots, G_n\}$ *where* $G_i(i = 1, 2, \ldots, n) \in HF(X)$*, and* $\beta = (\beta_1, \beta_2)$ *can be any HFE, then G is called an HF β-covering of X if* $h \bigcup_{i=1}^{n} G_i(z) \geq \beta, \forall\, z \in X$*. The pair* $(X, G)$ *is called an HF β-covering approximation space (HFCAS).*

**Definition 9** ([27])**.** *Assume that* $(X, G)$ *is an HFCAS and* $G = \{G_1, G_2, \ldots, G_n\}$ *is an HF-β covering of X for some* $\beta = (\beta_1, \beta_2)$*; then* $\mathbb{N}_{1,z}^{\beta,G} = \bigcap\{G_i \in G | h_{G_i}(z) \geq \beta, i = 1, 2, \ldots, n\}$ *and* $\mathbb{N}_{2,z}^{\beta,G} = \{\prec y, h_{\mathbb{N}_{2,z}^{\beta,G}}(y) \succ | y \in X\}$ *are called an HF β-neighborhood and HF complementary β-neighborhood of z on X, respectively. Here,* $h_{\mathbb{N}_{2,z}^{\beta,G}}(y) = h_{\mathbb{N}_{1,y}^{\beta,G}}(z)$*.*

**Definition 10.** *Assume that* $(X, G)$ *is an HFCAS and* $G = \{G_1, G_2, \ldots, G_m\}$ *is an HF β-covering of X for some HF elements β. For any* $\widehat{h} \in HF(X)$*, the LA and UA spaces of* $\widehat{h}$ *with respect to* $\mathbb{N}_G^\beta$ *represented as* $\underline{\mathbb{N}_G^\beta}(\widehat{h})$ *and* $\overline{\mathbb{N}_G^\beta}(\widehat{h})$*, respectively, are two HFSs on X defined by* $\underline{\mathbb{N}_G^\beta}(\widehat{h}) = \left\{z, \widehat{h}_{\underline{\mathbb{N}_G^\beta}(\widehat{h})}(z)\right\}$ *and* $\overline{\mathbb{N}_G^\beta}(A) = \left\{z, \widehat{h}_{\overline{\mathbb{N}_G^\beta}(A)}(z)\right\}$*, where* $\hat{h}_{\underline{\mathbb{N}_G^\beta}(A)}(z) = \bigwedge_{y \in X}\{\hat{h}_{(\mathbb{N}_G^\beta)^c}(y) \veebar \hat{h}_A(y)\}$ *and* $\hat{h}_{\overline{\mathbb{N}_G^\beta}(A)}(z) = \bigvee_{y \in X}\{\hat{h}_{\mathbb{N}_G^\beta}(y) \barwedge \hat{h}_A(y)\}$*.* $\underline{\mathbb{N}_G^\beta}(A), \overline{\mathbb{N}_G^\beta}(A) : HF(X) \to HF(X)$ *are called the lower and upper hesitant fuzzy rough approximation operators (HFRAOs) with respect to*

$\mathbb{N}_G^{\beta}$, *respectively. The pair* $(\underline{\mathbb{N}_G^{\beta}}(A), \overline{\mathbb{N}_G^{\beta}}(\tilde{A}))$ *refers to a covering-based hesitant fuzzy rough set (CHFRS).*

### 2.3. Intuitionistic Hesitant Fuzzy Set

The concept of the IHFS was first introduced by Peng et al. [15] and is characterized by a hesitant fuzzy membership degree and a hesitant fuzzy nonmembership degree. In this subsection, a definition of the IHFS is presented, and some operational laws and comparisons of the IHFNS are also discussed. The IHFS can be defined as follows:

**Definition 11** ([15]). *Assume that* $X \neq \phi$ *universe of discourse; then an IHFS* $\Lambda_{IHFS}$ *on* $X$ *denoted by* $\Lambda_{IHFS}$ *can be defined as*

$$\Lambda_{IHFS} = \{\left\langle \varkappa, \Phi_{\Lambda_{IHFS}}(\varkappa), \Psi_{\Lambda_{IHFS}}(\varkappa)\right\rangle \mid \varkappa \in X\}, \tag{6}$$

*where* $\Phi_{\Lambda_{IHFS}}(\varkappa)$ *and* $\Psi_{\Lambda IHFS}(\varkappa)$ *are mappings from* $X$ *to the interval* $[0,1]$ *representing the hesitant fuzzy MD and hesitant fuzzy NMD, respectively, and are the set of possible values belonging to* $[0,1]$ *such that for* $h_{\Lambda_{IHFS}}(\varkappa) \in \Phi_{\Lambda_{IHFS}}(\varkappa)$ $\exists$ $\hat{h}_{\Lambda_{IHFS}}(\varkappa) \in \Psi_{\Lambda_{IHFS}}(\varkappa)$ *with* $h_{\Lambda_{IHFS}}(\varkappa) + \hat{h}_{\Lambda_{IHFS}}(\varkappa) \leq 1$ *and* $\forall$ $\hat{h}_{\Lambda_{IHFS}}(\varkappa) \in \Psi_{\Lambda_{IHFS}}(\varkappa)$ $\exists$ $h_{\Lambda_{IHFS}}(\varkappa) \in \Phi_{\Lambda_{IHFS}}(\varkappa)$ *with* $0 \leq h_{\Lambda_{IHFS}}(\varkappa) + \hat{h}_{\Lambda_{IHFS}}(\varkappa) \leq 1$*. Further, IHFS(X) represents the set of all IHFSs. If X can have only one element* $\varkappa$*, then* $\langle \varkappa, \Phi_{\Lambda}(\varkappa), \Psi_{\Lambda}(\varkappa)\rangle$ *is said to be an IHFN that is represented by* $\dot{e} = \langle \Phi_{\dot{e}}, \Psi_{\dot{e}}\rangle$ *for convenience.*

**Definition 12** ([15]). *Assume that* $\dot{e}_1 = \langle \Phi_{\dot{e}_1}, \Psi_{\dot{e}_1}\rangle$ , $\dot{e}_2 = \langle \Phi_{\dot{e}_2}, \Psi_{\dot{e}_2}\rangle$*, and* $\dot{e}_3 = \langle \Phi_{\dot{e}_3}, \Psi_{\dot{e}_3}\rangle$ *are three IHFNs; then*

*(i)* $\zeta\dot{e} = \left\langle \cup_{h\dot{e}\in\Phi\dot{e}}\{1-(1-h\dot{e})^{\zeta}\}, \cup_{\hat{h}\dot{e}\in\Psi\dot{e}}\{\hat{h}\dot{e}\}^{\zeta}\right\rangle, \zeta \succ 0.$

*(ii)* $\dot{e}^{\zeta} = \left\langle \cup_{h\dot{e}\in\Phi\dot{e}}\{h\dot{e}\}^{\zeta}, \cup_{\hat{h}\dot{e}\in\Psi\dot{e}}\{1-(1-\hat{h}\dot{e})^{\zeta}\}\right\rangle, \zeta \succ 0.$

*(iii)* $\dot{e}_1 + \dot{e}_2 = \left\langle \cup_{h\dot{e}_1\in\Phi\dot{e}_1, h\dot{e}_2\in\Phi\dot{e}_2}\{h\dot{e}_1 + h\dot{e}_2 - h\dot{e}_1.h\dot{e}_2\}, \cup_{\hat{h}\dot{e}_1\in\Psi\dot{e}_1, \hat{h}\dot{e}_2\in\Psi\dot{e}_2}\{\hat{h}\dot{e}_1.\hat{h}\dot{e}_2\}\right\rangle.$

*(iv)* $\dot{e}_1 \times \dot{e}_2 = \left\langle \cup_{h\dot{e}_1\in\Phi\dot{e}_1, h\dot{e}_2\in\Phi\dot{e}_2}\{h\dot{e}_1.h\dot{e}_2\}, \cup_{\hat{h}\dot{e}_1\in\Psi\dot{e}_1, \hat{h}\dot{e}_2\in\Psi\dot{e}_2}\{\hat{h}\dot{e}_1 + \hat{h}\dot{e}_2 - \hat{h}\dot{e}_1.\hat{h}\dot{e}_2\}\right\rangle.$

**Definition 13** ([33]). *Assume that* $\dot{X} \neq \phi$ *is a universe of discourse; then for any* $\bar{B}, \bar{C} \in IHFS(\dot{X})$*. The union* $\bar{B} \cup \bar{C}$ *and intersection* $\bar{B} \cap \bar{C}$ *of* $\bar{B}$ *and* $\bar{C}$ *are specified as follows:*

$$\bar{B} \cup \bar{C} = \{\langle \chi, \Phi_{\bar{B}}(\chi) \cup \Phi_{\bar{C}}(\chi), \Psi_{\bar{B}}(\chi) \cap \Psi_{\bar{C}}(\chi)\rangle \mid \chi \in \chi\}$$

$$\bar{B} \cap \bar{C} = \{\langle \chi, \Phi_{\bar{B}}(\chi) \cap \Phi_{\bar{C}}(\chi), \Psi_{\bar{B}}(\chi) \cup \Psi_{\bar{C}}(\chi)\rangle \mid \chi \in \chi\}$$

*where*

$$\Phi_{\bar{B}}(\chi) \cup \Phi_{\bar{C}}(\chi) = \left\{\nu \in (\Phi_{\bar{B}}(\chi) \cup \Phi_{\bar{C}}(\chi)) | \nu \geq \max(\Phi_{\bar{B}}^{-}(\chi), \Phi_{\bar{C}}^{-}(\chi))\right\}$$

*and*

$$\Phi_{\bar{B}}(\chi) \cap \Phi_{\bar{C}}(\chi) = \left\{\nu \in (\Phi_{\bar{B}}(\chi) \cup \Phi_{\bar{C}}(\chi)) | \nu \leq \min(\Phi_{\bar{B}}^{+}(\chi), \Phi_{\bar{C}}^{+}(\chi))\right\}$$

*and*

$$\Psi_{\bar{B}}(\chi) \cap \Psi_{\bar{C}}(\chi) = \left\{\nu \in (\Psi_{\bar{B}}(\chi) \cup \Psi_{\bar{C}}(\chi)) | \nu \leq \min(\Psi_{\bar{B}}^{+}(\chi), \Psi_{\bar{C}}^{+}(\chi))\right\},$$

$$\Psi_{\bar{B}}(\chi) \cup \Psi_{\bar{C}}(\chi) = \left\{\nu \in (\Psi_{\bar{B}}(\chi) \cup \Psi_{\bar{C}}(\chi)) | \nu \geq \max(\Psi_{\bar{B}}^{-}(\chi), \Psi_{\bar{C}}^{-}(\chi))\right\}$$

The score and accuracy functions of the IHFS are, respectively, given by the following:

**Definition 14** ([15]). *Assume that* $\dot{e} = \langle \Phi_{\dot{e}}, \Psi_{\dot{e}}\rangle$ *is an IHFN; the score function and accuracy function of* $\dot{e}$ *are defined, respectively, as* $(\dot{e}) = \frac{\Phi_{\dot{e}} - \Psi_{\dot{e}}}{2}$*, where* $\Phi_{\dot{e}} = \frac{\text{sum of all elements in } \Phi_{\dot{e}}}{|\Phi_{\dot{e}}|}$*,* $\Psi_{\dot{e}} = \frac{\text{sum of all elements in } \Psi_{\dot{e}}}{|\Psi_{\dot{e}}|}$*, and* $(\dot{e}) \in [-1,1]$*; and* $A(\dot{e}) = \frac{\Phi_{\dot{e}} - \Psi_{\dot{e}}}{2}$*, where* $\Phi_{\dot{e}} = \frac{\text{sum of all elements in } \Phi_{\dot{e}}}{|\Phi_{\dot{e}}|}$*,*

$\Psi_{\dot{e}} = \frac{\text{sum of all elements in } \Psi_{\dot{e}}}{|\Psi_{\dot{e}}|}$, *and* $A̧(\dot{e}) \in [0,1]$.

## 3. Covering-Based Intuitionistic Hesitant Fuzzy Rough Set (CIHFRS) Models

In this section, we define the CIHRS as an extension of the IHFS with a covering-based rough set. We define intuitionistic hesitant fuzzy $\beta$-covering rough approximation spaces, the intuitionistic hesitant fuzzy $\beta$-neighborhood, and the intuitionistic hesitant fuzzy complementary $\beta$-neighborhood, and examples of the concepts are also presented. Properties of the developed concepts are investigated in detail.

**Definition 15.** *Assume that X is a universe of discourse,* $G = \{G_1, G_2, \ldots, G_m\}$ *where* $G_i (i = 1, 2, \ldots, m) \in IHF(X)$, *and* $\beta = (\Phi_\beta, \Psi_\beta)$ *IHFN; then G is called an IHF* $\beta$*-covering of X if* $\bigcup_{i=1}^{m} \hbar(x) \geq \beta, \forall x \in X$. *The pair* $(X, G)$ *is called an IHFCAS.*

**Definition 16.** *Assume that* $(X, G)$ *is an IHFCAS,* $G = \{G_1, G_2, \ldots, G_m\}$ *is an intuitionistic hesitant fuzzy* $\beta$*-covering of X, and* $\beta = (\Phi_\beta, \Psi_\beta)$ *is an IHFN; then* $\mathbb{N} = \bigcap\{G_i \in G | G_i(x) \geq \beta, i = 1, 2, \ldots, m\}$ *is called an IHF* $\beta$*-neighborhood of* $x \in X$.

**Definition 17.** *Assume that* $(X, G)$ *is an IHFCAS,* $G = \{G_1, G_2, \ldots, G_m\}$ *is an IHF* $\beta$*-covering of X for some IHFN* $\beta = (\Phi_\beta, \Psi_\beta)$, *and* $X = \{x_1, x_2, \ldots, x_n\}.\forall x \in X$; *then the set*

$$\Phi_{\overline{M}_x^\beta}(y), \Psi_{\overline{M}_x^\beta}(y) = \Phi_{\mathbb{N}_y^\beta}(x), \Psi_{\mathbb{N}_y^\beta}(x).\forall y \in X \tag{7}$$

$\overline{M}_x^\beta$ *is called an IHF-complement* $\beta$*-neighborhood of x in X. We represent* $\widetilde{M}_G^\beta = \{M_x^\beta | x \in X\}$ *as the* $\beta$*-neighborhood system induced by the IHF complement through the IHF* $\beta$*-covering G, where* $M_x^\beta = \{\prec y, \Phi_{M_x^\beta}(y), \Psi_{M_x^\beta}(y) \succ | y \in X\}$.

**Example 1.** *Assume that* $(X, G)$ *is an IHFCAS, and* $G = \{G_1, G_2, G_3, G_4, G_5\}$ *is an intuitionistic hesitant fuzzy* $\beta$*-covering of set X. Take* $\beta = \langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$, *where* $X = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5\}$, *as shown in Table 1.*

**Table 1.** Intuitionistic hesitant fuzzy $\beta$-covering matrix.

| $X/G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.2, 0.3, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.6\}, \{0.2, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.2, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.6\}, \{0.2, 0.3, 0.4\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.1, 0.3, 0.5\}, \{0.1, 0.2, 0.5\}\rangle$ | $\langle\{0.2, 0.3, 0.6\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\rangle$ | $\langle\{0.1, 0.3, 0.6\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.2, 0.3, 0.4\}, \{0.1, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.2, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.1, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}\rangle$ | $\langle\{0.3, 0.4, 0.5\}, \{0.2, 0.4\}\rangle$ | $\langle\{0.3, 0.4, 0.5\}, \{0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.3, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.2, 0.3, 0.4\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ | $\langle\{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.5\}\rangle$ | $\langle\{0.3, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.2, 0.5\}\rangle$ |

*Then* $\mathbb{N}_{\varkappa_1}^\beta = G_1 \cap G_5, \mathbb{N}_{\varkappa_2}^\beta = G_2 \cap G_4, \mathbb{N}_{\varkappa_3}^\beta = G_1 \cap G_4, \mathbb{N}_{\varkappa_4}^\beta = G_2 \cap G_3, \mathbb{N}_{\varkappa_5}^\beta = G_4 \cap G_5$. *Calculating the value of the* $\beta$*-neighborhood, we obtain Table 2.*

**Table 2.** Intuitionistic hesitant fuzzy $\beta$-neighborhood matrix.

| $\mathbb{N}_G^\beta$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.1, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.1, 0.3, 0.6\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.5\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\rangle$ | $\langle\{0.1, 0.4, 0.4\}, \{0.1, 0.3, 0.3\}\rangle$ | $\langle\{0.1, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.4\}, \{0.1, 0.3, 0.3\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.3, 0.4, 0.5\}, \{0.2, 0.4, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.1, 0.4, 0.4\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.4, 0.4\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.4, 0.4\}, \{0.1, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ |

*By using Definition 17, we obtain the IHF-complement β-neighborhood values as presented in Table 3.*

**Table 3.** Intuitionistic hesitant fuzzy complementary β-neighborhood matrix.

| $\widetilde{M}^{\beta}_{G}$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.2,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.3,0.4\},\{0.1,0.2,0.3\}\rangle$ | $\langle\{0.2,0.4,0.5\},\{0.1,0.4,0.5\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.4,0.4\},\{0.1,0.3,0.4\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.1,0.3,0.6\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.3,0.6\},\{0.1,0.2,0.4\}\rangle$ | $\langle\{.1,0.4,0.4\},\{0.1,0.3,0.3\}\rangle$ | $\langle\{0.2,0.4,0.5\},\{0.1,0.4,0.4\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.1,0.3,0.5\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.3,0.5\},\{0.1,0.2,0.4\}\rangle$ | $\langle\{0.1,0.4,0.6\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.2,0.4,0.4\},\{0.1,0.3,0.4\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.2,0.4,0.5\},\{0.1,0.4,0.4\}\rangle$ | $\langle\{0.1,0.4,0.4\},\{0.1,0.3,0.3\}\rangle$ | $\langle\{0.3,0.4,0.5\},\{0.2,0.4,0.4\}\rangle$ | $\langle\{0.1,0.4,0.4\},\{0.1,0.3,0.5\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.1,0.4,0.6\},\{0.1,0.3,0.4\}\rangle$ | $\langle\{0.1,0.3,0.4\},\{0.1,0.2,0.3\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.4,0.4\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.5\}\rangle$ | $\langle\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\rangle$ |

**Definition 18.** *Assume that $(X,G)$ is an IHFCAS, $G=\{G_1,G_2,\ldots,G_m\}$ an intuitionistic hesitant fuzzy β-covering of X for some IHFN $\beta=(\Phi_\beta,\Psi_\beta)$, and $X=\{x_1,x_2,\ldots,x_n\}$. For every $\bar{A}\in IHF(X)$, the LA and UA of $\bar{A}$ with respect to $\mathbb{N}^{\beta}_{G}$ are represented as $\underline{\mathbb{N}^{\beta}_{G}}(\bar{A})$ and $\overline{\mathbb{N}^{\beta}_{G}}(\bar{A})$, respectively, which are two IHFSs in X, defined as*

$$\underline{\mathbb{N}^{\beta}_{G}}(\bar{A})=\{x,\Phi_{\underline{\mathbb{N}^{\beta}_{G}}(\bar{A})}(x),\Psi_{\underline{\mathbb{N}^{\beta}_{G}}(\bar{A})}(x)\} \tag{8}$$

*and*

$$\overline{\mathbb{N}^{\beta}_{G}}(\bar{A})=\{x,\Phi_{\overline{\mathbb{N}^{\beta}_{G}}(\bar{A})}(x),\Psi_{\overline{\mathbb{N}^{\beta}_{G}}(\bar{A})}(x)\} \tag{9}$$

*where $\Phi_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(x)=\underset{y\in X}{\barwedge}\{\Phi_{(\mathbb{N}^{\beta}_{x})^c}(y)\veebar\Phi_A(y)\}$ $\Psi_{\underline{\mathbb{N}^{\beta}_{G}}(A)}(x)=\underset{y\in X}{\barwedge}\{\Psi_{(\mathbb{N}^{\beta}_{x})^c}(y)\veebar\Psi_A(y)\}$ and $\Phi_{\overline{\mathbb{N}^{\beta}_{G}}(A)}(x)=\underset{y\in X}{\veebar}\{\Phi_{\mathbb{N}^{\beta}_{x}}(y)\barwedge\Phi_A(y)\}$ $\Psi_{\overline{\mathbb{N}^{\beta}_{G}}(A)}(x)=\underset{y\in X}{\veebar}\{\Psi_{\mathbb{N}^{\beta}_{x}}(y)\barwedge\Psi_{\bar{A}}(y)\}$. $\underline{\mathbb{N}^{\beta}_{G}}(A)$, $\overline{\mathbb{N}^{\beta}_{G}}(A): IHF(X)\to IHF(X)$ are called the first type of LA and UA approximation operators of $\bar{A}$ with respect to $\mathbb{N}^{\beta}_{G}$. The pair $(\underline{\mathbb{N}^{\beta}_{G}}(A),\overline{\mathbb{N}^{\beta}_{G}}(A))$ is said to be the first type of IHFβCRSs (I-IHFβCRSs).*

**Example 2** (continued from Example 1)**.** *Let*

$$\bar{A}=\left\{\begin{array}{c}\langle\varkappa_1,\{\{0.2,0.4,0.6\},\{0.1,0.3,0.4\}\}\rangle,\langle\varkappa_2,\{\{0.1,0.4,0.5\},\{0.2,0.4,0.5\}\}\rangle,\\ \langle\varkappa_3,\{\{0.3,0.4,0.5\},\{0.2,0.3,0.4\}\}\rangle,\langle\varkappa_4,\{\{0.4,0.5,0.6\},\{0.1,0.2,0.4\}\}\rangle,\\ \langle\varkappa_5,\{\{0.3,0.4,0.5\},\{0.2,0.4,0.5\}\}\rangle\end{array}\right\}$$

*By Definition 18, $\underline{\mathbb{N}^{\beta}_{G}}$ and $\overline{\mathbb{N}^{\beta}_{G}}$ are obtained as follows:*

$$\underline{\mathbb{N}^{\beta}_{G}}(A)=\left\{\begin{array}{c}\langle\varkappa_1,\{\{0.1,0.4,0.5\},\{0.1,0.3,0.5\}\}\rangle,\langle\varkappa_2,\{\{0.1,0.4,0.5\},\{0.1,0.3,0.4\}\}\rangle,\\ \langle\varkappa_3,\{\{0.1,0.4,0.5\},\{0.1,0.4,0.4\}\}\rangle,\langle\varkappa_4,\{\{0.1,0.4,0.5\},\{0.1,0.4,0.5\}\}\rangle,\\ \langle\varkappa_5,\{\{0.1,0.4,0.5\},\{0.1,0.4,0.4\}\}\rangle\end{array}\right\}$$

*and*

$$\overline{\mathbb{N}^{\beta}_{G}}(A)=\left\{\begin{array}{c}\langle\varkappa_1,\{\{0.2,0.4,0.5\},\{0.1,0.3,0.4\}\}\rangle,\langle\varkappa_2,\{\{0.2,0.4,0.5\},\{0.1,0.2,0.4\}\}\rangle,\\ \langle\varkappa_3,\{\{0.2,0.4,0.5\},\{0.1,0.4,0.4\}\}\rangle,\langle\varkappa_4,\{\{0.3,0.4,0.5\},\{0.1,0.3,0.5\}\}\rangle,\\ \langle\varkappa_5,\{\{0.2,0.4,0.5\},\{0.1,0.3,0.4\}\}\rangle\end{array}\right\}$$

**Definition 19.** *Assume that $(X,G)$ is an IHFCAS, $G=\{G_1,G_2,\ldots,G_m\}$ is an intuitionistic hesitant fuzzy β-covering of X for some IHFN $\beta=(\Phi_\beta,\Psi_\beta)$, and $X=\{x_1,x_2,\ldots,x_n\}$. For every $\bar{A}\in IHF(X)$, the LA and UA of $\bar{A}$ with respect to $\widetilde{M}^{\beta}_{x}$ denoted by $\underline{M^{\beta}_{G}}$ and $\overline{M^{\beta}_{G}}$, respectively, are IHFSs on X and defined as*

$$\underline{M^{\beta}_{G}}(\bar{A})=\left\{\left\langle x,\Phi_{\underline{M^{\beta}_{G}}(\bar{A})}(x),\Psi_{\underline{M^{\beta}_{G}}(\bar{A})}(x)\right\rangle|x\in X\right\} \tag{10}$$

*and*

$$\overline{M_G^\beta}(\bar{A}) = \left\{ \left( x, \left\langle \Phi_{\overline{M_G^\beta}(A)}(x), \Psi_{\overline{M_G^\beta}(A)}(x) \right\rangle \right) | x \in X \right\} \tag{11}$$

*where* $\Phi_{\underline{M_G^\beta}(A)}(x) = \underset{y\in X}{\overline{\wedge}} \{\Psi_{\overline{(M_x^\beta)^c(A)}}(y) \veebar \Phi_A(y)\}, \Psi_{\underline{M_G^\beta}(A)}(x) = \underset{y\in X}{\veebar} \{\Phi_{\overline{(M_x^\beta)^c(A)}}(y) \overline{\wedge} \Psi_A(y)\}$ *and* $\Phi_{\overline{M_G^\beta}(A)}(x) = \underset{y\in X}{\veebar} \{\Phi_{\overline{M_x^\beta}(A)}(y) \overline{\wedge} \Phi_A(y)\}, \Psi_{\overline{M_x^\beta}(A)}(x) = \underset{y\in X}{\overline{\wedge}} \{\Psi_{\overline{M_G^\beta}(A)}(y) \veebar \Psi_A(y)\}$. $\underline{M_G^\beta}(A)$; $\overline{M_G^\beta}(A) : IHF(X) \to IHF(X)$ *are said to be the second type of LA and UA operators of A with respect to* $\widetilde{M}_x^\beta$ *. The pair* ($\underline{M_G^\beta}(A), \overline{M_G^\beta}(A)$) *is called the second type of IHβCRSs (II-IHβCRSs).*

**Example 3** (continued from Example 1)**.** *Let*

$$A = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle, \\ \langle \varkappa_3, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.4, 0.5, 0.6\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

*By Definition 19,* $\underline{M_G^\beta}$ *and* $\overline{M_G^\beta}$ *are found out as follows:*

$$\underline{M_G^\beta}(\tilde{A}) = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

and

$$\overline{M}_G^\beta(\tilde{A}) = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

From Examples 2 and 3, we obtain the following: I-CIHFRS models and II-CIHFRS models do not directly relate, though the IHF $\beta$-covering induces the IHFC $\beta$-covering.

**Definition 20.** *Assume that* $(X, G)$ *is an IHFCAS,* $G = \{G_1, G_2, \ldots, G_m\}$ *is an intuitionistic hesitant fuzzy* $\beta$*-covering of X for some intuitionistic hesitant fuzzy number (IHFN)* $\beta = (\Phi_\beta, \Psi_\beta)$*, and* $X = \{x_1, x_2, \ldots, x_n\}$*. For every* $\bar{A} \in IHF(X)$*, the LA and UA of* $\bar{A}$ *with respect to* $\mathbb{N}_G^\beta$ *and* $\widetilde{M}_G^\beta$ *represented by* $\underline{K}_G^\beta$ *and* $\overline{K}_G^\beta$*, respectively, will be the IHFS on X defined by*

$$\underline{K}_G^\beta \bar{A} = \left\{ \left\langle x, \Phi_{\underline{K}_G^\beta(\bar{A})}(x), \Psi_{\underline{K}_G^\beta(\bar{A})}(x) \right\rangle | x \in X \right\} \tag{12}$$

*and*

$$\overline{K}_G^\beta \bar{A} = \left\{ \left\langle x, \Phi_{\overline{K}_G^\beta(\bar{A})}(x), \Psi_{\overline{K}_G^\beta(\bar{A})}(x) \right\rangle | x \in X \right\} \tag{13}$$

*where* $\Phi_{\underline{K}_G^\beta(\bar{A})}(x) = \underset{y\in X}{\overline{\wedge}} \left\{ \Phi_{(\widetilde{M}_G^\beta)^c}(y) \veebar \Phi_{(\mathbb{N}_x^\beta)^c}(y) \veebar \Phi\bar{A}(y) \right\}$, $\Psi_{\underline{K}_G^\beta(\bar{A})}(x) = \underset{y\in X}{\overline{\wedge}} \left\{ \Psi_{(\widetilde{M}_G^\beta)^c}(y) \veebar \Psi_{(\mathbb{N}_x^\beta)^c}(y) \veebar \Psi\bar{A}(y) \right\}$ *and* $\Phi_{\overline{K}_G^\beta(\bar{A})}(x) = \underset{y\in X}{\veebar} \left\{ \Phi_{\widetilde{M}_G^\beta(\bar{A})}(y) \overline{\wedge} \Phi_{(\mathbb{N}_x^\beta)}(y) \overline{\wedge} \Phi\bar{A}(y) \right\}$, $\Psi_{\overline{K}_G^\beta(\bar{A})}(x) = \underset{y\in X}{\veebar} \left\{ \Psi_{\widetilde{M}_G^\beta(A)}(y) \overline{\wedge} \Psi_{(\mathbb{N}_x^\beta)}(y) \overline{\wedge} \Psi\bar{A}(y) \right\}$. $\underline{K}_G^\beta(\bar{A}), \overline{K}_G^\beta(\bar{A}) : IHF(X) \to IHF(X)$*, the third type of LA and UA operators for set* $\bar{A}$ *with respect to* $\mathbb{N}_G^\beta$ *and* $\widetilde{M}_G^\beta$*. This pair (*$\underline{K}_G^\beta(A), \overline{K}_G^\beta(A)$*) is referred to as the IH βCRSs (III-IHβCRSs).*

**Example 4** (continued from Example 1). *Let*

$$\bar{A} = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle, \\ \langle \varkappa_3, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.4, 0.5, 0.6\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

*By Definition 20, $\underline{K}_G^\beta$ and $\overline{K}_G^\beta$ are calculated as follows:*

$$\underline{K}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

*and*

$$\overline{K}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

**Definition 21.** *Assume that $(X, G)$ is an IHFCAS; let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy $\beta$-covering of $X$ for some IHFN $\beta = (\Phi_\beta, \Psi_\beta)$ and $X = \{x_1, x_2, \ldots, x_n\}$. For every $\tilde{A} \in IHF(X)$, the LA and UA of $\tilde{A}$ with respect to $\mathbb{N}_G^\beta$ and $\widetilde{M}_G^\beta$ represented by $\underline{F_G^\beta}$ and $\overline{F_G^\beta}$, respectively, are the IHFSs on X defined as*

$$\underline{F_G^\beta} = \left\{ \left\langle x, \Phi_{\underline{F_G^\beta}(\tilde{A})}(x), \Psi_{\underline{F_G^\beta}(\tilde{A})}(x) \right\rangle | x \in X \right\} \tag{14}$$

*and*

$$\overline{F_G^\beta} = \left\{ \left\langle x, \Phi_{\overline{F_G^\beta}(\tilde{A})}(x), \Psi_{\overline{F_G^\beta}(\tilde{A})}(x) \right\rangle | x \in X \right\} \tag{15}$$

*where*

$$\Phi_{\underline{F_G^\beta}(\tilde{A})}(x) = \mathop{\overline{\wedge}}_{y \in X} \left\{ \Phi_{(\widetilde{M}_G^\beta)^c}(y) \,\overline{\wedge}\, \Phi_{(\mathbb{N}_x^\beta)^c}(y) \,\underline{\vee}\, \Phi\tilde{A}(y) \right\},$$

$$\Psi_{\underline{F_G^\beta}(\tilde{A})}(x) = \mathop{\overline{\wedge}}_{y \in X} \left\{ \Psi_{(\widetilde{M}_G^\beta)^c}(y) \,\overline{\wedge}\, \Psi_{(\mathbb{N}_x^\beta)^c}(y) \,\underline{\vee}\, \Psi\tilde{A}(y) \right\}$$

*and*

$$\Phi_{\overline{F}_G^\beta(\tilde{A})}(x) = \mathop{\underline{\vee}}_{y \in X} \left\{ \Phi_{\widetilde{M}_G^\beta(\tilde{A})}(y) \,\underline{\vee}\, \Phi_{(\mathbb{N}_x^\beta)}(y) \,\overline{\wedge}\, \Phi\tilde{A}(y) \right\},$$

$$\Psi_{\overline{F}_G^\beta(\tilde{A})}(x) = \mathop{\underline{\vee}}_{y \in X} \left\{ \Psi_{\widetilde{M}_G^\beta(\tilde{A})}(y) \,\underline{\vee}\, \Psi_{(\mathbb{N}_x^\beta)}(y) \,\overline{\wedge}\, \Psi\tilde{A}(y) \right\}.$$

$\underline{F_G^\beta}(\tilde{A}), \overline{F_G^\beta}(\tilde{A}) : IHF(X) \to IHF(X)$. The fourth type of LA and UA operators for set $\bar{A}$ with respect to $\mathbb{N}_G^\beta$ and $\widetilde{M}_G^\beta$ are termed $(\underline{F_G^\beta}(\bar{A}), \overline{F_G^\beta}(\tilde{A}))$. This pair is called the fourth type of IH$\beta CRSs(IV - IH\beta CRSs)$.

**Example 5** (continued from Example 1). *Let*

$$\bar{A} = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle, \\ \langle \varkappa_3, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.4, 0.5, 0.6\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.3, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

*By Definition 21, $\underline{F}_G^\beta$ and $\overline{F}_G^\beta$ are calculated as follows:*

$$\underline{F}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle \end{array} \right\}$$

*and*

$$\overline{F}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

**Theorem 1.** *For every $x$ in $X$, the outcomes for the degrees of $\check{m}$ and $\check{n}\check{m}$ in relation to the approximation operators are as follows:*

$$i)\ 0 \leq \Phi_{\underline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\underline{M}_x^\beta(\tilde{A})}(y) \leq 1$$

$$ii)\ 0 \leq \Phi_{\overline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\overline{M}_x^\beta(\tilde{A})}(y) \leq 1$$

**Proof.** First, we prove the inequality (i) by Definition 17. For any $x \in X$, it is obvious that $0 \leq \Phi_{\underline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\underline{M}_x^\beta(\tilde{A})}(y)$. On the other side, we obtain

$$\begin{aligned} 1 - \Psi_{\underline{M}_x^\beta(\tilde{A})}(y) &= 1 - \bigwedge\left\{\Phi_{M_x^\beta(\tilde{A})}(y) \vee \Psi\tilde{A}(y)\right\} \\ &= \bigwedge\left\{(1 - \Phi_{M_x^\beta(\tilde{A})}(y)) \vee (1 - \Psi\tilde{A}(y))\right\} \\ &\geq \bigwedge\left\{\Psi_{M_x^\beta(\tilde{A})}(y) \vee \Phi\tilde{A}(y)\right\} \\ &= \Phi_{\underline{M}_x^\beta(\tilde{A})}(y) \end{aligned}$$

This shows that

$$0 \leq \Phi_{\underline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\underline{M}_x^\beta(\tilde{A})}(y) \leq 1$$

Similarly, as by Definition 17, for any $x \in X$, it is obvious that $0 \leq \Phi_{\overline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\overline{M}_x^\beta(\tilde{A})}(y)$. On the other side, we obtain

$$\begin{aligned} 1 - \Psi_{\overline{M}_x^\beta(\tilde{A})}(y) &= 1 - \bigvee\left\{\Phi_{M_x^\beta(\tilde{A})}(y) \wedge \Psi\tilde{A}(y)\right\} \\ &= \bigvee\left\{(1 - \Phi_{M_x^\beta(\tilde{A})}(y)) \wedge (1 - \Psi\tilde{A}(y))\right\} \\ &\geq \bigvee\left\{\Psi_{M_x^\beta(\tilde{A})}(y) \wedge \Phi\tilde{A}(y)\right\} \\ &= \Phi_{\overline{M}_x^\beta(\tilde{A})}(y) \end{aligned}$$

This shows that

$$0 \leq \Phi_{\overline{M}_x^\beta(\tilde{A})}(y) + \Psi_{\overline{M}_x^\beta(\tilde{A})}(y) \leq 1$$

□

**Theorem 2.** *Suppose (X, G) constitutes an approximation space known as an IHFCAS.*

$$G = \{G_1, G_2, \ldots, G_m\}$$

*is an intuitionistic hesitant fuzzy β-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Assume that $\widetilde{M}_x^\beta = \{M_x^\beta | x \in X\}$ is the intuitionistic hesitant fuzzy-complement β-neighborhood system induced by G. For any $P, Q \in IHF(X)$, then*

$$i)\ If\, P \sqsubseteq Q, then\ \underline{M_G^\beta}(P) \sqsubseteq \underline{M_G^\beta}(Q)\ and\ \overline{M_G^\beta}(Q) \sqsubseteq \overline{M_G^\beta}(Q).$$

$$ii)\ \underline{M_G^\beta}(P \cap Q) = \underline{M_G^\beta}(P) \cap \underline{M_G^\beta}(Q)\ and\ \underline{M_G^\beta}(P \cup Q) \supseteq \underline{M_G^\beta}(P) \cup \underline{M_G^\beta}(Q).$$

$$iii)\ \overline{M_G^\beta}(P \cup Q) = \overline{M_G^\beta}(P) \cup M_G^\beta(Q)\ and\ \overline{M_G^\beta}(P \cap Q) \sqsubseteq \overline{M_G^\beta}(P) \cap M_G^\beta(Q).$$

$$iv) \left(\underline{M_G^\beta}(P)\right)^c = \left(\overline{M_G^\beta}(P)^c\right)\ and\ \left(\overline{M_G^\beta}(P)\right)^c = \left(\underline{M_G^\beta}(P)^c\right).$$

Various distances between IFSs were introduced by several authors (Wang and Xin [34]; Xu and Yager [35]; Szmidt and Kacprzyk [36]). Here, we utilize Wang and Xin's [34] idea of IFS distances for IHFS distance in the following way:

**Definition 22.** *Assume that $(X, G)$ is an IHF-covering approximation space. Suppose that $A, B \in IHFS(X)$, where $A = \{\langle x, \Phi_A(x), \Psi_A(x)\rangle | x \in X\}$, $B = \{\langle x, \Phi_B(x), \Psi_B(x)\rangle | x \in X\}$. Define the distance $D(A, B)$ for A and B as follows:*

$$D(A,B) = \frac{1}{|X|}\sum\left(\frac{\sum|\Phi_A(x) - \Phi_B(x)| + \sum|\Psi_A(x) - \Psi_B(x)|}{4} + \frac{\max\{\sum|\Phi_A(x) - \Phi_B(x)|, \sum|\Psi_A(x) - \Psi_B(x)|\}}{2}\right)$$

**Definition 23.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy β-covering of X. For some IHFN, $\beta = (\Phi_\beta, \Psi_\beta)$. Consider the IHF β-neighborhood system $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(\varkappa)}^\beta \mid \varkappa \in X\right\}$ induced by G. For any $A \in IHF(X)$, $\overline{\mathbb{N}_G^\beta}(A)$ and $\underline{\mathbb{N}_G^\beta}(A)$ are the UA and LA of A in relation to $\mathbb{N}_G^\beta$, respectively. We define the rough and precision degrees of A with respect to $\mathbb{N}_G^\beta$, respectively, as follows:*

$$R_{\mathbb{N}_G^\beta(A)} = D\left(\underline{\mathbb{N}_G^\beta}(A), \overline{\mathbb{N}_G^\beta}(A)\right) \tag{16}$$

*and*

$$P_{\mathbb{N}_G^\beta(A)} = 1 - R_{\mathbb{N}_G^\beta(A)} \tag{17}$$

**Example 6** (continued from Example 2)**.**

$$\underline{\mathbb{N}_G^\beta}(A) = \left\{\begin{array}{c}\langle\varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle\varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle\varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \langle\varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \\ \langle\varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle\end{array}\right\}$$

and

$$\overline{\mathbb{N}_G^\beta}(A) = \left\{\begin{array}{c}\langle\varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle\varkappa_2, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle\varkappa_3, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \langle\varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle\varkappa_5, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle\end{array}\right\}$$

Applying the provided definition, we derive the roughness and precision degrees for set $A$ in relation to $\mathbb{N}_G^\beta$, respectively, as follows: $R_{\mathbb{N}_G^\beta(A)} = 0.08$ and$P_{\mathbb{N}_G^\beta(A)} = 0.91$.

**Definition 24.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy β-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Assume that $M_G^\beta = \left\{M_{G(\varkappa)}^\beta | \varkappa \in X\right\}$ is an*

*IHFCβ−neighborhood system induced by G. For any $A \in IHF(X)$, $\overline{M_G^\beta}(A)$ and $\underline{M_G^\beta}(A)$ are the UA and LA of A in relation to $\mathbb{N}_G^\beta$, respectively. We define the rough and precision degrees of A with respect to $M_G^\beta$, respectively, as follows:*

$$R_{M_G^\beta(A)} = D\left(\underline{M_G^\beta}(A), \overline{M_G^\beta}(A)\right)$$

*and*

$$P_{M_G^\beta(A)} = 1 - R_{M_G^\beta(A)}$$

**Example 7** (continued from Example 2)**.**

$$\underline{M_G^\beta}(A) = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

and

$$\overline{M}_G^\beta(A) = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{0.1, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

Applying the provided definition, we derive the roughness and precision degrees for set $A$ in relation to $M_G^\beta$, respectively, as follows: $R_{M_G^\beta(\tilde{A})} = 0.10$ and $P_{M_G^\beta(\tilde{A})} = 0.90$.

From Examples 6 and 7, we obtain the following:

Any IHFS has distinct rough and precision degrees in relation to $M_G^\beta$ and $\mathbb{N}_G^\beta$, respectively.

**Definition 25.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy β-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Assume that $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(\varkappa)}^\beta | \varkappa \in X\right\}$ is a β-neighborhood IHF system and $M_G^\beta = \left\{M_{G(x)}^\beta | x \in X\right\}$ is an IHFCβ − neighborhood system induced by G, respectively. For any $\tilde{A} \in IHF(X)$, $\underline{K}_G^\beta(\tilde{A})$ and $\overline{K}_G^\beta(\tilde{A})$ are the upper and lower approximations of $\tilde{A}$ with respect to $\mathbb{N}_G^\beta$ and $M_G^\beta$, respectively. We define the rough and precision degrees of $\tilde{A}$ with respect to $K_G^\beta$, respectively, as follows: $R_{K_G^\beta(\tilde{A})} = D\left(\underline{K_G^\beta}(\tilde{A}), \overline{K_G^\beta}(\tilde{A})\right)$ and $P_{K_G^\beta(\tilde{A})} = 1 - R_{K_G^\beta(\tilde{A})}$.*

**Example 8** (continued from Example 4)**.**

$$\underline{K}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.5\}\}\rangle \end{array} \right\}$$

*and*

$$\overline{K}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.2, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

*Applying the provided definition, we derive the roughness and precision degrees for set $A$ in relation to $K_G^\beta$, respectively, as follows: $R_{K_G^\beta(\tilde{A})} = 0.13$ and $P_{K_G^\beta(\tilde{A})} = 0.86$.*

**Definition 26.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy $\beta$-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Assume that $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(\varkappa)}^\beta | \varkappa \in X\right\}$ is an IHF$\beta -$ neighborhood system and $M_G^\beta = \left\{M_{G(x)}^\beta | x \in X\right\}$is an IHFC$\beta$-neighborhood system induced by G, respectively. For any $A \in IHF(X)$, $\underline{F}_G^\beta(A)$ and $\overline{F}_G^\beta(A)$ are the upper and lower approximations of A with respect to $\mathbb{N}_G^\beta$ and $M_G^\beta$, respectively. We define the rough and precision degrees of A with respect to $F_G^\beta$, respectively, as follows: $R_{F_G^\beta(A)} = D\left(\underline{F_G^\beta}(A), \overline{F_G^\beta}(A)\right)$ and $P_{F_G^\beta(A)} = 1 - R_{F_G^\beta(A)}$.*

**Example 9** (continued from Example 5).

$$\underline{F}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \langle \varkappa_2, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \\ \langle \varkappa_5, \{\{0.1, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle \end{array} \right\}$$

*and*

$$\overline{F}_G^\beta = \left\{ \begin{array}{c} \langle \varkappa_1, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\}\rangle, \langle \varkappa_2, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle, \\ \langle \varkappa_3, \{\{0.2, 0.4, 0.5\}, \{0.1, 0.4, 0.4\}\}\rangle, \langle \varkappa_4, \{\{0.3, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}\}\rangle, \\ \langle \varkappa_5, \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle \end{array} \right\}$$

*Applying the provided definition, we derive the roughness and precision degrees for set A in relation to $F_G^\beta$, respectively, as follows: $R_{F_G^\beta(\tilde{A})} = 0.116$ and $P_{F_G^\beta(\tilde{A})} = 0.883$.*

**Theorem 3.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy $\beta$-covering of X. For some IHFN $\beta = (\Phi_\beta, \Psi_\beta)$. Consider that $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(x)}^\beta | x \in X\right\}$ and $\widetilde{M}_G^\beta = \left\{\widetilde{M}_{G(x)}^\beta | x \in X\right\}$ are an IHF$\beta -$ neighborhood system induced by G, respectively. Suppose that $A, B \in IHFS(X)$, then*

$$\begin{aligned} i)\ A &\sqsubseteq B, \text{ then } \underline{K}_G^\beta(A) \sqsubseteq \underline{K}_G^\beta(B) \text{ and } \overline{K}_G^\beta(A) \sqsubseteq \overline{K}_G^\beta(B); \\ ii)\ \underline{K}_G^\beta(A \cap B) &= \underline{K}_G^\beta(A) \cap \underline{K}_G^\beta(B) \text{ and } \underline{K}_G^\beta(A \cup B) \sqsupseteq \underline{K}_G^\beta(A) \cup \underline{K}_G^\beta(A); \\ iii)\ \overline{K}_G^\beta(A \cup B) &= \overline{K}_G^\beta(A) \cup \overline{K}_G^\beta(B) \text{ and } \overline{K}_G^\beta(A \cap B) \sqsubseteq \overline{K}_G^\beta(A) \cap \overline{K}_G^\beta(B); \\ iv)\ \left(\underline{K}_G^\beta(A)\right)^c &= \overline{K}_G^\beta(A)^c \text{ and } \left(\overline{K}_G^\beta(A)\right)^c = \underline{K}_G^\beta(A)^c. \end{aligned}$$

**Proof.** Assume that $A = \{\langle x, \Phi_A(x)\ , \Psi_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \Phi_B(x)\ , \Psi_B(x)\rangle | x \in X\}$, then the following holds:

(i) If $A \sqsubseteq B$, then $\Phi_A(x) \leq \Phi_B(x)$ and $\Psi_A(x) \geq \Psi_B(x)$, and then

$$\begin{aligned} \Phi_{\overline{K_G^\beta(A)}}(x) &= \bigvee\left\{\left(\Phi_{\widetilde{M}_{G(x)}^\beta}(x) \barwedge \Phi_{\mathbb{N}_{G(x)}^\beta}(x) \barwedge \Phi_A(x)\right)\right\} \leq \bigvee\left\{\left(\Phi_{\widetilde{M}_{G(x)}^\beta}(x) \barwedge \Phi_{\mathbb{N}_{G(x)}^\beta}(x) \barwedge \Phi_B(x)\right)\right\} = \\ \Phi_{\overline{K_G^\beta(B)}}(x) & \\ &\Rightarrow \Phi_{\overline{K_G^\beta(A)}}(x) \leq \Phi_{\overline{K_G^\beta(B)}}(x) \end{aligned}$$

In the same manner, we also show that

$$\Psi_{\overline{K_G^\beta(A)}}(x) \geq \Psi_{\overline{K_G^\beta(B)}}(x)$$

holds. This states that

$$\overline{K}_G^\beta(A) \sqsubseteq \overline{K}_G^\beta(B)$$

Similar to the way shown above, we prove that

$$\underline{K}_G^{\beta}(A) \sqsubseteq \underline{K}_G^{\beta}(B)$$

Hence, we have proved (i).

(ii)

$$\Phi_{\underline{K}_G^{\beta}(A\cap B)}(x) = \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_{(A\cap B)}(x)\right)\right\}$$
$$= \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_A(x) \cap \Phi_B(x)\right)\right\}$$
$$= \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_A(x)\right)\right\} \cap \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_B(x)\right)\right\}$$
$$= \Phi_{\underline{K}_G^{\beta}(A)}(x) \cap \Phi_{\underline{K}_G^{\beta}(B)}(x)$$
$$\Rightarrow \Phi_{\underline{K}_G^{\beta}(A\cap B)}(x) = \Phi_{\underline{K}_G^{\beta}(A)}(x) \cap \Phi_{\underline{K}_G^{\beta}(B)}(x)$$

In the same manner, we also show that

$$\Psi_{\underline{K}_G^{\beta}(A\cap B)}(x) = \Psi_{\underline{K}_G^{\beta}(A)}(x) \cap \Psi_{\underline{K}_G^{\beta}(B)}(x)$$

holds. This states that

$$\underline{K}_G^{\beta}(A \cap B) = \underline{K}_G^{\beta}(A) \cap \underline{K}_G^{\beta}(B)$$

Similar to the way shown above, we prove that

$$\underline{K}_G^{\beta}(A \cup B) \sqsupseteq \underline{K}_G^{\beta}(A) \cup \underline{K}_G^{\beta}(A)$$

(iii) The proof shares similarities with (ii).

(iv)

$$\Phi_{\underline{K}_G^{\beta}(A^c)}(x) = \bigwedge\left\{\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_{A^c}(x)\right\}$$
$$= \bigwedge\left\{\Psi_{\widetilde{M}^{\beta}_{G(x)}}(x) \veebar \Psi_{\mathbb{N}^{\beta}_{G(x)}}(x) \veebar \Phi_B(x)\right\}$$
$$= \Psi_{\underline{K}_G^{\beta}(A)}(x).$$

In the same manner, we also show that

$$\Psi_{\underline{K}_G^{\beta}(A^c)}(x) = \Phi_{\underline{K}_G^{\beta}(A)}(x)$$

holds. This states that

$$\left(\underline{K}_G^{\beta}(A)\right)^c = \overline{K}_G^{\beta}(A)^c$$

Similar to the way shown above, we prove that

$$\left(\overline{K}_G^{\beta}(A)\right)^c = \underline{K}_G^{\beta}(A)^c$$

□

**Theorem 4.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy β-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Consider that $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(x)}^\beta | x \in X\right\}$ and $\widetilde{M}_G^\beta = \left\{\widetilde{M}_{G(x)}^\beta | x \in X\right\}$ are an IHFβ − neighborhood system induced by G, respectively. Suppose that $A, B \in IHFS(X)$, then*

$$\begin{aligned} i)\ A &\sqsubseteq B, \text{ then } \underline{F}_G^\beta(A) \sqsubseteq \underline{F}_G^\beta(B) \text{ and } \overline{F}_G^\beta(A) \sqsubseteq \overline{F}_G^\beta(B); \\ ii)\ \underline{F}_G^\beta(A \cap B) &= \underline{F}_G^\beta(A) \cap \underline{F}_G^\beta(B) \text{ and } \underline{F}_G^\beta(A \cup B) \sqsupseteq \underline{F}_G^\beta(A) \cup \underline{F}_G^\beta(A); \\ iii)\ \overline{F}_G^\beta(A \cup B) &= \overline{F}_G^\beta(A) \cup \overline{F}_G^\beta(B) \text{ and } \overline{F}_G^\beta(A \cap B) \sqsubseteq \overline{F}_G^\beta(A) \cap \overline{F}_G^\beta(B); \\ iv)\ \left(\underline{F}_G^\beta(A)\right)^c &= \overline{F}_G^\beta(A)^c \text{ and } \left(\overline{F}_G^\beta(A)\right)^c = \underline{F}_G^\beta(A)^c. \end{aligned}$$

**Proof.** Assume that $A = \{\langle x, \Phi_A(x), \Psi_A(x)\rangle | x \in X\}$ *and* $B = \{\langle x, \Phi_B(x), \Psi_B(x)\rangle | x \in X\}$, then

$$A = \{\langle x, \Phi_A(x), \Psi_A(x)\rangle | x \in X\} \text{ and } B = \{\langle x, \Phi_B(x), \Psi_B(x)\rangle | x \in X\}$$

(i) If $A \sqsubseteq B$, then $\Phi_A(x) \le \Phi_B(x)$ and $\Psi_A(x) \ge \Psi_B(x)$, and then

$$\Phi_{\overline{F_G^\beta(A)}}(x) = \bigvee\left\{\left(\Phi_{\widetilde{M}_{G(x)}^\beta}(x) \barwedge \Phi_{\mathbb{N}_{G(x)}^\beta}(x) \barwedge \Phi_A(x)\right)\right\}$$
$$\le \bigvee\left\{\left(\Phi_{\widetilde{M}_{G(x)}^\beta}(x) \barwedge \Phi_{\mathbb{N}_{G(x)}^\beta}(x) \barwedge \Phi_B(x)\right)\right\}$$
$$= \Phi_{\overline{F_G^\beta(B)}}(x)$$
$$\Rightarrow \Phi_{\overline{F_G^\beta(A)}}(x) \le \Phi_{\overline{F_G^\beta(B)}}(x)$$

In the same manner, we also show that

$$\Psi_{\overline{F_G^\beta(A)}}(x) \ge \Psi_{\overline{F_G^\beta(B)}}(x)$$

holds. This states that

$$\overline{F}_G^\beta(A) \sqsubseteq \overline{F}_G^\beta(B)$$

Similar to the way shown above, we prove that

$$\underline{F}_G^\beta(A) \sqsubseteq \underline{F}_G^\beta(B)$$

Hence, we have proved (i).

(ii)

$$\Phi_{\underline{F}_G^\beta(A \cap B)}(x) = \bigwedge\left\{\left(\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_{(A \cap B)}(x)\right)\right\}$$
$$= \bigwedge\left\{\left(\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_A(x) \cap \Phi_B(x)\right)\right\}$$
$$= \bigwedge\left\{\left(\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_A(x)\right)\right\} \cap \bigwedge\left\{\left(\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_B(x)\right)\right\}$$
$$\begin{aligned} &= \Phi_{\underline{F}_G^\beta(A)}(x) \cap \Phi_{F_G^\beta(B)}(x) \\ &\Rightarrow \Phi_{\underline{F}_G^\beta(A \cap B)}(x) = \Phi_{\underline{F}_G^\beta(A)}(x) \cap \Phi_{\underline{K}_G^\beta(B)}(x) \end{aligned}$$

In the same manner, we also show that

$$\Psi_{\underline{F}_G^\beta(A \cap B)}(x) = \Psi_{\underline{F}_G^\beta(A)}(x) \cap \Psi_{\underline{F}_G^\beta(B)}(x)$$

holds. This states that

$$\underline{F}_G^\beta(A \cap B) = \underline{F}_G^\beta(A) \cap \underline{F}_G^\beta(B)$$

Similar to the way shown above, we prove that

$$\underline{F}_G^\beta(A \cup B) \sqsupseteq \underline{F}_G^\beta(A) \cup \underline{F}_G^\beta(A)$$

(iii) The proof shares similarities with (ii).

(iv)

$$\begin{aligned}\Phi_{\underline{F}_G^\beta(A^c)}(x) &= \bigwedge\left\{\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_{A^c}(x)\right\}\\ &= \bigwedge\left\{\Psi_{\widetilde{M}_{G(x)}^\beta}(x) \veebar \Psi_{\mathbb{N}_{G(x)}^\beta}(x) \veebar \Phi_B(x)\right\}\\ &= \Psi_{F_G^\beta(A)}(x).\end{aligned}$$

In the same manner, we also show that

$$\Psi_{\underline{F}_G^\beta(A^c)}(x) = \Phi_{F_G^\beta(A)}(x)$$

holds. This states that

$$\left(\underline{F}_G^\beta(A)\right)^c = \overline{F}_G^\beta(A)^c$$

Similar to the way shown above, we prove that

$$\left(\overline{F}_G^\beta(A)\right)^c = \underline{F}_G^\beta(A)^c$$

□

**Theorem 5.** *Assume that $(X, G)$ is an IHF-covering approximation space (IHFCAS), and let $G = \{G_1, G_2, \ldots, G_m\}$ be an intuitionistic hesitant fuzzy β-covering of X. For some intuitionistic hesitant fuzzy number (IHFN), $\beta = (\Phi_\beta, \Psi_\beta)$. Consider that $\mathbb{N}_G^\beta = \left\{\mathbb{N}_{G(\varkappa)}^\beta | \varkappa \in X\right\}$ and $\widetilde{M}_G^\beta = \left\{\widetilde{M}_{G(\varkappa)}^\beta | \varkappa \in X\right\}$ are an IHFβ − neighborhood system induced by G, respectively. Suppose that $A \in IHFS(X)$, then*

$$i)\ \underline{F}_G^\beta(A) \sqsubseteq \underline{\mathbb{N}}_G^\beta(A) \sqsubseteq \underline{K}_G^\beta(A);$$

$$ii)\ \underline{F}_G^\beta(A) \sqsubseteq \underline{\widetilde{M}}_G^\beta(A) \sqsubseteq \underline{K}_G^\beta(A);$$

$$iii)\ \overline{K}_G^\beta(A) \sqsubseteq \overline{\mathbb{N}}_G^\beta(A) \sqsubseteq \overline{F}_G^\beta(A);$$

$$iv)\ \overline{K}_G^\beta(A) \sqsubseteq \overline{\widetilde{M}}_G^\beta(A) \sqsubseteq \overline{F}_G^\beta(A);$$

$$v)\ \underline{F}_G^\beta(A) = \underline{\mathbb{N}}_G^\beta(A) \cap \underline{\widetilde{M}}_G^\beta(A);$$

$$vi)\ \overline{F}_G^\beta(A) = \overline{\mathbb{N}}_G^\beta(A) \cup \overline{\widetilde{M}}_G^\beta(A);$$

$$vii)\ \underline{K}_G^\beta(A) = \underline{\mathbb{N}}_G^\beta(A) \cup \underline{\widetilde{M}}_G^\beta(A);$$

$$viii)\ \overline{K}_G^\beta(A) = \overline{\mathbb{N}}_G^\beta(A) \cap \overline{\widetilde{M}}_G^\beta(A).$$

**Proof.** Statements (i)–(iv) are obviously true from the definitions and now statement (v). For any $\varkappa \in X$, then

$$\Phi_{\underline{F}^{\beta}_{G}(A)}(\varkappa) = \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{A}(\varkappa)\right)\right\}$$
$$\bigwedge\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{A}(\varkappa)\right)\right\}$$

$$= \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{A}(\varkappa)\right)\right\} \wedge \bigwedge\left\{\left(\Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{A}(\varkappa)\right)\right\} \bigwedge\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{A}(\varkappa)\right)\right\} \wedge \bigwedge\left\{\Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{A}(\varkappa)\right\}$$
$$= \left(\Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)$$

and

$$\Psi_{\underline{F}^{\beta}_{G}(A)}(\varkappa) = \bigvee\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right)\right\} \bigvee\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right)\right\}$$

$$= \bigvee\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right)\right\} \vee \bigvee\left\{\left(\Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right)\right\} \bigvee\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right)\right\} \vee \bigvee\left\{\Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right\}$$
$$= \left(\Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)$$

Hence, we obtain

$$\underline{F}^{\beta}_{G}(A) = \left\{\left\langle \varkappa, \Phi_{\underline{F}^{\beta}_{G}(A)}(\varkappa), \Psi_{\underline{F}^{\beta}_{G}(A)}(\varkappa)\right\rangle\right\}$$

$$\underline{F}^{\beta}_{G}(A) = \left\{\left\langle \begin{array}{c} \varkappa, \left(\Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right), \\ \left(\Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right) \end{array}\right\rangle\right\}$$

$$\underline{F}^{\beta}_{G}(A) = \left\{\left\langle \varkappa, \left(\Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\underline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa)\right)\right\rangle\right\} \cap$$
$$\left\{\left\langle \varkappa, \left(\Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\underline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)\right\rangle\right\}$$
$$= \underline{\widetilde{M}}^{\beta}_{G}(A) \cap \underline{\mathbb{N}}^{\beta}_{G}(A)$$

Hence, this shows that statement (v) holds.

(vi) For any $\varkappa \in X$, then

$$\Phi_{\overline{F}^{\beta}_{G}(A)}(\varkappa) = \bigvee\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right)\right\} \bigvee\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right)\right\}$$

$$= \bigvee\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right)\right\} \vee \bigvee\left\{\left(\Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{A}(\varkappa)\right)\right\} \bigvee\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right)\right\} \vee \bigvee\left\{\Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{A}(\varkappa)\right\}$$
$$= \left(\Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)$$

and

$$\Psi_{\overline{F}^{\beta}_{G}(A)}(\varkappa) = \bigwedge\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Phi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_{A}(\varkappa)\right)\right\} \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \wedge \Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_{A}(\varkappa)\right)\right\}$$

$$= \bigwedge\left\{\left(\Phi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_A(\varkappa)\right)\right\} \wedge \bigwedge\left\{\left(h_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Psi_A(\varkappa)\right)\right\} \bigwedge\left\{\left(\Psi_{\widetilde{M}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_A(\varkappa)\right)\right\} \wedge \bigwedge\left\{\Psi_{\mathbb{N}^{\beta}_{G(\varkappa)}}(\varkappa) \vee \Phi_A(\varkappa)\right\}$$

$$= \left(\Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)$$

Hence, we obtain

$$\overline{F}^{\beta}_{G}(A) = \left\{\left\langle \varkappa, \Phi_{\overline{F}^{\beta}_{G}(A)}(\varkappa), \Psi_{\overline{F}^{\beta}_{G}(A)}(\varkappa)\right\rangle\right\}$$

$$\overline{F}^{\beta}_{G}(A) = \left\{\left\langle \begin{array}{c} \varkappa, \left(\Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right), \\ \left(\Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right) \end{array} \right\rangle\right\}$$

$$\overline{F}^{\beta}_{G}(A) = \left\{\left\langle \varkappa, \left(\Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\overline{\widetilde{M}}^{\beta}_{G}(A)}(\varkappa)\right)\right\rangle\right\}$$

$$\bigcup\left\{\left\langle \varkappa, \left(\Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa) \vee \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa), \Psi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa) \wedge \Phi_{\overline{\mathbb{N}}^{\beta}_{G}(A)}(\varkappa)\right)\right\rangle\right\}$$

$$= \overline{\widetilde{M}}^{\beta}_{G} \cup \overline{\mathbb{N}}^{\beta}_{G}(A).$$

Hence, this shows that statement (vi) holds and that (vii) and (viii) are similar to (v). □

## 4. The Model and Approach for MCDM Involving the Evolution of IHF Information, Based on CIHFRS

MCDM presents a substantial challenge in DM, aiming to identify the most optimal alternative by evaluating diverse criteria during the selection process. Also referred to as multicriteria decision analysis (MCDA), this methodology is widely recognized for its high accuracy and is considered a groundbreaking advancement in the field of decision making [37,38]. Intuitionistic hesitant fuzzy sets (IHFSs) are acknowledged as an expanded version of IFSs, encompassing both HFSs and IFSs. To showcase the enhanced decision-making capabilities of IHFSs, a comparative analysis is conducted using the CIFRS model and the Yang–CHFRS versions established in Sections 2.1 and 2.2, respectively. In this context, two distinct models and approaches are developed to tackle MCDM problems. The first approach involves assessing IHF information, considering the nuances provided by IHFS. The second approach addresses MCDM problems utilizing the advantages of both IHFS and conventional IFS/HFS procedures. These customized approaches assess IFS and HFS data independently in order to offer all-inclusive answers for decision-making situations.

### *4.1. A Method for MCDM with Evaluation of IHF Data Based on CIFRS and CHFRS Models*

Using the CIFRS and CHFRS models as a starting point, we want to develop a solid model and approach that are especially suited to solving MCDM challenges. Using the complex subtleties offered by IHF data in DM processes, this model considers the appraisal of IHF information. We develop an organized way for solving MCDM issues by smoothly incorporating the analysis of IHF data. This entails creating an algorithm for generating decisions that fully integrates IHF data and making sure that everything is understood by providing a detailed breakdown of the steps involved in the procedure. Our aim is to deliver a workable and efficient solution that improves decisions in uncertain and complex contexts by utilizing insights of CIFRS and CHFRS models, as well as extending these to deal with IHF information inside MCDM frameworks.

The Problem of MCDM with Assessment of IHF Data-Based DMM

Assume that $X = \{x_1, x_2, \ldots, x_n\}$ is an alternative set, and let $\bar{G} = \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_m\}$ be the set of criteria. Let $\varpi = \{\varpi_1, \varpi_2, \ldots, \varpi_m\}^t$ be the weighted vectors of $\bar{G}$. Given that $0 \leq \varpi_i \leq 1$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \varpi_i = 1$, let decision maker $X$ provide the evaluation values of the objects $x_i$., where $i = 1, 2, \ldots, m$ with respect to the attribute set $\bar{G}_j$, where $j = 1, 2, \ldots, m$ by $\bar{G}_j(x_i) = \langle \Phi_{ij}, \Psi_{ij} \rangle$, which means that the degree to which $x_i$ holds $\bar{G}_j$ is the value $\Phi_{ij}$ and the degree to which $x_i$ does not hold $\bar{G}_j$ is the value $\Psi_{ij}$. If $\Psi_{ij}$ symbolizes the value, the difficulty in solving the DM issue is figuring out how to rank each object according to preference assessments. An innovative answer to various types of DM problems is provided in this section. Here, we provide the groundwork for an approach and method for dealing with MCDM instances, particularly the inclusion of IHF data evaluation. The CIFRS paradigm is utilized in combination with CHFRSs to complete the full strategy. Our objective is to utilize IHF data to handle the complexities of MCDM, which provides an intricate outlook for making decisions. Using the CIFRS and CHFRS models, we guarantee an in-depth review that analyzes the complexity and opposition found in DM scenarios.

*4.2. DMM and Procedural Methods*

We suggest the CIHFRS model, which involves the assessment of IHF information and extends to the study of the MCDM problem. Three basic stages make up this strategy, which includes the analysis of IHF information while addressing MCDM problems efficiently. Finding the IHF DMÂ item among all of the potential options is the initial step in the CIHFRS paradigm. This first step is critical because it establishes the framework for the model's latter phases, which provide an organized method of making decisions in the face of ambiguity and hesitancy. Applying the IHF-TOPSIS approach, an IHF-positive information system (IHF-$\overline{PIS}$) is established:

$$\hat{P}^{+} = \{G_j, \max\{\c{S}(G_j(x_i))\} | 1 \leq j \leq m\}$$

and the IHF-negative information system (IHF-$\underline{NIS}$) is

$$\hat{P}^{-} = \{G_j, \min\{\c{S}(G_j(x_i))\} | 1 \leq j \leq m\}$$

We compute $\hat{\mathrm{D}}^{+}$ and $\hat{\mathrm{D}}^{-}$, the distances between the alternatives $\mathrm{x}_i$ and IHF-PIS $\mathrm{P}^{+}$ and IHF-$\underline{NIS}$ $\mathrm{P}^{-}$. Consequently, a new IHFS $\hat{D} = (\hat{D}^{+}, \hat{D}^{-})$ is formed. Thus, we develop the multicriteria IHF DM information system (MCIHFDMIS) (X, G, $\hat{\mathrm{P}}$ , $\hat{\mathrm{D}}$). To elaborate further, the IHF set representing the universe is determined based on evaluations provided by the decision maker $X$, including the optimal calculations. In the second step, we calculate the LA and UA of the IHF decision-making object within the universe. This calculation incorporates a precision parameter denoted as $\bar{P}$ $(0 < \beta \leq 1)$, where $\beta$ is the threshold set by decision maker $X$ for consistency consensus within the CIHFRS model in the context of MCDM.

As a result, a new IHFS is created. We therefore create the MCIHFDMIS (X, G, P, and D) multicriteria IHF DM database. To clarify, the decision maker X provides evaluations, including the best and worst optimum evaluations, from which the IHF set representing the universe is derived. In the second phase, we compute the IHF decision-making object's LA and UA within the universe. After the first phase of finding the IHF DM item in the universe of possible alternatives, the CIHFRS model generates a rating of all the alternatives. The ideal item in the DM issue is finally identified by using a DM principle that was derived from the previous phases. This ranking is based on that principle.

This paper introduces the foundational model and approach for addressing the MCDM problem, specifically incorporating the assessment of interval-valued hesitant fuzzy (IHF) information. The methodology operates within the framework of the CIHFRS model, showcasing a structured and effective way to handle DM scenarios with uncertainty and hesitation. The main strategy, called IHF-TOPSIS, is put forth with the supposition that the best option should be the one that is closest to the IHF-PIS and furthest from the IHF-NIS. This entails applying a score function to identify both the IHF-PIS and the IHF-NIS.

$$\hat{P}^{+} = \{G_j, \max\{\c{S}(G_j(x_i))\} | j = 1, 2, \ldots m\}(1 \leq i \leq n)$$
$$= \{\langle G_1, \Phi_1^{+}, \Psi_1^{+}\rangle, \langle G_2, \Phi_2^{+}, \Psi_2^{+}\rangle, \ldots \langle G_m, \Phi_m^{+}, \Psi_m^{+}\rangle\}$$

and

$$\hat{P}^{-} = \{G_j \min\{\c{S}(G_j(x_i))\} | j = 1, 2, \ldots m\}(1 \leq i \leq n)$$
$$= \{\langle G_1, \Phi_1^{-}, \Psi_1^{-}\rangle, \langle G_2, \Phi_2^{-}, \Psi_2^{-}\rangle, \ldots \langle G_m, \Phi_m^{-}, \Psi_m^{-}\rangle\}$$

Next, we calculate the weighted distances $\hat{D}^{+}$ and $\hat{D}^{-}$ among the available alternatives. $x_i$ and IHF-$\overline{PIS}$ $\hat{\mathrm{P}}^{+}$ as well as IHF-$\underline{NIS}$ $\hat{\mathrm{P}}^{-}$ are computed as follows:

$$\hat{D}^{+} = \sum_{i=1}^{m} \omega_i d(G_j(x_i), G_j(P^{+}))$$

$$= \sum_{i=1}^{m} \omega_i \frac{\sum|\Phi^{+}ij - \Phi^{+}j| + \sum|\Psi^{+}ij - \Psi^{+}j|}{4} + \frac{\max\{\sum|\Phi^{+}ij - \Phi^{+}j|, \sum|\Psi^{+}ij - \Psi^{+}j|\}}{2}(1 \leq i \leq n) \text{ and } \hat{D}^{-} = \sum_{i=1}^{m} \omega_i . d(G_j(x_i), G_j(\hat{P}^{-}))$$

$$= \sum_{i=1}^{m} \omega_i \frac{\sum|\Phi^{-}ij - \Phi^{-}j| + \sum|\Psi^{-}ij - \Psi^{-}j|}{4} + \frac{\max\{\sum|\Phi^{-}ij - \Phi^{-}j|, \sum|\Psi^{-}ij - \Psi^{-}j|\}}{2}(1 \leq i \leq n)$$

Here, we formulate a novel IHF structure. $\hat{D} = (\Phi_{\hat{D}}, \Psi_{\hat{D}}) = (\hat{D}^{+}, \hat{D}^{-})$. Next, we calculate the LA and UA of the optimal and least favorable IHF DM objects within the framework of the MCIHFDMIS, taking into account the $\bar{\mathrm{P}}$P $\beta$ $(0 < \beta \leq 1)$, individually:

$$\Phi_{\underline{M_G^{\beta}}(D)}(x_i) = \bigwedge_{j=1}^{n} \{\Psi_{M^{\beta}_{G(x_i)}}(x_i, x_j) \veebar \Phi_D(x_j)\}$$

$$\Psi_{\underline{M_G^{\beta}}(D)}(x_i) = \bigvee_{j=1}^{n} \{\Phi_{M^{\beta}_{G(x_i)}}(x_i, x_j) \barwedge \Psi_D(x_j)\}$$

and

$$\Phi_{\overline{M_G^{\beta}}(\tilde{A})}(x_i) = \bigvee_{j=1}^{n} \{\Phi_{M^{\beta}_{G(x_i)}}(x_i, x_j) \barwedge \Phi_D(x_j)\}$$

$$\Psi_{\overline{M_G^{\beta}}(D)}(x_i) = \bigwedge_{j=1}^{n} \{\Psi_{M^{\beta}_{G(x_i)}}(x_i, x_j) \veebar \Psi_D(x_j)\}$$

In the final step, we utilize ranking rules to evaluate all alternatives within the universe. This evaluation is based on the LA and UA of the IHF decision-making object. We adhere to the consistency consensus threshold throughout this process, ensuring that the rankings maintain a level of coherence and agreement for $\beta$ $(0 < \beta \leq 1)$.

**Definition 27** ([15])**.** *Assume that $X \neq \phi$ finite universe and suppose that $\ddot{e}_1 = \langle \Phi_{\ddot{e}_1}, \Psi_{\ddot{e}_1} \rangle$, $\ddot{e}_2 = \langle \Phi_{\ddot{e}_2}, \Psi_{\ddot{e}_2} \rangle$ are two IHFNs and $E = \{x, \Phi_E(x), \Psi_E(x) | x \in X\} \in IHFS(X)$; then, assume some basic operations as follows:*

*(i)* $\ddot{e}_1 + \ddot{e}_2 = \langle \cup_{\mu\ddot{e}_1 \in \Phi\ddot{e}_1, \mu\ddot{e}_2 \in \Phi\ddot{e}_2} \{\mu\ddot{e}_1 + \mu\ddot{e}_2 - \mu\ddot{e}_1 . \mu\ddot{e}_2\}, \cup_{\nu\ddot{e}_1 \in \Psi\ddot{e}_1, \nu\ddot{e}_2 \in \Psi\ddot{e}_2} \{\nu\ddot{e}_1 . \nu\ddot{e}_2\} \rangle$.

*(ii)* $\ddot{e}_1 \times \ddot{e}_2 = \langle \cup_{\mu\ddot{e}_1 \in \Phi\ddot{e}_1, \mu\ddot{e}_2 \in \Phi\ddot{e}_2} \{\mu\ddot{e}_1 . \mu\ddot{e}_2\}, \cup_{\nu\ddot{e}_1 \in \Psi\ddot{e}_1, \nu\ddot{e}_2 \in \Psi\ddot{e}_2} \{\nu\ddot{e}_1 + \nu\ddot{e}_2 - \nu\ddot{e}_1 . \nu\ddot{e}_2\} \rangle$.

Utilizing the aforementioned operators, we establish a ranking function applicable to any alternative in the context of the MCDM problem, incorporating the evaluation of IHF information.

Additionally, let us assume that (X, G, P̂, D̂) represents an MCIHFDMIS.

For an IHF DM object, $\hat{D} = (\hat{D}^+, \hat{D}^-) \in IHF(X)$, determined by the preference information of decision maker $X$.

We refer to $\delta(x_i) =Ş\left(\overline{M_G^\beta}(D)(x_i) + \underline{M_G^\beta}(D)(x_i)\right)$, the ranking function of the alternative $x_i$ $(i = 1, 2, \ldots n)$.

*4.3. An Algorithm Designed for MCDM with Assessment of IHF Data*

This subsection delineates the stages of MCDM while integrating the assessment of IHF information through the CIHFRS model, as illustrated below:

**Input:** MCIHFDMIS $(X, G, P, D)$.

**Output:** The arrangement of sorting for all alternatives.

**Step 1:** Build IHF-$\overline{PIS}$ P$^+$, where $\hat{P}^+ = \{G_j, \max\{Ş(G_j(x_i))\}|j = 1, 2, \ldots m\}(1 \le i \le n)$:

$$= \{\langle G_1, \Phi_1^+, \Psi_1^+\rangle, \langle G_2, \Phi_2^+, \Psi_2^+\rangle, \ldots \langle G_m, \Phi_m^+, \Psi_m^+\rangle\} \text{ and IHF-}\underline{NIS}P^-$$

where

$$\hat{P}^- = \{G_j \min\{Ş(G_j(x_i))\}|j = 1, 2, \ldots m\}(1 \le i \le n) = \{\langle G_1, \Phi_1^-, \Psi_1^-\rangle, \langle G_2, \Phi_2^-, \Psi_2^-\rangle, \ldots \langle G_m, \Phi_m^-, \Psi_m^-\rangle\}.$$

**Step 2:** Compute the distance $\Phi_D$ =D$^+$ and $\Psi_D$ =D$^-$ among the alternatives. $x_i$ and IHF-$\overline{PIS}$ $P^+$and IHF-$\underline{NIS}$ $P^-$ are computed individually.

**Step 3:** Compute the lower and upper approximations:

$$\Phi_{\underline{M_G^\beta}(D)}(x_i), \Psi_{\underline{M_G^\beta}(D)}(x_i) \text{ and } \Phi_{\overline{M_G^\beta}(\tilde{A})}(x_i), \Psi_{\overline{M_G^\beta}(D)}(x_i)$$

**Step 4:** Calculate the summation function: $\overline{M_G^\beta}(D)(x_i)$ and $\underline{M_G^\beta}(D)(x_i)$.

**Step 5:** Calculate the ranking function: $\delta(x_i)$.

**Step 6:** Assign a ranking to each value across all alternatives.

*4.4. A Demonstrative Example*

In this subsection, we apply the MCDM model and approach, integrating the evaluation of interval-valued hesitant fuzzy (IHF) information through CIHFRS models, to tackle the car-selection problem. This application aims to enrich the MCDM problem by incorporating IHF information assessment within its framework. Following this, we elucidate the IHF-TOPSIS principle and provide a detailed explanation of the procedure outlined in this paper. We elucidate each step, using a practical example for illustration purposes, to ensure a clear understanding of the methodology and its application in real-world scenarios. Let $X = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5\}$ be the five cars that are selected, Mahindra XUV ($\varkappa_1$), Scorpio ($\varkappa_2$), Duster ($\varkappa_3$), Vitara Breeza ($\varkappa_4$), and Fortuner ($\varkappa_5$), and compare this model using four criteria: cost ($G_1$), seating capacity ($G_2$), engine power ($G_3$), and maximum speed ($G_4$). The model of the car and comparison criteria are described in Table 4.

**Table 4.** The IHF $\beta-$Covering.

| $X/G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.1, 0.2, 0.6\}, \{0.2, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.2, 0.6\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.1, 0.2, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.2, 0.3, 0.5\}, \{0.1, 0.2, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.4\}, \{0.2, 0.3\}\rangle$ | $\langle\{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.1, 0.3, 0.5\}, \{0.2, 0.3, 0.4\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.1, 0.4, 0.5\}, \{0.2, 0.3, 0.5\}\rangle$ | $\langle\{0.1, 0.3, 0.6\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.3, 0.4\}, \{0.1, 0.2, 0.5\}\rangle$ | $\langle\{0.1, 0.2, 0.5\}, \{0.1, 0.2, 0.4\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.1, 0.3, 0.4\}, \{0.1, 0.2, 0.5\}\rangle$ | $\langle\{0.2, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.2, 0.3, 0.5\}, \{0.1, 0.3, 0.5\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.3, 0.5, \}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.1, 0.2, 0.5\}, \{0.1, 0.3, 0.4\}\rangle$ | $\langle\{0.2, 0.3, 0.5\}, \{0.1, 0.2, 0.3\}\rangle$ | $\langle\{0.3, 0.4, 0.5\}, \{0.2, 0.3, 0.5\}\rangle$ |

Here, apply the pessimistic approach for MCDM problems in an IHFS environment. When making decisions in risk-averse environments, where there is a need to rigorously restrict the possibility of negative consequences, the pessimistic approach is often used.

Decision maker $X$ evaluates the five cars based on their specialized knowledge, assigning real values for each alternative concerning each attribute outlined in the table above. Additionally, the weights for the attribute set are designated in the following order, respectively: $\varpi_1 = 0.3276836160, \varpi_2 = 0.2316384179, \varpi_3 = 0.2090395481$, and $\varpi_4 = 0.2316384181$. According to the IHF-$\overline{PIS}$ $\hat{P}^+$ and IHF-$\underline{NIS}$ $\hat{P}^-$, it follows that

$$\hat{P}^+ = \left\{ \begin{array}{l} \{\langle G_1, \{0.1, 0.5, 0.4\}, \{0.1, 0.3, 0.3\}\rangle\}, \{\langle G_2, \{0.1, 0.4, 0.4\}, \{0.1, 0.3, 0.3\}\rangle\}, \\ \{\langle G_3, \{0.1, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle\}, \{\langle G_4, \{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.4\}\rangle\} \end{array} \right\}$$

$$\hat{P}^- = \left\{ \begin{array}{l} \{\langle G_1, \{0.3, 0.2, 0.4\}, \{0.2, 0.2, 0.3\}\rangle\}, \{\langle G_2, \{0.2, 0.4, 0.4\}, \{0.2, 0.3, 0.3\}\rangle\}, \\ \{\langle G_3, \{0.2, 0.2, 0.4\}, \{0.1, 0.2, 0.3\}\rangle\}, \{\langle G_4, \{0.3, 0.2, 0.5\}, \{0.2, 0.2, 0.4\}\rangle\} \end{array} \right\}$$

Next, find out the weighted distances $\Phi_{\hat{D}} = \hat{D}^+$ and $\Psi_{\hat{D}} = \hat{D}^-$ among the alternatives $x_i$ and IHF-$\overline{PIS}\hat{P}^+$and IHF-$\underline{NIS}$ $\hat{P}^-$, respectively, as follows:

$$\Phi_{D^+} = D^+ = \frac{0.2500000000}{\varkappa_1} + \frac{0.1766949153}{\varkappa_2} + \frac{0.2199152542}{\varkappa_3} + \frac{0.1868644068}{\varkappa_4} + \frac{0.2175141243}{\varkappa_5}$$

and

$$\Psi_{D^-} = D^- = \frac{0.2138418079}{\varkappa_1} + \frac{0.1984463277}{\varkappa_2} + \frac{0.3108757062}{\varkappa_3} + \frac{0.2413841808}{\varkappa_4} + \frac{0.2468926554}{\varkappa_5}$$

Assume that the consistency consensus threshold IHFN $\beta = \langle 0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.5\}\rangle$; then G is an IHF $\beta$-covering of $X$, and then

$$\mathbb{N}^{\beta}_{\varkappa_1} = G_1 \cap G_3, \mathbb{N}^{\beta}_{\varkappa_2} = G_1 \cap G_2 \cap G_3, \mathbb{N}^{\beta}_{\varkappa_3} = G_2 \cap G_4, \mathbb{N}^{\beta}_{\varkappa_4} = G_2 \cap G_3 \cap G_4, \mathbb{N}_{\varkappa_5} = G_3 \cap G_4$$

computing LA and UA $\Phi_{\underline{M^{\beta}_{G}}(D)}(\varkappa_i), \Psi_{\underline{M^{\beta}_{G}}(D)}(\varkappa_i)$ and $\Phi_{\overline{M^{\beta}_{G}}(\tilde{A})}(\varkappa_i), \Psi_{\overline{M^{\beta}_{G}}(D)}(\varkappa_i)$

$$\Phi_{\underline{M^{\beta}_{G}}(D)}(\varkappa_i) = \frac{0.1766949153}{\varkappa_1} + \frac{0.1766949153}{\varkappa_2} + \frac{0.1868644068}{\varkappa_3} + \frac{0.1766949153}{\varkappa_4} + \frac{0.1766949153}{\varkappa_5}$$

and

$$\Psi_{\underline{M^{\beta}_{G}}(D)}(\varkappa_i) = \frac{0.5}{\varkappa_1} + \frac{0.5}{\varkappa_2} + \frac{0.4}{\varkappa_3} + \frac{0.4}{\varkappa_4} + \frac{0.4}{\varkappa_5}$$

$$\Phi_{\overline{M^{\beta}_{G}}(D)}(\varkappa_i) = \frac{0.5}{\varkappa_1} + \frac{0.5}{\varkappa_2} + \frac{0.4}{\varkappa_3} + \frac{0.4}{\varkappa_4} + \frac{0.4}{\varkappa_5}$$

and

$$\Psi_{\overline{M^{\beta}_{x}}(D)}(\varkappa_i) = \frac{0.1984463277}{\varkappa_1} + \frac{0.1984463277}{\varkappa_2} + \frac{0.2}{\varkappa_3} + \frac{0.1984463277}{\varkappa_4} + \frac{0.1984463277}{\varkappa_5}$$

Now, compute the sum function $\overline{M^{\beta}_{G}}(D)(\varkappa_i)$ and $\underline{M^{\beta}_{G}}(D)(\varkappa_i)$. Now, compute the ranking function function $\delta(\varkappa_i)$:

$$\delta = \frac{0.2500000000}{\varkappa_1} + \frac{0.2500000000}{\varkappa_2} + \frac{0.2186864407}{\varkappa_3} + \frac{0.2176694916}{\varkappa_4} + \frac{0.2176694916}{\varkappa_5}$$

Ultimately, we reveal the optimal ranking for all cars based on the values of the ranking function $\delta(\varkappa_i)(i = 1, 2, 3, 4, 5)$ as follows:

$$\varkappa_1 \approx \varkappa_2 \geq \varkappa_3 \geq \varkappa_4 \approx \varkappa_5$$

As a result, we conclude the decision-making process for optimal selection by applying multiple criteria decision making (MCDM) using CIHFRS models and incorporating an assessment of interval-valued hesitant fuzzy (IHF) data. Based on the numerical computations and outcomes, the ultimate optimal decision is to choose both the first and second cars. Furthermore, we leverage the model and methodology of MCDM by incorporating an assessment of IF data based on CIFRS models. This enhancement strengthens the selection process for cars within the MCDM framework, taking into account the assessment of IF data.

The respective weights of the attribute set are provided as follows: $\hat{w}_1 = 0.4563199563$, $\hat{w}_2 = 0.4150969151, \hat{w}_3 = 0.2090395481$, and $\hat{w}_4 = 0.06988806989$. In accordance with IF-$\overline{PIS}\hat{\mathrm{P}}^+$ and IF-$\underline{NIS}\hat{\mathrm{P}}^-$, as stated below,

$$\hat{P}^+ = \{\{C_1, 0.40, 0.20\}, \{C_2, 0.37, 0.25\}, \{C_3, 0.33, 0.20\}, \{C_4, 0.40, 0.23\}\}$$

$$\hat{P}^- = \{\{C_1, 0.23, 0.33\}, \{C_2, 0.27, 0.30\}, \{C_3, 0.33, 0.27\}, \{C_4, 0.27, 0.33\}\}$$

Next, find out the weighted distances $\tau_{\hat{D}} = \hat{D}^+$ and $\eta_{\hat{D}} = \hat{D}^-$ among the alternatives $x_i$ and IF-$\overline{PIS}$ $\hat{\mathrm{P}}^+$ and IF-$\underline{NIS}$ $\hat{\mathrm{P}}^-$, which are presented individually as follows:

$$\tau_{\hat{D}^+} = \hat{D}^+ = \frac{0.1489854627}{\varkappa_1} + \frac{0.1393366093}{\varkappa_2} + \frac{0.1456852307}{\varkappa_3} + \frac{0.1338049412}{\varkappa_4} + \frac{0.1323645236}{\varkappa_5}$$

$$\eta_{\hat{D}^-} = \hat{D}^- = \frac{0.07517267269}{\varkappa_1} + \frac{0.06637626263}{\varkappa_2} + \frac{0.06301904178}{\varkappa_3} + \frac{0.06609370736}{\varkappa_4} + \frac{0.08660865413}{\varkappa_5}$$

Suppose the threshold for consistency and consensus is assumed. IFN $\beta = \langle\{0.30\}, \{0.27\}\rangle$, then $G$ is an IF $\beta$ covering of $X$.

$$\mathbb{N}^{\beta}_{\varkappa_1} = G_1 \cap G_3, \mathbb{N}^{\beta}_{\varkappa_2} = G_1 \cap G_2 \cap G_3, \mathbb{N}^{\beta}_{\varkappa_3} = G_2 \cap G_4, \mathbb{N}^{\beta}_{\varkappa_4} = G_2 \cap G_3 \cap G_4, \mathbb{N}^{\beta}_{\varkappa_5} = G_1 \cap G_3 \cap G_4$$

Compute the lower and upper approximations $\tau_{\underline{M^{\beta}_G}(D)}(\varkappa_i), \eta_{\underline{M^{\beta}_G}(D)}(\varkappa_i)$ and $\tau_{\overline{M^{\beta}_G}(\tilde{A})}(\varkappa_i), \eta_{\overline{M^{\beta}_G}(D)}(\varkappa_i)$:

$$\tau_{\underline{M^{\beta}_G}(D)}(\varkappa) = \frac{0.23}{\varkappa_1} + \frac{0.25}{\varkappa_2} + \frac{0.27}{\varkappa_3} + \frac{0.20}{\varkappa_4} + \frac{0.27}{\varkappa_5}$$

$$\eta_{\underline{M^{\beta}_G}(D)}(\varkappa) = \frac{0.08660865413}{\varkappa_1} + \frac{0.08660865413}{\varkappa_2} + \frac{0.08660865413}{\varkappa_3} + \frac{0.08660865413}{\varkappa_4} + \frac{0.08660865413}{\varkappa_5}$$

$$\tau_{\overline{M^{\beta}_G}(D)}(\varkappa) = \frac{0.1323645236}{\varkappa_1} + \frac{0.1323645236}{\varkappa_2} + \frac{0.1323645236}{\varkappa_3} + \frac{0.1323645236}{\varkappa_4} + \frac{0.1323645236}{\varkappa_5}$$

$$\eta_{\overline{M^{\beta}_x}(D)}(\varkappa) = \frac{0.06301904178}{\varkappa_1} + \frac{0.06301904178}{\varkappa_2} + \frac{0.06301904178}{\varkappa_3} + \frac{0.06301904178}{\varkappa_4} + \frac{0.06301904178}{\varkappa_5}$$

Now, compute the sum function $\overline{M^{\beta}_G}(D)(\varkappa_i)$ and $\underline{M^{\beta}_G}(D)(\varkappa_i)$:

$$\underline{M^{\beta}_G}(D) + \overline{M^{\beta}_G}(D) = \left\{ \begin{array}{c} \{\varkappa_1, 0.3319206832, 0.03044384043\}, \{\varkappa_2, 0.3492733927, 0.03309113090\}, \\ \{\varkappa_3, 0.3666261022, 0.03573842137\}, \{\varkappa_4, 0.3058916189, 0.02647290472\}, \\ \{\varkappa_5, 0.3666261022, 0.03573842137\} \end{array} \right\}$$

Now, compute the ranking function $\delta(\varkappa_i)$:

$$\partial = \frac{0.3014768428}{\varkappa_1} + \frac{0.3161822618}{\varkappa_2} + \frac{0.2701531975}{\varkappa_3} + \frac{0.2701531975}{\varkappa_4} + \frac{0.3308876808}{\varkappa_5}$$

In the end, we unveil the optimal arrangement for categorizing all cars according to the criteria of the ranking function $\delta(\varkappa_i)(i = 1, 2, 3, 4, 5)$ as follows:

$$\varkappa_5 \geq \varkappa_2 \geq \varkappa_1 \geq \varkappa_3 \approx \varkappa_4$$

Therefore, the DM process for optimal selection is finalized by employing CIFRS model-based MCDM, which integrates the assessment of IF data. The quantitative computation outcomes indicate that the ultimate optimal decision identifies the fifth car. Subsequently, in the IHFS, the nonmembership component is terminated. Following this, the model and methodology of MCDM with HF information evaluation based on CHFRS models are applied to address car selection issues. This approach is designed to enhance problem solving within the MCDM framework, particularly in the context of HF information evaluation.

The weights for the attribute set are provided in the following order: $\hat{W}_1 = 0.3191489362$, $\hat{W}_2 = 0.2553191490$, $\hat{W}_3 = 0.2127659574$, and $\hat{W}_4 = 0.2127659574$. As per HF-PIS $\mathrm{P}^+$ and HF-NIS $\mathrm{P}^-$, the following is indicated:

$$P^+ = \{\{G_1, \{0.1, 0.2, 0.4\}\}, \{G_2, \{0.1, 0.2, 0.4\}\}, \{G_3, \{0.1, 0.2, 0.4\}\}, \{G_4, \{0.1, 0.2, 0.5\}\}\}$$

$$P^- = \{\{G_1, \{0.2, 0.5, 0.5\}\}, \{G_2, \{0.2, 0.4, 0.6\}\}, \{G_3, \{0.2, 0.3, 0.6\}\}, \{G_4, \{0.3, 0.4, 0.5\}\}\}$$

Next, find out the weighted distances $\hat{H}_{\hat{D}^+} = \hat{D}^+$ and $\hat{H}_{\hat{D}^-} = \hat{D}^-$ among the alternatives $x_i$ and HF-$\overline{PIS}$ $\hat{\mathrm{P}}^+$ and HF-$\underline{NIS}$ $\hat{\mathrm{P}}^-$, which are specified as follows, respectively:

$$\hat{H}_{D^+} = D^+ = \frac{0.1361702128}{\varkappa_1} + \frac{0.2361702128}{\varkappa_2} + \frac{0.1851063830}{\varkappa_3} + \frac{0.2191489362}{\varkappa_4} + \frac{0.3659574468}{\varkappa_5}$$

$$\hat{H}_{D^-} = D^- = \frac{0.3638297873}{\varkappa_1} + \frac{0.3638297873}{\varkappa_2} + \frac{0.3638297873}{\varkappa_3} + \frac{0.3638297873}{\varkappa_4} + \frac{0.3638297873}{\varkappa_5}$$

Assuming the consistency consensus threshold HFN $\beta = \{0.2, 0.3, 0.4\}$, then $G$ is an HF $\beta$-covering of $X$. $\mathbb{N}^{\beta}_{\varkappa_1} = G_2 \cap G_3, \mathbb{N}^{\beta}_{\varkappa_2} = G_1 \cap G_4, \mathbb{N}^{\beta}_{\varkappa_3} = G_1 \cap G_2, \mathbb{N}^{\beta}_{\varkappa_4} = G_2 \cap G_4, \mathbb{N}^{\beta}_{\varkappa_5} = G_3 \cap G_4$.

Compute the lower and upper approximations $\hat{H}_{\underline{M^{\beta}_{G}}(D)}(\varkappa)$ and $\hat{H}_{\overline{M^{\beta}_{G}}(D)}(\varkappa)$:

$$\hat{H}_{\underline{M^{\beta}_{G}}(D)}(\varkappa) = \frac{0.9}{\varkappa_1} + \frac{0.7}{\varkappa_2} + \frac{0.9}{\varkappa_3} + \frac{0.8}{\varkappa_4} + \frac{0.9}{\varkappa_5}$$

$$\hat{H}_{\overline{M^{\beta}_{G}}(D)}(\varkappa) = \frac{0.3638297873}{\varkappa_1} + \frac{0.3638297873}{\varkappa_2} + \frac{0.3638297873}{\varkappa_3} + \frac{0.3638297873}{\varkappa_4} + \frac{0.3638297873}{\varkappa_5}$$

Now , compute the sum function $\hat{H}_{\underline{M^{\beta}_{G}}(D)}$ and $\hat{H}_{\overline{M^{\beta}_{G}}(D)}$:

$$\hat{H}_{\underline{M^{\beta}_{G}}(D)} + \hat{H}_{\overline{M^{\beta}_{G}}(D)} = \left\{ \begin{array}{c} \{\varkappa_1, 0.9363829784, 0.3274468086\}, \{\varkappa_2, 0.8091489359, 0.2546808511\}, \\ \{\varkappa_3, 0.9363829784, 0.3274468086\}, \{\varkappa_4, 0.8727659572, 0.2910638298\}, \\ \{\varkappa_5, 0.9363829784, 0.3274468086\} \end{array} \right\}$$

Now, compute the ranking function $\delta(x_i)$:

$$\partial = \frac{0.6319148935}{x_1} + \frac{0.5319148935}{x_2} + \frac{0.6319148935}{x_3} + \frac{0.5819148935}{x_4} + \frac{0.6319148935}{x_5}$$

Ultimately, we reveal the ideal classification for all cars based on the values derived from the ranking function $\delta(x_i)(i = 1, 2, \ldots, 5)$ as follows:

$$\varkappa_1 \approx \varkappa_3 \approx \varkappa_5 \geq \varkappa_4 \geq \varkappa_2$$

Hence, we finalize the decision-making process for optimal selection through the application of CHFRS models in MCDM, incorporating the assessment of HF information. According to the numerical calculations, the ultimate optimal decision is to choose the first, third, and fifth cars.

## 5. Comparison Analysis

The proposed model offers improved handling of hesitation, improved approximation accuracy, and enhanced representation of uncertainty, which makes it a significant improvement over traditional fuzzy covering rough set models in decision making. Due to these advantages, it is an effective tool for complex decision-making situations where ambiguity and unpredictability are frequent. Also, the proposed model approach proposes a more comprehensive structure to deal with uncertainty and inaccuracy by integrating the advantages of hesitant and intuitionistic fuzzy sets. In order to demonstrate the accuracy and effectiveness of the proposed method, a comparative analysis is carried out using the approaches outlined by Huang et al. in CIFRS [30] and Zhou et al. in CHFRS [39], both of which represent specific cases within CIHFRS.

### *5.1. An Analysis Comparing the Current MCDM Method with CIFRS*

A CIFRS may be seen as a particular instance of a CIHFRS, occurring when there is only one element present in both the $\breve{m}$ and $\breve{n}\breve{m}$ degrees. To facilitate a comparison, transforming a CIHFRS to a CIFRS involves calculating the average values of $\breve{m}$ and $\breve{n}\breve{m}$ degrees. Once converted, the intuitionistic information is presented in Table 5. Subsequently, the broad assessment values can be computed using TOPSIS within an IF environment. The final ranking of alternatives is $\chi_5 \geq \chi_2 \geq \chi_1 \geq \chi_3 \approx \chi_4$, and $\chi_5$ is the desired alternative (in Table 6). Observably, the ranking being derived from the technique recommended by the outcome of the proposed technique contrasts with the CIFRS [30]. The primary objective is to take the average value of $\breve{m}$ and $\breve{n}\breve{m}$ degrees of the IFRS, which may result in information falsification and loss. CIHFRSs prove to be more practical compared to CIFRSs as they account for situations in which decision makers prefer to employ a multitude of potential values for precisely articulating both $\breve{m}$ and $\breve{n}\breve{m}$ degrees (in Table 7).

**Table 5.** The IF $\beta$-covering.

| $X/G$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.23\},\{0.20\}\rangle$ | $\langle\{0.30\},\{0.30\}\rangle$ | $\langle\{0.30\},\{0.23\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.33\},\{0.23\}\rangle$ | $\langle\{0.27\},\{0.25\}\rangle$ | $\langle\{0.23\},\{0.20\}\rangle$ | $\langle\{0.30\},\{0.30\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.33\},\{0.33\}\rangle$ | $\langle\{0.33\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.23\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.37\},\{0.27\}\rangle$ | $\langle\{0.23\},\{0.20\}\rangle$ | $\langle\{0.33\},\{0.30\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.40\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.33\},\{0.20\}\rangle$ | $\langle\{0.40\},\{0.33\}\rangle$ |

**Table 6.** The IF $\beta$-neighborhood.

| $\mathbb{N}_G^\beta$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.23\},\{0.23\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.30\}\rangle$ | $\langle\{0.23\},\{0.27\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.23\},\{0.23\}\rangle$ | $\langle\{0.23\},\{0.25\}\rangle$ | $\langle\{0.27\},\{0.20\}\rangle$ | $\langle\{0.23\},\{0.20\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.23\},\{0.27\}\rangle$ | $\langle\{0.23\},\{0.27\}\rangle$ | $\langle\{0.33\},\{0.30\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.33\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.33\},\{0.33\}\rangle$ |

**Table 7.** The IFC $\beta$-neighborhood.

| $\widetilde{M}_G^\beta$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\langle\{0.23\},\{0.23\}\rangle$ | $\langle\{0.23\},\{0.23\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.23\},\{0.27\}\rangle$ | $\langle\{0.33\},\{0.27\}\rangle$ |
| $\varkappa_2$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.23\},\{0.25\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.23\},\{0.27\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ |
| $\varkappa_3$ | $\langle\{0.27\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.20\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.33\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ |
| $\varkappa_4$ | $\langle\{0.27\},\{0.30\}\rangle$ | $\langle\{0.23\},\{0.20\}\rangle$ | $\langle\{0.27\},\{0.27\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ |
| $\varkappa_5$ | $\langle\{0.23\},\{0.27\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.27\},\{0.33\}\rangle$ | $\langle\{0.23\},\{0.30\}\rangle$ | $\langle\{0.33\},\{0.33\}\rangle$ |

*5.2. An Analytical Comparison between the Current MCDM Method and CHFRS*

A CHFRS can also be measured as a specific instance of a CIHFRS when decision makers solely consider $\breve{m}$ degrees during assessment. To make a comparison, the CIHFRS can be transformed into a CHFRS by excluding $\breve{n}\breve{m}$ degrees and taking only the $\breve{m}$ degrees. Hesitant information is depicted in Table 8, and the overall assessment values can be calculated using TOPSIS in a hesitant fuzzy environment. The conclusive ranking of alternatives is $\chi_1 \approx \chi_3 \approx \chi_5 \geq \chi_4 \geq \chi_2$, with $\chi_1$ identified as the optimal alternative (in Table 9). Notably, the ranking derived from the approach advocated by the CHFRS [39] aligns with the results of our proposed technique, affirming the validity of our approach. Moreover, the remaining alternatives are not comparative, such as $\varkappa_2$ and $\varkappa_3$ or $\varkappa_4$ and $\varkappa_5$ (in Table 10).

**Table 8.** The HF $\beta$-covering.

| $X/G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|---|---|---|---|
| $\varkappa_1$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.2, 0.6\}$ | $\{0.1, 0.2, 0.6\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_2$ | $\{0.2, 0.3, 0.5\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.5\}$ |
| $\varkappa_3$ | $\{0.1, 0.4, 0.5\}$ | $\{0.1, 0.3, 0.6\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_4$ | $\{0.1, 0.3, 0.4\}$ | $\{0.2, 0.4, 0.5\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.2, 0.3, 0.5\}$ |
| $\varkappa_5$ | $\{0.3, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.2, 0.3, 0.5\}$ | $\{0.3, 0.4, 0.5\}$ |

**Table 9.** The HF $\beta$-neighborhood.

| $\mathbb{N}_G^\beta$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\{0.1, 0.2, 0.6\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_2$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.5\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.4\}$ |
| $\varkappa_3$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.3, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.2, 0.2\}$ |
| $\varkappa_4$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.2, 0.3, 0.5\}$ | $\{0.1, 0.2, 0.4\}$ |
| $\varkappa_5$ | $\{0.1, 0.2, 0.5\}$ | $\{0.3, 0.4, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.2, 0.3, 0.5\}$ |

**Table 10.** The HFC $\beta$-neighborhood.

| $\widetilde{M}_G^\beta$ | $\varkappa_1$ | $\varkappa_2$ | $\varkappa_3$ | $\varkappa_4$ | $\varkappa_5$ |
|---|---|---|---|---|---|
| $\varkappa_1$ | $\{0.1, 0.2, 0.6\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_2$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.3, 0.4, 0.5\}$ |
| $\varkappa_3$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.3, 0.5\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_4$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.3, 0.4\}$ | $\{0.1, 0.2, 0.5\}$ | $\{0.2, 0.3, 0.5\}$ | $\{0.1, 0.2, 0.5\}$ |
| $\varkappa_5$ | $\{0.1, 0.2, 0.5\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.1, 0.2, 0.2\}$ | $\{0.1, 0.2, 0.4\}$ | $\{0.2, 0.3, 0.5\}$ |

Table 11 provides the ranking values for the preceding discussion. The advantages of our proposed approach, as outlined in the above comparative analysis, can be summarized. The IHFS is well suited for representing uncertain or fuzzy information in MCDM problems due to the availability of two sets of $\breve{m}$ and $\breve{n}\breve{m}$ degrees with various possible values, a feature not achievable by the IFS and HFS. The IHFS can also be employed for processing MCDM and comparison methods based on its inherent capabilities. After minor modification of the IFS and HFS, the IHFS uses the general form of the IFS and HFS. Because it is capable of supporting many degrees of membership and nonmembership simultaneously, the intuitionistic hesitant fuzzy set is seen as superior to the intuitionistic fuzzy set and the hesitant fuzzy set alone. This enables a fuller and more complex representation of uncertainty and hesitancy. This results in enhanced decision-making procedures and more accurate and flexible modeling of complex problems.

**Table 11.** Analytical comparison with current methodologies.

| Method | Ranking |
|---|---|
| CIHFRS | $\varkappa_1 \approx \varkappa_2 \geq \varkappa_3 \geq \varkappa_4 \approx \varkappa_5$ |
| CIFRS [30] | $\varkappa_5 \geq \varkappa_2 \geq \varkappa_1 \geq \varkappa_3 \approx \varkappa_4$ |
| CHRF [39] | $\varkappa_1 \approx \varkappa_3 \approx \varkappa_5 \geq \varkappa_4 \geq \varkappa_2$ |

## 6. Conclusions

The notion of the CIHFRS plays a vital role in order to deal with uncertainties in real-world scenarios (in Table 12). In this paper, we initiated the notion some novel CIHFRS models, namely, the IHF $\beta$-neighborhoods and the IHF complementary $\beta$-neighborhoods. We presented examples of these new notions and investigated some of their properties in detail. Particularly the rough and precision degrees of CIHFRS models were discussed in detail. The relationships among these models were also presented. Further, by means of the developed IHF $\beta$-neighborhoods and IHF complementary $\beta$-neighborhoods, we constructed four types of CIFRS models, and the relationship among these models was also discussed. Furthermore, we applied the proposed models to the MCDM problem under an IHF environment. The developed CIHFRS models not only enriched granular computing and CIHFRSs but also proposed a novel perspective for MCDM with IHF information. Moreover, we presented a numerical example for the application and effectiveness of the developed approach. Finally for the proposed approach, we compared it with existing studies. From this analysis, we found that the method developed in this paper is more effective to deal with an MCDM problem with IHF information based on CIHFRS models than an MCDM problem with the evaluation of IF and HF information based on CIFRS and CHFRS, respectively. In future, the developed approach will extend to the interval-valued intuitionistic hesitant fuzzy setting, Pythagorean hesitant fuzzy setting, and q-rung orthopair hesitant fuzzy setting.

**Table 12.** Summary of the current research work on the CIFS, CHFR and CIHFRS.

| Research Area | Objectives | Methodologies |
|---|---|---|
| **CIFS in MCDM** | | |
| Optimization of flow systems | Evaluate and rank flow system configurations | AHP, TOPSIS, ANP |
| **CFHS in MCDM** | | |
| Risk assessment | Evaluate risks in controlled flow heat systems | Fuzzy MCDM, risk analysis models |
| **CIHFRS in MCDM** | | |
| Integrated system optimization | Optimize combined heat and flow systems | Hybrid MCDM models, integrated assessment |

**Author Contributions:** Conceptualization, M.K.K., K., and M.S.A.K.; methodology, M.K.K., K., M.S.A.K.; software, M.S.A.K.; validation, M.K.K., K. and M.S.A.K.; formal analysis, A.A. and N.M.; investigation, K. and N.M.; resources, A.A. and N.M.; data curation, M.S.A.K. and M.K.K.; writing—original draft preparation, M.K.K. and M.S.A.K.; writing—review and editing, M.K.K., M.S.A.K., K., A.A. and N.M; visualization, A.A. and N.M.; supervision, K. and N.M.; project administration, A.A. and N.M.; funding acquisition, A.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** No data were used to support this work.

**Acknowledgments:** The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Abbreviations

**List of Symbols**

| | |
|---|---|
| $\breve{m}$ | Degrees of membership |
| $\breve{n}\breve{m}$ | Degrees of nonmembership |
| $X$ | Finite universe set |
| $\dot{m}$ | Intuitionistic fuzzy numbers |
| $\widehat{i}$ | Intuitionistic fuzzy set |
| $\hat{H}$ | Hesitant fuzzy set |
| $\Lambda_{IHFS}$ | Intuitionistic hesitant fuzzy set |
| $\dot{e}$ | Intuitionistic hesitant fuzzy numbers |
| $\mathbb{N}$ | Neighborhood |
| Ş | Score function |
| $\bar{A}$ | Accuracy function |
| $\overline{\wedge}$ | Minimum |
| $\overline{\wedge}$ | Maximum |

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