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Magneto Axisymmetric Vibration of FG-GPLs Reinforced Annular Sandwich Plates with an FG Porous Core Using DQM and a New Shear Deformation Theory

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Abstract: Based on the differential quadrature procedure (DQP), the vibrational response of functionally graded (FG) sandwich annular plates enhanced with graphene platelets (GPLs) and with an FG porous core is illustrated in this paper. The current annular plate is assumed to deform axisymmetrically and expose to a radial magnetic field. The Lorentz magnetic body force is deduced via Maxwell's relations. The effective physical properties of the upper and lower layers of the sandwich plate are obtained by employing the Halpin–Tsai model. Our technique depends on a new four-unknown shear deformation theory to depict the displacements. In addition, the motion equations are established via Hamilton's principle. The motion equations are solved by employing the DQP. In order to study the convergence of the DQ method, the minimum number of grid points needed for a converged solution is ascertained. In addition, the current theory's outcomes are compared with those of previous higher-order theories. The effects of the porosity distribution type, porosity factor, GPLs distribution pattern, GPLs weight fraction, inner-to-outer radius ratio, outer radius-to-thickness ratio, magnetic field parameters, core thickness, and elastic substrate parameters on the nondimensional vibration frequencies are discussed.

Keywords: axisymmetric vibration; annular sandwich plates; FG porous core; differential quadrature method; new shear deformation theory



Citation: Al-Ali, A. H. K.; Al Mukahal, F. H. H.; Sobhy, M. Magneto Axisymmetric Vibration of FG-GPLs Reinforced Annular Sandwich Plates with an FG Porous Core Using DQM and a New Shear Deformation Theory. *Symmetry* **2024**, *16*, 696. <https://doi.org/10.3390/sym16060696>

Academic Editor: Theodore E. Simos

Received: 28 April 2024

Revised: 30 May 2024

Accepted: 1 June 2024

Published: 5 June 2024



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1. Introduction

Reinforcements like graphene platelets (GPLs) are extensively used to improve the material's mechanical characteristics. Because of their excellent toughness-to-weight ratio, reinforced composite structures have found extensive use in a variety of engineering fields, including aerospace, automotive, and marine engineering. Graphene represents the best reinforcing nanofillers due to its remarkable physical and mechanical characteristics, such as the highest strength and rigidity, the best thermal and electrical conductivity, and so on [1–4]. Polymers, metals, and ceramics can be enhanced with GPLs, which gives them high resistance, improved mechanical strength, and excellent thermal properties [1]. In this context, FG graphene-enhanced nanocomposite plates are considered to be one of the most important recent innovations in this field. These plates are characterized by their graded composition, which gives them unique mechanical and thermal properties. Song et al. [5] implemented the buckling and bending of graphene/polymer composite plates via the first-order theory (FPT). On the other hand, Abazid [6] explored the thermal buckling of GPLs metal foam nanoplates under the impact of 2D magnetic fields and humid conditions. Zargaripoora et al. [7] investigated the free vibration behavior of nanoplates strengthened by GPLs by employing the finite element method. On the basis of quasi-three-dimensional theory, wave propagation in porous nanocomposite plates immersed in

an elastic media and subjected to mechanical and Lorentz forces was studied by Abazid et al. [8]. Zhang et al. [9] investigated the nonlinear analysis of rotating GPLs-reinforced metal foams annular plates resting on a viscoelastic foundation under pulse loads utilizing the FPT and von-Karman's nonlinearity. The nonlinear vibration and bending responses of GPLs-reinforced piezoelectric porous curved beams were illustrated by Yang et al. [10], who employed Hamilton's principle to deduce the nonlinear equations and used the DQP to solve them. The literature contains a large amount of other research on graphene-reinforced composites, such as that of Li et al. [11], Zhang et al. [12], Ding and She [13], Zhang et al. [9], Yee et al. [14], Sobhy [15], and Garg et al. [16].

Functionally graded materials (FGMs) exhibit progressive changes in their material structure and mechanical characteristics in a variety of directions [17]. This is a great choice for composite constructions that experience severe inconsistencies in structural and thermal properties along the boundaries of multiple materials [18]. Furthermore, the gradual alterations in composition may be customized to adapt for diverse operating situations while also meeting varying objectives. FGMs are often composed of several kinds of ceramics and metals or a variety of other materials. The ceramic material used in an FGM provides heat insulation benefits while also protecting the metal from damage caused by oxidation and corrosion. The metallic substance toughens and strengthens the FGM. FGMs are presently being developed for broad usage as construction components in severe high-temperature settings and other fields. In this particular circumstance, FGM structures have garnered substantial research efforts in recognition of their distinctive benefits and extensive prospective practical usage in the aviation industry, engineering, etc. Furthermore, recent studies that have examined the characteristics and behaviors of FG-GPLs-reinforced structures include [19–23].

A porous material is a type of material that contains small internal holes or pores [24,25]. These pores allow materials to absorb and pass fluids or gases through them. Porous materials such as metal foams in a graphene-FG microstructure are utilized as lightweight structures to ensure adequate rigidity stiffness relative to their weight. Correspondingly, numerous technical sectors, including civil construction, the automobile industry, and aerospace, frequently use porous structures [26]. Such materials' internal porosities may be evenly distributed, unevenly distributed, or FG across the thickness of the structure. The strength of such materials is decreased because of the interior pores. Porous materials may be strengthened with harder elements, such as graphene nanoplates or carbon nanotubes [8,27], or they may be sandwiched between two strong sheets to minimize this disadvantage [28]. Chen et al. [29] explored the effect of porosity of sandwich beams with FG porous cores. In order to maximize critical force while minimizing beam mass, Magnucka-Blandzi and Magnucki [30] developed a sandwich beam with a porous core. In a different work, Chen et al. [29] examined the vibration response of porous beams using various porosity patterns. They found that a decrease in stiffness caused by different porosity distributions had a considerable impact on frequencies of the plate. Wattanasakulpong and Eiadtrong [31] investigated the vibration behavior of sandwich panels with an FG metal foam core exposed to a dynamic load utilizing the FPT. Pham et al. [32] illustrated the effects of a blast load on the vibration of sandwich panels with an FG porous core using the mixed interpolation of tensorial components triangular element and the Newmark-beta approach. Sobhy [33] studied the impacts of moisture and external voltage on the thermal buckling of sandwich panels composed of FG piezoelectric faces and a porous core. Meanwhile, Sobhy [34] used a nonlocal strain gradient model and a refined four-variable theory to elucidate the static bending behavior of annular and circular sandwich nanoplates with a metal foam core and piezoelectric faces resting on elastic foundations and exposed to an electric voltage and mechanical load. Based on various shear deformation theories, different behaviors of sandwich beams [35–37] and sandwich shells [38,39] with porous cores have been studied.

Since the differential quadrature method (DQM) is a useful numerical technique for solving ordinary and partial differential equations with initial or boundary conditions, it is

used to solve the governing equations. Furthermore, in comparison to previous numerical approaches, the DQM predicts fairly accurate numerical outcomes with substantially fewer mesh points [40]. A weighted linear sum of the function's values throughout the whole domain is used to estimate the partial derivatives of a function at a specific discrete point in accordance with the DQM. Employing the DQ approach, linear free vibration and nonlinear vibration responses of FG-GPLs-reinforced annular and circular plates and cylindrical shells were studied by Nie and Zhong [41], Mercan et al. [42], Liu et al. [43], Liang et al. [44], Alsebai et al. [45] and Alali et al. [46].

As discussed in the above literature review, an axisymmetric vibrational analysis of FG sandwich annular plates strengthened with GPLs having an FG porous core and subjected to a radial magnetic field has not been considered in the available literature. Furthermore, a new improved four-unknown shear deformation plate theory that takes transverse shear strain into account to get the displacement field is used in this research. To discretize the equations of motion and convert them into an algebraic system of equations, the DQM is utilized. This system represents an eigenvalue problem that can be solved to obtain the eigenfrequencies of the sandwich annular plates. Next, the validity of the current theory is confirmed by comparing its results with the results of some other higher-order theories. In addition, the impacts of the graphene distribution type, GPLs volume fraction, magnetic field, and electric voltage on the eigenfrequencies of the FG porous sandwich annular plates are illustrated.

2. Mathematical Formulation

Figure 1 depicts a three-layered annular plate, two FG metal/GPLs face sheets, and a metal foam core placed on an elastic foundation. The face sheets are assumed to be perfectly bonded to the core. This annular plate has an inner radius of r_1 and an outer radius of r_2 . Each of the top and bottom layers has a thickness T_{GPL} , while the middle layer has a thickness T_{core} ; therefore, the total plate thickness is $T = 2T_{GPL} + T_{core}$. In this investigation, the cylindrical coordinate system (R, θ, Z) is used. The current approach takes into account both axial symmetry and geometry.

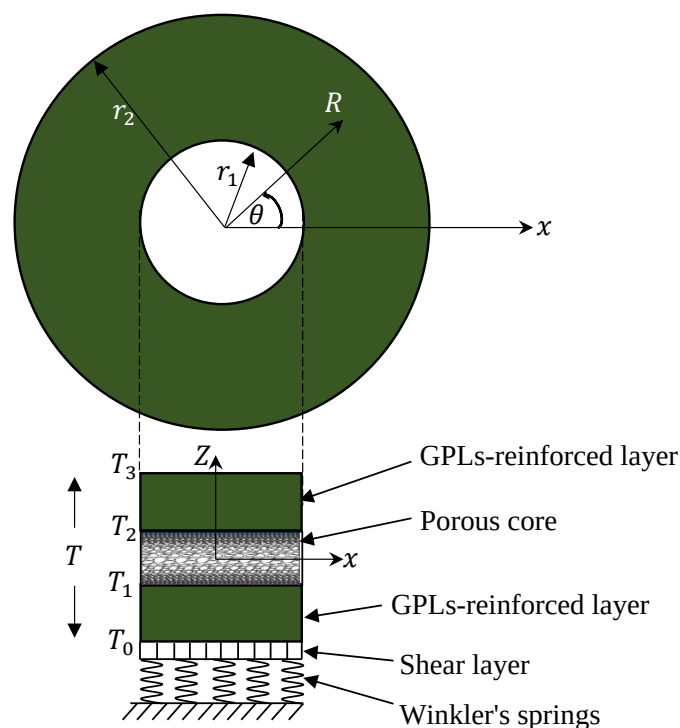


Figure 1. A schematic representation of FG-GPLs-reinforced annular sandwich plates with a porous core and lying on an elastic substrate.

2.1. Refined Four-Variable Shear and Normal Deformations Theory

Shimpi's theory [47] is utilized to model the present theory and includes the following assumptions:

1. Since the displacements are tiny, the strains involved are also very tiny.
2. The transverse normal stress σ_{ZZ} is negligible in comparison to the in-plane stresses σ_{RR} and $\sigma_{\theta\theta}$.
3. The transverse displacement u_Z contains two components: the bending component ϕ^b and the shear component ϕ^s . Both components are functions of R and t only.
4. The mid-plane radial displacement u is added to Shimpi's two-variable plate theory [47].

Therefore, the displacement field can be written as:

$$\begin{Bmatrix} u_R(R, Z, t) \\ u_Z(R, Z, t) \end{Bmatrix} = \begin{Bmatrix} u(R, t) \\ \phi^b(R, t) + \phi^s(R, t) \end{Bmatrix} - Z \begin{Bmatrix} \frac{\partial \phi^b}{\partial R} \\ 0 \end{Bmatrix} - G(Z) \begin{Bmatrix} \frac{\partial \phi^s}{\partial R} \\ 0 \end{Bmatrix}, \quad u_\theta(R, Z, t) = 0, \quad (1)$$

where $G(Z) = Z - \bar{G}(Z)$, for which

$$\bar{G}(Z) = \frac{T}{2} \sinh^{-1} \left(\frac{2Z}{T} \right) - \frac{2\sqrt{2}Z^3}{3T^2}. \quad (2)$$

The shape function $\bar{G}(Z)$ takes many forms such as the third-order plate theory (TPT) [48], the sinusoidal plate theory (SPT) [49], the hyperbolic plate theory (HPT) [50], and the exponential plate theory (EPT) [51] that can be respectively given as:

$$\begin{aligned} \bar{G}(Z) &= Z \left(1 - \frac{4Z^2}{3T^2} \right), \\ \bar{G}(Z) &= \frac{T}{\pi} \sin \left(\frac{\pi Z}{T} \right), \\ \bar{G}(Z) &= T \sinh \left(\frac{Z}{T} \right) - Z \cosh \left(\frac{1}{2} \right), \\ \bar{G}(Z) &= Z \exp \left(\frac{-2Z^2}{T^2} \right). \end{aligned} \quad (3)$$

The general form of the strain–displacement relationship in cylindrical coordinates is now given as:

$$\begin{aligned} \varepsilon_{RR} &= \frac{\partial u_R}{\partial R}, & \varepsilon_{\theta\theta} &= \frac{\partial u_\theta}{R\partial\theta} + \frac{u_R}{R}, & \varepsilon_{ZZ} &= \frac{\partial u_Z}{\partial Z}, \\ \varepsilon_{R\theta} &= \frac{\partial u_\theta}{\partial R} + \frac{\partial u_R}{R\partial\theta} - \frac{u_\theta}{R}, & \varepsilon_{\theta Z} &= \frac{\partial u_\theta}{\partial Z} + \frac{\partial u_Z}{R\partial\theta}, & \varepsilon_{RZ} &= \frac{\partial u_R}{\partial Z} + \frac{\partial u_Z}{\partial R}. \end{aligned} \quad (4)$$

Substituting Equation (1) into Equation (4) leads to:

$$\begin{aligned} \varepsilon_{RR} &= \frac{\partial u}{\partial R} - Z \frac{\partial^2 \phi^b}{\partial R^2} - G \frac{\partial^2 \phi^s}{\partial R^2}, \\ \varepsilon_{\theta\theta} &= \frac{1}{R} \left(u - Z \frac{\partial \phi^b}{\partial R} - G \frac{\partial \phi^s}{\partial R} \right), \\ \varepsilon_{ZZ} &= 0, & \varepsilon_{R\theta} &= 0, & \varepsilon_{\theta Z} &= 0, \\ \varepsilon_{RZ} &= \bar{G}' \frac{\partial \phi^s}{\partial R}, & \bar{G}' &= \frac{d\bar{G}}{dZ}. \end{aligned} \quad (5)$$

The stresses of each layer are given as [52]:

$$\begin{aligned} \sigma_{RR}^{(i)} &= C_{11}^{(i)} \varepsilon_{RR} + C_{12}^{(i)} \varepsilon_{\theta\theta}, \\ \sigma_{\theta\theta}^{(i)} &= C_{12}^{(i)} \varepsilon_{RR} + C_{11}^{(i)} \varepsilon_{\theta\theta}, \\ \sigma_{RZ}^{(i)} &= C_{55}^{(i)} \varepsilon_{RZ}, & i &= 1, 2, 3, \end{aligned} \quad (6)$$

where

$$C_{11}^{(i)} = \frac{E^{(i)}}{[1 - (\nu^{(i)})^2]}, \quad C_{12}^{(i)} = \frac{\nu^{(i)} E^{(i)}}{[1 - (\nu^{(i)})^2]}, \quad C_{55}^{(i)} = \frac{E^{(i)}}{2[1 + \nu^{(i)}]}. \quad (7)$$

2.2. The Magnetic Field

The relationships between the current density $\vec{J}$ and the distribution vector of the magnetic field $\vec{h}$ are given by Maxwell's equations as [53,54]:

$$\begin{aligned} \vec{J} &= \nabla \times \vec{h}, \\ \vec{h} &= \nabla \times (\vec{u} \times \vec{M}), \end{aligned} \quad (8)$$

where $\vec{M}(M_R, 0, 0)$ represents the in-plane magnetic field vector, and $\vec{u}(u_R, 0, u_Z)$ represents the displacement vector. The vector of the distributed magnetic field $\vec{h}$ can be expanded as:

$$\vec{h} = M_R \left(\frac{u_Z}{R} + \frac{\partial u_Z}{\partial R} \right) \hat{Z}. \quad (9)$$

By substituting Equation (9) into Equation (8), the current density $\vec{J}$ can be expressed as:

$$\vec{J} = M_R \left(\frac{\partial u_Z}{R^2 \partial \theta} + \frac{\partial^2 u_Z}{R \partial R \partial \theta} \right) \hat{R} - M_R \left(\frac{\partial u_Z}{R \partial R} + \frac{\partial^2 u_Z}{\partial R^2} - \frac{u_Z}{R^2} \right) \hat{\theta}. \quad (10)$$

The Lorentz magnetic force vector per unit volume $\vec{F}$ is given by [53]:

$$\vec{F} = \gamma (\vec{J} \times \vec{M}), \quad (11)$$

where γ is the permeability of the magnetic field. By substituting Equation (10) into Equation (11), we get:

$$F_R = 0, \quad F_\theta = 0, \quad F_Z = \gamma M_R^2 \left(\frac{\partial u_Z}{R \partial R} + \frac{\partial^2 u_Z}{\partial R^2} - \frac{u_Z}{R^2} \right). \quad (12)$$

2.3. Plate Properties

2.3.1. Nanocomposite Face Layers

The face layers are made of metal reinforced with GPLs that can be equally dispersed or smoothly modulated across the thickness of the layer. The Young's modulus and Poisson's ratio of the top and bottom layers have the following form in accordance with the modified Halpin–Tsai model and mixture law [55]:

$$E^{(j)}(Z) = \frac{1}{8} [3E_1^{(j)}(Z) + 5E_2^{(j)}(Z)], \quad E_K^{(j)}(Z) = \frac{1 + \xi_K^g \eta_K F_g^{(j)}}{1 - \eta_K F_g^{(j)}} E_m, \quad (13)$$

$$\eta_K = \frac{E_g - E_m}{E_g + \xi_K^g E_m}, \quad \xi_1^g = 2 \frac{L^g}{T^g}, \quad \xi_2^g = 2 \frac{W^g}{T^g}, \quad K = 1, 2, \quad j = 1, 3,$$

$$\begin{aligned} \nu^{(j)}(Z) &= F_m^{(j)}(Z) \nu_m + F_g^{(j)}(Z) \nu_g, \\ \rho^{(j)} &= F_m^{(j)}(Z) \rho_m + F_g^{(j)}(Z) \rho_g, \quad F_m^{(j)}(Z) = 1 - F_g^{(j)}(Z), \quad j = 1, 3, \end{aligned} \quad (14)$$

where E_g and ν_g are graphene's Young's modulus and Poisson's ratio, respectively; E_m and ν_m are metal's Young's modulus and Poisson's ratio, respectively; L^g , W^g , and T^g are the length, width, and thickness of the GPLs; ρ is the mass density of the layers; ρ_g and ρ_m are graphene's density and metal's density, respectively. There are three types of GPLs distributions, as shown in Figure 2:

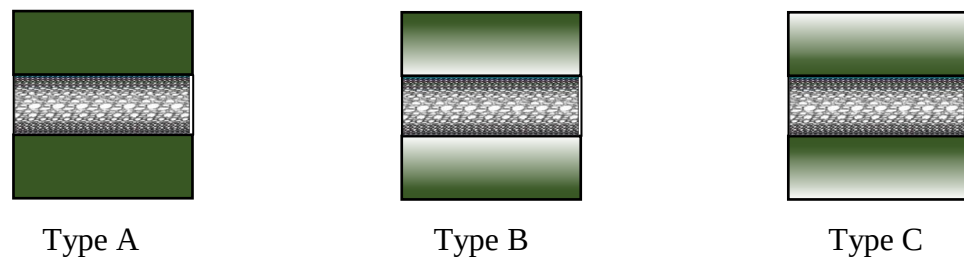


Figure 2. Types of GPLs distributions.

Type A:

In this type, the GPLs are evenly dispersed throughout the sandwich faces' thickness; one may write [5,56]:

$$F_g^{(j)}(Z) = V_g, \quad T_0 \leq Z \leq T_1 \quad \text{and} \quad T_2 \leq Z \leq T_3, \quad j = 1, 3, \text{UD}, \quad (15)$$

where $T_0 = -T/2$ and $T_3 = T/2$; T_1 and T_2 are the layer interface coordinates (see Figure 1). Meanwhile, V_g is provided as [5]:

$$V_g = \frac{W_G}{W_G + \left(\frac{\rho_g}{\rho_m}\right)(1 - W_G)}, \quad (16)$$

where W_G represents the weight fraction of the GPLs.

Type B:

In this state, the interfaces (T_1 and T_2) are graphene-free, while the upper and lower surfaces of the plate are graphene-rich ($\frac{\Delta - FG}{\Delta - FG}$); thus, we can type [57]:

$$\begin{aligned} F_g^{(1)}(Z) &= V_g \cos\left(\frac{(2Z - T_0 - T_1)\pi}{4(T_0 - T_1)} - \frac{\pi}{4}\right), \quad T_0 \leq Z \leq T_1, \\ F_g^{(3)}(Z) &= V_g \cos\left(\frac{(2Z - T_3 - T_2)\pi}{4(T_3 - T_2)} - \frac{\pi}{4}\right), \quad T_2 \leq Z \leq T_3. \end{aligned} \quad (17)$$

Type C:

In the final situation, the graphene distribution is the inverse of the preceding type: ($\frac{\Delta - FG}{\Delta - FG}$); thus, $F_g^{(j)}(Z)$ is given as [57]:

$$\begin{aligned} F_g^{(1)}(Z) &= V_g \cos\left(\frac{(2Z - T_0 - T_1)\pi}{4(T_0 - T_1)} + \frac{\pi}{4}\right), \quad T_0 \leq Z \leq T_1, \\ F_g^{(3)}(Z) &= V_g \cos\left(\frac{(2Z - T_3 - T_2)\pi}{4(T_3 - T_2)} + \frac{\pi}{4}\right), \quad T_2 \leq Z \leq T_3. \end{aligned} \quad (18)$$

It is worth noting that the distribution types of GPLs play a crucial role in determining the response of the plates. In particular, Type A might improve the mechanical properties of the plates more than the other types. On the other hand, Type B is characterized by a strong bond between the two face layers and the core layer, where the interfaces between the layers are only metal. Therefore, it is expected that the third type will be less efficient than the other two types.

2.3.2. Porous Core Layer

The metal foam core is considered to be constructed of aluminum with evenly dispersed or FG porosities throughout the core thickness (see Figure 3). The current study takes into account three alternative porosity distributions:

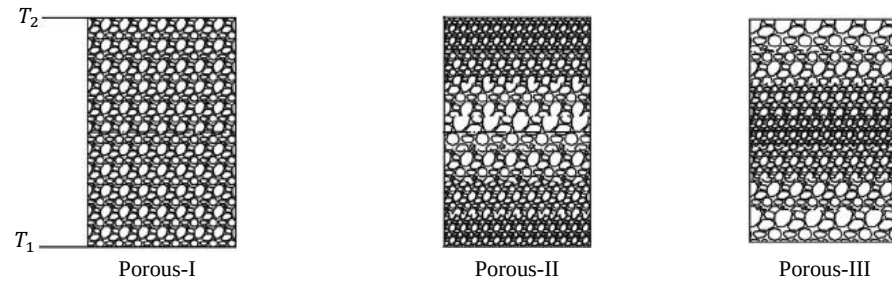


Figure 3. Types of porosity distributions.

Porous-I:

In this case, the pores are dispersed equally across the thickness of the core (UD) as follows [58]:

$$\begin{aligned} P^{(2)}(Z) &= P_m(1 - \beta_0\lambda), \quad T_1 \leq Z \leq T_2, \\ \rho^{(2)}(Z) &= \rho_m\sqrt{1 - \beta_0\lambda}, \quad T_1 \leq Z \leq T_2, \end{aligned} \quad (19)$$

where P denotes the mechanical properties, like Young's modulus E and shear modulus G , and β_0 denotes the porosity coefficient, where $0 \leq \beta_0 < 1$, and the parameter λ is defined as [58]:

$$\lambda = \frac{1}{\beta_0} - \frac{1}{\beta_0} \left(\frac{2}{\pi} \sqrt{1 - \beta_0\lambda} - \frac{2}{\pi} + 1 \right)^2. \quad (20)$$

Porous-II:

From the interfaces to the core's mid-plane, the volume fraction of the porosities increases (O-FG). As a result, the mid-plane has the lowest mechanical property values. As a result, the properties are as follows [57]:

$$\begin{aligned} P^{(2)}(Z) &= P_m \left\{ 1 - \beta_0 \left[1 - \left(\frac{2Z - T_1 - T_2}{T_1 - T_2} \right)^2 \right] \right\}, \quad T_1 \leq Z \leq T_2, \\ \rho^{(2)}(Z) &= \rho_m \left\{ 1 - (1 - \sqrt{1 - \beta_0}) \left[1 - \left(\frac{2Z - T_1 - T_2}{T_1 - T_2} \right)^2 \right] \right\}, \quad T_1 \leq Z \leq T_2. \end{aligned} \quad (21)$$

Porous-III:

The mid-plane of the core is pore-free in this instance, but the interfaces are pore-rich (X-FG). As a result, we can write [57]:

$$\begin{aligned} P^{(2)}(Z) &= P_m \left[1 - \beta_0 \left(\frac{2Z - T_1 - T_2}{T_1 - T_2} \right)^2 \right], \quad T_1 \leq Z \leq T_2, \\ \rho^{(2)}(Z) &= \rho_m \left\{ 1 - (1 - \sqrt{1 - \beta_0}) \left(\frac{2Z - T_1 - T_2}{T_1 - T_2} \right)^2 \right\}, \quad T_1 \leq Z \leq T_2. \end{aligned} \quad (22)$$

The Poisson's ratios of the cores in the preceding three categories are computed as follows [59]:

$$\nu^{(2)}(Z) = \nu_m \left[0.342 \left(1 - \frac{\rho^2(Z)}{\rho_m} \right)^2 - 1.21 \left(1 - \frac{\rho^2(Z)}{\rho_m} \right) + 1 \right] + 0.221 \left(1 - \frac{\rho^2(Z)}{\rho_m} \right). \quad (23)$$

Once again, the porosity distribution types significantly influence the reaction characteristics of the plates. Since the upper and lower surfaces of Type II are pore-free, the bond between the core and the face layers is enhanced. On the other hand, for porosity Type III, the thermal conductivity of the core at the interfaces might be substantially lower than that of the other types, reducing heat transfer to the core layer. Therefore, in high thermal environments, this type will be the best to use. The impact of distinct distribution types

of GPLs and porosity on the vibrational response of FG sandwich annular plates will be discussed in more detail in Section 5.

2.4. Elastic Foundation

The flexible foundation is a layer beneath the plate that helps to distribute the weight and stress. Polymers, rubber, and flexible metals can be used to create the flexible foundation. In this context, the flexible foundation has a significant impact on the plate's ability to absorb vibrations and shocks. Further, the flexible foundation better distributes stresses along the plate and absorbs vibration energy. The load is distributed across a broader area on the ground when employing a flexible foundation, which can minimize the pressure on the plate. This broader distribution can improve the overall performance of the plate by decreasing undesirable deformations or bending. For adding foundation stiffness, the Pasternak model has been defined as [8,60,61]:

$$S = K_w u_Z - K_P \left(\frac{\partial^2 u_Z}{\partial R^2} + \frac{1}{R} \frac{\partial u_Z}{\partial R} \right), \quad (24)$$

where S , K_w , u_Z , and K_P represent the density reaction of the foundations, the Winkler stiffness, the transverse displacement, and the shear (Pasternak) foundation stiffness, respectively.

3. Governing Equations

To find the equations of motion, Hamilton's principle is used and is stated as:

$$\int_{t_1}^{t_2} [\delta \pi_s - \delta \pi_f + \delta \pi_k] dt = 0, \quad (25)$$

where $\delta \pi_s$ is the variation of the strain energy, $\delta \pi_f$ is the variation of the work done by an external force, and $\delta \pi_k$ is the variation of the kinetic energy; these are given as:

$$\delta \pi_s = \sum_{i=1}^3 \int_0^{2\pi} \int_{r_1}^{r_2} \int_{T_{i-1}}^{T_i} \sigma_{ij}^{(i)} \delta \varepsilon_{ij} R dZ dR d\theta, \quad (26)$$

$$\delta \pi_f = \int_0^{2\pi} \int_{r_1}^{r_2} \int_{-\frac{T}{2}}^{\frac{T}{2}} \vec{F} \cdot \delta \vec{u} R dZ dR d\theta - \int_0^{2\pi} \int_{r_1}^{r_2} S \delta u_Z R dR d\theta, \quad (27)$$

$$\delta \pi_k = \sum_{i=1}^3 \int_0^{2\pi} \int_{r_1}^{r_2} \int_{T_{i-1}}^{T_i} \rho^{(i)} \ddot{u}_i \delta u_i R dZ dR d\theta. \quad (28)$$

By inserting Equations (26)–(28) into Equation (25) and integrating by parts, we obtain

$$\begin{aligned} \delta u : & \frac{\partial N_{RR}}{\partial R} + \frac{N_{RR}}{R} - \frac{N_{\theta\theta}}{R} - I_1 \ddot{u} + I_2 \frac{\partial \ddot{\phi}^b}{\partial R} + I_3 \frac{\partial \ddot{\phi}^s}{\partial R} = 0, \\ \delta \phi^b : & \frac{\partial^2 M_{RR}}{\partial R^2} + \frac{2}{R} \frac{\partial M_{RR}}{\partial R} - \frac{1}{R} \frac{\partial M_{\theta\theta}}{\partial R} - I_2 \frac{\partial \ddot{u}}{\partial R} - \frac{1}{R} I_2 \ddot{u} + I_4 \frac{\partial^2 \ddot{\phi}^b}{\partial R^2} + \frac{1}{R} I_4 \frac{\partial \ddot{\phi}^b}{\partial R} \\ & + I_5 \frac{\partial^2 \ddot{\phi}^s}{\partial R^2} + \frac{1}{R} I_5 \frac{\partial \ddot{\phi}^s}{\partial R} - I_1 \ddot{\phi}^b - I_1 \ddot{\phi}^s - \gamma M_R^2 \frac{u_Z}{R^2} T + \frac{1}{R} \gamma M_R^2 \frac{\partial u_Z}{\partial R} T + \gamma M_R^2 \frac{\partial^2 u_Z}{\partial R^2} T \\ & - S = 0, \\ \delta \phi^s : & \frac{\partial^2 S_{RR}}{\partial R^2} + \frac{2}{R} \frac{\partial S_{RR}}{\partial R} - \frac{1}{R} \frac{\partial S_{\theta\theta}}{\partial R} + \frac{\partial S_{RZ}}{\partial R} + \frac{S_{RZ}}{R} - I_3 \frac{\partial \ddot{u}}{\partial R} - \frac{1}{R} I_3 \ddot{u} + I_5 \frac{\partial^2 \ddot{\phi}^b}{\partial R^2} \\ & + \frac{1}{R} I_5 \frac{\partial \ddot{\phi}^b}{\partial R} + I_6 \frac{\partial^2 \ddot{\phi}^s}{\partial R^2} + \frac{1}{R} I_6 \frac{\partial \ddot{\phi}^s}{\partial R} - I_1 \ddot{\phi}^b - I_1 \ddot{\phi}^s - \gamma M_R^2 \frac{u_Z}{R^2} T + \frac{1}{R} \gamma M_R^2 \frac{\partial u_Z}{\partial R} T \\ & + \gamma M_R^2 \frac{\partial^2 u_Z}{\partial R^2} T - S = 0, \end{aligned} \quad (29)$$

where

$$\begin{aligned} N_{RR} &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{RR}^{(i)} dZ, \quad M_{RR} = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{RR}^{(i)} Z dZ, \quad S_{RR} = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{RR}^{(i)} G(Z) dZ, \\ N_{\theta\theta} &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{\theta\theta}^{(i)} dZ, \quad M_{\theta\theta} = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{\theta\theta}^{(i)} Z dZ, \quad S_{\theta\theta} = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{\theta\theta}^{(i)} G(Z) dZ, \\ S_{RZ} &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \sigma_{RZ}^{(i)} \bar{G}'(Z) dZ, \end{aligned} \quad (30)$$

and

$$\begin{aligned} I_1 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} dZ, \quad I_2 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} Z dZ, \\ I_3 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} G(Z) dZ, \quad I_4 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} Z^2 dZ, \\ I_5 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} Z G(Z) dZ, \quad I_6 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} \rho^{(i)} G^2(Z) dZ. \end{aligned} \quad (31)$$

By substituting Equations (5) and (6) into Equation (30) and then into Equation (29), the equations of motion are rewritten as:

$$\begin{aligned} & -A_2 \frac{\partial^3 \phi^b}{\partial R^3} - A_3 \frac{\partial^3 \phi^s}{\partial R^3} + A_1 \frac{\partial^2 u}{\partial R^2} - A_2 \frac{\partial^2 \phi^b}{R \partial R^2} - A_3 \frac{\partial^2 \phi^s}{R \partial R^2} + A_1 \frac{\partial u}{R \partial R} \\ & + A_2 \frac{\partial \phi^b}{R^2 \partial R} + A_3 \frac{\partial \phi^s}{R^2 \partial R} - A_1 \frac{u}{R^2} - I_1 \ddot{u} + I_2 \frac{\partial \ddot{\phi}^b}{\partial R} + I_3 \frac{\partial \ddot{\phi}^s}{\partial R} = 0, \\ & -A_4 \frac{\partial^4 \phi^b}{\partial R^4} - A_5 \frac{\partial^4 \phi^s}{\partial R^4} + A_2 \frac{\partial^3 u}{\partial R^3} - A_4 \frac{2 \partial^3 \phi^b}{R \partial R^3} - A_5 \frac{2 \partial^3 \phi^s}{R \partial R^3} + A_2 \frac{2 \partial^2 u}{R \partial R^2} + A_4 \frac{\partial^2 \phi^b}{R^2 \partial R^2} \\ & + A_5 \frac{\partial^2 \phi^s}{R^2 \partial R^2} - A_2 \frac{\partial u}{R^2 \partial R} - A_4 \frac{\partial \phi^b}{R^3 \partial R} - A_5 \frac{\partial \phi^s}{R^3 \partial R} + A_2 \frac{u}{R^3} - I_2 \frac{\partial \ddot{u}}{\partial R} - \frac{1}{R} I_2 \ddot{u} \\ & + I_4 \frac{\partial^2 \ddot{\phi}^b}{\partial R^2} + \frac{1}{R} I_4 \frac{\partial \ddot{\phi}^b}{\partial R} + I_5 \frac{\partial^2 \ddot{\phi}^s}{\partial R^2} + \frac{1}{R} I_5 \frac{\partial \ddot{\phi}^s}{\partial R} - I_1 \ddot{\phi}^b - I_1 \ddot{\phi}^s - \gamma M_R^2 \frac{u_Z}{R^2} T + \frac{1}{R} \gamma M_R^2 \frac{\partial u_Z}{\partial R} T \\ & + \gamma M_R^2 \frac{\partial^2 u_Z}{\partial R^2} T - S = 0, \\ & -A_5 \frac{\partial^4 \phi^b}{\partial R^4} - A_6 \frac{\partial^4 \phi^s}{\partial R^4} + A_3 \frac{\partial^3 u}{\partial R^3} - A_5 \frac{2 \partial^3 \phi^b}{R \partial R^3} - A_6 \frac{2 \partial^3 \phi^s}{R \partial R^3} + A_3 \frac{2 \partial^2 u}{R \partial R^2} + A_5 \frac{\partial^2 \phi^b}{R^2 \partial R^2} \\ & + \frac{\partial^2 \phi^s}{\partial R^2} \left[\frac{A_6}{R^2} + A_7 \right] - A_3 \frac{\partial u}{R^2 \partial R} - A_5 \frac{\partial \phi^b}{R^3 \partial R} + \frac{\partial \phi^s}{R \partial R} \left[-\frac{A_6}{R^2} + A_7 \right] + A_3 \frac{u}{R^3} - I_3 \frac{\partial \ddot{u}}{\partial R} \\ & - \frac{1}{R} I_3 \ddot{u} + I_5 \frac{\partial^2 \ddot{\phi}^b}{\partial R^2} + \frac{1}{R} I_5 \frac{\partial \ddot{\phi}^b}{\partial R} + I_6 \frac{\partial^2 \ddot{\phi}^s}{\partial R^2} + \frac{1}{R} I_6 \frac{\partial \ddot{\phi}^s}{\partial R} - I_1 \ddot{\phi}^b - I_1 \ddot{\phi}^s - \gamma M_R^2 \frac{u_Z}{R^2} T \\ & + \frac{1}{R} \gamma M_R^2 \frac{\partial u_Z}{\partial R} T + \gamma M_R^2 \frac{\partial^2 u_Z}{\partial R^2} T - S = 0, \end{aligned} \quad (32)$$

where

$$\begin{aligned} A_1 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} dZ, \quad A_2 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} Z dZ, \quad A_3 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} G(Z) dZ, \\ A_4 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} Z^2 dZ, \quad A_5 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} Z G(Z) dZ, \\ A_6 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{11}^{(i)} G^2(Z) dZ, \quad A_7 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{55}^{(i)} (\bar{G}'(Z))^2 dZ. \end{aligned} \quad (35)$$

By separating the variables, the displacement components can be expressed as:

$$u(R, t) = U(R)\mathcal{V}(t), \phi^b(R, t) = \Phi^b(R)\mathcal{V}(t), \phi^s(R, t) = \Phi^s(R)\mathcal{V}(t), \quad (36)$$

where $\mathcal{V}(t) = e^{i\omega t}$, in which $i = \sqrt{-1}$, and ω is the eigenfrequency. By substituting Equation (36) into Equations (32)–(34), we get the following ordinary differential equations:

$$\begin{aligned} & -A_2 \frac{d^3 \Phi^b}{dR^3} - A_3 \frac{d^3 \Phi^s}{dR^3} + A_1 \frac{d^2 U}{dR^2} - A_2 \frac{d^2 \Phi^b}{R dR^2} - A_3 \frac{d^2 \Phi^s}{R dR^2} + A_1 \frac{dU}{R dR} \\ & + \frac{d\Phi^b}{dR} \left(\frac{A_2}{R^2} - I_2 w^2 \right) + \frac{d\Phi^s}{dR} \left(\frac{A_3}{R^2} - I_3 w^2 \right) - U \left(\frac{A_1}{R^2} - I_1 w^2 \right) = 0, \end{aligned} \quad (37)$$

$$\begin{aligned} & -A_4 \frac{d^4 \Phi^b}{dR^4} - A_5 \frac{d^4 \Phi^s}{dR^4} + A_2 \frac{d^3 U}{dR^3} - A_4 \frac{2 d^3 \Phi^b}{R dR^3} - A_5 \frac{2 d^3 \Phi^s}{R dR^3} + A_2 \frac{2 d^2 U}{R dR^2} \\ & + \frac{d^2 \Phi^b}{dR^2} \left(\frac{A_4}{R^2} - I_4 w^2 + K_P + \gamma M_R^2 T \right) + \frac{d^2 \Phi^s}{dR^2} \left(\frac{A_5}{R^2} - I_5 w^2 + K_P + \gamma M_R^2 T \right) \\ & - \frac{dU}{dR} \left(\frac{A_2}{R^2} - I_2 w^2 \right) - \frac{d\Phi^b}{R dR} \left(\frac{A_4}{R^2} + I_4 w^2 - K_P - \gamma M_R^2 T \right) \\ & - \frac{d\Phi^s}{R dR} \left(\frac{A_5}{R^2} + I_5 w^2 - K_P - \gamma M_R^2 T \right) + \frac{U}{R} \left(\frac{A_2}{R^2} + I_2 w^2 \right) \\ & + \Phi^b \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R^2} \right) + \Phi^s \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R^2} \right) = 0, \end{aligned} \quad (38)$$

$$\begin{aligned} & -A_5 \frac{d^4 \Phi^b}{dR^4} - A_6 \frac{d^4 \Phi^s}{dR^4} + A_3 \frac{d^3 U}{dR^3} - A_5 \frac{2 d^3 \Phi^b}{R dR^3} - A_6 \frac{2 d^3 \Phi^s}{R dR^3} + A_3 \frac{2 d^2 U}{R dR^2} \\ & + \frac{d^2 \Phi^b}{dR^2} \left(\frac{A_5}{R^2} - I_5 w^2 + K_P + \gamma M_R^2 T \right) + \frac{d^2 \Phi^s}{dR^2} \left(\frac{A_6}{R^2} + A_7 - I_6 w^2 + K_P + \gamma M_R^2 T \right) \\ & - \frac{dU}{dR} \left(\frac{A_3}{R^2} - I_3 w^2 \right) - \frac{d\Phi^b}{R dR} \left(\frac{A_5}{R^2} + I_5 w^2 - K_P - \gamma M_R^2 T \right) \\ & - \frac{d\Phi^s}{R dR} \left(\frac{A_6}{R^2} - A_7 + I_6 w^2 - K_P - \gamma M_R^2 T \right) + \frac{U}{R} \left(\frac{A_3}{R^2} + I_3 w^2 \right) \\ & + \Phi^b \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R^2} \right) + \Phi^s \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R^2} \right) = 0. \end{aligned} \quad (39)$$

In the current analysis, the following simply supported conditions are used at $R = r_1$ and $R = r_2$:

$$\Phi^b = \Phi^s = N_{RR} = 0. \quad (40)$$

4. Solution Procedure

In this part, the equilibrium Equations (37)–(39) are solved using the DQM in the radial direction. The current annular plate is discretized by n grid points in the domain ($r_1 \leq R \leq r_2$). According to the DQM, the displacement component differentiations are approximated as a weighted linear sum of function values at all discrete sites as follows [40]:

$$\begin{aligned} \left. \frac{d^p U}{dR^p} \right|_{R=R_i} &= \sum_{j=1}^n C_{ij}^{(p)} U_j, \quad U_j = U(R_j), \\ \left. \frac{d^p \Phi^b}{dR^p} \right|_{R=R_i} &= \sum_{j=1}^n C_{ij}^{(p)} \Phi_j^b, \quad \Phi_j^b = \Phi^b(R_j), \\ \left. \frac{d^p \Phi^s}{dR^p} \right|_{R=R_i} &= \sum_{j=1}^n C_{ij}^{(p)} \Phi_j^s, \quad \Phi_j^s = \Phi^s(R_j), \quad i = 1, 2, \dots, n, \end{aligned} \quad (41)$$

where $C_{ij}^{(p)}$ denotes the weighting coefficients for the p th-order derivative. They are provided as [40]:

$$\begin{aligned} C_{ij}^{(1)} &= \frac{K(R_i)}{(R_i - R_j)K(R_j)}, \quad i, j = 1, 2, \dots, n; \quad i \neq j, \\ C_{ii}^{(1)} &= -\sum_{l=1}^n C_{il}^{(1)}, \quad i = 1, 2, \dots, n; \quad l \neq i, \\ K(R_i) &= \prod_{j=1}^n (R_i - R_j), \quad i \neq j. \end{aligned} \quad (42)$$

Furthermore, the weighting coefficients $C_{ij}^{(p)}$, ($p > 1$), for higher-order derivatives are defined as [40]:

$$C_{ij}^{(p)} = \sum_{l=1}^n C_{il}^{(1)} C_{lj}^{(p-1)}, \quad i, j = 1, 2, \dots, n. \quad (43)$$

However, the Gauss–Chebyshev–Lobatto approach is applied to estimate the mesh points R_i as [40]:

$$R_i = r_1 + \frac{r_2 - r_1}{2} \left[1 - \cos \left(\pi \frac{i-1}{n-1} \right) \right]. \quad (44)$$

The motion Equations (37)–(39) can be discretized using Equation (41), resulting in $(n-2)$ linear algebraic equations as follows:

$$\begin{aligned} & -A_2 \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^b - A_3 \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^s + A_1 \sum_{j=1}^n C_{ij}^{(2)} U_j - A_2 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b \\ & - A_3 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s + A_1 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} U_j + \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b \left(\frac{A_2}{R_i^2} - I_2 w^2 \right) \\ & + \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s \left(\frac{A_3}{R_i^2} - I_3 w^2 \right) + U_i \left(\frac{-A_1}{R_i^2} + I_1 w^2 \right) = 0, \\ & -A_4 \sum_{j=1}^n C_{ij}^{(4)} \Phi_j^b - A_5 \sum_{j=1}^n C_{ij}^{(4)} \Phi_j^s + A_2 \sum_{j=1}^n C_{ij}^{(3)} U_j - A_4 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^b \\ & - A_5 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^s + A_2 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(2)} U_j + \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b \left(\frac{A_4}{R_i^2} - I_4 w^2 + K_P + \gamma M_R^2 T \right) \\ & + \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s \left(\frac{A_5}{R_i^2} - I_5 w^2 + K_P + \gamma M_R^2 T \right) + \sum_{j=1}^n C_{ij}^{(1)} U_j \left(\frac{-A_2}{R_i^2} + I_2 w^2 \right) \\ & - \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b \left(\frac{A_4}{R_i^2} + I_4 w^2 - K_P - \gamma M_R^2 T \right) \\ & - \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s \left(\frac{A_5}{R_i^2} + I_5 w^2 - K_P - \gamma M_R^2 T \right) + \frac{U_i}{R_i} \left(\frac{A_2}{R_i^2} + I_2 w^2 \right) \\ & + \Phi_i^b \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R_i^2} \right) + \Phi_i^s \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R_i^2} \right) = 0, \end{aligned} \quad (45)$$

(46)

$$\begin{aligned}
& -A_5 \sum_{j=1}^n C_{ij}^{(4)} \Phi_j^b - A_6 \sum_{j=1}^n C_{ij}^{(4)} \Phi_j^s + A_3 \sum_{j=1}^n C_{ij}^{(3)} U_j - A_5 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^b \\
& - A_6 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(3)} \Phi_j^s + A_3 \frac{2}{R_i} \sum_{j=1}^n C_{ij}^{(2)} U_j + \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b \left(\frac{A_5}{R_i^2} - I_5 w^2 + K_P + \gamma M_R^2 T \right) \\
& + \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s \left(\frac{A_6}{R_i^2} + A_7 - I_6 w^2 + K_P + \gamma M_R^2 T \right) + \sum_{j=1}^n C_{ij}^{(1)} U_j \left(\frac{-A_3}{R_i^2} + I_3 w^2 \right) \\
& - \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b \left(\frac{A_5}{R_i^2} + I_5 w^2 - K_P - \gamma M_R^2 T \right) \\
& + \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s \left(\frac{-A_6}{R_i^2} + A_7 - I_6 w^2 + K_P + \gamma M_R^2 T \right) + \frac{U_i}{R_i} \left(\frac{A_3}{R_i^2} + I_3 w^2 \right) \\
& + \Phi_i^b \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R_i^2} \right) + \Phi_i^s \left(I_1 w^2 - K_w - \frac{\gamma M_R^2 T}{R_i^2} \right) = 0, \quad i = 2, 3, \dots, (n-1).
\end{aligned} \tag{47}$$

Furthermore, the boundary conditions can be represented in a discretized form as follows:

Simply supported (S):

$$\begin{aligned}
\Phi_i^b = \Phi_i^s &= A_1 \sum_{j=1}^n C_{ij}^{(1)} U_j - A_2 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b - A_3 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s + B_1 \frac{U_i}{R_i} - B_2 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b \\
& - B_3 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s = A_2 \sum_{j=1}^n C_{ij}^{(1)} U_j - A_4 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b - A_5 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s + B_2 \frac{U_i}{R_i} \\
& - B_4 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b - B_5 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s = A_3 \sum_{j=1}^n C_{ij}^{(1)} U_j - A_5 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^b \\
& - A_6 \sum_{j=1}^n C_{ij}^{(2)} \Phi_j^s + B_3 \frac{U_i}{R_i} - B_5 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^b - B_6 \frac{1}{R_i} \sum_{j=1}^n C_{ij}^{(1)} \Phi_j^s = 0, \quad i = 1, n,
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
B_1 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} dZ, \quad B_2 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} Z dZ, \\
B_3 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} G(Z) dZ, \quad B_4 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} Z^2 dZ, \\
B_5 &= \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} Z G(Z) dZ, \quad B_6 = \sum_{i=1}^3 \int_{T_{i-1}}^{T_i} C_{12}^{(i)} G^2(Z) dZ.
\end{aligned} \tag{49}$$

System (45)–(47) represents an eigenvalue problem that can be solved with the boundary conditions (48) to obtain the eigenfrequency.

5. Numerical Results

In the current section, various numerical examples are presented to demonstrate the effects of various parameters on the frequency of a GPLs-reinforced annular plate with an FG porous core, including the porosity factor β_0 , the porosity distribution types, the GPLs weight fraction W_G , the GPLs distribution type, the outer radius-to-thickness ratio r_2/T , the magnetic field parameter $\bar{M}_R$, the core thickness T_c/T , and the elastic substrate parameters g_1 and g_2 . The dimensionless formulations are defined by:

$$\Omega = \omega r_2^2 \sqrt{\frac{T \rho_m}{E_m}}, \quad \bar{M}_R = \frac{\gamma T r_2^2 M_R^2}{D_m}, \quad g_1 = \frac{K_w r_2^4}{D_m}, \quad g_2 = \frac{K_p r_2^2}{D_m}, \quad D_m = \frac{E_m T^3}{12(1 - \nu_m^2)}. \quad (50)$$

The material properties of the constituent materials are taken as: $E_m = 70$ GPa, $\nu_m = 0.3$, $\rho_m = 2700$ Kg/m³, $E_g = 1.01$ TPa, $\nu_g = 0.186$, and $\rho_g = 1060$ Kg/m³.

The numerical examples use the following fixed data (unless otherwise declared): $W_G = 0.1$, $\bar{M}_R = 50$, $T_c/T = 0.1$, $g_1 = 100$, $g_2 = 10$, $\beta_0 = 0.2$, $r_1/r_2 = 0.1$, $r_2/T = 10$, $T = 0.2$ m, $L^g = 15$ nm, $W^g = 9$ nm, and $T^g = 0.188$ nm.

Firstly, Table 1 shows a convergence analysis for the current outcomes of the FG porous annular plate enhanced by GPLs for various values of core thickness T_c/T . The minimal number of grid points needed for a converged solution in the DQM is calculated in this table. It is worth noting that 17 grid points are adequate to produce a converged solution.

Table 1. Convergence of the results of FG porous nanocomposite annular plates ($r_2/T = 8$, $T_R = T_c/T$).

n	$T_R = 0.1$	Error	$T_R = 0.2$	Error	$T_R = 0.3$	Error	$T_R = 0.4$	Error
7	1.2917	-	1.2795	-	1.2637	-	1.2424	-
9	2.9013	1.6096	2.7686	1.4891	2.6314	1.3677	2.4888	1.2454
11	2.8621	0.0393	2.7674	0.0012	2.6302	0.0011	2.4877	0.0011
13	2.8996	0.0375	2.7669	0.0005	2.6297	0.0005	2.4872	0.0005
15	2.8994	0.0002	2.7667	0.0002	2.6296	0.0002	2.4871	0.0002
17	2.8993	0.0001	2.7667	0.0001	2.6295	0.0001	2.4870	0.0001

Error _{i} = $\Omega_i - \Omega_{i-1}$.

Secondly, to ensure the accuracy of the current theory, the frequency obtained via the present theory is compared with that estimated using the third-order plate theory (TPT) [48], the sinusoidal plate theory (SPT) [49], the hyperbolic plate theory (HPT) [50], and the exponential plate theory (EPT) [51] for various plate geometry parameter values as shown in Table 2. It should be observed that the frequencies offered by our theory are extremely consistent with those obtained by the other, higher-order theories. Furthermore, this table shows the results for various values of the outer radius-to-thickness ratio r_2/T and the inner-to-outer radius ratio r_1/r_2 . As shown in this table, the eigenfrequency increases as the ratio r_2/T increases and the ratio r_1/r_2 decreases because the increase in the inner-to-outer radius ratio r_1/r_2 leads to a decrease in the plate width; hence, the annular plate becomes weaker.

Table 2. Comparison of the results of the present theory for the frequency Ω of a GPLs-reinforced annular plate with an FG porous core with the results of other theories.

r_2/T	r_1/r_2	TPT [48]	SPT [49]	HPT [50]	EPT [51]	Present
10	0.1	3.62419	3.62419	3.62419	3.62419	3.62419
	0.2	3.31653	3.31653	3.31653	3.31653	3.31653
	0.3	2.68767	2.68631	2.68781	2.68524	2.68662
	0.4	2.17241	2.17182	2.17247	2.17135	2.17195
	0.5	2.05191	2.05112	2.05191	2.05049	2.05130
20	0.1	7.24838	7.24838	7.24838	7.24838	7.24838
	0.2	6.63306	6.63306	6.63306	6.63306	6.63306
	0.3	2.80871	2.80831	2.80874	2.80815	2.80847
	0.4	2.23078	2.23064	2.23071	2.23053	2.23067
	0.5	2.17024	2.16992	2.17027	2.16967	2.16999
30	0.1	10.87256	10.87256	10.87256	10.87256	10.87256
	0.2	9.94959	9.94959	9.94959	9.94959	9.94959
	0.3	2.82940	2.82927	2.82942	2.82916	2.82930
	0.4	2.24134	2.24128	2.24135	2.24123	2.24129
	0.5	2.19826	2.19810	2.19827	2.19798	2.19814

Table 2. Cont.

r_2/T	r_1/r_2	TPT [48]	SPT [49]	HPT [50]	EPT [51]	Present
40	0.1	12.62866	12.62634	12.62890	12.62451	12.62687
	0.2	12.76462	12.76230	12.76485	12.76049	12.76283
	0.3	2.83644	2.83637	2.83645	2.83631	2.83638
	0.4	2.24501	2.24498	2.24501	2.24495	2.24498
	0.5	2.20858	2.20849	2.20859	2.20842	2.20851
50	0.1	12.74444	12.74289	12.74460	12.74167	12.74324
	0.2	12.89102	12.88947	12.89118	12.88825	12.88982
	0.3	2.83966	2.83961	2.83966	2.83957	2.83962
	0.4	2.24671	2.24668	2.24671	2.24667	2.24669
	0.5	2.21345	2.21339	2.21345	2.21334	2.21340

Tables 3–5 depict the effects of the outer radius-to-thickness ratio r_2/T , the inner-to-outer radius ratio r_1/r_2 , and the GPLs weight fraction W_G on the frequency of a GPLs-reinforced annular plate with an FG porous core. It can be noted that, irrespective of the graphene distribution type, the frequency gradually increases with an increasing outer radius-to-thickness ratio r_2/T and a decreasing inner-to-outer radius ratio r_1/r_2 . On the other hand, the frequency increases as the graphene weight fraction W_G increases because the plate becomes stiffer by increasing the graphene component.

Table 3. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of inner-to-outer radius ratio r_1/r_2 , outer radius-to-thickness ratio r_2/T , and weight fraction of GPLs W_G (type A) ($T_c/T = 0.2$).

W_G	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	3.45840	6.91671	10.37520	12.50054	12.63196
	0.2	3.16457	6.32914	9.49371	12.63682	12.77847
	0.3	2.64327	2.78527	2.80948	2.81768	2.82143
	0.4	2.14791	2.21516	2.22724	2.23144	2.23337
	0.5	2.02688	2.15425	2.18558	2.19721	2.20272
0.3	0.1	5.25929	10.51858	15.77788	19.06438	19.27266
	0.2	4.83757	9.67515	14.51272	19.16868	19.38888
	0.3	3.78301	4.05868	4.11357	4.13300	4.14203
	0.4	2.98221	3.11984	3.14735	3.15713	3.16167
	0.5	2.57020	2.64910	2.66782	2.67471	2.67811
0.5	0.1	6.46262	12.92523	19.38785	23.54438	23.79938
	0.2	5.96345	11.92689	17.89034	23.64136	23.90945
	0.3	4.58008	4.92612	4.99636	5.02136	5.03301
	0.4	3.56945	3.74259	3.77768	3.79011	3.79603
	0.5	3.00713	3.08113	3.09749	3.10348	3.10630

Table 4. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of inner-to-outer radius ratio r_1/r_2 , outer radius-to-thickness ratio r_2/T , and weight fraction of GPLs W_G (type B) ($T_c/T = 0.2$).

W_G	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	2.93543	5.87086	8.80621	11.41827	11.57085
	0.2	2.68318	5.36635	8.04953	10.73270	11.72338
	0.3	2.40694	2.56998	2.59556	2.60392	2.60768
	0.4	1.99251	2.06498	2.07729	2.08141	2.08342
	0.5	1.91637	2.06641	2.10484	2.11912	2.12589
0.3	0.1	4.13848	8.27695	12.41543	16.39444	16.65699
	0.2	3.79834	7.59667	11.39501	15.19334	16.77831
	0.3	3.19026	3.53593	3.60621	3.63108	3.64263
	0.4	2.58143	2.75033	2.78485	2.79713	2.80286
	0.5	2.29262	2.39291	2.42024	2.43087	2.43591

Table 4. *Cont.*

W_G	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.5	0.1	4.87654	9.75307	14.62961	19.47708	19.80334
	0.2	4.48737	8.97474	13.46211	17.94948	19.91742
	0.3	3.68636	4.12317	4.21542	4.24846	4.26387
	0.4	2.95577	3.17110	3.21653	3.23284	3.24046
	0.5	2.57556	2.66715	2.68980	2.69841	2.70252

Table 5. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of inner-to-outer radius ratio r_1/r_2 , outer radius-to-thickness ratio r_2/T , and weight fraction of GPLs W_G (type C) ($T_c/T = 0.2$).

W_G	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	2.93543	5.87086	8.80621	9.65646	9.73211
	0.2	2.68318	5.36635	8.04953	9.83057	9.91635
	0.3	2.13381	2.14971	2.14907	2.14859	2.14832
	0.4	1.77801	1.79123	1.79341	1.79416	1.79451
	0.5	1.89657	1.99483	2.01435	2.02127	2.02441
0.3	0.1	4.13848	8.27695	12.41543	13.07747	13.17954
	0.2	3.79834	7.59667	11.39501	13.21485	13.32873
	0.3	2.80084	2.90316	2.92066	2.92662	2.92935
	0.4	2.25143	2.30241	2.31166	2.31488	2.31637
	0.5	2.12881	2.24002	2.26534	2.27458	2.27893
0.5	0.1	4.87654	9.75307	14.62961	15.25590	15.37391
	0.2	4.48737	8.97474	13.46211	15.37919	15.51052
	0.3	3.20118	3.33752	3.36254	3.37123	3.37524
	0.4	2.54227	2.61137	2.62437	2.62893	2.63105
	0.5	2.29633	2.39271	2.41537	2.42374	2.42761

Tables 6–8 depict the effects of the porosity factor β_0 on the frequency of a GPLs-reinforced circular/annular plate with an FG porous core for various values of the outer radius-to-thickness ratio r_2/T and the inner-to-outer radius ratio r_1/r_2 . It is found that, regardless of the variations of the ratios r_2/T and r_1/r_2 , the frequency gradually increases with an increasing porosity factor β_0 since the plate becomes lighter. It is also noted that the difference between the results of the three porosity types is slight.

The influence of the magnetic field parameter $\bar{M}_R$ on the frequency of a GPLs-reinforced annular plate with or without an FG porous core is depicted in Tables 9–11 for different values of the outer radius-to-thickness ratio r_2/T and the inner-to-outer radius ratio r_1/r_2 . It can be seen that the frequency monotonically increases with an increase in the magnetic field parameter $\bar{M}_R$. One can conduct that the presence of the magnetic field enhances the plate strength.

Tables 12–14 display the effects of the Winkler g_1 and shear elastic substrate g_2 stiffnesses on the frequency of a GPLs-reinforced annular plate with an FG porous core for various values of the outer radius-to-thickness ratio r_2/T and the inner-to-outer radius ratio r_1/r_2 . As expected, the presence of the elastic foundation enhances the plate stiffness, so the frequency gradually increases with an increase in the elastic substrate parameters g_1 and g_2 .

To explain the effects of various parameters on the frequency of a GPLs-reinforced (type A) annular plate with a porous core (porous-I) in graphical form, Figures 4–8 are presented. As shown in Figure 4, the eigenfrequency Ω of the GPLs-reinforced annular plate with a porous core increases linearly when $r_2/T \in [5, 34]$. After that, it increases slowly as the outer radius-to-thickness ratio r_2/T increases. As expected and as mentioned above in Tables 3–5, a noticeable increment in the frequency Ω occurs with an increasing in the GPLs weight fraction W_G .

Table 6. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the inner-to-outer radius ratio r_1/r_2 and the outer radius-to-thickness ratio r_2/T (Porous-I) ($T_c/T = 0.6$).

β_0	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	2.72497	5.44993	8.17490	10.89987	11.26156
	0.2	2.49151	4.98303	7.47454	9.96605	11.41602
	0.3	2.23661	2.50761	2.53736	2.54687	2.55101
	0.4	1.93981	2.02395	2.03781	2.04249	2.04462
	0.5	1.84099	2.03665	2.08098	2.09765	2.10556
0.3	0.1	2.73089	5.46178	8.19267	10.92356	11.45363
	0.2	2.49809	4.99618	7.49428	9.99237	11.61191
	0.3	2.24354	2.55254	2.58533	2.59572	2.60032
	0.4	1.97081	2.06336	2.07847	2.08354	2.08585
	0.5	1.84745	2.07738	2.12569	2.14393	2.15261
0.6	0.1	2.75916	5.51833	8.27749	11.03665	11.84989
	0.2	2.52529	5.05059	7.57588	10.10118	12.01566
	0.3	2.26905	2.64491	2.68498	2.69748	2.70296
	0.4	2.03245	2.14456	2.16263	2.16863	2.17135
	0.5	1.86941	2.15875	2.21583	2.23765	2.24806

Table 7. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the inner-to-outer radius ratio r_1/r_2 and the outer radius-to-thickness ratio r_2/T (porous-II) ($T_c/T = 0.6$).

β_0	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	2.72531	5.45062	8.17592	10.90123	11.27346
	0.2	2.49184	4.98369	7.47553	9.96737	11.42781
	0.3	2.23701	2.51017	2.54036	2.55002	2.55430
	0.4	1.94074	2.02598	2.04003	2.04477	2.04693
	0.5	1.84125	2.03739	2.08208	2.09889	2.10688
0.3	0.1	2.73431	5.46879	8.20318	10.93758	11.49275
	0.2	2.50127	5.00254	7.50380	10.00507	11.65068
	0.3	2.24637	2.56091	2.59522	2.60610	2.61092
	0.4	1.97382	2.07005	2.08581	2.09101	2.09351
	0.5	1.84975	2.07992	2.12939	2.14814	2.15707
0.6	0.1	2.77761	5.55521	8.33282	11.11042	11.94643
	0.2	2.54181	5.08371	7.62569	10.16759	12.11144
	0.3	2.28374	2.66534	2.70911	2.72280	2.72881
	0.4	2.04035	2.16095	2.18060	2.18713	2.19009
	0.5	1.88132	2.16586	2.22549	2.24845	2.25944

Table 8. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the inner-to-outer radius ratio r_1/r_2 and the outer radius-to-thickness ratio r_2/T (porous-III) ($T_c/T = 0.6$).

β_0	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.1	0.1	2.72479	5.44957	8.17436	10.89915	11.21445
	0.2	2.49104	4.98207	7.47311	9.96414	11.36829
	0.3	2.23602	2.49678	2.52556	2.53478	2.53888
	0.4	1.93307	2.01465	2.02810	2.03264	2.03472
	0.5	1.84021	2.02824	2.07149	2.08772	2.09542
0.3	0.1	2.72891	5.45783	8.18674	10.91566	11.29909
	0.2	2.49541	4.99081	7.48622	9.98163	11.45526
	0.3	2.24042	2.51690	2.54624	2.55559	2.55974
	0.4	1.94907	2.03251	2.04618	2.05079	2.05289
	0.5	1.84426	2.04960	2.09410	2.11081	2.11873

Table 8. Cont.

β_0	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0.6	0.1	2.74774	5.49548	8.24322	10.99096	11.48015
	0.2	2.51333	5.02666	7.53999	10.05332	11.64057
	0.3	2.25708	2.55922	2.58967	2.59930	2.60356
	0.4	1.98222	2.06927	2.08339	2.08813	2.09029
	0.5	1.85847	2.09156	2.13834	2.15591	2.16426

Table 9. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the magnetic field parameter $\bar{M}_R$, the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($T_c/T = 0$).

$\bar{M}_R$	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0	0.1	3.78511	7.57022	11.35533	12.32216	12.42809
	0.2	3.46402	6.92804	10.39205	12.34207	12.45619
	0.3	2.34392	2.48982	2.51952	2.53012	2.53507
	0.4	1.75904	1.83196	1.84672	1.85199	1.85444
	0.5	1.39447	1.41808	1.42271	1.42448	1.42526
100	0.1	3.78511	7.57022	11.35533	13.10209	13.20503
	0.2	3.46402	6.92804	10.39205	13.34898	13.46559
	0.3	2.88701	2.89661	2.89333	2.89184	2.89101
	0.4	2.42077	2.43442	2.43667	2.43746	2.43782
	0.5	2.56396	2.74667	2.77239	2.78151	2.78575
200	0.1	3.78511	7.57022	11.35533	13.84635	13.94750
	0.2	3.46402	6.92804	10.39205	13.85607	14.42084
	0.3	2.79490	2.69504	2.67677	2.67045	2.66755
	0.4	2.73538	2.74577	2.74929	2.75065	2.75121
	0.5	2.56396	3.39930	3.41867	3.42558	3.42880

Table 10. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the magnetic field parameter $\bar{M}_R$, the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($T_c/T = 3$).

$\bar{M}_R$	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0	0.1	3.29501	6.59002	9.88503	11.97895	12.13395
	0.2	3.01502	6.03004	9.04506	12.00375	12.16475
	0.3	2.21325	2.42378	2.46854	2.48467	2.49221
	0.4	1.68716	1.79144	1.81341	1.82132	1.82502
	0.5	1.36481	1.39534	1.40156	1.40379	1.40483
100	0.1	3.29501	6.59002	9.88503	12.77211	12.92070
	0.2	3.01502	6.03004	9.04506	12.06008	13.18007
	0.3	2.70851	2.84856	2.84548	2.84349	2.84241
	0.4	2.37977	2.40221	2.40479	2.40565	2.40604
	0.5	2.23097	2.70291	2.73938	2.75235	2.75840
200	0.1	3.29501	6.59002	9.88503	13.18003	13.67082
	0.2	3.01502	6.03004	9.04506	12.06008	14.13854
	0.3	2.70851	2.66906	2.64006	2.62994	2.62527
	0.4	2.44472	2.71928	2.72263	2.72406	2.72476
	0.5	2.23097	3.35626	3.38256	3.39199	3.39641

The influences of the porosity factor β_0 , Winkler g_1 and shear elastic substrate g_2 stiffnesses, magnetic field parameter $\bar{M}_R$, and core thickness ratio T_c/T on the eigenfrequency Ω of a GPLs-reinforced annular plate with a porous core are illustrated in Figures 5–8, respectively. It can be noted that the eigenfrequency Ω increases with increasing β_0 , g_1 , g_2 , and $\bar{M}_R$. However, it decreases as the ratio T_c/T increases.

Table 11. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the magnetic field parameter $\bar{M}_R$, the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($T_c/T = 0.6$).

$\bar{M}_R$	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
0	0.1	2.73089	5.46178	8.19267	10.82260	11.02036
	0.2	2.49809	4.99618	7.49428	9.99237	11.05634
	0.3	1.94258	2.20352	2.26182	2.28306	2.29303
	0.4	1.51377	1.64233	1.67076	1.68111	1.68597
	0.5	1.25866	1.29468	1.30224	1.30499	1.30627
100	0.1	2.73089	5.46178	8.19267	10.92356	11.87347
	0.2	2.49809	4.99618	7.49428	9.99237	12.14720
	0.3	2.24354	2.59419	2.57964	2.57274	2.56925
	0.4	2.02467	2.25650	2.25682	2.25693	2.25698
	0.5	1.84745	2.60073	2.64365	2.65898	2.66613
200	0.1	2.73089	5.46178	8.19267	10.92356	12.67992
	0.2	2.49809	4.99618	7.49428	3.55904	3.45135
	0.3	2.24354	2.46485	2.42912	2.41691	2.41148
	0.4	2.02467	2.62179	2.62684	2.62906	2.63016
	0.5	1.84745	3.24021	3.27054	3.28156	3.28675

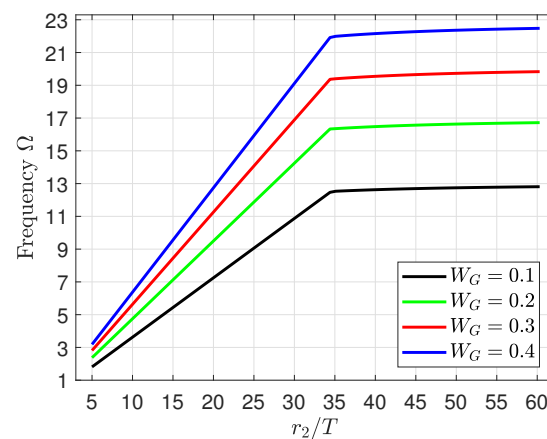


Figure 4. The frequency Ω of a GPLs-reinforced annular plate with a porous core vs. the outer radius-to-thickness ratio r_2/T for different values of the graphene weight fraction W_G .

Table 12. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the shear elastic substrate stiffness g_2 , the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($g_1 = 0$).

g_2	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
20	0.1	3.45840	6.91671	10.37520	12.57104	12.70205
	0.2	3.16457	6.32914	9.49371	12.65827	12.87656
	0.3	2.67273	2.79571	2.81476	2.82099	2.82371
	0.4	2.17469	2.23203	2.24195	2.24536	2.24693
	0.5	2.11098	2.26067	2.29559	2.30835	2.31436
40	0.1	3.45840	6.91671	10.37520	12.71755	12.84778
	0.2	3.16457	6.32914	9.49371	12.65827	13.07717
	0.3	2.74371	2.81842	2.82535	2.82719	2.82794
	0.4	2.25592	2.29301	2.29907	2.30112	2.30206
	0.5	2.31699	2.48732	2.52325	2.53611	2.54211
60	0.1	3.45840	6.91671	10.37520	12.86270	12.99219
	0.2	3.16457	6.32914	9.49371	12.65827	13.27517
	0.3	2.78841	2.80649	2.80131	2.79891	2.79770
	0.4	2.32685	2.34765	2.35101	2.35219	2.35274
	0.5	2.34169	2.67632	2.71006	2.72202	2.72759

Table 13. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the shear elastic substrate stiffness g_2 , the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($g_1 = 200$).

g_2	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
20	0.1	3.45840	6.91671	10.37520	12.57759	12.70857
	0.2	3.16457	6.32914	9.49371	12.65827	12.88296
	0.3	2.70335	2.82538	2.84418	2.85036	2.85314
	0.4	2.21218	2.26893	2.27875	2.28213	2.28369
	0.5	2.14844	2.29674	2.33137	2.34404	2.34991
40	0.1	3.45840	6.91671	10.37520	12.72404	12.85422
	0.2	3.16457	6.32914	9.49371	12.65827	13.08347
	0.3	2.77378	2.84785	2.85461	2.85652	2.85726
	0.4	2.29218	2.32905	2.33498	2.33702	2.33795
	0.5	2.34169	2.52018	2.55587	2.56864	2.57461
60	0.1	3.45840	6.91671	10.37520	12.86911	12.99856
	0.2	3.16457	6.32914	9.49371	12.65827	13.28139
	0.3	2.81827	2.83610	2.83094	2.82854	2.82734
	0.4	2.36208	2.38278	2.38614	2.38732	2.38787
	0.5	2.34169	2.70693	2.74048	2.75237	2.75791

Table 14. The frequency Ω of a GPLs-reinforced annular plate with an FG porous core for different values of the shear elastic substrate stiffness g_2 , the inner-to-outer radius ratio r_1/r_2 , and the outer radius-to-thickness ratio r_2/T ($g_1 = 400$).

g_2	r_1/r_2	$r_2/T = 10$	$r_2/T = 20$	$r_2/T = 30$	$r_2/T = 40$	$r_2/T = 50$
20	0.1	3.45840	6.91671	10.37520	12.58415	12.71508
	0.2	3.16457	6.32914	9.49371	12.65827	12.88936
	0.3	2.73362	2.85465	2.87330	2.87943	2.88219
	0.4	2.24905	2.30524	2.31497	2.31832	2.31986
	0.5	2.18526	2.33225	2.36662	2.37919	2.38511
40	0.1	3.45840	6.91671	10.37520	12.73052	12.86065
	0.2	3.16457	6.32914	9.49371	12.65827	13.08978
	0.3	2.80352	2.87697	2.88375	2.88554	2.88627
	0.4	2.32787	2.36445	2.37035	2.37237	2.37321
	0.5	2.34169	2.55261	2.58808	2.60077	2.60661
60	0.1	3.45840	6.91671	10.37520	12.87552	13.00492
	0.2	3.16457	6.32914	9.49371	12.65827	13.28760
	0.3	2.84289	2.86541	2.86026	2.85787	2.85667
	0.4	2.39671	2.41740	2.42076	2.42194	2.42249
	0.5	2.34169	2.73719	2.77056	2.78239	2.78781

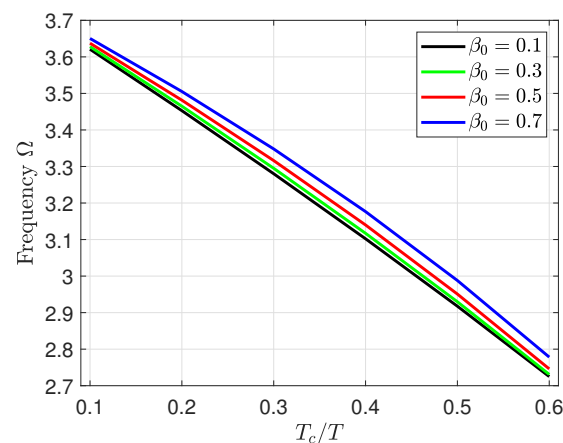


Figure 5. The frequency Ω of a GPLs-reinforced annular plate with a porous core vs. the core thickness ratio T_c/T for various values of the porosity factor β_0 .

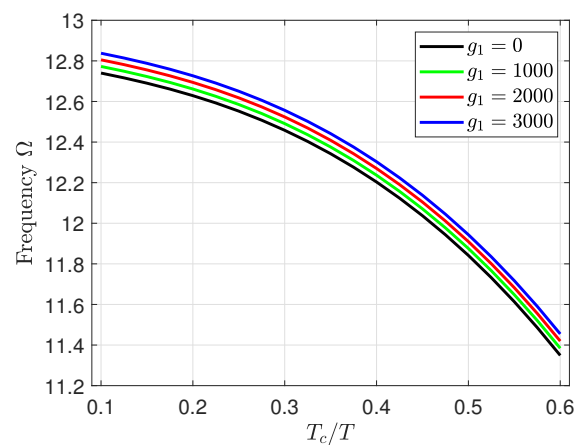


Figure 6. Effect of Winkler elastic substrate stiffness g_1 on the frequency Ω of a GPLs-reinforced annular plate with a porous core ($r_2/T = 50$).

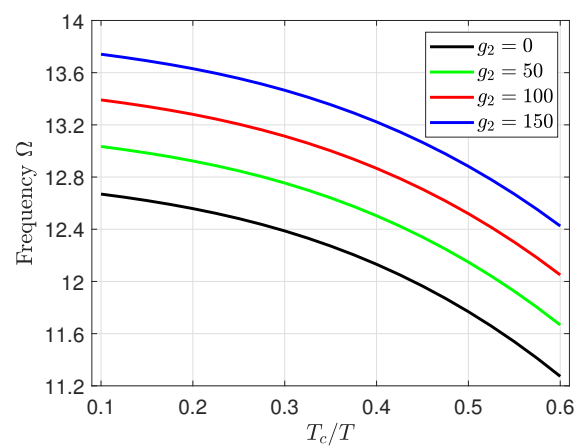


Figure 7. Effect of the shear elastic substrate stiffness g_2 on the frequency Ω of a GPLs-reinforced annular plate with a porous core ($r_2/T = 50$).

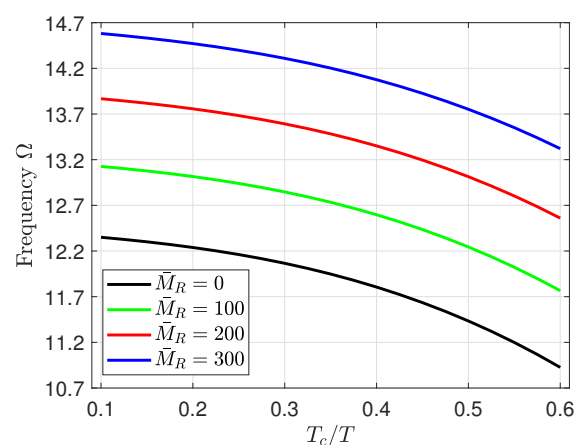


Figure 8. Effect of the magnetic field parameter $\bar{M}_R$ on the frequency Ω of a GPLs-reinforced annular plate with a porous core ($r_2/T = 50$).

6. Conclusions

The current article discusses the vibrational behavior of FG-GPLs-strengthened sandwich annular plates with an FG porous core lying on an elastic medium and exposed to a radial magnetic field. Based on three graphene distribution types, the volume fraction of

the GPLs is varied through the thickness. Further, the porosities are graded across the core thickness according to three patterns of distribution. The Lorentz magnetic body force is deduced via Maxwell's relations. The displacements are modeled based on a novel shear deformation theory, while the motion equations are depicted from Hamilton's principle. The DQM is employed to convert the motion equations into a linear algebraic system that can be solved to obtain the minimum eigenfrequency. Convergence analysis is done to find the minimum mesh points that are needed to obtain reliable findings. The results of the current theory are in a good agreement with those obtained by other theories. Moreover, numerous parametric examples have been carried out to show the impacts of different parameters on the eigenfrequency of the composite annular plate. Accordingly, the ensuing items can be concluded:

- The presence of the elastic foundation and the magnetic field boosts the strength of the plate, so the eigenfrequency increases by increasing the elastic foundation parameters and the magnetic field parameter.
- A noticeable decrement in the eigenfrequency occurs as the inner-to-outer radius ratio increases and the outer radius-to-thickness ratio decreases.
- It is known that the addition of graphene components raises the plate stiffness, so the eigenfrequency grows as the GPLs weight fraction increases.
- Since increasing the porosities in the core layer makes the plate lighter, increasing the porosity factor leads to an increase in frequency.

Author Contributions: Conceptualization, F.H.H.A.M., A.H.K.A.-A., and M.S.; methodology, F.H.H.A.M., A.H.K.A.-A., and M.S.; software, M.S.; validation, F.H.H.A.M. and M.S.; formal analysis, F.H.H.A.M., A.H.K.A.-A., and M.S.; investigation, F.H.H.A.M., A.H.K.A.-A., and M.S.; resources, F.H.H.A.M., A.H.K.A.-A., and M.S.; data curation, F.H.H.A.M., A.H.K.A.-A., and M.S.; writing—original draft preparation, F.H.H.A.M., A.H.K.A.-A., and M.S.; writing—review and editing, F.H.H.A.M., A.H.K.A.-A., and M.S.; visualization, F.H.H.A.M., A.H.K.A.-A., and M.S.; supervision, M.S.; project administration, F.H.H.A.M., A.H.K.A.-A., and M.S.; funding acquisition, F.H.H.A.M., A.H.K.A.-A., and M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (grant No. KFU241076).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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