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On Generalized Fibospinomials: Generalized Fibonacci Polynomial Spinors

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Abstract: Spinors are important objects in physics, which have found their place more and more after the discovery that particles have an intrinsic angular momentum shape and Cartan's mathematical expression of this situation. Recent studies using special number sequences have also revealed a new approach to the use of spinors in mathematics and have provided a different perspective for spinor research that can be used as a source for future physics studies. The purpose of this work is to expand the generalized Fibonacci quaternion polynomials to the generalized Fibonacci polynomial spinors by associating spinors with quaternions, and to introduce and investigate a new polynomial sequence that can be used to benefit from the potential advantages of spinors in physical applications, and thus, to provide mathematical arguments, such as new polynomials, for studies using spinors and quaternions in quantum mechanics. Starting from this point of view, in this paper we introduce and investigate a new family of sequences called generalized Fibospinomials (or generalized Fibonacci polynomial spinors or Horadam polynomial spinors). Being particular cases, we use (r,s)-Fibonacci and (r,s)-Lucas polynomial spinors. We present Binet's formulas, generating functions and the summation formulas for these polynomials. In addition, we obtain some special identities of these new sequences and matrices related to these polynomials. The importance of this study is that generalized Fibospinomials are currently the most generalized sequence in the literature when moving from Fibonacci quaternions to spinor structure, and that a wide variety of new spinor sequences can be obtained from this particular polynomial sequence.

Keywords: Fibonacci numbers; Fibonacci polynomials; Fibonacci spinors; generalized Fibonacci spinors; generalized Fibonacci polynomial spinors

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1. Introduction

Classical mechanics, whose laws were given by Newton in the late 1600s, was not sufficient to interpret the discoveries concerning the electronic structure of atoms and the nature of light. It became a necessity to develop a different type of mechanics known as quantum mechanics or wave mechanics to explain such situations [1]. Heisenberg and Schrödinger discovered matrix mechanics and wave mechanics, respectively. Later, Schrödinger and Eckart tried to prove that these two theories are mathematically equivalent. In Heisenberg's mechanics, a physical quantity is represented by a matrix, while in Schrödinger's mechanics it is represented by a linear operator [2]. In quantum mechanics, three different equations, often called Dirac, Pauli, and Schrödinger equations, are used to describe the movement of an electron [3]. Schrödinger defined a wave equation in 1925 based on the suggestion of the famous physicist Peter Debye and the work of de Broglie. The equation did not match the observations of real atoms, as Schrödinger did not include electron spin in his work, which began with trying to find a wave equation that would characterize the behavior of an electron in hydrogen [4]. In 1926, Schrödinger presented his papers [1] in which he wrote the full mathematical basis of non-relativistic quantum mechanics. In 1926, Klein, Fock, and Gordon independently formulated the relativistic

Schrödinger equation for the free spin-zero particle. Meanwhile, in 1928, Dirac discovered the relativistic wave equation for the free electron, a spin-1/2 particle [5]. According to Bandyopadhyay and Cahay, the eigenvectors of Pauli spin matrices are examples of spinors, which are 2×1 column vectors that represent the spin state of an electron [2]. Physicists initially introduced spinors under this designation within the realm of quantum mechanics. But this concept in its most general mathematical form was defined much earlier, by Cartan in 1913 [6]. In four-dimensional space, spinors appear in Dirac's famous electron equations, and the components of a spinor are four wave functions indicated by Cartan [7]. The primary reason spinors enter physics is the presence of spin. Spinors, a concept still not fully understood [8] and requiring extensive study, are the subject of focus for mathematicians in algebraic and geometrical research, while physicists delve deeply into their implications in quantum physics. Upon examining the literature, it becomes evident that the study of spinors devoid of any geometric significance has contributed to the complexity surrounding attempts to extend Dirac's equations to general relativity. As a result, the concept of spinors remains incomprehensible [7]. Spinors and Dirac equations in general relativity theory on Riemannian spaces have been studied independently by Weyl [9], Schrödinger [10], and Fock [11], and then, many studies have been performed in terms of space-time geometry [12]. Vaz and Rocha proposed three distinct definitions of spinors, each delineated by various researchers, highlighting diverse perspectives. Among these, two definitions enjoy broader acceptance, while the third is emerging in scholarly discourse [13]. They classified these definitions as algebraic, classical, and operatorial.

Now, the definition of $SU(2)$ as given in Westra's notes [14] will be recalled. Let U be a 2×2 matrix and let $U^{\dagger}$ represent the conjugate transpose of U. $SU(2)$ is the group that provides the following properties:

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, U^{\dagger}U = UU^{\dagger} = I \text{ and } \det U = 1 \text{ where } a, b, c, d \in \mathbb{C}.$$

Then, the most general element of $SU(2)$ is written as [14]

$$U(x,y) = \begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}, |x|^2 + |y|^2 = 1.$$

The elements of the representation space of $SU(2)$, obtained with the help of Cayley–Klein parameters, which are the inputs of a unitary matrix A belonging to the $SU(2)$, are called classical spinors [13]. In quantum mechanics, particle spin, defined in Lie group theory as n-dimensional $Spin(n)$ with elements known as spinors, is used to represent quaternions. Because spinors change sign when rotated 360°, it is advantageous to use spinors instead of vectors and tensors to describe the spin angular characteristic of the electron. Three-dimensional spinors of group $Spin(3)$ are $SU(2)$. The most important application of spinors in quantum physics is to provide mathematical representations of energy transfer in EM fields [15].

Now, let us give the definition of spinors in this sense with the notation used in the book written by Nagashima [16]. In two-dimensional complex variable space, the spinor is defined as the base vector of the group representation $SU(2)$. The notation $\varphi = \begin{bmatrix} \varphi_1, \varphi_2 \end{bmatrix}^T$, where φ_1, φ_2 are generally complex numbers, represents a two-component column vector and is commonly used to denote spinors. On the other hand, the representation matrices are stated as 2×2 unitary matrices with unit determinants. φ transforms under $SU(2)$ as follows:

$$\varphi \longrightarrow \varphi' = U\varphi, \text{ i.e., } \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = U(x,y) \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \quad (1)$$

where $U(x,y) \in SU(2)$ and

$$\varphi_1' = x\varphi_1 + y\varphi_2, \varphi_2' = -\overline{y}\varphi_1 + \overline{x}\varphi_2. \quad (2)$$

Furthermore, there are three independent parameters in $SU(2)$ [16].

In the algebraic definition, in physics, the spinor space is defined as a member of the minimal left ideal of Clifford's algebra [13]. Cartan created the mathematical form of spinors while investigating linear representations of simple groups [7].

The concept of spinors, initially named by the quantum physicist P. Ehrenfest, has emerged as a pivotal tool in numerous physical theories, notably in the realm of quantum mechanics [17].

In addition, there are important studies in which spinors are used in the application of mathematics in the field of physics. On the other hand, spinors, also studied geometrically by E. Cartan, are elements of complex vector space and are used in mathematics and physics to extend the concepts of rotation and space vectors. Spinors, which consist of two complex components in terms of vectors, were obtained in three-dimensional Euclidean space by Cartan [7]. The properties of spinors have been studied in different dimensions by different authors. In 2004, Castillo and Barrales gave some properties of spinors in three-dimensional real space [12]. Later, Castillo defined a spinor formulation in four-dimensional space [18].

Let us explain this concept better with the representation of spinors established with the help of the orthonormal base. The homomorphic groups $SO(3)$ and $SU(2)$ are the rotation group around the origin in $\mathbb{R}^3$ and unitary complex 2×2 matrices group with unit determinant, respectively. Here, the elements of $SU(2)$ move on two vectors with complex structure named spinors [12].

Every spinor $\varphi = \begin{bmatrix} \varphi_1, \varphi_2 \end{bmatrix}^T$ defines vectors $d, e, f \in \mathbb{R}^3$ through $d + ie = \varphi^t \sigma \varphi$, $f = \check{\varphi} \sigma \varphi$. Here, σ is a vector whose $\dot{\sigma}_1, \dot{\sigma}_2, \dot{\sigma}_3$ components are complex symmetric 2×2 matrices; $\widetilde{\varphi}$ and $\check{\varphi}$ denote the conjugate and mate of φ, respectively, where

$$\dot{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \dot{\sigma}_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \dot{\sigma}_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{3}$$

$$\widetilde{\varphi} = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\varphi} = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \begin{bmatrix} i\overline{\varphi_2} \\ -i\overline{\varphi_1} \end{bmatrix}. \tag{4}$$

and

$$\check{\varphi} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\varphi} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \begin{bmatrix} -\overline{\varphi_2} \\ \overline{\varphi_1} \end{bmatrix}. \tag{5}$$

On the other hand, the d, e, and $f \in \mathbb{R}^3$ vectors are defined by

$$d + ie = \left(\varphi_1^2 - \varphi_2^2, i\left(\varphi_1^2 + \varphi_2^2\right), -2\varphi_1\varphi_2\right) \tag{6}$$

and

$$f = \left(\overline{\varphi_2}\varphi_1 + \overline{\varphi_1}\varphi_2, i\overline{\varphi_2}\varphi_1 - i\overline{\varphi_1}\varphi_2, |\varphi_1|^2 - |\varphi_2|^2\right). \tag{7}$$

Also, $|d| = |e| = |f| = \overline{\varphi^t}\varphi$ and $d \times e.f > 0$. Let φ and ϕ be two arbitrary spinors and d, e be complex numbers. In this case, $\overline{\varphi^t \sigma \phi} = -\check{\varphi}^t \sigma \check{\phi}$, $(d\varphi \check{+} e\phi) = \overline{d}\check{\phi} + \overline{e}\check{\varphi}$, and $\check{\check{\phi}} = -\varphi$. Furthermore, for nonzero spinor φ, $\{\varphi, \check{\varphi}\}$ is linear independent and the spinors corresponding to $\{d, e, f\}, \{e, f, d\}, \{f, e, d\}$ are different [12].

Now, let us give place to one of the important concepts in this field: Pauli matrices. Hermitian, involutory, and unitary Pauli matrices are 2×2 matrices (8). However, all of the Pauli matrices can be compacted into a single expression. In addition, every 2×2 Hermitian matrix is uniquely written as a linear combination of Pauli matrices, where all coefficients are real numbers. Now, we recall the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8}$$

Clearly, $\dot{\sigma}_1, \dot{\sigma}_2, \dot{\sigma}_3$ matrices in (3) are obtained by multiplying the $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ matrix and the Pauli matrices (8). Pauli used spinors, thought to be elements of $\mathbb{C}^2$, to reveal the behavior of an electron by taking the spin of the electron into account in quantum mechanics. In physics, spinors arose as a product of Pauli's theory of non-relativistic quantum mechanics (1926) and Dirac's (1928) theory of relativistic quantum mechanics [19,20]. These matrices, which appear in the Pauli equation, that takes into calculus the interaction of a particle's spin with an external electromagnetic field, are named after the physicist Wolfgang Pauli [20] and these matrices have a very important place in nuclear physics studies. Dirac gave important formulas about Pauli matrices [19]. The space of Dirac spinors is a complex four-dimensional vector space, and turns out to split as the sum of a complex two-dimensional vector space S, called a spin space; and its complex conjugate $\overline{S}$. Since spinors are complex objects, both the space of complex conjugate spinors and its dual must be given. Vivarelli [21] was involved in this area in the geometrical aspect. He showed an injective and linear correspondence between spinors and quaternions and in three-dimensional Euclidean space he gave spinor representations of rotations. Thus, a more concise and simpler depiction of quaternions can be reached by the concept of spinors. Quaternions, being applied to the fields of mathematics, physics, robotics, engineering and chemistry, can be worked through spinors with the help of the correspondence given by Vivarelli [21].

Recent studies using special numbers like Fibonacci and Lucas numbers have brought a different perspective to the use of spinors in mathematics. The Fibonacci sequence and the Lucas sequence are sequences whose first terms are 0 and 2, respectively, and whose second term is 1, and each term after the second term is written as the sum of the two preceding terms. Fibonacci numbers and the golden ratio associated with these special numbers are the focus of attention in mathematics as well as in physics, chemistry, biology, technology, and astrophysics. Looking at quantum physics, one can find an example in [22] showing that the golden ratio can be seen in the anyon, being a two-dimensional quasi-particle. If we look at the polynomial sequences, the Fibonacci polynomial sequence is a generalization of the Fibonacci sequence. Polynomials generated in a similar way from Lucas numbers are called Lucas polynomials. Soykan [23] exhibited the generalized Fibonacci polynomials in many aspects. Over the years, new number and polynomial sequences have also been defined and several of their properties examined. The most prominent ones are Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas numbers and polynomials [24,25]. Horadam extends the Fibonacci integer numbers to the complex numbers [26], then extends complex Fibonacci and Lucas sequences to the Fibonacci and Lucas quaternions studied in various fields [26]. The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$ are division algebras and by cause of commutativity several arguments cannot be expanded from complex numbers to quaternions. As a result of this property, various results cannot be extended from Fibonacci and Horadam numbers to Fibonacci and Horadam quaternions, the same holds for polynomials. We can see some recently studied special quaternion sequences, for example, bicomplex (p, q)-Fibonacci quaternions [27], dual-generalized complex fibonacci quaternions [28], quantum quaternion polynomials [29], Gaussian Pell quaternions [30] and Pauli–Fibonacci quaternions [31].

Being extensions of Fibonacci quaternions, Erişir and Güngör [32], in 2020, first introduced Fibonacci and Lucas spinors with the help of the Fibonacci quaternions. Later, in 2023, Kumari et al. [33] examined the k-Fibonacci and k-Lucas spinors via k-Fibonacci and k-Lucas quaternions, which were worked out by Ramirez [34]. Erişir [35] give Horadam spinors as an extension of Horadam quaternions. This study is also a generalization of [32,33]. On the part of polynomial sequences, the generalized Fibonacci polynomials (Horadam polynomials) and the generalized Fibonacci polynomial spinors (Horadam polynomial spinors) are polynomial sequences that can be thought as an extension of the generalized Fibonacci numbers (Horadam numbers) and of the generalized Fibonacci spinors (Horadam spinors), respectively. No more work on spinor sequences and polynomial spinor sequences has been brought to the literature yet. For this reason, this study will fill the gap about special polynomial spinor sequences.

In this work, by associating spinors with quaternions, the spinor correspondences of Horadam quaternion polynomials, which are a generalization of quaternion polynomial sequences, are analyzed, and thus, it is shown that the given Horadam polynomial spinors are a generalization of the spinor representations of all these quaternion polynomial sequences. The motivation of this work is to define a new and special sequence that can be used in physics and exploit the advantages of spinors in physical applications.

In Sections 4 and 5 in the present paper, Simson's identity and Catalan's identity of the generalized Fibonacci polynomial spinors and their related corollaries and remarks, Theorem 4 and its corollaries and remark are completely new results which differ from the classical case of generalized Fibonacci polynomials. Apart from this, looking at all the results in this paper from Section 3 onwards, all the findings obtained for both sequences are different since the generalized Fibonacci polynomials are extended to polynomial matrices, primarily because the product is defined very differently.

Highlights of this research are:

- The relations between quaternions and spinors lead us to understand more about spinor sequences.
- The steps in the literature for the examination process of sequences and polynomials are also valid for the new sequence of polynomials we have defined.
- This special kind of sequence of polynomial spinors gives aesthetic results in quantum mechanics.
- Generalized Fibospinomials give a wide perspective about the other polynomial spinor sequences and spinor sequences.

Trying to see the beauty of these sequences in quantum mechanics will be interesting and worth examining. In our paper, we will work with generalized Fibonacci polynomial spinors through the correspondence between generalized Fibonacci spinors with generalized Fibonacci quaternions like Erişir and Güngör [32]. We will investigate several properties of this new polynomial spinor sequence such as Binet's formula, etc. In addition, as particular cases, we will obtain these features for (r,s)-Fibonacci polynomial spinors, (r,s)-Lucas polynomial spinors, Fibonacci polynomial spinors, and Lucas polynomial spinors, and we will reveal the relations between these polynomial spinors.

From this point of view, the outline of the present paper is as follows: In Section 2, some necessary definitions are introduced and preliminaries of the Horadam polynomial sequence, quaternions, Horadam quaternion polynomials, spinors, and correspondence between quaternions and spinors are given. Also, a few terms of the sequence of the generalized Fibonacci polynomials and its special cases are recalled in Table 1 in this section. Section 3 is reserved for introducing a new polynomial sequence, the generalized Fibospinomials, and its special cases, which are (r,s)-Fibonacci polynomial spinors, (r,s)-Lucas polynomial spinors, Fibonacci polynomial spinors, and Lucas polynomial spinors, and for exhibiting their important features such as the generating function and Binet's formula. Some terms of this new polynomial sequence and its special cases are listed in Tables 2–4. Apart from this, using the terms of the sequence we have obtained in Tables 3 and 4, some equalities come up. In Section 4, the Simson's identity of this new sequence and its special cases, an alternative demonstration of this identity, and an interesting identity demonstrating the relation between the terms of the sequences are focused on. Section 5 presents several identities revealing the relation between the generalized Fibospinomials, (r,s)-Fibonacci polynomial spinors, and (r,s)-Lucas polynomial spinors, and Catalan's identity and its special cases and an alternative description. Section 6 provides the sum of the first n terms of the sequence. In Section 7, the last section before the conclusion, matrices associated with the sequence of this special polynomial spinor are displayed.

2. Preliminaries

The Horadam polynomial sequence, or the generalized Fibonacci polynomial sequence, $\{W_n(x)\}$ was introduced by Horadam [36] with

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \;\; n \geq 2 \tag{9}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$. See also Ref. [23].

Binet's formula of generalized Fibonacci polynomials can be calculated using its characteristic equation, which is given as

$$z^2 - r(x)z - s(x) = 0. \tag{10}$$

The roots of the characteristic equation are

$$\alpha(x) = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \;\; \beta(x) = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2} \tag{11}$$

and the sum and product of the these roots are as follows:

$$\alpha(x) + \beta(x) = r(x), \tag{12}$$

$$\alpha(x)\beta(x) = -s(x). \tag{13}$$

If $\alpha(x) \neq \beta(x)$, then $r^2(x) + 4s(x) \neq 0$, and if $\alpha(x) = \beta(x)$, then (10) can be written as

$$z^2 - r(x)z - s(x) = (z - \alpha(x))^2 = z^2 - 2\alpha(x)z + \alpha^2(x) = 0 \tag{14}$$

and, in this case,

$$r(x) = 2\alpha(x), \tag{15}$$

$$s(x) = -\alpha^2(x) = -\frac{r^2(x)}{4}, \tag{16}$$

$$r^2(x) + 4s(x) = 0. \tag{17}$$

The Horadam polynomial sequence can be expanded to negative subscripts through defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)} W_{-(n-1)}(x) + \frac{1}{s(x)} W_{-(n-2)}(x) \tag{18}$$

for $n = 1, 2, 3, \ldots$, where $s(x) \neq 0$. Thus, recurrence (9) holds for all integers n. Soykan examined many aspects of the Horadam polynomials in detail [23].

Now, we recall the definitions of two special cases of the generalized Fibonacci polynomials $W_n(x)$ according to their first two values, denoted by $G_n(x)$ and $H_n(x)$, respectively, via the second-order recurrence relations [23]

$$G_{n+2}(x) = r(x)G_{n+1}(x) + s(x)G_n(x), \;\; G_0(x) = 0, G_1(x) = 1, \tag{19}$$

$$H_{n+2}(x) = r(x)H_{n+1}(x) + s(x)H_n(x), \;\; H_0(x) = 2, H_1(x) = r(x). \tag{20}$$

The sequences of polynomials $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ are named the (r,s)-Fibonacci polynomial sequence and (r,s)-Lucas polynomial sequence. The Fibonacci polynomial sequence $\{F_n(x)\}_{n\geq 0}$ and the Lucas polynomial sequence $\{L_n(x)\}_{n\geq 0}$ coming from $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ are given as a special case when $r(x) = x$ and $s(x) = 1$, respectively, as follows:

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad F_0(x) = 0, F_1(x) = 1, \tag{21}$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \quad L_0(x) = 2, L_1(x) = x. \tag{22}$$

Refs. [37–39] can be checked for the properties of Fibonacci polynomials, Lucas polynomials, and some generalizations of these sequences. Furthermore, the sequence of polynomials $\{W_n(x)\}_{n\geq 0}$, which is the generalized Fibonacci polynomial sequence, becomes the sequence of numbers $\{W_n\}_{n\geq 0}$, which we call the generalized Fibonacci numbers if $r(x)$ and $s(x)$ are the sequence of numbers as in the following:

$$W_n = rW_{n-1} + sW_{n-2}, \ \ n \geq 2 \tag{23}$$

with initial values W_0, W_1 not all being zero integers, where r, s are integers with $r \neq 0$, $s \neq 0$. As a special case, (r,s)-Fibonacci numbers, denoted by $\{G_n\}_{n\geq 0}$, and (r,s)-Lucas numbers, denoted by $\{H_n\}_{n\geq 0}$, are given by the following recurrence relations regarding initial values.

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \tag{24}$$
$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r. \tag{25}$$

In particular, the sequences of polynomials $\{F_n(x)\}_{n\geq 0}$ and $\{L_n(x)\}_{n\geq 0}$ are the extensions of the Fibonacci and Lucas numbers, respectively, so that if $r(x) = 1$ and $s(x) = 1$, then we have Fibonacci numbers $\{F_n\}$ and Lucas numbers $\{L_n\}$ as follows:

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, F_1 = 1, \tag{26}$$
$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, L_1 = 1. \tag{27}$$

As indicated above, the sequence of polynomials generalize the sequence of numbers. Now, let us have a look at the generalized Fibonacci quaternion polynomials and the generalized Fibonacci quaternions. For this, let us take a quick look at the definition of quaternions.

Quaternions extend the complex numbers and are defined in the form $q = q_0 + iq_1 + jq_2 + kq_3$, where q_0, q_1, q_2, q_3 are real numbers and $1, i, j, k$ are basis vectors satisfying $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. We denote the set of quaternions by $\mathbb{H}$. Multiplication of quaternions is not commutative. Hamilton defined that quaternions consist of a scalar part, given above as q_0, and a vector part, given above as $iq_1 + jq_2 + kq_3$. The conjugate and norm of a quaternion q are given by $q^* = q_0 - iq_1 - jq_2 - kq_3$ and $\|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, respectively.

Catarino defined the $h(x)$-Fibonacci quaternion polynomials generalizing the k-Fibonacci quaternion numbers and examined this polynomial sequence with its several properties [40,41]. Then, Özkoç and Porsuk [42] examined the generalized Fibonacci quaternion polynomials generalizing the generalized Fibonacci quaternion numbers defined by

$$QW_n(x) = W_n(x) + iW_{n+1}(x) + jW_{n+2}(x) + kW_{n+3}(x) \tag{28}$$

where $\{W_n(x)\}$ is a Horadam polynomial sequence, and they presented the Binet's formula, generating function and some identities for this polynomial sequence. From this point of view, it can be easily seen that the generalized Fibonacci quaternion polynomials sequence $\{QW_n(x)\}$ can be written with a second-order recurrence relation [42], as in the following:

$$QW_n(x) = r(x)QW_{n-1}(x) + s(x)QW_{n-2}(x), \ \ n \geq 2. \tag{29}$$

It can be noted that the characteristic equation for generalized Fibonacci polynomial quaternions will be the same as that of generalized Fibonacci polynomials since they have the same linear recurrence relation. In addition, by changing the coefficients $r(x)$ and $s(x)$, we obtain generalizations of different sequences such as Fibonacci quaternion polynomials, Lucas quaternion polynomials, Pell quaternion polynomials, Pell–Lucas quaternion polynomials, Jacobsthal quaternion polynomials, and Jacobsthal–Lucas quaternion polynomials, which are among the most well-known sequences.

The Horadam quaternions polynomial sequence also generalizes the sequence of numbers which we will recall for Horadam quaternions below. Horadam [26] defined the Fibonacci quaternions $\{QF_n\}$ and Lucas quaternions $\{QL_n\}$ and showed a few relations regarding the Fibonacci quaternions. Later, Iyer exhibited some relations between Fibonacci quaternions and Lucas quaternions [43] and Swamy found new properties between the generalized Fibonacci quaternion sequence and Fibonacci quaternion sequence [44]. Then, Halıcı introduced the Binet's formulas for the Fibonacci and Lucas quaternions, and also exhibited generating functions and some sum formulas for these sequences [45]. Later on, İpek introduced (r,s)-Fibonacci quaternions [46] and Patel and Ray [47] introduced (r,s)-Lucas quaternions and exhibited some identities about (r,s)-Fibonacci quaternions and (r,s)-Lucas quaternions, such as Catalan's identity, d'Ocagne's identity, etc. Then, Cerda-Morales presented these well-known identities for (r,s)-Fibonacci quaternions and (r,s)-Lucas quaternions using their Binet's formulas [48] and Szynal-Liana and Włoch worked on generalized commutative Fibonacci quaternions [49]. Halıcı and Karataş gave the most generalized versions of these series as follows:

The nth generalized Fibonacci quaternion was exhibited by Halıcı and Karataş [50] such that

$$QW_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} \tag{30}$$

where $\{W_n\}$ is a generalized Fibonacci or Horadam sequence satisfying (23). The generalized Fibonacci quaternion sequence, another way of saying Horadam quaternion sequence, is a second-order linear recurrence relation, so that for $n \geq 0$

$$QW_{n+2} = rQW_{n+1} + sQW_n \tag{31}$$

where $r, s \in \mathbb{Z}$.

For the sake of simplicity, throughout the rest of the paper for all integers n we use

$$W_n, G_n, H_n, F_n, L_n, r, s, \alpha, \beta$$

instead of

$$W_n(x), G_n(x), H_n(x), F_n(x), L_n(x), r(x), s(x), \alpha(x), \beta(x)$$

respectively, unless otherwise stated.

We can see the first few values of the sequence of polynomials $\{W_n\}, \{G_n\}, \{H_n\}, \{F_n\}$, and $\{L_n\}$ in Table 1 [23].

Next, we present the first few values of the sequence of polynomials $\{QW_n\}$ as follows:

$$\begin{aligned} QW_0 &= W_0 + iW_1 + j(rW_0 + sW_1) + k\Big(r^2W_0 + s(r+1)W_1\Big), \\ QW_1 &= W_1 + i(rW_0 + sW_1) + j\Big(r^2W_0 + s(r+1)W_1\Big) \\ &\quad + k\Big((r^3 + rs)W_0 + (s^2 + rs + r^2s)W_1\Big), \\ QW_2 &= rQW_1 + sQW_0. \end{aligned} \tag{32}$$

We now recall the spinors obtained with the help of the quaternions. Consider a spinor φ given by

$$\varphi = (\varphi_1, \varphi_2) \cong \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \tag{33}$$

where $\varphi_1, \varphi_2 \in \mathbb{C}$. We denote the set of spinors by $\mathbb{S}$. Vivarelli [21] pointed out that there is a correspondence between any quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ and a spinor $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ such that

$$f:\mathbb{H}\to\mathbb{S},\, f(q_0+iq_1+jq_2+kq_3)=\begin{bmatrix} q_3+iq_0 \\ q_1+iq_2 \end{bmatrix}\equiv\varphi. \tag{34}$$

Since $f(q+p)=f(q)+f(p)$, $f(\lambda q)=\lambda f(q)$, where $\lambda\in\mathbb{C}$, $p,q\in\mathbb{H}$ and $\ker f=\{0\}$, f is linear and injective. Under this correspondence f, the conjugate of q, i.e., q^*, is mapped to φ^* as below:

$$f(q^*)=f(q_0-iq_1-jq_2-kq_3)=\begin{bmatrix} -q_3+iq_0 \\ -q_1-iq_2 \end{bmatrix}\equiv\varphi^*. \tag{35}$$

Given two spinors $\varphi_a=\begin{bmatrix} \varphi_{1_a} \\ \varphi_{2_a} \end{bmatrix}$ and $\varphi_b=\begin{bmatrix} \varphi_{1_b} \\ \varphi_{2_b} \end{bmatrix}$, $\varphi_a=\varphi_b$ are equal if and only if $\varphi_{1_a}=\varphi_{1_b}$ and $\varphi_{2_a}=\varphi_{2_b}$.

Table 1. Some values of generalized Fibonacci, (r,s)-Fibonacci, (r,s)-Lucas, Fibonacci, and Lucas polynomials.

n	0	1	2	3
W_n	W_0	W_1	rW_0+sW_1	$r^2W_0+s(r+1)W_1$
W_{-n}		$\frac{(W_1-rW_0)}{s}$	$\frac{((r^2+s)W_0-rW_1)}{s^2}$	$\frac{(-(r^3+2rs)W_0+(r^2+s)W_1)}{s^3}$
G_n	0	1	r	$s+r^2$
G_{-n}		$\frac{1}{s}$	$-\frac{r}{s^2}$	$\frac{1}{s^3}(s+r^2)$
H_n	2	r	$2s+r^2$	$r(3s+r^2)$
H_{-n}		$-\frac{r}{s}$	$\frac{2s+r^2}{s^2}$	$-r\frac{3s+r^2}{s^3}$
F_n	0	1	x	$1+x^2$
F_{-n}		1	$-x$	$1+x^2$
L_n	2	x	$2+x^2$	x^3+3x
L_{-n}		$-x$	$2+x^2$	$-x^3-3x$

The product of quaternions $q\times p$ has been shown by Vivarelli [21] in relation to a spinor matrix product as follows:

$$q\times p\to -i\widehat{U}f(p)=-i\begin{bmatrix} q_3+iq_0 & q_1-iq_2 \\ q_1+iq_2 & -q_3+iq_0 \end{bmatrix}\begin{bmatrix} p_3+ip_0 \\ p_1+ip_2 \end{bmatrix} \tag{36}$$

where $\widehat{U}$ is complex, unitary 2×2 matrix taking in $SU(2)$. Here, $\widehat{U}$ can be written by means of Pauli matrices (8):

$$\widehat{U}=\begin{bmatrix} q_3+iq_0 & q_1-iq_2 \\ q_1+iq_2 & -q_3+iq_0 \end{bmatrix}=q_0iI+q_1\sigma_1+q_2\sigma_2+q_3\sigma_3 \tag{37}$$

where σ_1,σ_2, and σ_3 are Pauli matrices and I is the unit square matrix of type 2×2. The connection between a spinor φ and a 2×2 matrix $\widehat{U}$ is given [21] by

$$\varphi=\widehat{U}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{38}$$

The conjugate of a spinor φ is given by Cartan [7] as $\widetilde{\varphi}$ and the mate of a spinor φ is presented by Castillo and Barrales [12] as $\check{\varphi}$, as in the following identities:

$$\widetilde{\varphi} = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\overline{\varphi},\ \check{\varphi} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\overline{\varphi} \tag{39}$$

where $\overline{\varphi}$ is the complex conjugate of φ.

3. Generalized Fibonacci (Horadam) Polynomial Spinors

In the following, the definition of the sequence of the generalized Fibospinomials, which is a generalization of the sequence of Horadam spinors defined by Erişir [35], will be given.

Definition 1. *For integer n, the nth generalized Fibonacci polynomial spinor sequence of W_n is defined by*

$$SW_n = \begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} \tag{40}$$

where W_n is the nth generalized Fibonacci polynomial.

The generalized Fibonacci polynomial spinor sequence $\{SW_n\}$ satisfies the second-order linear recurrence sequence from the recurrence (9). We can see this in the next lemma.

Lemma 1. *The generalized Fibonacci polynomial spinor sequence $\{SW_n\}$ has the following identity for all integers n:*

$$SW_{n+2} = rSW_{n+1} + sSW_n \tag{41}$$

where $SW_0 = \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix}, SW_1 = \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix}$ are arbitrary polynomial spinors with real coefficients.

Proof. For all integers n, by using the recurrence (9), we can easily have the required identity:

$$\begin{aligned} rSW_{n+1} + sSW_n &= r\begin{bmatrix} W_{n+4} + iW_{n+1} \\ W_{n+2} + iW_{n+3} \end{bmatrix} + s\begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} W_{n+5} + iW_{n+2} \\ W_{n+3} + iW_{n+4} \end{bmatrix} = SW_{n+2}. \end{aligned} \tag{42}$$

□

Remark 1. *Note that first we can define the generalized Fibonacci polynomial spinor sequence $\{SW_n\}$ as (41), then we obtain (40).*

We can see a correspondence between the generalized Fibonacci polynomial quaternions and the generalized Fibonacci polynomial spinors by adapting from the transformation between quaternions and spinors with the following linear and injective transformation given by Vivarelli [21]:

$$\begin{aligned} f &: \mathbb{W} \to \mathbb{S}, \\ f(QW_n) &= f(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}) \\ &= \begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} = SW_n. \end{aligned} \tag{43}$$

If $QW_n^* = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$ is the conjugate of the nth generalized Fibonacci quaternion polynomial QW_n, then the nth generalized Fibonacci polynomial spinor SW_n corresponding to QW_n^* is

$$SW_n^* = \begin{bmatrix} -W_{n+3} + iW_n \\ -W_{n+1} - iW_{n+2} \end{bmatrix}. \tag{44}$$

We can write the complex conjugate of SW_n as

$$\overline{SW_n} = \begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix}, \tag{45}$$

the spinor conjugate to the SW_n as

$$\widetilde{SW_n} = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix} = \begin{bmatrix} W_{n+2} + iW_{n+1} \\ -W_n - iW_{n+3} \end{bmatrix}, \tag{46}$$

and the mate of SW_n as

$$S\check{W}_n = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+3} - iW_n \\ W_{n+1} - iW_{n+2} \end{bmatrix} = \begin{bmatrix} -W_{n+1} + iW_{n+2} \\ W_{n+3} - iW_n \end{bmatrix}. \tag{47}$$

Now, we define two special cases of the polynomial spinors SW_n. (r,s)-Fibonacci polynomial spinors, or shortly SG_n, and (r,s)-Lucas polynomial spinors, or shortly SH_n, are the special cases of (40).

Definition 2. *For integer n, the nth sequence of polynomials SG_n is defined by*

$$SG_n = \begin{bmatrix} G_{n+3} + iG_n \\ G_{n+1} + iG_{n+2} \end{bmatrix} \tag{48}$$

where G_n is the nth (r,s)-Fibonacci polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SG_{n+2} = rSG_{n+1} + sSG_n, \tag{49}$$

with

$$SG_0 = \begin{bmatrix} r^2 + s \\ 1 + ir \end{bmatrix}, SG_1 = \begin{bmatrix} r^3 + 2rs + i \\ r + i(r^2 + s) \end{bmatrix}, \tag{50}$$

and the nth sequence of polynomials SH_n is defined by

$$SH_n = \begin{bmatrix} H_{n+3} + iH_n \\ H_{n+1} + iH_{n+2} \end{bmatrix} \tag{51}$$

where H_n is an nth (r,s)-Lucas polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SH_{n+2} = rSH_{n+1} + sSH_n, \tag{52}$$

with

$$SH_0 = \begin{bmatrix} r(3s + r^2) + 2i \\ r + i(2s + r^2) \end{bmatrix}, SH_1 = \begin{bmatrix} 4r^2s + r^4 + 2s^2 + ir \\ 2s + r^2 + ir(3s + r^2) \end{bmatrix}. \tag{53}$$

When $r = x$ and $s = 1$, we have the special case of generalized Fibonacci polynomial spinors, which are called Fibonacci polynomial spinors, denoted as $\{SF_n\}$, and Lucas polynomial spinors, denoted as $\{SL_n\}$, as in the next definition. These polynomial sequences are a generalization of the sequence of Fibonacci spinors and the sequence of Lucas spinors, respectively, defined by Erişir and Güngör [32].

Definition 3. *For integer n, the nth Fibonacci polynomial spinors SF_n are defined as by*

$$SF_n = \begin{bmatrix} F_{n+3} + iF_n \\ F_{n+1} + iF_{n+2} \end{bmatrix} \tag{54}$$

where F_n is the nth Fibonacci polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SF_{n+2} = xSF_{n+1} + SF_n, \tag{55}$$

with

$$SF_0 = \begin{bmatrix} x^2 + 1 \\ 1 + ix \end{bmatrix}, SF_1 = \begin{bmatrix} x^3 + 2x + i \\ x + i(x^2 + 1) \end{bmatrix}. \tag{56}$$

and the nth Lucas polynomial spinors SL_n are defined by

$$SL_n = \begin{bmatrix} L_{n+3} + iL_n \\ L_{n+1} + iL_{n+2} \end{bmatrix} \tag{57}$$

where L_n is the nth Lucas polynomial. From Lemma 1, it can be written equivalently by the second-order recurrence relations

$$SL_{n+2} = xSL_{n+1} + SL_n, \tag{58}$$

with

$$SL_0 = \begin{bmatrix} x + x^3 + 2i \\ x + i(2 + x^2) \end{bmatrix}, SL_1 = \begin{bmatrix} x^4 + 4x^2 + 2 + ix \\ 2 + x^2 + i(3x + x^3) \end{bmatrix}. \tag{59}$$

If we take $r = x = 1$, then the sequences of Fibonacci polynomial spinors $\{SF_n\}$ and $\{SL_n\}$ become the number sequences of Fibonacci spinors and Lucas spinors, respectively. Erişir and Güngör exhibited some algebraic definitions for Fibonacci and Lucas spinors in addition to giving some significant formulas like Binet's and Cassini's formulas for these sequences of numbers [32]. One can also see Cartan [7] and Vivarelli [21] for some algebraic properties of spinors.

Now, we list a few values of generalized fibospinomials in Table 2.

Table 2. The first few values of generalized Fibonacci polynomial spinors with negative and positive subscripts.

n	0	1	2
SW_n	$\begin{bmatrix} (r^2+s)W_1 \\ +rsW_0 + iW_0 \\ \\ W_1 \\ +i(rW_1 + sW_0) \end{bmatrix}$	$\begin{bmatrix} (r^3+2rs)W_1 \\ +(r^2s+s^2)W_0 \\ +iW_1 \\ \\ rW_1 + sW_0 \\ +i((r^2+s)W_1 \\ +rsW_0) \end{bmatrix}$	$\begin{bmatrix} (r^4+3r^2s+s^2)W_1 \\ +(r^3s+2rs^2)W_0 \\ +(irW_1 + sW_0) \\ \\ (r^2+s)W_1 + rsW_0 \\ +i((r^3+2rs)W_1 \\ +(r^2s+s^2)W_0) \end{bmatrix}$
SW_{-n}		$\begin{bmatrix} rW_1 + sW_0 \\ +i\frac{1}{s}(W_1 - rW_0) \\ \\ W_0 + iW_1 \end{bmatrix}$	$\begin{bmatrix} W_1 + i\frac{1}{s}(W_0 \\ -\frac{r}{s}(W_1 - rW_0)) \\ \\ \frac{1}{s}(W_1 - rW_0) \\ +iW_0 \end{bmatrix}$

Next, we present the first few values of the special polynomial spinors of second order with negative and positive subscripts in Table 3:

Table 3. The first few values of (r,s)-Fibonacci and (r,s)-Lucas polynomial spinors with negative and positive subscripts.

n	0	1	2
SG_n	$\begin{bmatrix} r^2+s \\ 1+ir \end{bmatrix}$	$\begin{bmatrix} r^3+2rs+i \\ r+i(r^2+s) \end{bmatrix}$	$\begin{bmatrix} r^4+3r^2s \\ +s^2+ir \\ \\ r^2+s \\ +i(r^3+2rs) \end{bmatrix}$
SG_{-n}		$\begin{bmatrix} r+i\frac{1}{s} \\ i \end{bmatrix}$	$\begin{bmatrix} 1-i\frac{r}{s^2} \\ \frac{1}{s} \end{bmatrix}$
SH_n	$\begin{bmatrix} r(3s+r^2)+2i \\ r+i(2s+r^2) \end{bmatrix}$	$\begin{bmatrix} 4r^2s+r^4 \\ +2s^2+ir \\ \\ 2s+r^2 \\ +ir(3s+r^2) \end{bmatrix}$	$\begin{bmatrix} r(r^4+5r^2s+5s^2) \\ +i(2s+r^2) \\ \\ r(3s+r^2) \\ +i(4r^2s+r^4+2s^2) \end{bmatrix}$
SH_{-n}		$\begin{bmatrix} (2s+r^2)-i\frac{r}{s} \\ 2+ir \end{bmatrix}$	$\begin{bmatrix} r+i\frac{2s+r^2}{s^2} \\ -\frac{r}{s}+2i \end{bmatrix}$

Using the recurrence relation of the Fibonacci and Lucas polynomial spinors $\{SF_n\}$ and $\{SL_n\}$, we can write the first few terms of these sequences of polynomials, respectively. See Table 4.

Table 4. The first few values of Fibonacci and Lucas polynomial spinors with positive and negative subscripts.

n	0	1	2
SF_n	$\begin{bmatrix} x^2+1 \\ 1+ix \end{bmatrix}$	$\begin{bmatrix} x^3+2x+i \\ x+i(x^2+1) \end{bmatrix}$	$\begin{bmatrix} x^4+3x^2 \\ +1+ix \\ \\ x^2+1 \\ +i(x^3+2x) \end{bmatrix}$
SF_{-n}		$\begin{bmatrix} x+i \\ i \end{bmatrix}$	$\begin{bmatrix} 1-ix \\ 1 \end{bmatrix}$
SL_n	$\begin{bmatrix} x(3+x^2)+2i \\ x+i(2+x^2) \end{bmatrix}$	$\begin{bmatrix} 4x^2+x^4 \\ +2+ix \\ \\ 2+x^2 \\ +ix(3+x^2) \end{bmatrix}$	$\begin{bmatrix} x(x^4+5x^2+5) \\ +i(2+x^2) \\ \\ x(3+x^2) \\ +i(4x^2+x^4+2) \end{bmatrix}$
SL_{-n}		$\begin{bmatrix} (2+x^2)-ix \\ 2+ix \end{bmatrix}$	$\begin{bmatrix} x+i(2+x^2) \\ -x+2i \end{bmatrix}$

We can show the product of the unitary complex matrix $\widehat{SW_n}$ obtained by SW_n with a generalized Fibonacci polynomial spinor sequence SW_n as follows:

$$\begin{aligned} \widehat{SW_n} SW_n &= \begin{bmatrix} W_{n+3} + iW_n & W_{n+1} - iW_{n+2} \\ W_{n+1} + iW_{n+2} & -W_{n+3} + iW_n \end{bmatrix} \begin{bmatrix} W_{n+3} + iW_n \\ W_{n+1} + iW_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 + i(W_n W_{n+3} + W_n) \\ -2W_n W_{n+2} + i2W_n W_{n+1} \end{bmatrix}. \end{aligned} \tag{60}$$

Considering this product, the following identities can easily be seen:

Lemma 2. *For all integers n, the next identities hold:*

(i)

$$\widehat{SW_n}\, SW_n^* = \widehat{SW_n^*}\, SW_n. \tag{61}$$

(ii)

$$\widehat{SW_n^*}\, SW_n^* = -(\widehat{SW_n}\, SW_n)^*. \tag{62}$$

(iii)

$$\overline{\widehat{SW_n}\, SW_n} = \widehat{\overline{SW_n}}\, \overline{SW_n}. \tag{63}$$

(iv)

$$\widetilde{\widehat{SW_n}\, SW_n} = -\widehat{SW_n}\, \widetilde{SW_n}. \tag{64}$$

(v)

$$\check{\widehat{SW_n} SW_n} = -\widehat{SW_n} \check{SW_n}. \tag{65}$$

Lemma 3. *The following equalities are true:*

(i)

$$s\left(\widehat{SG_0} SG_1 - \widehat{SG_1} SG_0\right) = \widehat{SG_2} SG_1 - \widehat{SG_1} SG_2. \tag{66}$$

(ii)

$$r\left(\widehat{SG_0} SG_1 - \widehat{SG_1} SG_0\right) = \widehat{SG_0} SG_2 - \widehat{SG_2} SG_0. \tag{67}$$

(iii)

$$s\left(\widehat{SH_0} SH_1 - \widehat{SH_1} SH_0\right) = \widehat{SH_2} SH_1 - \widehat{SH_1} SH_2. \tag{68}$$

(iv)

$$r\left(\widehat{SH_0} SH_1 - \widehat{SH_1} SH_0\right) = \widehat{SH_0} SH_2 - \widehat{SH_2} SH_0. \tag{69}$$

Proof. Once we take the values of n as $0, 1,$ and 2 in Table 3, we can easily obtain the required equalities by (60). □

It can be noted that the characteristic equation for generalized Fibonacci polynomial spinors is the same that of generalized Fibonacci polynomials.

Now, we can give the Binet's formula of SW_n using the roots α, β in (11) and recurrence relation (41) as follows:

Theorem 1. *For all integers n, the Binet's formula for the generalized Fibonacci polynomial spinor SW_n is given by the following formula:*

$$SW_n = \begin{cases} \dfrac{1}{\alpha - \beta}((SW_1 - \beta SW_0)\alpha^n - (SW_1 - \alpha SW_0)\beta^n), & \alpha \neq \beta \\ (SW_0 + (\dfrac{1}{\alpha}(SW_1 - \alpha SW_0)n)\alpha^n, & \alpha = \beta. \end{cases} \tag{70}$$

Proof. When the roots α, β of the characteristic Equation (10) are distinct, one can write the general formula of SW_n as follows:

$$SW_n = p_1\alpha^n + q_1\beta^n \tag{71}$$

where the coefficients p_1 and q_1 are determined by the system of linear equations

$$\begin{aligned} SW_0 &= p_1 + q_1 \\ SW_1 &= p_1\alpha + q_1\beta. \end{aligned} \tag{72}$$

Solving these two simultaneous equations for SW_0 and SW_1, we obtain

$$\begin{aligned} p_1 &= \frac{1}{\alpha-\beta}(SW_1 - \beta SW_0) \\ q_1 &= \frac{-1}{\alpha-\beta}(SW_1 - \alpha SW_0). \end{aligned} \tag{73}$$

If the roots α, β are equal, then we can write SW_n as follows:

$$SW_n = (p_2 + q_2 n)\alpha^n \tag{74}$$

where the coefficients p_2 and q_2 are the polynomials whose values are determined by SW_0 and any other known value of the sequence. By using the values SW_0 and SW_1, we obtain

$$\begin{aligned} SW_0 &= p_2 \\ SW_1 &= (p_2 + q_2)\alpha. \end{aligned} \tag{75}$$

Solving these two simultaneous equations for SW_0 and SW_1, we obtain

$$\begin{aligned} p_2 &= SW_0 \\ q_2 &= \frac{1}{\alpha}(SW_1 - \alpha SW_0). \end{aligned} \tag{76}$$

□

Now, let us calculate the values of $SW_1 - \beta SW_0$ and $SW_1 - \alpha SW_0$, which are in the Binet's formula, by using (9), (12) and (13), as in the following:

$$\begin{aligned} SW_1 - \beta SW_0 &= \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} - \beta \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha^3(W_1 - \beta W_0) + i(W_1 - \beta W_0) \\ \alpha(W_1 - \beta W_0) + i\alpha^2(W_1 - \beta W_0) \end{bmatrix} \\ &= (W_1 - \beta W_0)\begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} \end{aligned} \tag{77}$$

and

$$\begin{aligned} SW_1 - \alpha SW_0 &= \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} - \alpha \begin{bmatrix} W_3 + iW_0 \\ W_1 + iW_2 \end{bmatrix} \\ &= \begin{bmatrix} \beta^3(W_1 - \alpha W_0) + i(W_1 - \alpha W_0) \\ \beta(W_1 - \alpha W_0) + i\beta^2(W_1 - \alpha W_0) \end{bmatrix} \\ &= (W_1 - \alpha W_0)\begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix}. \end{aligned} \tag{78}$$

We can also find the Binet's formula of the generalized Fibonacci polynomial spinors $\{SW_n\}$ by using the Binet's formula of the generalized Fibonacci polynomial $\{W_n\}$ given by Soykan [23] as

$$W_n = \begin{cases} \dfrac{1}{\alpha-\beta}(c_1\alpha^n - c_2\beta^n)), & \alpha \neq \beta \\ (d_1 + d_2 n)\alpha^n, & \alpha = \beta. \end{cases} \tag{79}$$

where

$$c_1 = W_1 - \beta W_0,\ c_2 = W_1 - \alpha W_0, \tag{80}$$

and

$$d_1 = W_0,\ d_2 = \frac{1}{\alpha}(W_1 - \alpha W_0). \tag{81}$$

Hence, we present an alternative method for finding the Binet's formula of SW_n as follows:

For $\alpha \neq \beta$, we obtain that

$$\begin{aligned} SW_n &= \frac{1}{\alpha-\beta}\begin{bmatrix} (c_1\alpha^{n+3} - c_2\beta^{n+3}) + i(c_1\alpha^n - c_2\beta^n) \\ (c_1\alpha^{n+1} - c_2\beta^{n+1}) + i(c_1\alpha^{n+2} - c_2\beta^{n+2}) \end{bmatrix} \\ &= \frac{1}{\alpha-\beta}\begin{bmatrix} c_1\alpha^{n+3} + ic_1\alpha^n \\ c_1\alpha^{n+1} + ic_1\alpha^{n+2} \end{bmatrix} - \frac{1}{\alpha-\beta}\begin{bmatrix} c_2\beta^{n+3} + ic_2\beta^n \\ c_2\beta^{n+1} + ic_2\beta^{n+2} \end{bmatrix} \\ &= \frac{1}{\alpha-\beta}((W_1 - \beta W_0)\alpha^n \begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} - (W_1 - \alpha W_0)\beta^n \begin{bmatrix} \beta^3 + i \\ \beta + i\beta^2 \end{bmatrix}) \end{aligned} \tag{82}$$

and for $\alpha = \beta$ by using (9), (15), (16), and (17), we obtain that

$$\begin{aligned} SW_n &= \begin{bmatrix} (d_1 + d_2(n+3))\alpha^{n+3} + i(d_1 + d_2 n)\alpha^n \\ (d_1 + d_2(n+1))\alpha^{n+1} + i(d_1 + d_2(n+2))\alpha^{n+2} \end{bmatrix} \\ &= \alpha^n \begin{bmatrix} (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+3))\alpha^3 + i(W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)n) \\ (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+1))\alpha + i(W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)(n+2))\alpha^2 \end{bmatrix} \\ &= \alpha^n((\frac{1}{\alpha}W_1 - W_0)n \begin{bmatrix} \alpha^3 + i \\ \alpha + i\alpha^2 \end{bmatrix} + SW_0). \end{aligned} \tag{83}$$

Next , we present a corollary for special cases of generalized Fibonacci polynomial spinors.

Corollary 1. *For all integers n, the Binet's formula for (r,s)-Fibonacci polynomial spinors $\{SG_n\}$, (r,s)-Lucas polynomial spinors $\{SH_n\}$, Fibonacci polynomial spinors $\{SF_n\}$, and Lucas polynomial spinors $\{SL_n\}$ is as follows:*

(i)

$$SG_n = \begin{cases} \dfrac{1}{\alpha-\beta}(\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n), & \alpha \neq \beta \\ (SG_0 + (\dfrac{1}{\alpha}\widehat{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{84}$$

where $\widehat{\alpha} = \begin{bmatrix} \alpha(r^2+s) + rs + i \\ \alpha + i(\alpha r + s) \end{bmatrix}, \widehat{\beta} = \begin{bmatrix} \beta(r^2+s) + rs + i \\ \beta + i(\beta r + s) \end{bmatrix}$ *and* $SG_0 = \begin{bmatrix} r^2+s \\ 1+ir \end{bmatrix}.$

(ii)

$$SH_n = \begin{cases} \dfrac{1}{\alpha-\beta}(\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n), & \alpha \neq \beta \\ (SH_0 + (\dfrac{1}{\alpha}\widetilde{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{85}$$

$$\text{where } \widetilde{\alpha} = \begin{bmatrix} \alpha(r^3+3rs)+r^2s+2s^2+i(-r+2\alpha) \\ \alpha r+2s+i(\alpha(r^2+2s)+rs) \end{bmatrix},$$
$$\widetilde{\beta} = \begin{bmatrix} \beta(r^3+sr+2rs)+r^2s+2s^2+i(-r+2\beta) \\ \beta r+2s+i(\beta(r^2+2s)+rs) \end{bmatrix} \text{ and } SH_0 = \begin{bmatrix} r^3+3rs+2i \\ r+i(r^2+2s) \end{bmatrix}.$$

(iii)

$$SF_n = \begin{cases} \dfrac{1}{\alpha-\beta}(\overline{\alpha}\alpha^n - \overline{\beta}\beta^n), & \alpha \neq \beta \\ (SF_0 + \dfrac{1}{\alpha}\overline{\beta}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{86}$$

$$\text{where } \overline{\alpha} = \begin{bmatrix} \alpha x^2+x+\alpha+i \\ \alpha+i(\alpha x+1) \end{bmatrix}, \overline{\beta} = \begin{bmatrix} \beta x^2+x+\beta+i \\ \beta+i(\beta x+1) \end{bmatrix} \text{ and } SF_0 = \begin{bmatrix} x^2+1 \\ 1+ix \end{bmatrix}.$$

(iv)

$$SL_n = \begin{cases} \dfrac{1}{\alpha-\beta}(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n), & \alpha \neq \beta \\ (SL_0 + \dfrac{1}{\alpha}\underline{\alpha}n)\alpha^n, & \alpha = \beta. \end{cases} \tag{87}$$

$$\text{where } \underline{\alpha} = \begin{bmatrix} \alpha(x^3+3x)+x^2+2+i(-x+2\alpha) \\ \alpha x+2+i(\alpha(x^2+2)+x) \end{bmatrix},$$
$$\underline{\beta} = \begin{bmatrix} \beta(x^3+3x)+x^2+2+i(-x+2\beta) \\ \beta x+2+i(\beta(x^2+2)+x) \end{bmatrix} \text{ and } SL_0 = \begin{bmatrix} x^3+3x+2i \\ x+i(x^2+2) \end{bmatrix}$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} SW_n y^n$ of the sequence $\{SW_n\}$.

Lemma 4. *Suppose that $f_{SW_n}(y) = \sum_{n=0}^{\infty} SW_n y^n$ is the ordinary generating function of the generalized Fibonacci polynomial spinors $\{SW_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} SW_n y^n$ is given by*

$$\sum_{n=0}^{\infty} SW_n y^n = \frac{SW_0 + (SW_1 - rSW_0)y}{1 - ry - sy^2}. \tag{88}$$

Proof. Using the definition of generalized Fibonacci polynomial spinors, and subtracting $ry\sum_{n=0}^{\infty} SW_n y^n$ and $sy^2\sum_{n=0}^{\infty} SW_n y^n$ from $\sum_{n=0}^{\infty} SW_n y^n$, we obtain

$$\begin{aligned}
(1-ry-sy^2)\sum_{n=0}^{\infty} SW_n y^n &= \sum_{n=0}^{\infty} SW_n y^n - ry\sum_{n=0}^{\infty} SW_n y^n - sy^2\sum_{n=0}^{\infty} SW_n y^n \\
&= \sum_{n=0}^{\infty} SW_n y^n - r\sum_{n=0}^{\infty} SW_n y^{n+1} - s\sum_{n=0}^{\infty} SW_n y^{n+2} \\
&= \sum_{n=0}^{\infty} SW_n y^n - r\sum_{n=1}^{\infty} SW_{n-1} y^n - s\sum_{n=2}^{\infty} SW_{n-2} y^n \\
&= (SW_0 + SW_1 y) - rSW_0 y \\
&\quad + \sum_{n=2}^{\infty}(SW_n - rSW_{n-1} - sSW_{n-2})y^n \\
&= SW_0 + (SW_1 - rSW_0)y.
\end{aligned} \tag{89}$$

Rearranging the above equation, we obtain (88). □

Lemma 4 gives the following results as particular examples.

Corollary 2. *The generating functions of* (r,s)*-Fibonacci,* (r,s)*-Lucas, Fibonacci, and Lucas polynomial spinors are given by the following formulas:*

(i)

$$\sum_{n=0}^{\infty} SG_n y^n = \frac{1}{1-ry-sy^2}\begin{bmatrix} r^2+s+rsy+iy \\ 1+i(r+sy) \end{bmatrix}. \tag{90}$$

(ii)

$$\sum_{n=0}^{\infty} SH_n y^n = \frac{1}{1-ry-sy^2}\begin{bmatrix} r^3+3rs+\left(2s^2+r^2s\right)y+i(2-ry) \\ r+2sy+i(r^2+2s+rsy) \end{bmatrix}. \tag{91}$$

(iii)

$$\sum_{n=0}^{\infty} SF_n y^n = \frac{1}{1-xy-y^2}\begin{bmatrix} x(x+y)+1+iy \\ 1+i(x+y) \end{bmatrix}. \tag{92}$$

(iv)

$$\sum_{n=0}^{\infty} SL_n y^n = \frac{1}{1-xy-y^2}\begin{bmatrix} x^3+3x+\left(2+x^2\right)y+i(2-xy) \\ x+2y+i(x(x+y)+2) \end{bmatrix}. \tag{93}$$

respectively.

Proof. In Lemma 4, take SW_n as SG_n, SH_n, SF_n, and SL_n, respectively. Use the first two terms of these sequences of polynomials by taking $n=0,1$ in Tables 3 and 4 for the formula. □

4. Simson's Formulas

We start with by defining the generalized Fibonacci polynomial spinor matrix with the help of Fibonacci spinor matrix definition given by Erişir and Güngör [32]. In order to define required matrix, we need to recall the Fibonacci quaternion matrix defined by Halıcı [45] as follows:

$$Q = \begin{bmatrix} QF_2 & QF_1 \\ QF_1 & QF_0 \end{bmatrix} \tag{94}$$

where

$$\begin{aligned} QF_0 &= F_0+iF_1+jF_2+kF_3 = i+j+2k, \\ QF_1 &= F_1+iF_2+jF_3+kF_4 = 1+i+2j+3k, \\ QF_2 &= F_2+iF_3+jF_4+kF_5 = 1+2i+3j+5k. \end{aligned} \tag{95}$$

From this point of view, we can define the generalized Fibonacci quaternion polynomial matrix as follows:

$$QW(x) = \begin{bmatrix} QW_2 & QW_1 \\ QW_1 & QW_0 \end{bmatrix}. \tag{96}$$

In addition, we can define the way the determinant of a given matrix is calculated as follows:

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{22}a_{11} - a_{12}a_{21}. \tag{97}$$

We prefer to use this formula of a determinant of a 2×2 matrix throughout the paper.

Erişir and Güngör [32] gave the Fibonacci spinor matrix for Fibonacci spinors, helping to obtain the Simson's identity. Now, we will present the generalized Fibonacci polynomial spinor matrix in the next theorem and we will denote this matrix with S_W. Then, we will use the matrix S_W to find several identities such as Simson's identity, and so on.

Theorem 2. *Let $QW(x)$ be the generalized Fibonacci quaternion polynomial matrix. Then, the following equality holds from quaternion products via spinors:*

$$\det QW(x) \equiv i \det S_W$$

where

$$S_W = -i \begin{bmatrix} SW_2 & \widehat{SW_1} \\ SW_1 & \widehat{SW_0} \end{bmatrix} \tag{98}$$

with

$$\widehat{SW_0} = \begin{bmatrix} W_3 + iW_0 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 + iW_0 \end{bmatrix}, \widehat{SW_1} = \begin{bmatrix} W_4 + iW_1 & W_2 - iW_3 \\ W_2 + iW_3 & -W_4 + iW_1 \end{bmatrix} \tag{99}$$

and

$$SW_1 = \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix}, SW_2 = \begin{bmatrix} W_5 + iW_2 \\ W_3 + iW_4 \end{bmatrix}. \tag{100}$$

Proof. Let the nth generalized Fibonacci polynomial spinor SW_n correspond to the nth generalized Fibonacci quaternion polynomial QW_n. Given a generalized Fibonacci quaternion polynomial matrix $QW(x)$, we obtain by (36) that

$$\begin{aligned} \det QW(x) &= QW_0 QW_2 - QW_1 QW_1 \\ &\equiv -i\widehat{SW_0} SW_2 + i\widehat{SW_1} SW_1 \equiv -i(\widehat{SW_0} SW_2 - \widehat{SW_1} SW_1) \\ &= -i \det \begin{bmatrix} SW_2 & \widehat{SW_1} \\ SW_1 & \widehat{SW_0} \end{bmatrix} = i \det S_W. \end{aligned} \tag{101}$$

Hence, we can write the generalized Fibonacci polynomial spinor matrix corresponding to its quaternion version by using the formula of the determinant as follows:

$$S_W = -i \begin{bmatrix} SW_2 & \widehat{SW_1} \\ SW_1 & \widehat{SW_0} \end{bmatrix} \tag{102}$$

where $\widehat{SW_0}$, $\widehat{SW_1}$, SW_1, and SW_2 are given in (99) and (100). Thus, we have the following matrix:

$$S_W = -i \begin{bmatrix} \begin{bmatrix} W_5 + iW_2 \\ W_3 + iW_4 \end{bmatrix} & \begin{bmatrix} W_4 + iW_1 & W_2 - iW_3 \\ W_2 + iW_3 & -W_4 + iW_1 \end{bmatrix} \\ \begin{bmatrix} W_4 + iW_1 \\ W_2 + iW_3 \end{bmatrix} & \begin{bmatrix} W_3 + iW_0 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 + iW_0 \end{bmatrix} \end{bmatrix}. \tag{103}$$

□

From Theorem 2, we can present the Simson's identity in two different forms in the next theorem.

Theorem 3. *(Simson's Identity) For all integers n, we have*

$$\widehat{SW_{n-1}} SW_{n+1} - \widehat{SW_n} SW_n = (-s)^n (\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0). \tag{104}$$

i.e.,

$$\det \begin{bmatrix} SW_{n+1} & \widehat{SW_n} \\ SW_n & \widehat{SW_{n-1}} \end{bmatrix} = (-s)^n \det \begin{bmatrix} SW_1 & \widehat{SW_0} \\ SW_0 & \widehat{SW_{-1}} \end{bmatrix}.$$

Proof. For $\alpha \neq \beta$, using (82), the Binet's formula of SW_n, and (13) we obtain the following identities:

$$\begin{aligned} & \widehat{SW_{n-1}} SW_{n+1} - \widehat{SW_n} SW_n \\ = \quad & -(-s)^{n-1} c_1 c_2 \begin{bmatrix} (\beta - \alpha - \alpha^3\beta^4 + \alpha^4\beta^3 - \alpha^2\beta^3 \\ +\alpha^3\beta^2 - \alpha\beta^2 + \alpha^2\beta + i(\alpha^4 - \beta^4)) \\ \\ (\beta^3 - \alpha^3 + \alpha^4\beta - \alpha\beta^4 + \alpha^2\beta - \alpha\beta^2 \\ +\alpha^3\beta^2 - \alpha^2\beta^3 + i(\alpha^2 - \beta^2 - \alpha^2\beta^4 + \alpha^4\beta^2)) \end{bmatrix} \end{aligned} \tag{105}$$

where $c_1 = W_1 - \beta W_0$ and $c_2 = W_1 - \alpha W_0$. On the other hand,

$$\begin{aligned} & \widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0 \\ = \quad & -\frac{1}{s} c_1 c_2 \begin{bmatrix} (\beta - \alpha - \alpha^3\beta^4 + \alpha^4\beta^3 - \alpha^2\beta^3 \\ +\alpha^3\beta^2 - \alpha\beta^2 + \alpha^2\beta + i(\alpha^4 - \beta^4)) \\ \\ (\beta^3 - \alpha^3 + \alpha^4\beta - \alpha\beta^4 + \alpha^2\beta - \alpha\beta^2 + \alpha^3\beta^2 \\ -\alpha^2\beta^3 + i(\alpha^2 - \beta^2 - \alpha^2\beta^4 + \alpha^4\beta^2)) \end{bmatrix}. \end{aligned} \tag{106}$$

From (105) and (106), we have the required identity:

$$\begin{aligned} \widehat{SW_{n-1}} SW_{n+1} - \widehat{SW_n} SW_n &= (-s)^{n-1} s \left(\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0 \right) \\ &= (-s)^n (\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0). \end{aligned}$$

Now, let us see when the roots are equal, i.e., for $\alpha = \beta$, that (104) holds for all integers n. Using (83), we obtain that

$$\widehat{SW_{n-1}} SW_{n+1} - \widehat{SW_n} SW_n = -\alpha^{2n} d_2^2 \begin{bmatrix} (\alpha^6 + \alpha^4 + \alpha^2 - 1 + 4i\alpha^3) \\ -4\alpha^4 + 2\alpha^2 - 2\alpha i(\alpha^4 + 1) \end{bmatrix} \tag{107}$$

where $d_1 = W_0$ and $d_2 = \frac{1}{\alpha}(W_1 - \alpha W_0)$. On the other hand,

$$\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0 = -d_2^2 \begin{bmatrix} (\alpha^6 + \alpha^4 + \alpha^2 - 1 + 4i\alpha^3) \\ 2\alpha(i\alpha^4 + 2\alpha^3 - \alpha + i) \end{bmatrix}. \tag{108}$$

From (107), (108) and (16) we obtain the required identity as follows:

$$\begin{aligned} \widehat{SW_{n-1}} SW_{n+1} - \widehat{SW_n} SW_n &= \alpha^{2n} \left(\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0 \right) \\ &= (-s)^n \left(\widehat{SW_{-1}} SW_1 - \widehat{SW_0} SW_0 \right). \end{aligned}$$

□

The previous theorem gives the following results as particular examples.

Corollary 3. *For all integers n, Simson's formula of (r,s)-Fibonacci, (r,s)-Lucas, Fibonacci, and Lucas polynomial spinors is given as*

(i)

$$\begin{aligned} & \widehat{SG_{n+1}} SG_{n-1} - \widehat{SG_n} SG_n \\ = \quad & (-1)^n s^n (\widehat{SG_{-1}} SG_1 - \widehat{SG_0} SG_0) \\ = \quad & (-1)^n s^{n-1} \begin{bmatrix} (ir^3 + 2irs - s^3 + s^2 - s - 1) \\ (-r^2 s - r^2 + irs^2 + ir - 2s) \end{bmatrix}. \end{aligned} \tag{109}$$

(ii)

$$\begin{aligned}
&\widehat{SH_{n+1}}SH_{n-1} - \widehat{SH_n}SH_n \\
=\ & (-1)^n s^n (\widehat{SH_{-1}}SH_1 - \widehat{SH_0}SH_0) \\
=\ & (-1)^{n-1}s^{n-1}\left(r^2+4s\right)\begin{bmatrix} (ir^3+2irs-s^3+s^2-s-1) \\ (-r^2s-r^2+irs^2+ir-2s) \end{bmatrix}.
\end{aligned} \tag{110}$$

(iii)

$$\widehat{SF_{n+1}}SF_{n-1} - \widehat{SF_n}SF_n = (-1)^n \begin{bmatrix} (-2+i(x^3+2x)) \\ (-2x^2-2+2ix) \end{bmatrix}. \tag{111}$$

(iv)

$$\widehat{SL_{n+1}}SL_{n-1} - \widehat{SL_n}SL_n = (-1)^{n-1}\left(x^2+4\right)\begin{bmatrix} (-2+i(x^3+2x)) \\ (-2x^2-2+2ix) \end{bmatrix}. \tag{112}$$

respectively.

Proof. The proof can be obtained from (60) and Tables 3 and 4. □

If one compares results (i) and (ii) in Corollary 3, the result presenting the relation between Simson's identities for SG_n and SH_n can quickly be seen as follows:

Corollary 4. *For all integers n, the next identities hold:*

(i)

$$\widehat{SH_{n+1}}SH_{n-1} - \widehat{SH_n}SH_n = \left(r^2+4s\right)\left(\widehat{SG_{n+1}}SG_{n-1} - \widehat{SG_n}SG_n\right). \tag{113}$$

(ii)

$$\widehat{SL_{n+1}}SL_{n-1} - \widehat{SL_n}SL_n = \left(x^2+4\right)\left(\widehat{SF_{n+1}}SF_{n-1} - \widehat{SF_n}SF_n\right). \tag{114}$$

The next theorem exhibits the relation of generalized Fibonacci polynomial spinor transforms with different terms.

Theorem 4. *For all integers n, the following identities hold:*

(i)

$$\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 = (-s)^{-n}\left(\widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n\right). \tag{115}$$

(ii)

$$\begin{aligned}
&\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\
=\ & \left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0\right).
\end{aligned} \tag{116}$$

(iii)

$$\begin{aligned}
&\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\
=\ & \left(W_1^2 - sW_0^2 - rW_0W_1\right)(-s)^{-n}\left(\widehat{SG_0}SG_{2n} - \widehat{SG_n}SG_n\right).
\end{aligned} \tag{117}$$

(iv)

$$\widehat{SH_{-n}}SH_n - \widehat{SH_0}SH_0 = -\left(r^2+4s\right)\left(\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0\right). \tag{118}$$

(v)

$$\widehat{SH_0}SH_{2n} - \widehat{SH_n}SH_n = -(-s)^n\left(r^2+4s\right)\left(\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0\right). \tag{119}$$

Proof. (i) For $\alpha \neq \beta$, using (82) and (13) we obtain the following identities:

$$\begin{aligned} & \widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\ = & -\frac{1}{(-s)^n} c_1 c_2 (\alpha^n - \beta^n) \left[\begin{array}{c} (\beta^n - \alpha^n - \alpha\beta^{n+1} + \alpha^{n+1}\beta - \alpha^2\beta^{n+2} \\ +\alpha^{n+2}\beta^2 - \alpha^3\beta^{n+3} + \alpha^{n+3}\beta^3 \\ -i\alpha\beta^{n+2} + i\alpha^{n+2}\beta - i\alpha^3\beta^n + i\alpha^n\beta^3 \\ +i\alpha^{n+3} - i\beta^{n+3} + i\alpha^2\beta^{n+1} - i\alpha^{n+1}\beta^2) \\ \\ (\alpha^{n+3}\beta - \alpha\beta^{n+3} + \alpha^2\beta^n - \alpha^n\beta^2 + \beta^{n+2} \\ -\alpha^{n+2} + \alpha^3\beta^{n+1} - \alpha^{n+1}\beta^3 + i\alpha^{n+1} \\ -i\beta^{n+1} - i\alpha^2\beta^{n+3} + i\alpha^3\beta^{n+2} \\ -i\alpha^{n+2}\beta^3 + i\alpha^{n+3}\beta^2 - i\alpha\beta^n + i\alpha^n\beta) \end{array} \right]. \end{aligned} \tag{120}$$

where $c_1 = W_1 - \beta W_0$ and $c_2 = W_1 - \alpha W_0$. On the other hand,

$$\begin{aligned} & \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n \\ = & -c_1 c_2 (\alpha^n - \beta^n) \left[\begin{array}{c} (\beta^n - \alpha^n - \alpha\beta^{n+1} + \alpha^{n+1}\beta - \alpha^2\beta^{n+2} \\ +\alpha^{n+2}\beta^2 - \alpha^3\beta^{n+3} + \alpha^{n+3}\beta^3 \\ -i\alpha\beta^{n+2} + i\alpha^{n+2}\beta - i\alpha^3\beta^n + i\alpha^n\beta^3 \\ +i\alpha^{n+3} - i\beta^{n+3} + i\alpha^2\beta^{n+1} - i\alpha^{n+1}\beta^2) \\ \\ (\alpha^{n+3}\beta - \alpha\beta^{n+3} + \alpha^2\beta^n - \alpha^n\beta^2 \\ +\alpha^3\beta^{n+1} - \alpha^{n+2} + \beta^{n+2} - \alpha^{n+1}\beta^3 \\ +i\alpha^{n+1} - i\beta^{n+1} - i\alpha^2\beta^{n+3} + i\alpha^3\beta^{n+2} \\ -i\alpha^{n+2}\beta^3 + i\alpha^{n+3}\beta^2 - i\alpha\beta^n + i\alpha^n\beta) \end{array} \right]. \\ = & (-s)^n \left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \right). \end{aligned} \tag{121}$$

Now, let us see when the roots are equal, i.e., for $\alpha = \beta$, that (115) holds for all integers n. Using (83), we obtain that

$$\begin{aligned} & \widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \\ = & -nd_2^2 \left[\begin{array}{c} (-n + n\alpha^2 + 2in\alpha^3 + n\alpha^4 + n\alpha^6 + 2i\alpha^3) \\ 2\alpha(i\alpha^4 + 2\alpha^3 - n\alpha + in) \end{array} \right] \end{aligned} \tag{122}$$

where $d_1 = W_0$ and $d_2 = \frac{1}{\alpha}(W_1 - \alpha W_0)$. On the other hand,

$$\begin{aligned} & \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n \\ = & -n\alpha^{2n}d_2^2 \left[\begin{array}{c} (-n + n\alpha^2 + 2in\alpha^3 + n\alpha^4 + n\alpha^6 + 2i\alpha^3) \\ 2\alpha(i\alpha^4 + 2\alpha^3 - n\alpha + in) \end{array} \right]. \end{aligned} \tag{123}$$

Therefore, (122) and (123) give us the next required equality:

$$\begin{aligned} \widehat{SW_0}SW_{2n} - \widehat{SW_n}SW_n &= \alpha^{2n} \left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \right) \\ &= (-s)^n \left(\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 \right). \end{aligned} \tag{124}$$

(ii) For $\alpha \neq \beta$, since $c_1 = G_1 - \beta G_0 = 1$ and $c_2 = G_1 - \alpha G_0 = 1$, we can immediately obtain the following identity using (120) and taking $W_n = G_n$:

$$\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0 = -\frac{1}{(-s)^n}(\alpha^n - \beta^n)\begin{bmatrix} (\beta^n - \alpha^n - \alpha\beta^{n+1} + \alpha^{n+1}\beta - \alpha^2\beta^{n+2} \\ +\alpha^{n+2}\beta^2 - \alpha^3\beta^{n+3} + \alpha^{n+3}\beta^3 \\ -i\alpha\beta^{n+2} + i\alpha^{n+2}\beta - i\alpha^3\beta^n + i\alpha^n\beta^3 \\ +i\alpha^{n+3} - i\beta^{n+3} + i\alpha^2\beta^{n+1} - i\alpha^{n+1}\beta^2) \\ \\ (\alpha^{n+3}\beta - \alpha\beta^{n+3} + \alpha^2\beta^n - \alpha^n\beta^2 \\ +\beta^{n+2} - \alpha^{n+2} + \alpha^3\beta^{n+1} - \alpha^{n+1}\beta^3 \\ +i\alpha^{n+1} - i\beta^{n+1} - i\alpha^2\beta^{n+3} + i\alpha^3\beta^{n+2} \\ -i\alpha^{n+2}\beta^3 + i\alpha^{n+3}\beta^2 - i\alpha\beta^n + i\alpha^n\beta) \end{bmatrix}. \tag{125}$$

Hence, the product c_1c_2 in (120) equals $\left(W_1^2 - sW_0^2 - rW_0W_1\right)$ by (12) and (13) and by comparing the identities (120) and (125) we have the desired identity:

$$\widehat{SW_{-n}}SW_n - \widehat{SW_0}SW_0 = \left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0\right).$$

We now prove the identity holds for all integers n for equal roots, i.e, for $\alpha = \beta$. By taking $W_n = G_n$ in (122), since $d_1 = 0$ and $d_2 = \frac{1}{\alpha}$ we arrive at the following identity.

$$\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0 = -\frac{n}{\alpha^2}\begin{bmatrix} (-n + n\alpha^2 + 2in\alpha^3 + n\alpha^4 + n\alpha^6 + 2i\alpha^3) \\ 2\alpha(i\alpha^4 + 2\alpha^3 - n\alpha + in) \end{bmatrix}. \tag{126}$$

Therefore, the product d_2^2 in (122) equals $\left(W_1^2 - rW_0W_1 - sW_0^2\right)$ by (15) and (16) and by comparing the identities (122) and (126), we have the desired identity.

(iii) It is clear from (i) and (ii).

(iv) Considering $SW_n = SH_n$ for all integers n and setting the value of $\left(W_1^2 - sW_0^2 - rW_0W_1\right)$ in (ii) from Table 1, we reach the desired identity.

(v) It is obvious from (i) and (iii).

□

Through Theorem 4, by taking $SW_n = SG_n$ and $SW_n = SH_n$, we can see a more general formula of (r,s)-Fibonacci polynomial spinors SG_n and (r,s)-Lucas polynomial spinors SH_n in the next corollary.

Corollary 5. *For all integers n, the following equalities hold:*

(i)

$$\widehat{SG_{-n}}SG_n - \widehat{SG_0}SG_0 = (-s)^{-n}\left(\widehat{SG_0}SG_{2n} - \widehat{SG_n}SG_n\right). \tag{127}$$

(ii)

$$\widehat{SH_{-n}}SH_n - \widehat{SH_0}SH_0 = (-s)^{-n}\left(\widehat{SH_0}SH_{2n} - \widehat{SH_n}SH_n\right). \tag{128}$$

The next corollary is the result of Corollary 5 for special cases.

Corollary 6. *For all integers n, the following equalities hold:*

(i)

$$\widehat{SF_{-n}}SF_n - \widehat{SF_0}SF_0 = (-1)^n\left(\widehat{SF_0}SF_{2n} - \widehat{SF_n}SF_n\right). \tag{129}$$

(ii)

$$\widehat{SL_{-n}}SL_n - \widehat{SL_0}SL_0 = (-1)^n\left(\widehat{SL_0}SL_{2n} - \widehat{SL_n}SL_n\right). \tag{130}$$

We can rewrite the Simson's formula in five different ways using the previous corollaries and theorem.

Remark 2. *(Simson's Identity) For all integers n, Simson's identity of generalized Fibonacci polynomial spinors can be given by the next five different formulas:*

$$\begin{aligned} & \widehat{SW_{n-1}}SW_{n+1} - \widehat{SW_n}SW_n \\ = \ & (-s)^{n-1}\left(\widehat{SW_0}SW_2 - \widehat{SW_1}SW_1\right) \\ = \ & (-s)^n\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_{-1}}SG_1 - \widehat{SG_0}SG_0\right) \\ = \ & (-s)^{n-1}\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_0}SG_2 - \widehat{SG_1}SG_1\right). \end{aligned} \tag{131}$$

and

$$\begin{aligned} & \left(r^2+4s\right)\left(\widehat{SW_{n-1}}SW_{n+1} - \widehat{SW_n}SW_n\right) \\ = \ & (-s)^{n-1}\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_{-1}}SH_1 - \widehat{SH_0}SH_0\right) \\ = \ & -\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_0}SH_2 - \widehat{SH_1}SH_1\right). \end{aligned} \tag{132}$$

5. Some Identities

In this section, we obtain some identities of generalized Fibonacci polynomial spinors, (r,s)-Fibonacci polynomial spinors and (r,s)-Lucas polynomial spinors. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{SW_n\}$.

Theorem 5. *For all integers m, n we have*

$$SW_{n+m} = SW_nG_{m+1} + sSW_{n-1}G_m \tag{133}$$

i.e.,

$$SW_{n+m} = SW_mG_{n+1} + sSW_{m-1}G_n \tag{134}$$

Proof. For $m \geq 1$ and $m \leq 0$, we use induction on m. First, we assume that $m \geq 1$. The equation is true for $m = 1$ since

$$\begin{aligned} SW_{n+1} &= rSW_n + sSW_{n-1} \\ &= SW_nG_2 + sSW_{n-1}G_1 \end{aligned} \tag{135}$$

where $G_2 = r$ and $G_1 = 1$. For $m = 2$, the equation is also true, which we can see below, because using definition of SW_n and the values $G_2 = r, G_3 = s + r^2$, we obtain

$$\begin{aligned} SW_{n+2} &= rSW_{n+1} + sSW_n = r(rSW_n + sSW_{n-1}) + sSW_n \\ &= (s+r^2)SW_n + rsSW_{n-1} = SW_nG_3 + sSW_{n-1}G_2. \end{aligned} \tag{136}$$

Assume now that the equation holds for all m with $0 \leq m \leq k+1$. Then, by assumption, for $m = k$ and $m = k+1$ we have, respectively,

$$sSW_{n+k} = s(SW_nG_{k+1} + sSW_{n-1}G_k), \tag{137}$$

and

$$rSW_{n+k+1} = r(SW_nG_{k+2} + sSW_{n-1}G_{k+1}). \tag{138}$$

By adding up these two equations we obtain

$$rSW_{n+k+1} + sSW_{n+k} = r(SW_nG_{k+2} + sSW_{n-1}G_{k+1}) + s(SW_nG_{k+1} + sSW_{n-1}G_k), \quad (139)$$

i.e.,

$$\begin{aligned} SW_{n+k+2} &= SW_n(rG_{k+2} + sG_{k+1}) + sSW_{n-1}(rG_{k+1} + sG_k) \\ &= SW_nG_{k+3} + sSW_{n-1}G_{k+2} \end{aligned} \quad (140)$$

which yields the equation for $m = k + 2$.

Now, we proceed by induction on $|m| = -m = v$ when $m \leq 0$. For $v = 0$, that is, $m = 0$, the equation is true because

$$SW_n = SW_nG_1 + sSW_{n-1}G_0 \quad (141)$$

where $G_0 = 0$ and $G_1 = 1$. For $v = 1$, that is, $m = -1$, it is true because

$$SW_{n-1} = SW_nG_0 + sSW_{n-1}G_{-1} \quad (142)$$

where $G_0 = 0$ and $G_{-1} = \frac{1}{s}$. Suppose that it holds for all $v = |m| = -m$ with $1 \leq v \leq k+1$. Then, by assumption, for $v = k$ and $v = k + 1$ we have, respectively,

$$\frac{1}{s}SW_{n-k} = \frac{1}{s}(SW_nG_{-k+1} + sSW_{n-1}G_{-k}) \quad (143)$$

and

$$\frac{-r}{s}SW_{n-k-1} = \frac{-r}{s}(SW_nG_{-k} + sSW_{n-1}G_{-k-1}). \quad (144)$$

By summing up these two equations we obtain

$$\begin{aligned} &\frac{-r}{s}SW_{n-k-1} + \frac{1}{s}G_{n-k} \\ = \ &\frac{-r}{s}(SW_nG_{-k} + sSW_{n-1}G_{-k-1}) + \frac{1}{s}(SW_nG_{-k+1} + sSW_{n-1}G_{-k}), \end{aligned} \quad (145)$$

i.e.,

$$\begin{aligned} SW_{n-k-2} &= SW_n(-\frac{r}{s}G_{-k} + \frac{1}{s}G_{-k+1}) + sSW_{n-1}(-\frac{r}{s}G_{-k-1} + \frac{1}{s}G_{-k}) \\ &= SW_nG_{-k-1} + sSW_{n-1}G_{-k-2}; \end{aligned} \quad (146)$$

thus we obtain the equation for $v = |m| = k + 2$. □

We then have the next result via Theorem 5 as a corollary.

Corollary 7. *For $n \in \mathbb{Z}$, the following equalities are true:*

(i)

$$SW_n = SW_0G_{n+1} + (SW_1 - rSW_0)G_n \quad (147)$$

(ii)

$$(4s + r^2)SW_n = (2SW_1 - rSW_0)H_{n+1} + (-rSW_1 + (2s + r^2)SW_0)H_n \quad (148)$$

(iii)

$$SW_{n+m} = G_nSW_{m+1} + sG_{n-1}SW_m \quad (149)$$

(iv)

$$SG_{n+m} = SG_nG_{m+1} + sSG_{n-1}G_m \quad (150)$$

(v)

$$SH_{n+m} = SH_nG_{m+1} + sSH_{n-1}G_m \tag{151}$$

(vi)

$$SG_n = SG_0G_{n+1} + (SG_1 - rSG_0)G_n \tag{152}$$

(vii)

$$SH_n = SH_0G_{n+1} + (SH_1 - rSH_0)G_n \tag{153}$$

(viii)

$$(4s + r^2)SH_n = (2SH_1 - rSH_0)H_{n+1} + (-rSH_1 + (2s + r^2)SH_0)H_n \tag{154}$$

(ix)

$$(4s + r^2)SG_n = (2SG_1 - rSG_0)H_{n+1} + (-rSG_1 + (2s + r^2)SG_0)H_n \tag{155}$$

(x)

$$SG_{n+m} = G_nSG_{m+1} + sG_{n-1}SG_m \tag{156}$$

(xi)

$$SH_{n+m} = G_nSH_{m+1} + sG_{n-1}SH_m \tag{157}$$

Proof. *(i)* If we take m equals 0 in (134), then we obtain the required equation.
(ii) If we use the following equations in (a) coming from [23],

$$\begin{aligned} (r^2 + 4s)G_n &= 2H_{n+1} - rH_n, \\ (r^2 + 4s)G_n &= rH_n + 2sH_{n-1} \end{aligned} \tag{158}$$

then we have the required equation.
(iii) Choose $n - 1$ instead of n and $m + 1$ instead of m in (134).
(iv)–(v) Take SG_{n+m} and SH_{n+m} instead of SW_{n+m} in (133).
(vi)–(vii) Take SG_n and SH_n instead of SW_n in (147).
(viii)–(ix) Replace $SW_n = SG_n$ and $SW_n = SH_n$, respectively, in (148).
(x)–(xi) Replace $SW_{n+m} = SG_{n+m}$ and $SW_{n+m} = SH_{n+m}$, respectively, in (149). □

We obtain the next corollary from Theorem 5 and Corollary 7 (iii). The next identities will be useful for us in the last section.

Corollary 8. *Let $n \in \mathbb{Z}$. The following equalities are true:*

(i)

$$SW_n = G_nSW_1 + sG_{n-1}SW_0. \tag{159}$$

(ii)

$$SW_{n+1} = G_nSW_2 + sG_{n-1}SW_1. \tag{160}$$

(iii)

$$SW_{n+1} = G_{n+1}SW_1 + sG_nSW_0. \tag{161}$$

(iv)

$$SW_{n+2} = G_{n+1}SW_2 + sG_nSW_1. \tag{162}$$

Theorem 6. *(Catalan's identity) Let n and m be any integers. Then, the following identity is true:*

$$\widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n = (-s)^n\left(\widehat{SW_{-m}}SW_m - \widehat{SW_0}SW_0\right) \tag{163}$$

i.e.,

$$\begin{vmatrix} SW_{n+m} & \widehat{SW_n} \\ SW_n & \widehat{SW_{n-m}} \end{vmatrix} = (-s)^n \begin{vmatrix} SW_m & \widehat{SW_0} \\ SW_0 & \widehat{SW_{-m}} \end{vmatrix}. \tag{164}$$

Proof. Let n and m be any integers. Using the Binet's formula of SW_n, SW_{n-m}, and SW_{n+m}, we obtain the desired identity by (13) and (115) in Theorem 5 (i)

$$\begin{aligned}
&\widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n \\
= \; & -\alpha^{n-m}\beta^{n-m}c_1c_2(\alpha^m-\beta^m)\begin{bmatrix} (\beta^m-\alpha^m-\alpha\beta^{m+1}+\alpha^{m+1}\beta-\alpha^2\beta^{m+2} \\ +\alpha^{m+2}\beta^2-\alpha^3\beta^{m+3}+\alpha^{m+3}\beta^3 \\ -i\alpha\beta^{m+2}+i\alpha^{m+2}\beta-i\alpha^3\beta^m+i\alpha^m\beta^3 \\ +i\alpha^{m+3}-i\beta^{m+3}+i\alpha^2\beta^{m+1}-i\alpha^{m+1}\beta^2) \\ \\ (\alpha^{m+3}\beta-\alpha\beta^{m+3}+\alpha^2\beta^m-\alpha^m\beta^2 \\ +\beta^{m+2}-\alpha^{m+2}+\alpha^3\beta^{m+1}-\alpha^{m+1}\beta^3 \\ +i\alpha^{m+1}-i\beta^{m+1}-i\alpha^2\beta^{m+3}+i\alpha^3\beta^{m+2} \\ -i\alpha^{m+2}\beta^3+i\alpha^{m+3}\beta^2-i\alpha\beta^m+i\alpha^m\beta) \end{bmatrix} \\
= \; & \alpha^m\beta^m\alpha^{n-m}\beta^{n-m}\left(\widehat{SW_{-m}}SW_m-\widehat{SW_0}SW_0\right) \\
= \; & (-s)^n\left(\widehat{SW_{-m}}SW_m-\widehat{SW_0}SW_0\right). \qquad (165)
\end{aligned}$$

□

When we take SG_n, SH_n, SF_n, SL_n instead of SW_n, we have the Catalan's identity for (r,s)-Fibonacci, (r,s)-Lucas polynomial spinors, Fibonacci polynomial spinors, and Lucas polynomial spinors, respectively.

Corollary 9. *Let n and m be any integers. Then, the following identities are true:*

(i)

$$\widehat{SG_{n-m}}SG_{n+m}-\widehat{SG_n}SG_n=(-s)^n\left(\widehat{SG_{-m}}SG_m-\widehat{SG_0}SG_0\right) \qquad (166)$$

i.e.,

$$\begin{vmatrix} SG_{n+m} & \widehat{SG_n} \\ SG_n & \widehat{SG_{n-m}} \end{vmatrix} = (-s)^n\begin{vmatrix} SG_m & \widehat{SG_0} \\ SG_0 & \widehat{SG_{-m}} \end{vmatrix}. \qquad (167)$$

(ii)

$$\widehat{SH_{n-m}}SH_{n+m}-\widehat{SH_n}SH_n=(-s)^n\left(\widehat{SH_{-m}}SH_m-\widehat{SH_0}SH_0\right) \qquad (168)$$

i.e.,

$$\begin{vmatrix} SH_{n+m} & \widehat{SH_n} \\ SH_n & \widehat{SH_{n-m}} \end{vmatrix} = (-s)^n\begin{vmatrix} SH_m & \widehat{SH_0} \\ SH_0 & \widehat{SH_{-m}} \end{vmatrix}. \qquad (169)$$

(iii)

$$\widehat{SF_{n-m}}SF_{n+m}-\widehat{SF_n}SF_n=(-1)^n\left(\widehat{SF_{-m}}SF_m-\widehat{SF_0}SF_0\right) \qquad (170)$$

i.e.,

$$\begin{vmatrix} SF_{n+m} & \widehat{SF_n} \\ SF_n & \widehat{SF_{n-m}} \end{vmatrix} = (-1)^n\begin{vmatrix} SF_m & \widehat{SF_0} \\ SF_0 & \widehat{SF_{-m}} \end{vmatrix}. \qquad (171)$$

(iv)

$$\widehat{SL_{n-m}}SL_{n+m}-\widehat{SL_n}SL_n=(-1)^n\left(\widehat{SL_{-m}}SL_m-\widehat{SL_0}SL_0\right) \qquad (172)$$

i.e.,

$$\begin{vmatrix} SL_{n+m} & \widehat{SL_n} \\ SL_n & \widehat{SL_{n-m}} \end{vmatrix} = (-1)^n\begin{vmatrix} SL_m & \widehat{SL_0} \\ SL_0 & \widehat{SL_{-m}} \end{vmatrix}. \qquad (173)$$

We next take an example of Catalan's formula for a special n and m.

Remark 3. *We have the Catalan's identity for $n = 2$ and $m = 1$,*

$$\widehat{SW_1}SW_3 - \widehat{SW_2}SW_2 = s^2\left(\widehat{SW_{-1}}SW_1 - \widehat{SW_0}SW_0\right). \tag{174}$$

We can also exhibit the Catalan's identity in other forms via Theorem 5 as in the next corollary.

Corollary 10. *(Catalan's Identity) For all integers n and m, Catalan's identity of generalized Fibonacci polynomial spinors can be given with the next five different formulas:*

$$\begin{aligned} & \widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n \\ = \; & (-s)^{n-m}\left(\widehat{SW_0}SW_{2m} - \widehat{SW_m}SW_m\right) \\ = \; & (-s)^n\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_{-m}}SG_m - \widehat{SG_0}SG_0\right) \\ = \; & \left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right) \end{aligned} \tag{175}$$

and

$$\begin{aligned} & \left(r^2 + 4s\right)\left(\widehat{SW_{n-m}}SW_{n+m} - \widehat{SW_n}SW_n\right) \\ = \; & -(-s)^n\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_{-m}}SH_m - \widehat{SH_0}SH_0\right) \\ = \; & -\left(W_1^2 - sW_0^2 - rW_0W_1\right)\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right). \end{aligned} \tag{176}$$

Remark 4. *Due to Theorem 5, Catalan's identity of (r,s)-Fibonacci polynomial spinors, (r,s)-Lucas polynomial spinors, Fibonacci polynomial spinors, and Lucas polynomial spinors can be written in many different ways as a result of Corollary 10.*

(i)

$$\begin{aligned} & \widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n \\ = \; & (-s)^{n-m}\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right) \end{aligned} \tag{177}$$

$$\begin{aligned} & \left(r^2 + 4s\right)\left(\widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n\right) \\ = \; & -(-s)^n\left(\widehat{SH_{-m}}SH_m - \widehat{SH_0}SH_0\right) \end{aligned} \tag{178}$$

$$\begin{aligned} & \left(r^2 + 4s\right)\left(\widehat{SG_{n-m}}SG_{n+m} - \widehat{SG_n}SG_n\right) \\ = \; & -\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right). \end{aligned} \tag{179}$$

(ii)

$$\begin{aligned} & \widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = \; & (-s)^{n-m}\left(\widehat{SH_0}SH_{2m} - \widehat{SH_m}SH_m\right) \end{aligned} \tag{180}$$

$$\begin{aligned} & \widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = \; & -(-s)^n\left(r^2 + 4s\right)\left(\widehat{SG_{-m}}SG_m - \widehat{SG_0}SG_0\right) \end{aligned} \tag{181}$$

$$\begin{aligned} & \widehat{SH_{n-m}}SH_{n+m} - \widehat{SH_n}SH_n \\ = \; & -\left(r^2 + 4s\right)\left(\widehat{SG_0}SG_{2m} - \widehat{SG_m}SG_m\right). \end{aligned} \tag{182}$$

(iii)

$$\begin{aligned} & \widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n \\ = & (-1)^{n-m}\left(\widehat{SF_0}SF_{2m} - \widehat{SF_m}SF_m\right) \end{aligned} \tag{183}$$

$$\begin{aligned} & \left(x^2+4\right)\left(\widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n\right) \\ = & -(-1)^n\left(\widehat{SL_{-m}}SL_m - \widehat{SL_0}SL_0\right) \end{aligned} \tag{184}$$

$$\begin{aligned} & \left(x^2+4\right)\left(\widehat{SF_{n-m}}SF_{n+m} - \widehat{SF_n}SF_n\right) \\ = & -\left(\widehat{SL_0}SL_{2m} - \widehat{SL_m}SL_m\right). \end{aligned} \tag{185}$$

(iv)

$$\begin{aligned} & \widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = & (-1)^{n-m}\left(\widehat{SL_0}SL_{2m} - \widehat{SL_m}SL_m\right) \end{aligned} \tag{186}$$

$$\begin{aligned} & \widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = & -(-1)^n\left(x^2+4\right)\left(\widehat{SF_{-m}}SF_m - \widehat{SF_0}SF_0\right) \end{aligned} \tag{187}$$

$$\begin{aligned} & \widehat{SL_{n-m}}SL_{n+m} - \widehat{SL_n}SL_n \\ = & -\left(x^2+4\right)\left(\widehat{SF_0}SF_{2m} - \widehat{SF_m}SF_m\right). \end{aligned} \tag{188}$$

6. Sum Formulas

In this section, we present sum formulas of generalized Fibonacci polynomial spinors.

Theorem 7. *For generalized Fibonacci polynomial spinors, we have the following sum formula:*
If $-s-r+1 \neq 0$, then

$$\sum_{k=0}^{n} SW_k = \frac{(-s-r)SW_n - sSW_{n-1} + SW_1 + (1-r)SW_0}{-s-r+1}. \tag{189}$$

Proof. Using the recurrence relation

$$SW_n = rSW_{n-1} + sSW_{n-2} \tag{190}$$

i.e.,

$$-sSW_{n-2} = rSW_{n-1} - SW_n \tag{191}$$

we obtain

$$\begin{aligned} -sSW_0 &= rSW_1 - SW_2 \\ -sSW_1 &= rSW_2 - SW_3 \\ -sSW_2 &= rSW_3 - SW_4 \\ &\vdots \\ -sSW_{n-2} &= rSW_{n-1} - SW_n \end{aligned} \tag{192}$$

If we add the equations side by side, we obtain

$$-s\sum_{k=0}^{n-2} SW_k = r\sum_{k=1}^{n-1} SW_k - \sum_{k=2}^{n} SW_k. \tag{193}$$

Since

$$\begin{aligned} & -s\left(\sum_{k=0}^{n} SW_k - SW_{n-1} - SW_n\right) \\ = \; & r\left(\sum_{k=0}^{n} SW_k - SW_n - SW_0\right) - \left(\sum_{k=0}^{n} SW_k - SW_0 - SW_1\right) \end{aligned} \tag{194}$$

we obtain

$$\begin{aligned} & (-s-r+1)\sum_{k=0}^{n} SW_k \\ = \; & -sSW_{n-1} - sSW_n - rSW_n - rSW_0 + SW_0 + SW_1 \end{aligned} \tag{195}$$

and so,

$$\sum_{k=0}^{n} SW_k = \frac{1}{-s-r+1}(-(r+s)SW_n - sSW_{n-1} + SW_1 + (1-r)SW_0). \tag{196}$$

□

Corollary 11. *For (r,s)-Fibonacci, (r,s)-Lucas, Fibonacci, and Lucas polynomial spinors, we have the following sum formulas: If $-s-r+1 \neq 0$ and $x \neq 0$, then*

(i)

$$\sum_{k=0}^{n} SG_k = \frac{1}{-s-r+1}((-s-r)SG_n - sSG_{n-1} + \begin{bmatrix} r^2+sr+s+i \\ 1+i(r+s) \end{bmatrix}). \tag{197}$$

(ii)

$$\begin{aligned} \sum_{k=0}^{n} SH_k = \; & \frac{1}{-s-r+1}((-s-r)SH_n - sSH_{n-1} \\ & + \begin{bmatrix} r^3+r^2s+3rs+2s^2+i(r-2) \\ r+2s+i(rs+r^2+2s) \end{bmatrix}). \end{aligned} \tag{198}$$

(iii)

$$\sum_{k=0}^{n} SF_k = -\frac{1}{x}((-1-x)SF_n - SF_{n-1} + \begin{bmatrix} x^2+x+1+i \\ 1+i(x+1) \end{bmatrix}). \tag{199}$$

(iv)

$$\sum_{k=0}^{n} SL_k = -\frac{1}{x}((-1-x)SL_n - SL_{n-1} + \begin{bmatrix} x^3+x^2+3x+2+i(x-2) \\ x+2+i(x+x^2+2) \end{bmatrix}). \tag{200}$$

7. Matrices Associated with Generalized Fibonacci Polynomial Spinors

Let $A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$ and $N_{SW} = \begin{pmatrix} SW_2 & SW_1 \\ SW_1 & SW_0 \end{pmatrix}$. Then, we know by Soykan [23] that

$$A^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}. \tag{201}$$

Hence, we obtain the next theorem from the matrix product by using the identities in Corollary 8.

Theorem 8. *For all integers n, we have*

$$A^n N_{SW} = \begin{pmatrix} SW_{n+2} & SW_{n+1} \\ SW_{n+1} & SW_n \end{pmatrix}. \tag{202}$$

Proof. For all integers n, we product the required matrices and use the identities in Corollary 8.

$$\begin{aligned} A^n N_{SW} &= \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \begin{pmatrix} SW_2 & SW_1 \\ SW_1 & SW_0 \end{pmatrix} \\ &= \begin{pmatrix} G_{n+1}SW_2 + sG_nSW_1 & G_{n+1}SW_1 + sG_nSW_0 \\ G_nSW_2 + sG_{n-1}SW_1 & G_nSW_1 + sG_{n-1}SW_0 \end{pmatrix} \\ &= \begin{pmatrix} SW_{n+2} & SW_{n+1} \\ SW_{n+1} & SW_n \end{pmatrix}. \end{aligned} \tag{203}$$

□

Note that by taking SW_n as SG_n, SH_n, SF_n, and SL_n we obtain the following matrices.

$$\begin{aligned} N_{SG} &= \begin{pmatrix} SG_2 & SG_1 \\ SG_1 & SG_0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} r^4 + 3r^2s + s^2 + ir \\ r^2 + s + i(r^3 + 2rs) \end{bmatrix} & \begin{bmatrix} r^3 + 2rs + i \\ r + i(r^2 + s) \end{bmatrix} \\ \begin{bmatrix} r^3 + 2rs + i \\ r + i(r^2 + s) \end{bmatrix} & \begin{bmatrix} r^2 + s \\ 1 + ir \end{bmatrix} \end{pmatrix}, \end{aligned} \tag{204}$$

$$\begin{aligned} N_{SH} &= \begin{pmatrix} SH_2 & SH_1 \\ SH_1 & SH_0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} 2r^4 + r^3s + 5r^2s \\ +2rs^2 + s^2 + i(2r + s) \\ \\ (2r^2 + rs + s) \\ +i(2r^3 + r^2s + 3rs + s^2) \end{bmatrix} & \begin{bmatrix} 2r^3 + r^2s \\ +3rs + s^2 + i \\ \\ (2r + s) \\ +i(2r^2 + rs + s) \end{bmatrix} \\ \begin{bmatrix} 2r^3 + r^2s \\ +3rs + s^2 + i \\ \\ (2r + s) \\ +i(2r^2 + rs + s) \end{bmatrix} & \begin{bmatrix} 2r^2 + rs + s + 2i \\ 1 + i(2r + s) \end{bmatrix} \end{pmatrix}, \end{aligned} \tag{205}$$

$$\begin{aligned} N_{SF} &= \begin{pmatrix} SF_2 & SF_1 \\ SF_1 & SF_0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} x^4 + 3x^2 + 1 + ix \\ x^2 + 1 + i(x^3 + 2x) \end{bmatrix} & \begin{bmatrix} x^3 + 2x + i \\ x + i(x^2 + 1) \end{bmatrix} \\ \begin{bmatrix} x^3 + 2x + i \\ x + i(x^2 + 1) \end{bmatrix} & \begin{bmatrix} x^2 + 1 \\ 1 + ix \end{bmatrix} \end{pmatrix}, \end{aligned} \tag{206}$$

$$N_{SL} = \begin{pmatrix} SL_2 & SL_1 \\ SL_1 & SL_0 \end{pmatrix}$$
$$= \begin{pmatrix} \begin{bmatrix} 2x^4+x^3+5x^2+2x \\ +1+i(2x+1) \\ \\ (2x^2+x+1) \\ +i(2x^3+x^2+3x+1) \end{bmatrix} & \begin{bmatrix} 2x^3+x^2+3x \\ +1+i \\ \\ (2x+1) \\ +i(2x^2+x+1) \end{bmatrix} \\ \begin{bmatrix} 2x^3+x^2 \\ +3x+1+i \\ \\ (2x+1) \\ +i(2x^2+x+1) \end{bmatrix} & \begin{bmatrix} 2x^2+x+1+2i \\ \\ 1+i(2x+1) \end{bmatrix} \end{pmatrix}. \quad (207)$$

Theorem 8 gives the following results as particular examples.

Corollary 12. *For all integers n, we have*

(i)

$$A^n N_{SG} = \begin{pmatrix} SG_{n+2} & SG_{n+1} \\ SG_{n+1} & SG_n \end{pmatrix}. \quad (208)$$

(ii)

$$A^n N_{SH} = \begin{pmatrix} SH_{n+2} & SH_{n+1} \\ SH_{n+1} & SH_n \end{pmatrix}. \quad (209)$$

(iii)

$$A^n N_{SF} = \begin{pmatrix} SF_{n+2} & SF_{n+1} \\ SF_{n+1} & SF_n \end{pmatrix}. \quad (210)$$

(iv)

$$A^n N_{SL} = \begin{pmatrix} SL_{n+2} & SL_{n+1} \\ SL_{n+1} & SL_n \end{pmatrix}. \quad (211)$$

8. Conclusions

Fibonacci polynomials, which are also encountered in physics, were carried to the spinor structure, which tries to explain complex situations in physics with the help of concepts in mathematics. Since these polynomial spinor sequences can be used in many areas where spinors can be applied, we think that this is a work that will attract the attention of researchers working in the field of physics. In addition to this, aesthetic results of the sequences we worked on in quantum mechanics seem to be of interest. In this study, we obtained the closed formula of the sequence of the generalized Fibonacci polynomial spinors using its generating function, explicit formulas obtaining the nth term of this polynomial sequence via Binet's formula, and the sum of the first n terms of this polynomial sequence. Looking at Tables 2–4, the summation of the first three terms can be found easily, but finding the summation of the first n terms needs more application, which we have executed in Section 6. We also obtained the Catalan's identity and its special cases and various identities, which reveal the relations between the specific terms. These identities might be exercised for enhancing the new identities of polynomials. In addition, we revealed the relation between the generalized Fibonacci polynomial spinors and (r,s)-Fibonacci and (r,s)-Lucas polynomial spinors via several identities. Lastly, we exhibited the matrices associated with the generalized Fibonacci polynomial spinors. Referring to Tables 3 and 4, we can obtain all the results indicated above for the generalized Fibonacci polynomial spinors for the (r,s)-Fibonacci, (r,s)-Lucas, and Fibonacci and Lucas polynomial spinors. This led to us obtaining results for spinor sequences, which are a special case of polynomial spinor sequences. Considering here the most generalized form of this special sequence makes it easier to comment on all other cases. From this point of view, this work not only introduces generalized Fibonacci polynomial spinors, (r,s)-Fibonacci polynomial spinors, (r,s)-Lucas polynomial spinors, Fibonacci polynomial spinors, and Lucas polynomial spinors, but also gives a wide perspective for sequences of polynomial spinors. When changing r and s, known different sequences of polynomial spinors are found. For instance, taking $r = 2x$ and $s = 1$ in (41), generalized Pell polynomial spinors can be defined; and taking $r = 1$ and $s = 2x$ in (41), generalized Jacobsthal polynomial spinors can be described. Thus, as a corollary, generalized Pell spinors and generalized Jacobsthal spinors can be introduced.

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