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Existence Results for Tempered-Hilfer Fractional Differential Problems on Hölder Spaces

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Abstract: This paper considers a nonlinear fractional-order boundary value problem ${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu}x(t) + f(t, x(t), {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu}x(t)) = 0$, for $t \in [a, b]$, $\alpha_1 \in (1, 2]$, $\alpha_2 \in (0, 1]$, $\beta \in [0, 1]$ with appropriate integral boundary conditions on the Hölder spaces. Here, f is a real-valued function that satisfies the Hölder condition, and ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ represents the tempered-Hilfer fractional derivative of order $\alpha > 0$ with parameter $\mu \in \mathbb{R}^+$ and type $\beta \in [0, 1]$. The corresponding integral problem is introduced in the study of this issue. This paper addresses a fundamental issue in the field, namely the circumstances under which differential and integral problems are equivalent. This approach enables the study of differential problems using integral operators. In order to achieve this, tempered fractional calculus and the equivalence problem of the studied problems are introduced and studied. The selection of an appropriate function space is of fundamental importance. This paper investigates the applicability of these operators on Hölder spaces and provides a comprehensive rationale for this choice.

Keywords: fractional operator; tempered-Hilfer fractional differential and integral problems; Hölder spaces

MSC: 26A33; 26A16; 26B35; 34A08



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1. Introduction

The primary objective of this paper is to provide an effective approach to solving a variety of nonlinear boundary problems with fractional orders. This is accomplished by transforming them into equivalent integral problems and subsequently finding their solutions through the application of the fixed-point method to the constructed class of tempered fractional-order integral operators. In this paper, we examine a method based on the study of a general class of fractional-order integral operators. We demonstrate the equivalence of differential and integral problems in a specific class of function spaces under certain assumptions about functions. Furthermore, we present counterexamples that illustrate the limitations of this approach and the potential for asymmetry in such problems.

The objective of this study is to examine the following nonlinear fractional-order boundary value problem

$${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu}x(t) + f(t, x(t), {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu}x(t)) = 0, \quad t \in [a, b], \quad \alpha_1 \in (1, 2], \quad \alpha_2 \in (0, 1], \quad \beta \in [0, 1],$$

which is combined with appropriate integral boundary conditions, both local and nonlocal (for instance, see [1,2]).

Due to the lack of invertibility of differential and integral operators [3], this method currently seems to be the most efficient. Nevertheless, achieving this goal would require a

thorough reconstruction of tempered fractional calculus for operators that act in various function spaces (cf. [4,5]). This study commences with an examination of function spaces consisting of absolutely continuous functions, followed by an investigation into the significance of studying such operators on Hölder spaces. The theory is further developed to address previous limitations, thereby enabling the study to deal with not only the proposed initial/boundary value problems but also other fractional-order problems.

The structure of the paper is as follows: We will begin by examining fractional-order integral operators in both the $C[a, b]$ and $AC[a, b]$ spaces to adequately study initial/boundary value problems. We will then demonstrate the significance of their study in Hölder spaces. The properties of these operators will be demonstrated when they are used in Hölder spaces, with an expansion upon the current known results. The necessity of studying in Hölder spaces will be demonstrated through the use of appropriate examples or counterexamples. It is crucial to recognize that studying in Hölder spaces confers additional properties upon the solutions, thereby providing a foundation for the qualitative theory of such equations. In particular, we have demonstrated the validity of a Gronwall-type lemma for tempered integrals of fractional order (cf. [6]).

The following section of the paper is dedicated to defining fractional-order derivatives in a manner that permits the acquisition of differential and integral operators that are, to some extent, mutually inverse (cf. [3,4]). This study examines the problem by demonstrating the properties of derivatives in different function spaces. It is of paramount importance to exercise great care in selecting the domains of integral operators. Consequently, in the subsequent section, we demonstrate the equivalence between differential and integral problems, which serves as the foundation for our research on boundary problems. This section also builds upon previous papers by other authors, where this problem was either overlooked or treated as obvious, despite the assumptions made about functions and function spaces.

Finally, the concluding section of the paper addresses the investigation of boundary problems for nonlinear fractional-order equations. The equivalent integral problem method is employed, building upon previously obtained results, to streamline the proof and serve as a model for future studies. Furthermore, we illustrate how to extend the scope of previously considered problems. An additional objective is to illustrate the applicability of various fixed-point theorems, adapted to the assumptions of the question under study, which simplifies the proofs and allows for the referencing of previous results of a similar nature. The paper places particular emphasis on the role of Hölder spaces and suggests a number of open problems for future research.

2. Preliminaries

For the sake of clarity, we shall designate the set of real numbers by the symbol $\mathbb{R}$, the set of positive real numbers by $\mathbb{R}^+$, and the set of natural numbers by $\mathbb{N}$. For $\gamma \in (0, 1)$, the Hölder space $\mathbb{H}^\gamma[a, b]$ consists of all functions $f : [a, b] \rightarrow \mathbb{R}$ such that, for $t, s \in [a, b]$, there exists a constant $M > 0$ (independent on t, s) for which $|f(t) - f(s)| \leq M|t - s|^\gamma$. Furthermore, if $f(a) = 0$, we write $f \in \mathbb{H}_0^\gamma[a, b]$. When equipped with the following norm, it becomes a Banach space [7]:

$$\|x\|_\gamma = \|x\|_\infty + \omega_\gamma(x) = \|x\|_\infty + \sup_{t \neq s} \left\{ \frac{|x(t) - x(s)|}{|t - s|^\gamma} \right\}.$$

It is essential to recognize the interconnection between the Hölder space and nowhere differentiable functions. It is possible to find functions that are Hölderian but are not absolutely continuous, such as the Weierstrass function. Conversely, there are also functions that are absolutely continuous but not Hölderian, for example, the function $f : [0, 1] \rightarrow \mathbb{R}$, defined by $f(t) := t^\alpha$, $0 < \alpha < \gamma < 1$, is absolutely continuous on $[0, 1]$, but $f \notin \mathbb{H}^\gamma[0, 1]$. However, it is not difficult to show, for $0 < \gamma_1 < \gamma_2 < 1$, that

$$C^1[a, b] \subset \mathbb{H}^1[a, b] \subset \mathbb{H}^{\gamma_2}[a, b] \subset \mathbb{H}^{\gamma_1}[a, b] \subset C[a, b].$$

There are numerous modifications and generalizations of the classical fractional integration operators that are widely used in both theoretical and applied contexts. The following definition unifies various fractional integrals for integrable functions, thus allowing for the solution of initial and/or boundary value problems with different types of fractional integrals and derivatives in a unified manner.

Definition 1 (tempered-Riemann–Liouville fractional integral). Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a, b]$.

The tempered-Riemann–Liouville g -fractional integral of a given function $f \in L_1[a, b]$ of order $\alpha > 0$ and with parameter $\mu \in \mathbb{R}^+$ is defined by

$$\mathfrak{S}_{a,g}^{\alpha,\mu} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} f(s) g'(s) ds, \quad (-\infty \leq a < b \leq \infty). \quad (1)$$

For completeness, we define $\mathfrak{S}_{a,g}^{\alpha,\mu} f(a) := 0$. Similarly, we define

$$\mathfrak{I}_{a,g}^{\alpha,\mu} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (g(s) - g(t))^{\alpha-1} e^{-\mu(g(s)-g(t))} f(s) g'(s) ds, \quad (-\infty \leq a < b \leq \infty).$$

It is clear that

$$\mathfrak{S}_{a,g}^{\alpha,0} f(t) = \mathfrak{S}_{a,g}^{\alpha} f(t) \text{ and } \mathfrak{S}_{a,g}^{\alpha,\mu} f(t) = e^{-\mu g(t)} \mathfrak{S}_{a,g}^{\alpha,0} (e^{\mu g(t)} f(t)),$$

where $\mathfrak{S}_{a,g}^{\alpha}$ denotes the standard Riemann–Liouville fractional integral of order $\alpha > 0$ defined by

$$\mathfrak{S}_{a,g}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} f(s) g'(s) ds, \quad \mathfrak{S}_{a,g}^0 f := f.$$

For the sake of simplicity and without loss of generality, we will assume that $g(a) = 0$ in the following pages. According to the previously established definition, namely that $\mathfrak{S}_{a,g}^0 f := f$, the definition can be completed by setting $\mathfrak{S}_{a,g}^{0,\mu} f = f$.

In accordance with the methodology outlined in (Lemma 3.1, [6]) and (Theorem 4.4, [8]), with the help of $\mathfrak{S}_{a,g}^{\alpha,\mu} f(t) = e^{-\mu g(t)} \mathfrak{S}_{a,g}^{\alpha,0} (e^{\mu g(t)} f(t))$, we can demonstrate the following lemma:

Lemma 1. Suppose $\alpha > 0$, $\mu \in \mathbb{R}^+$ and let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a, b]$. Then,

1. $\mathfrak{S}_{a,g}^{\alpha,\mu} : L_p[a, b] \rightarrow L_p[a, b]$ is a bounded operator for every $p \in [1, \infty]$. In particular, if $p > \max\{(1/\alpha), 1\}$, then $\mathfrak{S}_{a,g}^{\alpha,\mu} : L_p[a, b] \rightarrow C[a, b]$ is compact.
2. $\mathfrak{S}_{a,g}^{\alpha,\mu} : C[a, b] \rightarrow C[a, b]$ is a bounded operator.
3. (cf. Lemma 2, [7]) $\mathfrak{S}_{a,g}^{\alpha,\mu} : \mathbb{H}_0^{\gamma}[a, b] \rightarrow \mathbb{H}_0^{\gamma+\alpha}[a, b] \subset \mathbb{H}_0^{\gamma}[a, b]$, for any $\gamma, \alpha \in (0, 1)$ with $0 < \gamma + \alpha < 1$.

Remark 1. Fractional-order Riemann–Liouville operators are naturally considered on Hölder spaces. A natural question that arises is whether for the entire family of operators $\mathfrak{S}_{a,g}^{\alpha,\mu}$ indexed by the corresponding function g the domain should be the same or not (cf. [7,9]). We will now provide a brief overview of the problem and consider generalized Hölder spaces.

Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a, b]$. For a continuous increasing function $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\vartheta(0) = 0$, $\lim_{t \rightarrow 0+} \frac{\vartheta(t)}{t} = +\infty$, we say that a function x satisfies the (generalized) Hölder condition, which is denoted by $x \in \mathbb{H}_g^{\vartheta}[a, b]$, provided that

$$|x(t) - x(s)| \leq L \vartheta(|g(t) - g(s)|), \quad L > 0, \quad x \in C[a, b].$$

Define a seminorm on this space:

$$\omega_{\vartheta}^g(x) = \sup_{t \neq s} \left\{ \frac{|x(t) - x(s)|}{\vartheta(|g(t) - g(s)|)} \right\},$$

Obviously, a norm on $\mathbb{H}_g^{\vartheta}[a, b]$ can be defined classically:

$$\|x\|_{g, \vartheta} = \|x\|_{\infty} + \omega_{\vartheta}^g(x).$$

In the case of tempered operators, we can choose $\vartheta(x) = x^{\gamma} \cdot e^{-\mu x}$.

Lemma 2. Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a, b]$. Moreover, put $\vartheta(x) = x^{\gamma} \cdot e^{-\mu x}$ for some fixed $\gamma \in (0, 1)$ and $\mu \geq 0$. Then,

$$\mathbb{H}_g^{\vartheta}[a, b] \subset \mathbb{H}^{\gamma}[a, b].$$

Proof. For the sake of simplicity, we will assume that $s < t$. Since $g(t) - g(s) = g'(\xi) \cdot (t - s)$ for some $\xi \in (s, t)$, we may estimate

$$\begin{aligned} \vartheta(|g(t) - g(s)|) &= |g(t) - g(s)|^{\gamma} \cdot e^{\mu \cdot |g(t) - g(s)|} \leq |g'(\xi)|^{\gamma} \cdot |t - s|^{\gamma} \cdot e^{\mu |g'(\xi)| \cdot |t - s|} \\ &\leq \left(\|g'\|_{\infty} \cdot e^{\mu \|g'\|_{\infty}} \right) \cdot |t - s|^{\gamma}. \end{aligned}$$

Consequently,

$$\begin{aligned} \omega_{\vartheta}^g(x) &= \sup_{t \neq s} \left\{ \frac{|x(t) - x(s)|}{\vartheta(|g(t) - g(s)|)} \right\} \geq \frac{1}{\|g'\|_{\infty} \cdot e^{\mu \|g'\|_{\infty}}} \cdot \sup_{t \neq s} \left\{ \frac{|x(t) - x(s)|}{|t - s|^{\gamma}} \right\} \\ &= \frac{1}{\|g'\|_{\infty} \cdot e^{\mu \|g'\|_{\infty}}} \cdot \omega_{\gamma}(x) \end{aligned}$$

and then

$$\|x\|_{g, \vartheta} \geq \frac{1}{\|g'\|_{\infty} \cdot e^{\mu \|g'\|_{\infty}}} \cdot \|x\|_{\gamma}.$$

In this case, we obtain $\mathbb{H}_g^{\vartheta}[a, b] \subset \mathbb{H}^{\gamma}[a, b]$. $\square$

It is therefore advisable to consider Hölder spaces that match the given function g precisely when the aim is to obtain estimates of the norm and to ascertain the uniqueness of solutions. However, as will be demonstrated in the forthcoming paper, the selection of a universal Hölder space may be sufficient if we employ compactness arguments. Nevertheless, we suggest further investigation into generalized Hölder spaces (cf. [7], for instance), which we leave as an open problem for the present [9].

Now, by reasoning as in (Exercise 86(b), page 353, [10]), we know that, if f is a bounded measurable function on $[a, b]$ such that $\lim_{t \rightarrow a} f(t)$ exists (see also (Lemma 3.1, [11]), when $f \in C[a, b]$), then

$$\lim_{t \rightarrow a} (g(t))^{-\alpha} \mathfrak{S}_{a, g}^{\alpha, 0} f(t) = \frac{1}{\Gamma(1 + \alpha)} \lim_{t \rightarrow a} f(t), \quad \alpha > 0.$$

Consequently, for all $\alpha > 0$ and $\mu \in \mathbb{R}^+$ we have

$$\lim_{t \rightarrow a} \mathfrak{S}_{a, g}^{\alpha, 0} f(t) = \lim_{t \rightarrow a} (g(t))^{\alpha} (g(t))^{-\alpha} \mathfrak{S}_{a, g}^{\alpha, 0} f(t) = 0.$$

Since

$$\left| \mathfrak{S}_{a, g}^{\alpha, 0} (e^{\mu g(t)} f(t)) - 0 \right| \leq e^{\mu g(b)} \left| \mathfrak{S}_{a, g}^{\alpha, 0} (f(t)) - 0 \right| \rightarrow 0 \text{ as } t \rightarrow a,$$

we infer

$$\lim_{t \rightarrow a} \mathfrak{S}_{a,g}^{\alpha,\mu} f(t) = \lim_{t \rightarrow a} e^{-\mu g(t)} \lim_{t \rightarrow a} \mathfrak{S}_{a,g}^{\alpha,0} \left(e^{\mu g(t)} f(t) \right) = \lim_{t \rightarrow a} \mathfrak{S}_{a,g}^{\alpha,0} \left(e^{\mu g(t)} f(t) \right) = 0. \quad (2)$$

Accordingly, our definition saying that $\mathfrak{S}_{a,g}^{\alpha,\mu} f(a) := 0$ is precise for any $f \in C[a, b]$. However, as a direct consequence of part 1 of Lemma 1, we have

Corollary 1. *If $\alpha > 0$ and $f \in L_p[a, b]$, $p > \max\{1/\alpha, 1\}$, then*

$$\mathfrak{S}_{a,g}^{\alpha,\mu} f(t) = \frac{1}{\Gamma(\alpha)} \begin{cases} \int_a^t (g(t) - g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} f(s) g'(s) ds & , t \in (a, b]. \\ 0 & , t = a. \end{cases}$$

For the sake of convenience, we may utilize the substitution $u = \frac{g(s)}{g(t)}$, to verify that

$$\mathfrak{S}_{a,g}^{\alpha,\mu} \left\{ e^{-\mu g(t)} (g(t))^{\beta-1} \right\} = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} e^{-\mu g(t)} (g(t))^{\alpha+\beta-1}, \quad \alpha, \beta > 0, t > a \quad (3)$$

(cf. also [12] for the lack of equivalence of differential and integral problems in this case). Moreover, as previously noted in [13],

$$\mathfrak{S}_{a,g}^{\alpha,\mu} \left\{ (g(t))^{\beta-1} \right\} = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} (g(t))^{\alpha+\beta-1} {}_1F_1(\alpha, \alpha + \beta; -\mu g(t)), \quad \alpha, \beta > 0, t > a, \quad (4)$$

where ${}_1F_1(\cdot, \cdot; \cdot)$ denotes the well-known confluent hypergeometric function [14,15] having the integral representation

$${}_1F_1(a, b; t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{ts} s^{a-1} (1-s)^{b-a-1} ds, \quad b > a.$$

Obviously,

$${}_1F_1(a, a+1; t) = t^{-a}, e^t {}_1F_1(a, b; -t) = {}_1F_1(b-a, b; t), \quad {}_1F_1(a, b; 0) = 1.$$

Also, for any $\lambda > 0$, by the aid of (3), we obtain

$$\begin{aligned} \mathfrak{S}_{a,g}^{\alpha,\mu} e^{(\lambda-\mu)g(t)} &= \sum_{n=0}^{\infty} \mathfrak{S}_{a,g}^{\alpha,\mu} \left(\frac{e^{-\mu g(t)} (\lambda g(t))^n}{\Gamma(1+n)} \right) = e^{-\mu g(t)} \sum_{n=0}^{\infty} \frac{\lambda^n (g(t))^{n+\alpha}}{\Gamma(1+n+\alpha)} \\ &= e^{-\mu g(t)} (g(t))^\alpha \mathcal{E}_{1,1+\alpha}(\lambda g(t)), \end{aligned} \quad (5)$$

where $\mathcal{E}_{\alpha,\beta}(\cdot)$ denotes the two-parameter Mittag-Leffler function (see e.g., [16,17]).

In a similar fashion to the proof presented in (Theorem 2.5, [18]) (see also [19]), we can demonstrate the following:

Proposition 1 (semi-group property of $\mathfrak{S}_{a,g}^{\alpha,\mu}$). *For any $\alpha, \beta > 0$, $\mu \in \mathbb{R}^+$ and a positive increasing function $g \in C^1[a, b]$, with $g'(t) \neq 0$ for any $f \in L_1[a, b]$, the following estimation*

$$\mathfrak{S}_{a,g}^{\alpha,\mu} \mathfrak{S}_{a,g}^{\beta,\mu} f = \mathfrak{S}_{a,g}^{\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} f = \mathfrak{S}_{a,g}^{\alpha+\beta,\mu} f$$

holds true.

Lemma 3 (generalized linearity of $\mathfrak{S}_{a,g}^{\alpha,\mu}$). Let (f_n) be a sequence of continuous functions $f_n : [a, b] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ (here $[a, b]$ is a finite or infinite interval). If the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[a, b]$, then, for all $\alpha, \mu > 0$, we have

$$\mathfrak{S}_{a,g}^{\alpha,\mu} \left(\sum_{n=1}^{\infty} f_n(t) \right) = \left(\sum_{n=1}^{\infty} \mathfrak{S}_{a,g}^{\alpha,\mu} f_n(t) \right). \quad (6)$$

Proof. Define $f := \sum_{n=1}^{\infty} f_n$ as a uniformly convergent series of continuous functions $f \in C[a, b]$. So, for any $\alpha > 0$ and any fixed integer $m \geq 1$ we have

$$\begin{aligned} \Delta_m^\alpha &:= \left| \mathfrak{S}_{a,g}^{\alpha,\mu} f - \sum_{n=1}^m \mathfrak{S}_{a,g}^{\alpha,\mu} f_n \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_a^t (g(t) - g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} \left(\sum_{n=1}^{\infty} f_n(s) \right) g'(s) ds \right. \\ &\quad \left. - \sum_{n=1}^m \int_a^t (g(t) - g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} f_n(s) g'(s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} \left| \sum_{n=m+1}^{\infty} f_n(s) \right| g'(s) ds \end{aligned}$$

and since the remainders of the series are uniformly bounded, $\Delta_m^\alpha \rightarrow 0$ as $m \rightarrow \infty$. We obtain the desired thesis. $\square$

Example 1. Let $\alpha \in (0, 1)$ and define the continuous $f : [a, b] \rightarrow \mathbb{R}$ by

$$f(t) := \sum_{n=0}^{\infty} i e^{-\mu g(t)} (g(t))^{1-\alpha} \mathcal{E}_{1,2-\alpha}(i b^n g(t)) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} i e^{-\mu g(t)} (g(t))^{1-\alpha} \frac{(i b^n g(t))^k}{\Gamma(k+2-\alpha)},$$

for $b > 1$. Then,

$$\begin{aligned} \mathfrak{S}_{a,g}^{\alpha,\mu} f(t) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{i^{k+1} b^{nk}}{\Gamma(k+2-\alpha)} \mathfrak{S}_{a,g}^{\alpha,\mu} \left(e^{-\mu g(t)} (g(t))^{1-\alpha+k} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{i^{k+1} b^{nk}}{\Gamma(2+k)} e^{-\mu g(t)} (g(t))^{1+k} \\ &= \sum_{n=0}^{\infty} i e^{-\mu g(t)} (g(t)) \mathcal{E}_{1,2}(i b^n g(t)) \\ &= \sum_{n=0}^{\infty} \frac{1}{b^n} e^{-\mu g(t)} \left[e^{i b^n g(t)} - 1 \right] = e^{-\mu g(t)} [\mathcal{W}((g(t))) - \mathcal{W}(0)], \end{aligned}$$

where $\mathcal{W}(\cdot)$ denotes the classical Weierstrass singular function. Since it is continuous (see, e.g., [20]), we conclude that $\mathfrak{S}_{a,g}^{\alpha,\mu} f \in C[a, b]$ as expected from Lemma 1.

Example 2. Let $\alpha \in (0, 1)$ and define the continuous $f : [a, b] \rightarrow \mathbb{R}$ by

$$f(t) := e^{-\mu g(t)} [\mathcal{W}((g(t))) - \mathcal{W}(0)] = \sum_{n=0}^{\infty} \frac{1}{b^n} e^{-\mu g(t)} \left[e^{i b^n g(t)} - 1 \right], \quad b > 1.$$

Then,

$$\begin{aligned}\mathfrak{S}_{a,g}^{\alpha,\mu} f(t) &= \sum_{n=0}^{\infty} \frac{1}{b^n} \mathfrak{S}_{a,g}^{\alpha,\mu} \left(e^{(ib^n - \mu)g(t)} - e^{-\mu g(t)} \right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\mu g(t)} (g(t))^\alpha}{b^n} \left(\mathcal{E}_{1,1+\alpha}(ib^n g(t)) - \frac{1}{\Gamma(1+\alpha)} \right) \\ &= \sum_{n=0}^{\infty} ie^{-\mu g(t)} (g(t))^{\alpha+1} \mathcal{E}_{1,2+\alpha}(ib^n g(t)).\end{aligned}$$

In the future, it would be beneficial to define the symbol $\mathfrak{d} := \frac{1}{g'} \frac{d}{dt} + \mu$. Interestingly,

$$\mathfrak{d}f(t) = e^{-\mu g(t)} \frac{1}{g'} \frac{d}{dt} \left(e^{\mu g(t)} f(t) \right), \text{ or } \frac{1}{g'(t)} \frac{d}{dt} \left(f(t) e^{\mu g(t)} \right) = e^{\mu g(t)} \mathfrak{d}f(t), \quad (7)$$

where

$$\mathfrak{d}\mathfrak{S}_{a,g}^{1,\mu} x(t) = \begin{cases} \left(\frac{1}{g'(t)} \frac{d}{dt} + \mu \right) \int_a^t e^{-\mu(g(t)-g(s))} x(s) g'(s) ds = x, \\ \quad \forall t \in [a, b] \text{ holds for any } x \in C[a, b], \\ \left(\frac{1}{g'(t)} \frac{d}{dt} + \mu \right) \int_a^t e^{-\mu(g(t)-g(s))} x(s) g'(s) ds = x, \\ \quad a.e. t \in [a, b] \text{ holds for any } x \in L_1[a, b]. \end{cases}$$

Also, if $x \in AC[a, b]$ and $\mathfrak{d}x = y$, then

$$\frac{dx(t)}{dt} + \mu g'(t)x(t) = g'(t)y(t).$$

Hence,

$$x(t) = \mathfrak{S}_{a,g}^{1,\mu} y(t) + x(a)e^{-\mu g(t)}, \text{ or } \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{d}x(t) = x(t) - x(a)e^{-\mu g(t)}. \quad (8)$$

This paper will examine the distinctions between fractional calculus on $AC[a, b]$ spaces and on Hölder spaces. This establishes the objective of the paper and highlights the necessity of developing the theory in Hölder spaces. We will begin with the following:

Lemma 4. For arbitrary $\alpha > 0, \mu \in \mathbb{R}^+$ and a positive increasing function $g \in C^1[a, b]$, with $g'(t) \neq 0$, the space $AC[a, b]$ is invariant for the operator $\mathfrak{S}_{a,g}^{\alpha,\mu}$:

$$\mathfrak{S}_{a,g}^{\alpha,\mu} : AC[a, b] \rightarrow AC[a, b].$$

Proof. We will consider only the most interesting case where $\alpha \in (0, 1]$. First, it should be recalled that the absolute continuity of $F(\cdot) := \exp\{\mu g(\cdot)\}f(\cdot)$ on $[a, b]$ is equivalent to the following equation: $F(t) = F(a) + \int_a^t F'(\theta) d\theta$. This is because the derivative F' exists almost everywhere on $[a, b]$. Consequently, the interchange of integration order yields

$$\begin{aligned}\mathfrak{S}_{a,g}^{\alpha,0}(F(t)) &= \mathfrak{S}_{a,g}^{\alpha,0} \left[F(a) + \int_a^t F'(\theta) d\theta \right] \\ &= \frac{(g(t))^\alpha F(a)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} \left[\int_a^t F'(\theta) d\theta \right] g'(s) ds \\ &= \frac{(g(t))^\alpha F(a)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_a^t (g(t) - g(s))^\alpha F'(s) ds.\end{aligned}$$

Noticing that, for almost every $t \in [a, b]$, we have

$$F'(t) = e^{\mu g(t)} (\mu g'(t)f(t) + f'(t)) = e^{\mu g(t)} g'(t) \mathfrak{d}f(t),$$

then, we obtain

$$\mathfrak{I}_{a,g}^{\alpha,0}(e^{\mu g(t)} f(t)) = \frac{(g(t))^\alpha f(a)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_a^t e^{\mu g(s)} (g(t) - g(s))^\alpha g'(s) \mathfrak{d}f(s) ds.$$

Therefore,

$$\begin{aligned} \mathfrak{I}_{a,g}^{\alpha,\mu} f(t) &= e^{-\mu g(t)} \mathfrak{I}_{a,g}^{\alpha,0}(e^{\mu g(t)} f(t)) \\ &= \frac{e^{-\mu g(t)} (g(t))^\alpha f(a)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_a^t e^{-\mu(g(t)-g(s))} (g(t) - g(s))^\alpha g'(s) \mathfrak{d}f(s) ds \\ &= \frac{e^{-\mu g(t)} (g(t))^\alpha f(a)}{\Gamma(1+\alpha)} + \mathfrak{I}_{a,g}^{1+\alpha,\mu} \mathfrak{d}f(t) = \frac{e^{-\mu g(t)} (g(t))^\alpha f(a)}{\Gamma(1+\alpha)} + \mathfrak{I}_{a,g}^{1,\mu} \mathfrak{I}_{a,g}^{\alpha,\mu} \mathfrak{d}f(t). \end{aligned} \quad (9)$$

Since both terms on the right-hand side of (9) are absolutely continuous on $[a, b]$, the left-hand side of (9) is also absolutely continuous on $[a, b]$. Consequently, the proof is complete. $\square$

The following example illustrates that the domain $AC[a, b]$ of the operator $\mathfrak{I}_{a,g}^{\alpha,\mu}$ in Lemma 4 can also be extended to encompass a function f with a point of discontinuity. The issue of selecting an appropriate, potentially invariant domain on which the operators can be inverted arises when attempting to enhance the properties of functions through the use of integral operators.

Example 3. Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$. Define the following function $f : [a, b] \rightarrow \mathbb{R}$ with a point of discontinuity:

$$f(t) := e^{-\mu g(t)} \begin{cases} g(t), & t \in [a, c), \\ g(c) - g(t), & t \geq c. \end{cases} \quad \text{for some } c \in (a, b). \quad (10)$$

A simple substitution $u = g(t) - g(s)$ yields

$$\mathfrak{I}_{a,g}^{\alpha,\mu} f(t) = \begin{cases} \frac{e^{-\mu g(t)}}{\Gamma(\alpha+2)} (g(t))^{\alpha+1}, & t \in [a, c), \\ \frac{e^{-\mu g(t)}}{\Gamma(\alpha+2)} \left[(g(t))^{\alpha+1} - (g(t) - g(c))^\alpha ((g(t) - g(c))(1-\alpha) + g(t)(1+\alpha)) \right], & t \geq c. \end{cases}$$

Consequently, although f is not continuous at $t = c$, the map $\mathfrak{I}_{a,g}^{\alpha,\mu} f$ is continuous at this point. Indeed, $\mathfrak{I}_{a,g}^{\alpha,\mu} f \in AC[a, b]$.

The space $AC[a, b]$ appears to be excessively large, prompting the suggestion of an alternative approach for functions from the Hölder spaces.

Lemma 5. Assume that $\lambda, \mu > 0$. Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$ for all $t \in [a, b]$. Then, the fractional integral equation

$$x(t) = f(t) - \lambda \mathfrak{I}_{a,g}^{\alpha,\mu} x(t), \quad t \in [a, b], \quad \alpha \in (0, 1) \quad (11)$$

admits a solution that is Hölder continuous, i.e., $x \in \mathbb{H}_0^\gamma[a, b]$ (resp. absolutely continuous solution $x \in AC[a, b]$), if $f \in \mathbb{H}_0^\gamma[a, b]$ with $\gamma + \alpha < 1$ (resp. $f \in AC[a, b]$). When $f \in AC[a, b]$, then, for sufficiently small λ ,

$$x(t) = f(a) \mathcal{E}_\alpha(-\lambda(g(t))^\alpha) + \int_a^t e^{-\mu(g(t)-g(s))} \mathcal{E}_\alpha(-\lambda(g(t) - g(s))^\alpha) \mathfrak{d}f(s) g'(s) ds.$$

Proof. Define $Ty := \mathfrak{S}_{a,g}^{\alpha,\mu} y$. Then,

$$\begin{aligned} x(t) &= f(t) - \lambda T x(t) = f(t) - \lambda T[f(t) - \lambda T x(t)] = f(t) - \lambda T f(t) + \lambda^2 T^2 x(t) \\ &= f(t) - \lambda T f(t) + \lambda^2 T^2[f(t) - \lambda T x(t)] = f(t) - \lambda T f(t) + \lambda^2 T^2 f(t) - \lambda^3 T^3 x(t) \\ &= \sum_{k=0}^{k=2} (-\lambda)^k T^k f(t) + (-\lambda)^3 T^3 x(t) \\ &= \dots \\ &= \sum_{k=0}^{n-1} (-\lambda)^k T^k f(t) + (-\lambda)^n T^n x(t). \end{aligned}$$

According to Proposition 1, $T^n x = \mathfrak{S}_{a,g}^{n\alpha,\mu} x$. Therefore,

$$\begin{aligned} |T^n x(t)| &\leq \frac{1}{\Gamma(n\alpha)} \int_a^t (g(t) - g(s))^{n\alpha-1} e^{-\mu(g(t)-g(s))} |x(s)| g'(s) ds \\ &\leq \frac{\max_{t \in [a,b]} |x(t)|}{\Gamma(n\alpha + 1)} (g(t))^{n\alpha}. \end{aligned}$$

Since $|g(t)^{n\alpha}| \leq |(g'(t))^\alpha|^n$, we can deduce from the Stirling formula for the Γ function [21] that

$$|T^n x(t)| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Hence, for sufficiently small λ the series $\sum_{k=0}^{\infty} (-\lambda)^k T^k f(t)$ is convergent and then

$$x(t) = \sum_{k=0}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k, \mu} f(t).$$

Now, for any $f \in \mathbb{H}_0^\gamma[a, b]$ with $0 < \gamma + \alpha < 1$, we conclude that $\left\lceil \frac{1-\gamma}{\alpha} \right\rceil \geq 1$ and for each positive integer $k < \left\lceil \frac{1-\gamma}{\alpha} \right\rceil$ we have $\gamma + \alpha k < 1$. Since

$$\begin{aligned} x(t) &= \sum_{k=0}^{k < \left\lceil \frac{1-\gamma}{\alpha} \right\rceil} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k, \mu} f(t) + \sum_{k=\left\lceil \frac{1-\gamma}{\alpha} \right\rceil}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{1, \mu} \mathfrak{S}_{a,g}^{\alpha k-1, \mu} f(t) \\ &= \sum_{k=0}^{k < \left\lceil \frac{1-\gamma}{\alpha} \right\rceil} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k, \mu} f(t) + \mathfrak{S}_{a,g}^{1, \mu} \left(\sum_{k=\left\lceil \frac{1-\gamma}{\alpha} \right\rceil}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k-1, \mu} f(t) \right). \end{aligned}$$

Therefore, since $\mathfrak{S}_{a,g}^{\alpha k-1, \mu} f \in C[a, b]$ for any $k \geq \left\lceil \frac{1-\gamma}{\alpha} \right\rceil$, and

$$\sum_{k=\left\lceil \frac{1-\gamma}{\alpha} \right\rceil}^{\infty} |(-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k-1, \mu} f(t)| \rightarrow 0 \text{ uniformly as } k \rightarrow \infty.$$

It follows

$$\sum_{k=\left\lceil \frac{1-\gamma}{\alpha} \right\rceil}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k-1, \mu} f \in C[a, b] \implies \mathfrak{S}_{a,g}^{1, \mu} \left(\sum_{k=\left\lceil \frac{1-\gamma}{\alpha} \right\rceil}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{\alpha k-1, \mu} f(t) \right) \in C_0^1[a, b].$$

In conclusion, it can be stated that

$$x(t) = \sum_{k=0}^{k < \left\lfloor \frac{1-\gamma}{\alpha} \right\rfloor} (-\lambda)^k \underbrace{\mathfrak{S}_{a,g}^{ak,\mu} f(t)}_{\in \mathbb{H}_0^{\gamma+\alpha k}[a,b]} + \underbrace{\mathfrak{S}_{a,g}^{1,\mu} \left(\sum_{k=\left\lfloor \frac{1-\gamma}{\alpha} \right\rfloor}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{ak-1,\mu} f(t) \right)}_{\in \mathbb{H}_0^{\gamma}[a,b]}.$$

Since $C_0^1 \subset \mathbb{H}_0^{\gamma_2} \subset \mathbb{H}_0^{\gamma_1}$ holds for any $0 < \gamma_1 < \gamma_2 < 1$, we conclude that $x \in \mathbb{H}_0^{\gamma}[a, b]$ as required.

Now, let $f \in AC[a, b]$. Since

$$\begin{aligned} x(t) &= \sum_{k=0}^{k \leq \left\lfloor \frac{1}{\alpha} \right\rfloor} (-\lambda)^k \mathfrak{S}_{a,g}^{ak,\mu} f(t) + \sum_{k > \left\lfloor \frac{1}{\alpha} \right\rfloor}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{S}_{a,g}^{ak-1,\mu} f(t) \\ &= \sum_{k=0}^{k \leq \left\lfloor \frac{1}{\alpha} \right\rfloor} (-\lambda)^k \mathfrak{S}_{a,g}^{ak,\mu} f(t) + \mathfrak{S}_{a,g}^{1,\mu} \left(\sum_{k > \left\lfloor \frac{1}{\alpha} \right\rfloor}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{ak-1,\mu} f(t) \right). \end{aligned}$$

In light of Lemma 4, for $f \in AC[a, b]$ the series on the right-hand side is convergent. All the functions $\mathfrak{S}_{a,g}^{ak-1,\mu} f(t)$ are a.e. differentiable, say outside the negligible set D_k , so the series is a.e. differentiable outside $\cup_{k=\left\lfloor \frac{1}{\alpha} \right\rfloor}^{\infty} D_k$ still being negligible. It follows that

$$x = \sum_{k=0}^{\infty} (-\lambda)^k \mathfrak{S}_{a,g}^{ak,\mu} f \in AC[a, b].$$

However, if $f \in AC[a, b]$, an explicit calculation using integration by parts and the assertion of Lemma 3 demonstrate that

$$\begin{aligned} x(t) &= f(t) + \sum_{k=1}^{\infty} (-\lambda)^k T^k f(t) \\ &= f(t) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{\Gamma(k\alpha)} \int_a^t \underbrace{e^{-\mu(g(t)-g(s))} f(s)}_u \underbrace{(g(t)-g(s))^{k\alpha-1} g'(s)}_{dv} ds \\ &= f(t) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{\Gamma(k\alpha)} \left(\left[-e^{-\mu(g(t)-g(s))} f(s) \frac{(g(t)-g(s))^{k\alpha}}{k\alpha} \right]_a^t \right. \\ &\quad \left. + \int_a^t \frac{(g(t)-g(s))^{k\alpha}}{k\alpha} e^{-\mu(g(t)-g(s))} \mathfrak{D}f(s) g'(s) ds \right) \\ &= f(t) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{\Gamma(k\alpha)} \left(f(a) e^{-\mu g(t)} \frac{(g(t))^{k\alpha}}{k\alpha} \right. \\ &\quad \left. + \int_a^t \frac{(g(t)-g(s))^{k\alpha}}{k\alpha} e^{-\mu(g(t)-g(s))} \mathfrak{D}f(s) g'(s) ds \right) \\ &= f(t) + f(a) e^{-\mu g(t)} \sum_{k=1}^{\infty} \frac{(-\lambda (g(t))^\alpha)^k}{\Gamma(k\alpha + 1)} \\ &\quad + \int_a^t e^{-\mu(g(t)-g(s))} \mathfrak{D}f(s) \left[\sum_{k=1}^{\infty} \frac{(-\lambda (g(t)-g(s))^\alpha)^k}{\Gamma(k\alpha + 1)} \right] g'(s) ds \\ &= f(t) + f(a) e^{-\mu g(t)} [\mathcal{E}_\alpha(-\lambda (g(t))^\alpha) - 1] \\ &\quad + \int_a^t e^{-\mu(g(t)-g(s))} [\mathcal{E}_\alpha(-\lambda (g(t)-g(s))^\alpha) - 1] \mathfrak{D}f(s) g'(s) ds. \end{aligned}$$

Since $\int_a^t e^{\mu g(s)} \mathfrak{D}f(s) g'(s) ds = f(t) e^{\mu g(t)} - f(a)$, we conclude that

$$x(t) = f(a) \mathcal{E}_\alpha(-\lambda(g(t))^\alpha) + \int_a^t e^{-\mu(g(t)-g(s))} \mathcal{E}_\alpha(-\lambda(g(t)-g(s))^\alpha) \mathfrak{D}f(s) g'(s) ds.$$

Our result can be verified by direct substitution. $\square$

A crucial aspect of studying differential and integral equations using the operators under examination is obtaining a priori estimates for the solutions. This requires an extension of the classical Gronwall lemma to the fractional-order integrals being studied.

A Gronwall-type inequality for the tempered Riemann–Liouville fractional integral is presented.

Theorem 1. Assume that $\alpha > 0$, $\mu \in \mathbb{R}^+$ and let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$ for all $t \in [a, b]$. Let C, B be positive constants. If f is a non-negative continuous function on $[a, b]$ such that

$$f(t) \leq C + B \mathfrak{I}_{a,g}^{\alpha,\mu} f(t),$$

then, for any $t \in [a, b]$, we have

$$f(t) \leq C e^{\mu g(t)} \mathcal{E}_\alpha(B).$$

Proof. Let u be a nondecreasing and non-negative function on $[a, b]$ and v be a non-negative, integrable function on $[a, b]$. According to (Theorem 2, [22]) (see also (Corollary 3.10, [23])), the following implication holds

$$\begin{aligned} e^{\mu g(t)} f(t) &\leq v(t) + u(t) \mathfrak{I}_{a,g}^{\alpha,0} (e^{\mu g(t)} f(t)) \\ \implies e^{\mu g(t)} f(t) &\leq v(t) + \int_a^t \sum_{k=1}^{\infty} (g(t) - g(s))^{\alpha k - 1} \frac{(u(t))^k}{\Gamma(\alpha k)} v(s) g'(s) ds, \end{aligned}$$

where $\mathcal{E}_\alpha(\cdot)$ denotes the Mittag-Leffler function [16–18]. Since $f(t) \leq C + B \mathfrak{I}_{a,g}^{\alpha,\mu} f(t)$, we obtain

$$e^{\mu g(t)} f(t) \leq C e^{\mu g(t)} + B e^{\mu g(t)} \mathfrak{I}_{a,g}^{\alpha,\mu} f(t) = C e^{\mu g(t)} + B \mathfrak{I}_{a,g}^{\alpha,0} (e^{\mu g(t)} f(t)).$$

And, thus, it can be concluded that

$$\begin{aligned} e^{\mu g(t)} f(t) &\leq C e^{\mu g(t)} + \int_a^t \sum_{k=1}^{\infty} (g(t) - g(s))^{\alpha k - 1} \frac{B^k}{\Gamma(\alpha k)} C e^{\mu g(s)} g'(s) ds \\ &= C e^{\mu g(t)} + C \sum_{k=1}^{\infty} B^k \mathfrak{I}_{a,g}^{\alpha k,0} (e^{\mu g(t)}). \end{aligned}$$

This statement can be interpreted as follows:

$$\begin{aligned} f(t) &\leq C + \sum_{k=1}^{\infty} B^k \mathfrak{I}_{a,g}^{\alpha k,\mu} C = C + C \sum_{k=1}^{\infty} \frac{B^k (g(t))^{\alpha k}}{\Gamma(\alpha k + 1)} {}_1F_1(\alpha k, \alpha k + 1; -\mu g(t)) \\ &= C \left(\sum_{k=0}^{\infty} \frac{B^k e^{\mu g(t)}}{\Gamma(\alpha k + 1)} \right) = C e^{\mu g(t)} \mathcal{E}_\alpha(B). \end{aligned}$$

$\square$

The definition of tempered-g-fractional derivatives is presented in this section. This definition is essential for solving fractional-order differential problems. A precise definition of the derivative is essential for applying fractional-order integral operators to initial and/or boundary value problems and for determining the invertibility of these operators

in different spaces with varying definitions of the integral and derivative. This concept has numerous applications in the study of differential equations.

Definition 2 (tempered-Riemann–Liouville fractional derivative). Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$ for all $t \in [a, b]$. The tempered- g –Riemann–Liouville fractional derivative of order $n + \alpha$, $\alpha \in (0, 1)$, $n \in \mathbb{N}$, and with parameter $\mu \in \mathbb{R}^+$, applied to the function $f \in L_1[a, b]$ is defined as

$$\mathfrak{D}_{a,g}^{n+\alpha,\mu} f(t) = \mathfrak{D}_{a,g}^n \mathfrak{D}_{a,g}^{\alpha,\mu} f(t), \text{ where } \mathfrak{D}_{a,g}^{\alpha,\mu} := \mathfrak{D} \mathfrak{S}_{a,g}^{1-\alpha,\mu}.$$

For any $\alpha \in (0, 1)$, given (7), we have

$$\begin{aligned} \mathfrak{D}_{a,g}^{\alpha,0} \left(e^{\mu g(t)} f(t) \right) &= \frac{1}{g'(t)} \frac{d}{dt} \mathfrak{S}_{a,g}^{1-\alpha,0} \left(e^{\mu g(t)} f(t) \right) = \frac{1}{g'(t)} \frac{d}{dt} \left[e^{\mu g(t)} \mathfrak{S}_{a,g}^{1-\alpha,\mu} f(t) \right] \\ &= e^{\mu g(t)} \mathfrak{D} \mathfrak{S}_{a,g}^{1-\alpha,\mu} f(t). \end{aligned}$$

Thus,

$$\mathfrak{D}_{a,g}^{\alpha,\mu} f(t) = e^{-\mu g(t)} \mathfrak{D}_{a,g}^{\alpha,0} \left(e^{\mu g(t)} f(t) \right), \quad \alpha \in (0, 1),$$

Similarly,

$$\mathfrak{D}_{a,g}^{n+\alpha,\mu} f(t) = e^{-\mu g(t)} \mathfrak{D}_{a,g}^{n+\alpha,0} \left(e^{\mu g(t)} f(t) \right), \quad \alpha \in (0, 1), \quad n \in \mathbb{N}.$$

Definition 3 (tempered-Caputo fractional derivative). Let $g \in C^1[a, b]$ be a positive increasing function such that $g'(t) \neq 0$ for all $t \in [a, b]$. The tempered- g –Caputo fractional derivative of order $n + \alpha$, $\alpha \in (0, 1)$, $n \in \mathbb{N}$, and with parameter $\mu \in \mathbb{R}^+$, applied to the function $f \in AC[a, b]$ is defined as

$$\frac{d_{a,g}^{n+\alpha,\mu}}{dt^{n+\alpha}} f(t) = \mathfrak{D}^n \frac{d_{a,g}^{\alpha,\mu}}{dt^\alpha} f(t), \text{ where } \frac{d_{a,g}^{\alpha,\mu}}{dt^\alpha} := \mathfrak{S}_{a,g}^{1-\alpha,\mu} \mathfrak{D}.$$

Obviously,

$$\frac{d_{a,g}^{n+\alpha,\mu}}{dt^{n+\alpha}} f(t) = e^{-\mu g(t)} \frac{d_{a,g}^{n+\alpha,0}}{dt^{n+\alpha}} \left(e^{\mu g(t)} f(t) \right), \quad \alpha \in (0, 1), \quad n \in \mathbb{N}.$$

Note that the tempered fractional integral and differential operators defined by Definitions 2 and 3 generalize several existing fractional integral operators. Our study highlights the possibility of inverting the integral operator, but not only on the image of the Hölder space $\mathbb{H}_0^\zeta[a, b]$. This is an important result for its particular subspace $\mathbb{H}_0^{\zeta+\alpha}[a, b]$ (see claim 2 below).

Lemma 6 ((Lemma 5.1 and Proposition 5.2, [6]) (see also (Theorem 5, [7])). For any $\alpha \in (0, 1)$, we have

1. For any $f \in L_1[a, b]$ (resp. $f \in C[a, b]$), for any $\alpha_1, \alpha_2 \in (0, 1)$ and almost all $t \in [a, b]$ (resp. for all $t \in [a, b]$) we have

$$\mathfrak{D}_{a,g}^{\alpha_1,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} f = \mathfrak{D} \mathfrak{S}_{a,g}^{1+\alpha_2-\alpha_1,\mu} f = \begin{cases} \mathfrak{D} \mathfrak{S}_{a,g}^{1-(\alpha_1-\alpha_2),\mu} f, & \alpha_2 \leq \alpha_1, \\ \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} f, & \alpha_1 \leq \alpha_2. \end{cases} = \begin{cases} \mathfrak{D}_{a,g}^{\alpha_1-\alpha_2,\mu} f, & \alpha_2 \leq \alpha_1, \\ \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} f, & \alpha_1 \leq \alpha_2. \end{cases}.$$

2. Let $\zeta \in (0, 1)$ such that $\zeta + \alpha < 1$. Then, for any $f \in \mathbb{H}_0^{\zeta+\alpha}[a, b]$ we have

$$\mathfrak{S}_{a,g}^{\alpha,\mu} \mathfrak{D}_{a,g}^{\alpha,\mu} f = f.$$

3. For any $f \in AC[a, b]$ one obtains

$$\frac{d_{a,g}^{\alpha,\mu}}{dt^\alpha} f = \mathfrak{S}_{a,g}^{1-\alpha,\mu} \mathfrak{D} f = \mathfrak{D}_{a,g}^{\alpha,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} (\mathfrak{S}_{a,g}^{1-\alpha,\mu} \mathfrak{D} f) = \mathfrak{D}_{a,g}^{\alpha,\mu} \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{D} f = \mathfrak{D}_{a,g}^{\alpha,\mu} f - \frac{(g(t))^{-\alpha} e^{-\mu g(t)}}{\Gamma(1-\alpha)} f(a).$$

Remark 2. It is important to note that Definition 3 of the Caputo derivative becomes meaningless if f is not absolutely continuous.

Therefore, we utilize the third property of Lemma 6 to provide a comprehensive definition of the Caputo derivative. This definition coincides with Definition 3 in the space $AC[a, b]$. Thus, we observe a displacement of $\mathfrak{D}_{a,g}^{\alpha,\mu} f$:

$$\frac{d_{a,g}^{\alpha,\mu}}{dt^\alpha} f := \mathfrak{D}_{a,g}^{\alpha,\mu} f - \frac{(g(t))^{-\alpha} e^{-\mu g(t)}}{\Gamma(1-\alpha)} f(a), \quad \alpha \in (0, 1). \quad (12)$$

Besides the tempered fractional derivatives in the Riemann–Liouville and the Caputo sense, a new definition of the fractional derivative was introduced by Hilfer. He called it the left-sided (right-sided) generalized Riemann–Liouville derivative. Thus, many authors (see [1,2,24]) call it the Hilfer derivative. An important feature of this definition is parametrized and is that such a derivative is, in a sense, an interpolation between the Riemann–Liouville derivative operator and the Caputo derivative operator, which will be clarified later:

Definition 4. [tempered-Hilfer fractional derivative] The tempered-g-Hilfer fractional derivative of order $n + \alpha$, $\alpha \in (0, 1)$, $n \in \mathbb{N} := \{1, 2, \dots\}$ with parameter $\mu \in \mathbb{R}^+$ and type $\beta \in [0, 1]$ applied to the function $f \in AC[a, b]$ is defined as

$${}^H\mathbb{D}_{a,g}^{n+\alpha,\beta,\mu} f(t) := \mathfrak{D}^n {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} f(t), \text{ where } {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} := \mathfrak{S}_{a,g}^{\beta(1-\alpha),\mu} \mathfrak{D} \mathfrak{S}_{a,g}^{(1-\beta)(1-\alpha),\mu}.$$

It is clear that

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} f(t) = e^{-\mu g(t)} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,0} (e^{\mu g(t)} f(t)).$$

For completeness, we define ${}^H\mathbb{D}_{a,g}^{0,\beta,\mu} f := \mathfrak{S}_{a,g}^{\beta,\mu} \mathfrak{D} \mathfrak{S}_{a,g}^{1-\beta,\mu} f$. Then, for any $f \in AC[a, b]$, we have (in view of (9))

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{0,\beta,\mu} f(t) &= \begin{cases} \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{D} f(t), & \beta = 1, \\ \mathfrak{S}_{a,g}^{\beta,\mu} \mathfrak{D} \left[\frac{e^{-\mu g(t)} (g(t))^{1-\beta} f(a)}{\Gamma(2-\beta)} + \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{S}_{a,g}^{1-\beta,\mu} \mathfrak{D} \right] f(t), & \beta \in [0, 1) \end{cases} \\ &= \begin{cases} f(t) - e^{-\mu g(t)} f(a), & \beta = 1, \\ e^{-\mu g(t)} f(a) + \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{D} f(t), & \beta \in [0, 1) \end{cases} \\ &= \begin{cases} f(t) - e^{-\mu g(t)} f(a), & \beta = 1, \\ f(t), & \beta \in [0, 1). \end{cases} \end{aligned} \quad (13)$$

As we noted earlier, the tempered-g-Hilfer fractional derivative operator interpolates the tempered-Riemann–Liouville and Caputo fractional derivative operators.

Indeed, the tempered-g-Hilfer fractional derivative of type $\beta = 0$ (resp. $\beta = 1$) is in fact the tempered-g-Riemann–Liouville (resp. Caputo) fractional derivative. However, the class of fractional integrals and fractional derivatives, based on the arbitrary choice of g , is presented (in detail) in, e.g., [24]. By a direct verification with the help of (3), it is easy to show that for $\eta > \alpha + n$ (see [24–26])

$${}^H\mathbb{D}_{a,g}^{n+\alpha,\beta,\mu} \left\{ e^{-\mu g(t)} (g(t))^{\eta-1} \right\} = \frac{\Gamma(\eta)}{\Gamma(\eta - \alpha - n)} e^{-\mu g(t)} (g(t))^{\eta-\alpha-(n+1)}, \quad t > a, \eta > 0. \quad (14)$$

The following proposition can be easily demonstrated through direct verification (see, for example, [11,24–26]).

Proposition 2. For any $\alpha \in (0, 1)$ and $\beta \in [0, 1]$, we have

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \left\{ e^{-\mu g(t)} (g(t))^{(1-\alpha)(\beta-1)} \right\} = 0.$$

Also (see (Lemma 6, [24])),

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} (\mathcal{E}_\alpha(\lambda(g(t))^\alpha)) = \lambda \mathcal{E}_\alpha(\lambda(g(t))^\alpha), \quad \alpha, \lambda > 0.$$

That is, $\mathcal{E}_\alpha(\cdot)$ solves differential problem ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} f = \lambda f$ for all $\alpha, \lambda > 0$, so it is a kind of fractional exponential function.

The relationship between the tempered- g -Hilfer fractional derivative and the classical g -Riemann–Liouville fractional derivative is well-known and straightforward to observe.

$${}^H\mathbb{D}_{a,g}^{n+\alpha,\beta,\mu} f = \mathfrak{d}^n \mathfrak{S}_{a,g}^{\gamma-\alpha,\mu} \mathfrak{D}_{a,g}^{\gamma,\mu} f, \quad \text{where } \gamma = \alpha + \beta(1-\alpha) \in [\alpha, 1] \text{ (equivalently) } \beta \in [0, 1].$$

Obviously, ${}^H\mathbb{D}_{a,g}^{n+\alpha,1,\mu} f = \frac{d_{a,g}^{n+\alpha,\mu}}{dt^{n+\alpha}} f$. Also,

$${}^H\mathbb{D}_{a,g}^{n+\alpha,\beta,\mu} f = \mathfrak{d}^n \frac{d_{a,g}^{\gamma,\mu}}{dt^\gamma} \mathfrak{S}_{a,g}^{\gamma-\alpha,\mu} f = \frac{d_{a,g}^{n+\gamma,\mu}}{dt^{n+\gamma}} \mathfrak{S}_{a,g}^{\gamma-\alpha,\mu} f,$$

where $\gamma = 1 - \beta(1-\alpha) \in [\alpha, 1]$ (equivalently) $\beta \in (0, 1]$. Obviously, ${}^H\mathbb{D}_{a,g}^{n+\alpha,0,\mu} f = \mathfrak{D}_{a,g}^{n+\alpha,\mu} f$.

Now, Lemma 6 can be combined with the semi-group property (Proposition 1) in order to ensure the subsequent lemma.

Lemma 7. Let $\beta \in [0, 1)$ and $\alpha_1, \alpha_2, \zeta \in (0, 1)$, $\gamma_2 = \alpha_2 + \beta(1-\alpha_2)$. Then,

1. For any $f \in L_1[a, b]$ (resp. $f \in C[a, b]$), any $\alpha_1, \alpha_2 \in (0, 1)$, $\alpha_1 \leq \alpha_2$, and almost all $t \in [a, b]$ (resp. for all $t \in [a, b]$) we have

$${}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1,\mu} f = \mathfrak{S}_{a,g}^{\gamma_2-\alpha_2,\mu} \left(\mathfrak{D}_{a,g}^{\gamma_2,\mu} \mathfrak{S}_{a,g}^{\alpha_1,\mu} \right) f = \mathfrak{S}_{a,g}^{\gamma_2-\alpha_2,\mu} \mathfrak{D}_{a,g}^{\gamma_2-\alpha_1,\mu} f.$$

2. For any $f \in \mathbb{H}_0^{\zeta+\alpha}[a, b]$, with $\zeta + \gamma_2 < 1$ and $\alpha_1 \geq \alpha_2$, we obtain

$$\mathfrak{S}_{a,g}^{\alpha_1,\mu} {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} f = \mathfrak{S}_{a,g}^{\alpha_1,\mu} \mathfrak{S}_{a,g}^{\gamma_2-\alpha_2,\mu} \mathfrak{D}_{a,g}^{\gamma_2,\mu} f = \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \left(\mathfrak{S}_{a,g}^{\gamma_2,\mu} \mathfrak{D}_{a,g}^{\gamma_2,\mu} \right) f = \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f.$$

In particular, when $\alpha_1 = \alpha_2 = \alpha$ and $\beta \in [0, 1)$, the above assertions together with our observations that $\mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b] \subset \mathbb{H}_0^{\zeta+\beta(1-\alpha)}[a, b]$ and $\mathfrak{S}_{a,g}^{0,\mu} f = f$, we obtain the following fact:

The operators ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu}$ and $\mathfrak{S}_{a,g}^{\alpha,\mu} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ are defined on $\mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$, and ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ is the left-right inverse of $\mathfrak{S}_{a,g}^{\alpha,\mu}$.

The result is similar to the criterion for the invertibility of integral operators on the classical space $AC[a, b]$ (see [24,27]). We will now prove it for the general class of integral operators under consideration.

Lemma 8. Let $\alpha, \alpha_1, \alpha_2 \in (0, 1)$, $\beta \in [0, 1]$, $\mu \in [0, \infty)$ and $f \in AC[a, b]$. Then,

1. $\mathfrak{S}_{a,g}^{\alpha,\mu} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} f(t) = \begin{cases} f(t) - e^{-\mu g(t)} f(a), & \beta = 1, \\ f(t), & \beta \in [0, 1). \end{cases}$
2. ${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} f(t) = \begin{cases} \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} f(t), & \alpha_2 \geq \alpha_1, \\ {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} f(t), & \alpha_1 > \alpha_2 \end{cases}$.

In particular, when $\alpha_1 = \alpha_2 = \alpha$, $\beta \in [0, 1)$, the above assertions along with our observation that ${}^H\mathbb{D}_{a,g}^{0,\beta,\mu} f = \mathfrak{S}_{a,g}^{0,\mu} f = f$ (see (13)) give the following fact:

The operators ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}\mathfrak{S}_{a,g}^{\alpha,\mu}$ and $\mathfrak{S}_{a,g}^{\alpha,\mu}{}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ are defined on $AC[a,b]$ and ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ is the left-right inverse of $\mathfrak{S}_{a,g}^{\alpha,\mu}$. This is true for all $\beta \in [0,1]$ on $AC_0[a,b]$.

Proof. Let us prove assertion 1. Arguing similarly to the proof of Lemma 4 and using integration by parts, we conclude that for any $\beta \in [0,1]$ we have

$$\begin{aligned} \mathfrak{S}_{a,g}^{\alpha,\mu} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} f(t) &= \mathfrak{S}_{a,g}^{\alpha+\beta(1-\alpha),\mu} \mathfrak{D}_{a,g}^{(1-\beta)(1-\alpha),\mu} f(t) \\ &= \mathfrak{S}_{a,g}^{\alpha+\beta(1-\alpha),\mu} \mathfrak{D} \left[\frac{e^{-\mu g(t)} (g(t))^{(1-\beta)(1-\alpha)} f(a)}{\Gamma(1+(1-\beta)(1-\alpha))} + \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{S}_{a,g}^{(1-\beta)(1-\alpha),\mu} \mathfrak{D} f(t) \right] \\ &= \mathfrak{S}_{a,g}^{\alpha+\beta(1-\alpha),\mu} \left[\frac{e^{-\mu g(t)} (g(t))^{-\alpha-\beta(1-\alpha)} f(a)}{\Gamma((1-\beta)(1-\alpha))} + \mathfrak{S}_{a,g}^{(1-\beta)(1-\alpha),\mu} \mathfrak{D} f(t) \right] \\ &= e^{-\mu g(t)} f(a) + \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{D} f(t) = e^{-\mu g(t)} f(a) + f(t) - e^{-\mu g(t)} f(a) = f(t). \end{aligned}$$

In particular, if $\beta = 1$, given (8), it is not difficult to conclude that

$$\begin{aligned} \mathfrak{S}_{a,g}^{\alpha,\mu} {}^H\mathbb{D}_{a,g}^{\alpha,1,\mu} f(t) &= \mathfrak{S}_{a,g}^{\alpha,\mu} \mathfrak{S}_{a,g}^{1-\alpha,\mu} \left[\frac{1}{g'(t)} \frac{df(t)}{dt} + \mu f(t) \right] = \mathfrak{S}_{a,g}^{1,\mu} \left[\frac{1}{g'(t)} \frac{df(t)}{dt} + \mu f(t) \right] \\ &= f(t) - f(a)e^{-\mu g(t)}. \end{aligned}$$

Next, we prove assertion 2: for any $\beta \in [0,1]$ and $\alpha_2 \geq \alpha_1$, we have

$$\begin{aligned} &{}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} f(t) + \mathfrak{S}_{a,g}^{\beta(1-\alpha_1),\mu} \mathfrak{D}_{a,g}^{1+\alpha_2-\alpha_1-\beta(1-\alpha_1),\mu} f(t) \\ &= \mathfrak{S}_{a,g}^{\beta(1-\alpha_1),\mu} \mathfrak{D} \left[\frac{e^{-\mu g(t)} (g(t))^{1+\alpha_2-\alpha_1-\beta(1-\alpha_1)} f(a)}{\Gamma(2+\alpha_2-\alpha_1-\beta(1-\alpha_1))} + \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{S}_{a,g}^{1+\alpha_2-\alpha_1-\beta(1-\alpha_1),\mu} \mathfrak{D} f(t) \right] \\ &= \mathfrak{S}_{a,g}^{\beta(1-\alpha_1),\mu} \left[\frac{e^{-\mu g(t)} (g(t))^{\alpha_2-\alpha_1-\beta(1-\alpha_1)} f(a)}{\Gamma(1+\alpha_2-\alpha_1-\beta(1-\alpha_1))} + \mathfrak{S}_{a,g}^{1+\alpha_2-\alpha_1-\beta(1-\alpha_1),\mu} \mathfrak{D} f(t) \right] \\ &= \frac{e^{-\mu g(t)} (g(t))^{\alpha_2-\alpha_1} f(a)}{\Gamma(1+\alpha_2-\alpha_1)} + \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} \mathfrak{S}_{a,g}^{1,\mu} \mathfrak{D} f(t) \\ &= \frac{e^{-\mu g(t)} (g(t))^{\alpha_2-\alpha_1} f(a)}{\Gamma(1+\alpha_2-\alpha_1)} + \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} (f(t) - f(a)e^{-\mu g(t)}) = \mathfrak{S}_{a,g}^{\alpha_2-\alpha_1,\mu} f(t). \end{aligned}$$

Furthermore, for all $\beta \in [0,1]$ and $\alpha_2 < \alpha_1$, we have

$${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} f(t) = \frac{e^{-\mu g(t)} (g(t))^{\alpha_2-\alpha_1} f(a)}{\Gamma(1+\alpha_2-\alpha_1)} + \mathfrak{S}_{a,g}^{1-(\alpha_1-\alpha_2),\mu} \mathfrak{D} f(t).$$

Since

$$\begin{aligned} &{}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} f(t) = \mathfrak{S}_{a,g}^{\beta(1+\alpha_2-\alpha_1),\mu} \mathfrak{D}_{a,g}^{1+\alpha_2-\alpha_1-\beta(1+\alpha_2-\alpha_1),\mu} f(t) \\ &= \mathfrak{S}_{a,g}^{\beta(1+\alpha_2-\alpha_1),\mu} \left[\frac{e^{-\mu g(t)} (g(t))^{(1-\beta)(1+\alpha_2-\alpha_1)-1} f(a)}{\Gamma((1-\beta)(1+\alpha_2-\alpha_1))} + \mathfrak{S}_{a,g}^{(1-\beta)(1+\alpha_2-\alpha_1),\mu} \mathfrak{D} f(t) \right] \\ &= \frac{e^{-\mu g(t)} (g(t))^{\alpha_2-\alpha_1} f(a)}{\Gamma(1+\alpha_2-\alpha_1)} + \mathfrak{S}_{a,g}^{1-(\alpha_1-\alpha_2),\mu} \mathfrak{D} f(t), \end{aligned}$$

we have, therefore,

$${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} f = {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} f, \quad \alpha_2 < \alpha_1, \quad \beta \in [0,1] \quad f \in AC[a,b].$$

□

Lemma 7 can be combined with Lemma 8 to point out the following:

Remark 3. For any $\alpha \in (0, 1)$, $\beta \in [0, 1)$, $\mu \in [0, \infty)$, the operators ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu}$ and $\mathfrak{S}_{a,g}^{\alpha,\mu} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ are defined on $AC[a, b] \cup \mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$ with $0 < \zeta + \alpha + \beta(1 - \alpha) < 1$ and ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}$ is the left–right inverse of $\mathfrak{S}_{a,g}^{\alpha,\mu}$.

Unfortunately, our observations in Remark 3 for $\beta \in (0, 1]$ are no longer necessarily true outside the space $AC[a, b] \cup \mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$. In fact, for $\beta \in (0, 1]$, Lemmas 7 and 8 have no analog in the space $C[a, b]$. We will show that ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu}$ is not necessarily defined on $C[a, b]$ (but is still outside $AC[a, b] \cup \mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$). The problem of finding invariant spaces for integral order operators, which enable the definition of integral and derivative on the same space or the use of iterative methods, remains unsolved. This question pertains to the selection of generalized Hölder spaces based on the g function. Integral operators typically have values in spaces that are narrower than the domain, making it impossible to define the derivative from such a function. However, this is feasible even in the case of the classical Hölder spaces considered here. An instructive counterexample is the following:

Counterexample 1. Define $\mathcal{W}^*(\cdot) := e^{-\mu(\cdot)}[\mathcal{W}(\cdot) - \mathcal{W}(0)]$, where

$$\mathcal{W}(x) := \sum_{n=0}^{\infty} \frac{e^{-\mu x}}{b^n} e^{ib^n x}, \quad b > 1, x \in [0, \infty).$$

denotes the well-known Weierstrass function. According to (Lemma 1, [28]) (see also [20]), $\mathcal{W}^*(\cdot)$ satisfies the Hölder condition of any order less than 1. Clearly, ${}^H\mathbb{D}_{a,g}^{\alpha,0,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} f = \mathfrak{D}\mathfrak{S}_{a,g}^{1,\mu} f = f$. Now, in the same spirit as in the proof of (Theorem 2, [28]) with $\beta \in (0, 1]$, it can easily be seen that $\mathcal{W}^*(g(t)), t \in [a, b]$ has a continuous Riemann–Liouville fractional derivative ${}^H\mathbb{D}_{a,g}^{\alpha,0,\mu} \mathcal{W}^*(g(\cdot)) = \mathfrak{D}_{a,g}^{\alpha,\mu} \mathcal{W}^*(g(\cdot))$ of any order $\alpha \in (0, 1)$.

Now, define $f := \mathfrak{D}_{a,g}^{1-\beta(1-\alpha),\mu} \mathcal{W}^*(g(\cdot)) \in C[a, b]$. Then,

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} f(t) &= \mathfrak{S}_{a,g}^{\beta(1-\alpha),\mu} \mathfrak{D}\mathfrak{S}_{a,g}^{1-\beta(1-\alpha),\mu} f(t) \\ &= \mathfrak{S}_{a,g}^{\beta(1-\alpha),\mu} \mathfrak{D}\mathfrak{S}_{a,g}^{1-\beta(1-\alpha),\mu} \mathfrak{D}_{a,g}^{1-\beta(1-\alpha),\mu} \mathcal{W}^*(g(t)). \end{aligned}$$

Since $1 - \beta(1 - \alpha) < 1$ for any $\beta \in (0, 1]$, $\alpha \in (0, 1)$ and $\mathcal{W}^*(g(\cdot))$ is a Hölderian function of all orders less than 1, we conclude, in view of Lemma 7, that

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} f(t) = \mathfrak{S}_{a,g}^{\beta(1-\alpha),\mu} \mathfrak{D}\mathcal{W}^*(g(t)).$$

Since the Weierstrass function is nowhere differentiable, ${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha,\mu} f$ is meaningless.

3. Equivalence between Differential and Integral Forms of the Tempered- g -Hilfer Fractional Differential Problems

The equivalence between the differential and integral problems is a crucial matter in the study of differential equations, including fractional order. However, this problem is sometimes not proven and is merely referenced to previous results. To gain a comprehensive understanding of the topic, we must study various definitions of derivatives for different integral operators, as introduced earlier in the paper. Additionally, we need to determine the function spaces or domains of the operators under study.

In this section we discuss the problem of equivalence between the tempered- g -Hilfer fractional differential and the corresponding integral forms for different values of $\alpha > 0$ and $\beta \in [0, 1]$. The said problem was studied in the case when $\beta = 1$ in [4], where the authors showed that, even for the Hölderian functions, but off the $AC[a, b]$ s, the equivalence between the Caputo fractional differential problems and their corresponding integral forms may be lost. However, with the help of Counter-Example 1, it is clear that, for any $\beta \in (0, 1]$ and $\alpha \in (0, 1)$, outside of $AC[a, b] \cup \mathbb{H}_0^{\zeta}[a, b]$ for some $\zeta \in (0, 1)$, the equivalence

between the tempered- g -Hilfer fractional differential problems and their corresponding integral forms may also be lost. In the light of this fact, the main existence results of many authors (see, e.g., [1,22,23,26,29,30]) are not quite correct and actually apply only to integral equations.

In order to discuss the problem of equivalence between the fractional differential and the corresponding integral problems in $C[a, b]$, let us consider the following miscellaneous examples:

Example 4. Consider the tempered- g -Hilfer fractional differential problem

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}x(t) = f(t), \quad \mu \in [0, \infty), \text{ and } f \in C[a, b], \quad (15)$$

combined with appropriate initial or boundary conditions. In view of Proposition 2, formally, the fractional integral form corresponding to (15) reads as

$$x(t) = e^{-\mu g(t)}C_0' + e^{-\mu g(t)}(g(t))^{(\beta-1)(1-\alpha)}C_0 + \mathfrak{S}_{a,g}^{\alpha,\mu}f(t), \quad (16)$$

for $t \in [a, b]$, $\alpha \in (0, 1)$ and $\beta \in [0, 1]$, where C_0, C_0' are constants depending only on the initial/boundary conditions.

1. According to Remark 3 and Proposition 2, for any $\beta \in [0, 1)$ and for $\zeta + \alpha + \beta(1 - \alpha) < 1$, for $f \in AC[a, b] \cup \mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$, problems (15) and (16) are equivalent.
2. According to Counterexample 1 and Proposition 2, for any $\beta \in (0, 1]$ and f outside of the space $AC[a, b] \cup \mathbb{H}_0^{\zeta+\alpha+\beta(1-\alpha)}[a, b]$, $\zeta + \alpha + \beta(1 - \alpha) < 1$, the equivalence between problems (15) and (16) is no longer necessarily true.
3. According to Lemma 8 and Proposition 2, for any $\beta \in [0, 1]$ and $f \in AC[a, b]$, the equivalence between problems (15) and (16) is true.
4. Let us consider problem (15) with $\beta = 1$ and $f \in \mathbb{H}_0^{\zeta+\alpha}[a, b]$, with $\zeta + \alpha < 1$ but outside $\in AC[a, b]$:

$${}^H\mathbb{D}_{a,g}^{\alpha,1,\mu}x(t) = f(t), \quad \mu \in [0, \infty), \text{ and } f \in \mathbb{H}_0^{\zeta+\alpha}[a, b] - AC[a, b], \quad \zeta + \alpha < 1. \quad (17)$$

If we insert (12) into all occurrences of the Caputo derivative, in problem (17), using always the corresponding initial or boundary values, we arrive at the new problem:

$${}^H\mathbb{D}_{a,g}^{\alpha,0,\mu}x(t) = f(t) + \frac{(g(t))^{-\alpha}e^{-\mu g(t)}}{\Gamma(1-\alpha)}x(a).$$

This reads as

$$x(t) = e^{-\mu g(t)}(g(t))^{\alpha-1}C_0 + \mathfrak{S}_{a,g}^{\alpha,\mu}f(t) + e^{-\mu g(t)}x(a), \quad t \in [a, b], \quad \alpha \in (0, 1). \quad (18)$$

Since $f \in \mathbb{H}_0^{\zeta+\alpha}[a, b]$, the equivalence between problems (15) and (16) is allowed in view of Lemma 7.

Differential problem (15) with $\alpha > 1$ is not complicated:

Example 5. Consider differential problem (15) with $\alpha \in (n, n + 1)$, $n \in \mathbb{N}$, namely,

$${}^H\mathbb{D}_{a,g}^{\alpha,\beta,\mu}x(t) = f(t), \quad \alpha \in (n, n + 1), \beta \in [0, 1], \mu \in [0, \infty), f(\cdot) \in C[a, b]. \quad (19)$$

This reads as

$$\mathfrak{D}^n \left[{}^H\mathbb{D}_{a,g}^{\alpha-n,\beta,\mu}x(t) \right] = f(t) \implies {}^H\mathbb{D}_{a,g}^{\alpha-n,\beta,\mu}x(t) = \sum_{k=1}^n c_k e^{-\mu g(t)}(g(t))^{k-1} + \mathfrak{S}_{a,g}^{n,\mu}f(t), \quad (20)$$

where c_k , $k = 1, 1, \dots, n$ are constants depend only on the initial/boundary conditions. Operating by the fractional integral operator $\mathfrak{S}_{a,g}^{\alpha-n,\mu}$ on both sides of (20) and applying Lemma 8 enable us to show (at least formally) that

$$\begin{aligned} x(t) &= x(a)e^{-\mu g(t)} + \sum_{k=1}^n c_k \mathfrak{S}_{a,g}^{\alpha-n,\mu} e^{-\mu g(t)} (g(t))^{k-1} + \mathfrak{S}_{a,g}^{\alpha-n,\mu} \mathfrak{S}_{a,g}^{n,\mu} f(t) \\ &= x(a)e^{-\mu g(t)} + \sum_{k=1}^n \frac{c_k \Gamma(k) e^{-\mu g(t)}}{\Gamma(k + \alpha - n)} (g(t))^{k + \alpha - n} + \mathfrak{S}_{a,g}^{\alpha-n,\mu} \mathfrak{S}_{a,g}^{n,\mu} f(t). \end{aligned} \quad (21)$$

Conversely, since $\mathfrak{S}_{a,g}^{n,\mu} f \in AC[a, b]$, by Lemma 4, we infer $\mathfrak{S}_{a,g}^{\alpha-n,\mu} \mathfrak{S}_{a,g}^{n,\mu} f \in AC[a, b]$. Thus, according to Lemma 8, it is easy to show that

$${}^H\mathbb{D}_{a,g}^{\alpha-n,\beta,\mu} x(t) = \mathfrak{S}_{a,g}^{n,\mu} f(t).$$

Hence, ${}^H\mathbb{D}_{a,g}^{\alpha-n,\beta,\mu} x = \mathfrak{d}^n {}^H\mathbb{D}_{a,g}^{\alpha-n,\beta,\mu} x = f$.

The following examples serve to illustrate similar results observed in Examples 4 and 5:

Example 6. Let us consider the following problem

$${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} + \lambda {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t) = f(t), \quad \beta \in [0, 1], \quad \alpha_1, \alpha_2 \in (1, 2], \quad \alpha_2 \leq \alpha_1, \quad t \in [a, b], \quad (22)$$

combined with appropriate initial/boundary conditions, where $\lambda > 0$ is sufficiently small and $f \in C[a, b]$. Obviously,

$$\mathfrak{d} \left[{}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} + \lambda {}^H\mathbb{D}_{a,g}^{\alpha_2-1,\beta,\mu} \right] x(t) = f(t).$$

Therefore,

$$\begin{aligned} \left[{}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} + \lambda {}^H\mathbb{D}_{a,g}^{\alpha_2-1,\beta,\mu} \right] x(t) &= e^{-\mu g(t)} \left[\int_a^t e^{\mu g(s)} f(s) \right] g'(s) ds + C_1 e^{-\mu g(t)} \\ &= \mathfrak{S}_{a,g}^{1,\mu} f(t) + C_1 e^{-\mu g(t)}. \end{aligned}$$

Formally, by Lemma 8, we obtain

$${}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left[\mathfrak{S}_{a,g}^{0,\mu} + \lambda \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \right] x(t) = \mathfrak{S}_{a,g}^{1,\mu} f(t) + C_1 e^{-\mu g(t)}.$$

By direct verification, it is evident (cf. Proposition 2) that the formal integral form (any solution) of (15) is the following:

$$\begin{aligned} \left[\mathfrak{S}_{a,g}^{0,\mu} + \lambda \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \right] x(t) &= C'_0 e^{-\mu g(t)} + \frac{e^{-\mu g(t)} (g(t))^{(\alpha_1-2)(1-\beta)}}{\Gamma(\alpha_1-1+\beta(2-\alpha_1))} C_0 + \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t) \\ &\quad + C_1 \frac{e^{-\mu g(t)}}{\Gamma(\alpha_1)} (g(t))^{\alpha_1-1}. \end{aligned} \quad (23)$$

Consequently, for any $x \in C[a, b]$, it follows that $C_0 = 0$ for any $\beta \in [0, 1)$. Furthermore, since $\mathfrak{S}_{a,g}^{0,\mu} f(t) = e^{-\mu g(t)} \mathfrak{S}_{a,g}^{0,0} (e^{\mu g(t)} f(t)) = f$, it follows $C'_0 = x(a)$. Moreover, for any $x \in C[a, b]$ and $\beta = 1$ we obtain $C'_0 + C_0 = x(a)$. Thus, the formal integral form of the differential Equation (22) is as follows:

$$x(t) = F(t) - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x(t), \quad (24)$$

where $F := x(a)e^{-\mu g(t)} + \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t) + C_1 \frac{e^{-\mu g(t)}}{\Gamma(\alpha_1)} (g(t))^{\alpha_1-1}$, $\beta \in [0, 1]$. Let us undertake a rigorous investigation of the equivalence between properly defined solutions of fractional differential problem (22) and the solutions of corresponding integral Equation (24) in the space $C[a, b]$.

It is evident that, for any $f \in C[a, b]$, we have $\mathfrak{S}_{a,g}^{1,\mu} f \in C^1[a, b]$ with $\mathfrak{S}_{a,g}^{1,\mu} f(a) = 0$. Therefore, $\mathfrak{S}_{a,g}^{1,\mu} f \in \mathbb{H}_0^\gamma[a, b]$ for any $\gamma \in (0, 1)$ such that $\gamma + \alpha_1 - \alpha_2 < 1$. Consequently, in view of part 3 of Lemma 1, we have $\mathfrak{S}_{a,g}^{\alpha_1,\mu} f = \mathfrak{S}_{a,g}^{\alpha_1-1,\mu} \mathfrak{S}_{a,g}^{1,\mu} f \in \mathbb{H}_0^{\gamma+\alpha_1-1}[a, b]$. Since $(g(t))^{\alpha_1-1} \in \mathbb{H}_0^1 \subset \mathbb{H}_0^{\gamma+\alpha_1-1}[a, b]$, it follows that $F \in \mathbb{H}_0^{\gamma+\alpha_1-1}[a, b]$.

Let us consider the following:

1. Case I when $\beta \in [0, 1]$: If $f(\cdot) \in \mathbb{H}_0^\gamma[a, b]$ with $\gamma \in (0, 1)$ such that $\gamma + \alpha_1 - \alpha_2 < 1$ (resp. $f \in AC[a, b]$), then (by Lemma 5), integral Equation (24) admits a Hölder continuous solution $x \in \mathbb{H}_0^\gamma[a, b]$ (resp. an absolutely continuous solution $x \in AC[a, b]$). Conversely, if $x \in \mathbb{H}_0^\gamma[a, b]$ solves (24), it follows from Lemma 7 that

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} x &= \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} x = \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left(F - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x \right) \\ &= \mathfrak{D} \left({}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-1,\mu} \left[\underbrace{\mathfrak{S}_{a,g}^{1,\mu} f}_{\in AC_0[a,b]} \right] + C_1 e^{-\mu g(t)} \right) - \lambda \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} x \\ &= \mathfrak{D} \mathfrak{S}_{a,g}^{1,\mu} f - \lambda \mathfrak{D} \mathfrak{S}_{a,g}^{\beta(2-\alpha_1),\mu} \mathfrak{D}_{a,g}^{\alpha_2-1+\beta(2-\alpha_1),\mu} x \\ &= f - \lambda \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_2-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2-1,\mu} \left[\underbrace{\mathfrak{S}_{a,g}^{\beta(2-\alpha_1),\mu} \mathfrak{D}_{a,g}^{\alpha_2-1+\beta(2-\alpha_1),\mu} x}_{\in \mathbb{H}_0^\gamma[a,b]} \right] \\ &= f - \lambda \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_2-1,\beta,\mu} \left(\mathfrak{S}_{a,g}^{\alpha_2-1+\beta(2-\alpha_1),\mu} \mathfrak{D}_{a,g}^{\alpha_2-1+\beta(2-\alpha_1),\mu} x \right) \\ &= f - \lambda {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x. \end{aligned}$$

Thus, we arrive at differential Equation (22). Moreover, if $x \in AC[a, b]$ solves (24), then, in view of Lemma 8, it follows that

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} x &= \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} x = \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left(F - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x \right) \\ &= \mathfrak{D} \left({}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-1,\mu} \left[\underbrace{\mathfrak{S}_{a,g}^{1,\mu} f}_{\in AC_0[a,b]} \right] + C_1 e^{-\mu g(t)} \right) - \lambda \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} x \\ &= \mathfrak{D} \mathfrak{S}_{a,g}^{1,\mu} f - \lambda \mathfrak{D} {}^H\mathbb{D}_{a,g}^{\alpha_2-1,\beta,\mu} x = f - \lambda {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x. \end{aligned}$$

Thus, we arrive at differential Equation (22).

2. Case II $\beta = 1$: If $f \in AC[a, b]$, it can be demonstrated in a manner analogous to that employed in Case I that integral Equation (24) admits an absolutely continuous solution $x \in AC[a, b]$ that solves differential Equation (22).

Now, let $f(\cdot) \in \mathbb{H}_0^\gamma[a, b]$ with $\gamma \in (0, 1)$ such that $\gamma + \alpha_1 - \alpha_2 < 1$. By Lemma 5, it can be shown that integral Equation (24) admits a Hölder continuous solution $x \in \mathbb{H}_0^\gamma[a, b]$. In light of Counterexample 1, it is evident that the equivalence between (24) and (22) may be lost outside $AC[a, b]$ even for the Hölderian functions. Consequently, we proceed to

$$\frac{d_{a,g}^{\alpha,\mu}}{dt^\alpha} x(t) := \mathfrak{D}_{a,g}^{\alpha,\mu} x(t) - \frac{(g(t))^{-\alpha} e^{-\mu g(t)}}{\Gamma(1-\alpha)} x(a), \quad \alpha \in (0, 1).$$

Let us note that problem (22) with $\beta = 1$ reads as

$$\frac{d_{a,g}^{\alpha_1,\mu}}{dt^{\alpha_1}}x + \lambda \frac{d_{a,g}^{\alpha_2,\mu}}{dt^{\alpha_2}}x(t) = f(t), \quad (25)$$

for $t \in [a, b]$. It follows that

$$\frac{d_{a,g}^{\alpha_1-1,\mu}}{dt^{\alpha_1-1}}x + \lambda \frac{d_{a,g}^{\alpha_2-1,\mu}}{dt^{\alpha_2-1}}x(t) = \mathfrak{S}_{a,g}^{1,\mu}f(t) + C_1 e^{-\mu g(t)}.$$

Then,

$$\begin{aligned} \left[\mathfrak{D}_{a,g}^{\alpha_1-1,\mu} + \lambda \mathfrak{D}_{a,g}^{\alpha_2-1,\mu} \right] x(t) &= \mathfrak{S}_{a,g}^{1,\mu}f(t) + C_1 e^{-\mu g(t)} + \frac{(g(t))^{1-\alpha_1} e^{-\mu g(t)}}{\Gamma(2-\alpha_1)} x(a) \\ &\quad + \lambda \frac{(g(t))^{1-\alpha_2} e^{-\mu g(t)}}{\Gamma(2-\alpha_2)} x(a). \end{aligned}$$

In this case, we are arguing as in Case (I) with $\beta = 0$, which leads to the following conclusion:

$$x(t) = F(t) - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x(t), \quad (26)$$

where

$$F = \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t) + C_1 \frac{e^{-\mu g(t)}}{\Gamma(\alpha_1)} (g(t))^{\alpha_1-1} + x(a) e^{-\mu g(t)} + \lambda \frac{(g(t))^{\alpha_1-\alpha_2} e^{-\mu g(t)}}{\Gamma(1+\alpha_1-\alpha_2)} x(a),$$

$$\begin{aligned} \frac{d_{a,g}^{\alpha_1,\mu}}{dt^{\alpha_1}}x &= \mathfrak{D}_{a,g}^{\alpha_1-1,\mu}x = \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \left(F - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x \right) \\ &= \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \left(F(t) - \lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x(t) \right) - \frac{(g(t))^{1-\alpha_1} e^{-\mu g(t)}}{\Gamma(2-\alpha_1)} \left(\underbrace{F(a)}_{=x(a)} - \underbrace{\lambda \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x(a)}_{=0} \right) \\ &= \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \left[\frac{x(a) e^{-\mu g(t)}}{\Gamma(2-\alpha_1)} (g(t))^{1-\alpha_1} + \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t) + C_1 e^{-\mu g(t)} - \lambda \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \mathfrak{S}_{a,g}^{(\alpha_1-\alpha_2),\mu} x(t) \right. \\ &\quad \left. - \frac{(g(t))^{1-\alpha_1} e^{-\mu g(t)}}{\Gamma(2-\alpha_1)} x(a) \right] \\ &= \mathfrak{D}_{a,g}^{\alpha_1-1,\mu} \left[\mathfrak{S}_{a,g}^{1,\mu} f(t) + C_1 e^{-\mu g(t)} - \lambda \left(\mathfrak{D}_{a,g}^{\alpha_2-1,\mu} x(t) - \frac{x(a) e^{-\mu g(t)}}{\Gamma(2-\alpha_2)} (g(t))^{1-\alpha_2} \right) \right] \\ &= f(t) - \lambda \mathfrak{D}_{a,g}^{\alpha_2-1,\mu} x(t) = f(t) - \lambda \frac{d_{a,g}^{\alpha_2,\mu}}{dt^{\alpha_2}} x(t). \end{aligned}$$

Consequently, we arrive at differential Equation (25).

Example 7. Let us consider the following problem:

$${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \left({}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t) + \lambda x(t) \right) = f(t), \quad (27)$$

where $\beta \in [0, 1]$, $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, $\lambda > 0$, $f \in C[a, b]$, for $t \in [a, b]$, combined with appropriate initial/boundary conditions. The cases where $\alpha_2 = 1$ and $\alpha_1 = 2$ involve ordinary derivatives and offer little novelty; therefore, we will concentrate on the cases where $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$. It should be noted that it is not true, in general, that ${}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \neq {}^H\mathbb{D}_{a,g}^{\alpha_1+\alpha_2,\beta,\mu}$ even when $\alpha_1 + \alpha_2 < 1$.

A considerable number of mathematical papers have been published on the topic of the existence of solutions to Equation (27) with $\beta = 0$ and $\beta = 1$. For further reading, please see, e.g., [31–37]. Obviously,

$$\mathfrak{D}^H \mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left(H \mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t) + \lambda x(t) \right) = f(t).$$

Hence,

$$H \mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left(H \mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t) + \lambda x(t) \right) = \mathfrak{I}_{a,g}^{1,\mu} f(t) + C_1 e^{-\mu g(t)}.$$

So,

$$\begin{aligned} H \mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t) + \lambda x(t) &= e^{-\mu g(t)} C_0' + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1-2(1-\beta)}}{\Gamma(\alpha_1-1+\beta(2-\alpha_1))} C_0 + \mathfrak{I}_{a,g}^{\alpha_1,\mu} f(t) \\ &\quad + C_1 \frac{e^{-\mu g(t)}}{\Gamma(\alpha_1)} (g(t))^{\alpha_1-1}. \end{aligned}$$

This reads as

$$\begin{aligned} x(t) &= \frac{e^{-\mu g(t)} (g(t))^{\alpha_2-1(1-\beta)}}{\Gamma(\alpha_2+\beta(1-\alpha_2))} A - \lambda \mathfrak{I}_{a,g}^{\alpha_2,\mu} x(t) + \frac{e^{-\mu g(t)} (g(t))^{\alpha_2+(\alpha_1-2)(1-\beta)}}{\Gamma(\alpha_2+1+(\alpha_1-2)(1-\beta))} C_0 \\ &\quad + \mathfrak{I}_{a,g}^{\alpha_1+\alpha_2,\mu} f(t) + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} C_1 + \frac{(g(t))^{\alpha_2} e^{-\mu g(t)}}{\Gamma(1+\alpha_2)} C_0' + B e^{-\mu g(t)}. \end{aligned}$$

Consequently, for any $x \in C[a, b]$ and $\beta \in [0, 1)$, it can be demonstrated that $A = 0$. Furthermore,

$$\begin{cases} C_0 = 0, B = x(a) & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) < 0, \\ B = x(a) & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) > 0, \\ C_0 + B = x(a) & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) = 0. \end{cases}$$

In this case,

$$\begin{aligned} x(t) &= -\lambda \mathfrak{I}_{a,g}^{\alpha_2,\mu} x(t) + \mathfrak{I}_{a,g}^{\alpha_1+\alpha_2,\mu} f(t) + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} C_1 + \frac{(g(t))^{\alpha_2} e^{-\mu g(t)}}{\Gamma(1+\alpha_2)} C_0' \\ &\quad + \begin{cases} x(a) e^{-\mu g(t)} & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) < 0, \\ x(a) e^{-\mu g(t)} + \frac{e^{-\mu g(t)} (g(t))^{\alpha_2+(\alpha_1-2)(1-\beta)}}{\Gamma(\alpha_2+1+(\alpha_1-2)(1-\beta))} C_0 & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) > 0, \\ x(a) e^{-\mu g(t)} & \text{when } \alpha_2 + (\alpha_1 - 2)(1 - \beta) = 0. \end{cases} \end{aligned}$$

Thus, for any $x \in C[a, b]$ and $\beta = 1$,

$$\begin{aligned} x(t) &= x(a) e^{-\mu g(t)} - \lambda \mathfrak{I}_{a,g}^{\alpha_2,\mu} x(t) + \frac{e^{-\mu g(t)} (g(t))^{\alpha_2}}{\Gamma(\alpha_2+1)} K + \mathfrak{I}_{a,g}^{\alpha_1+\alpha_2,\mu} f(t) \\ &\quad + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} C_1. \end{aligned}$$

Consequently, for any $\beta \in [0, 1]$, the formal integral form to differential Equation (27) is as follows:

$$x(t) = F(t) - \lambda \mathfrak{I}_{a,g}^{\alpha_2,\mu} x(t), \text{ for some } F \in AC[a, b]. \quad (28)$$

Accordingly, by Lemma 5, fractional integral Equation (28) admits a solution $x \in AC[a, b]$.

Conversely, let $x \in AC[a, b]$ be a solution to fractional integral Equation (28). It can be shown, in view of

$$\mathfrak{I}_{a,g}^{\alpha_1+\alpha_2,\mu} f = \mathfrak{I}_{a,g}^{\alpha_2,\mu} \overbrace{\mathfrak{I}_{a,g}^{\alpha_1-1,\mu} \mathfrak{I}_{a,g}^{1,\mu} f}^{\in AC_0[a,b]}$$

that ${}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} F = \mathfrak{S}_{a,g}^{1,\mu} f$. By Lemma 8, it follows that

$${}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \left(F - \lambda \mathfrak{S}_{a,g}^{\alpha_2,\mu} x \right) = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} F(t) - \lambda x(t).$$

Consequently, we have that ${}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \left[{}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x + \lambda x \right] = {}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} F = \mathfrak{S}_{a,g}^{1,\mu} f$. This is differential Equation (27).

We need to remark that, unlike problem (22), the existence of a continuous solution of (27) holds for any $f \in C[a, b]$.

4. Nonlinear Tempered-Hilfer Fractional Boundary/Initial Value Problems

The preceding considerations are essential for the study of boundary/initial values in fractional-order problems. A precise study is essential for correctly solving differential problems with natural assumptions.

In this section, we consider problem (22) in the case when $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$ in a more general setting. We consider the following nonlinear fractional-order fractional boundary/initial value problem with integral boundary conditions:

$$\begin{cases} {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} x(t) + f(t, x(t), {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(t)) = 0, \\ \begin{cases} x(a) = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(a) = 0, & \text{when } \alpha_1 - \alpha_2 < 1, \text{ or} \\ x(b) + \int_a^b \psi(\tau) x(\tau) d\tau = \ell \in \mathbb{R}, \quad x(a) = 0, & \text{when } \alpha_1 - \alpha_2 \geq 1. \end{cases} \end{cases} \quad (29)$$

for $t \in [a, b]$, $\beta \in [0, 1]$, $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, and where $\psi \in L_1[a, b]$. The cases where $\alpha_2 = 1$, $\alpha_1 = 2$ involve ordinary derivatives and offer little novelty. Consequently, our attention will be concentrated on the cases where $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$.

The purpose of this paper is to provide an explanation for our intentions. Subsequently, we will examine the historical results that have been previously established. Some notable contributions, for $\alpha_1 \in (1, 2]$, $\alpha_2 \in (0, 1)$, are presented in [8,38]. They analyzed the following problem using the Leray–Schauder fixed-point theorem:

$$\begin{cases} x'' + A {}^H\mathbb{D}_{a,g}^{\alpha_1,1,\mu} x(t) = f(t, x(t), {}^H\mathbb{D}_{a,g}^{\alpha_2,1,\mu} x(t), x'), & A \in \mathbb{R}, t \in [a, b]. \\ x'(a) = 0, \quad x(b) + Cx'(b) = 0, \end{cases}$$

A similar problem, but with Neumann boundary conditions,

$$\begin{cases} x'' + A {}^H\mathbb{D}_{a,g}^{\alpha_1,1,\mu} x(t) = f(t, x(t), x'), & A \in \mathbb{R}, t \in [a, b]. \\ x'(a) = x'(b) = 0, \end{cases}$$

has been studied in [39] again using the Leray–Schauder fixed-point theorem (cf. [38], with the simple version of this problem and the use of the Schauder fixed-point theorem). In [40], the authors examined the problem (utilizing fixed points for mappings with the mixed monotone property on cones):

$$\begin{cases} x'' + A {}^H\mathbb{D}_{a,g}^{\alpha_1,1,\mu} x(t) = f(t) - Bx(t), & A, B \in \mathbb{R}, t \in [a, b]. \\ x(a) = x_0, x'(a) = x_1. \end{cases}$$

In a recent study [5,41], the problem of nonlocal three-point boundary conditions was investigated under certain assumptions expressed in terms of weak topology:

$$\begin{cases} {}^H\mathbb{D}_{a,g}^{\alpha_1,1,\mu} x(t) = \lambda f(t, x(t), {}^H\mathbb{D}_{a,g}^{\alpha_2,1,\mu} x(t)), & \lambda > 0, t \in [a, b], \\ x(a) = x_0, x(b) - px(\xi) = \ell, & \xi \in (a, b), \quad p \in \mathbb{R}^+. \end{cases}$$

As previously noted, various fixed-point theorems have been employed in this field. This paper presents new results concerning the existence of solutions to the problems under consideration. Furthermore, we will endeavor to elucidate the significance of these theorems in the context of varying boundary conditions. The additional objective of this study is to apply a number of fixed-point theorems in sequence, and to illustrate the evolving techniques of proof employed in each case.

We will start with the initial and/or boundary value problems:

$$\begin{cases} {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu}y(t) + f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)) = 0, & t \in [a, b], \alpha_1 \in (1, 2), \alpha_2 \in (0, 1), \\ \begin{cases} y(a) = y_0, & \text{when } \alpha_1 - \alpha_2 < 1, \text{ or} \\ \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(b) + \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(\tau) d\tau = \ell, & \text{when } \alpha_1 - \alpha_2 \geq 1. \end{cases} \end{cases} \quad (30)$$

Let us make some simple, but very general, assumptions about the functions in the problem we are examining. Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following assumptions:

(A1) For each $x, y \in \mathbb{R}$, $f(\cdot, x, y)$ is measurable, and for almost every $t \in [a, b]$, $f(t, \cdot, \cdot)$ is continuous.

(A2) For each $x, y \in C[a, b]$, $f(\cdot, x(\cdot), y(\cdot)) \in L_p[a, b]$, with some $p > \max\{1, \frac{1}{\alpha_1 - \alpha_2}\}$.

(A3) There exist positive constants M_0, M_1, M_2 such that

$$|f(t, x, y)| \leq M_0 + M_1|x| + M_2|y|, \text{ for all } (t, x, y) \in [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(A4) Let $\psi \in L_q[a, b]$ be a non-negative real-valued function, $q = \frac{p}{p-1}$.

To obtain (formally) the fractional integral equation modeled off problem (30), we note that, for any $\beta \in [0, 1]$:

1. If $\alpha_1 - \alpha_2 < 1$ and $\beta \in [0, 1]$ we have

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu}y + f &= 0 \implies \\ y(t) &= y(a)e^{-\mu g(t)} + \frac{e^{-\mu g(t)}(g(t))^{\alpha_1-\alpha_2-1}(1-\beta)C_0}{\Gamma(\alpha_1-\alpha_2+\beta(1+\alpha_2-\alpha_1))} - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu}f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)). \end{aligned}$$

Suppose we are looking for a continuous solution $y \in C[a, b]$. Then, in view of Lemma 1, we infer that $C_0 = 0$. Thus, problem (30) (with $\alpha_1 - \alpha_2 < 1$ and $\beta \in [0, 1]$) can be expressed in the form of an integral equation

$$y(t) = y(a)e^{-\mu g(t)} - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu}f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)). \quad (31)$$

2. If $\alpha_1 - \alpha_2 \geq 1$, we have

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2-1,\beta,\mu}y(t) + f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)) &= 0 \implies \\ {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2-1,\beta,\mu}y(t) &= C_1e^{-\mu g(t)} - \mathfrak{S}_{a,g}^{1,\mu}f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y(a)e^{-\mu g(t)} + \frac{e^{-\mu g(t)}(g(t))^{\alpha_1-\alpha_2-2}(1-\beta)C_0}{\Gamma(\alpha_1-\alpha_2-1+\beta(2+\alpha_2-\alpha_1))} \\ &\quad - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu}f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)) + C_1 \frac{e^{-\mu g(t)}(g(t))^{\alpha_1-\alpha_2-1}}{\Gamma(\alpha_1-\alpha_2)}. \end{aligned}$$

Suppose we are looking for a solution $y \in C[a, b]$. In view of Lemma 1, we can infer that $C_0 = 0$. Therefore, the integral equation form of problem (30) with $\alpha_1 - \alpha_2 \geq 1$ is the following:

$$y(t) = F(t) - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu}f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu}y(t), y(t)), \quad (32)$$

where $F(t) := y(a)e^{-\mu g(t)} + C_1 \frac{e^{-\mu g(t)}(g(t))^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)}$.

Let us evaluate C_1 . By the semi-group property (cf. Proposition 1)), we obtain

$$\begin{aligned}\mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t) &= \mathfrak{S}_{a,g}^{\alpha_2,\mu} F(t) - \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) \\ &= -\mathfrak{S}_{a,g}^{\alpha_1,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) + y(a) \frac{e^{-\mu g(t)}(g(t))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + C_1 \frac{e^{-\mu g(t)}(g(t))^{\alpha_1 - 1}}{\Gamma(\alpha_1)}.\end{aligned}$$

Now, we would like to pay our attention to solving Equation (32) for C_1 when $\alpha_1 - \alpha_2 \geq 1$ and $\beta \in [0, 1]$ by

$$\mathfrak{S}_{a,g}^{\alpha_2,\mu} y(b) + \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\tau) d\tau = \ell.$$

It follows that

$$\begin{aligned}&\mathfrak{S}_{a,g}^{\alpha_2,\mu} F(b) - \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(b, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(b), y(b)) \\ &= \ell - \int_a^b \psi(\tau) \left(\mathfrak{S}_{a,g}^{\alpha_2,\mu} F(\tau) - \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(\tau, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\tau), y(\tau)) \right) d\tau \\ &= \ell - \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_2,\mu} F(\tau) d\tau + \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(\tau, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\tau), y(\tau)) d\tau.\end{aligned}$$

Therefore, for $\beta \in [0, 1]$, it is evident that

$$\frac{C_1}{\Gamma(\alpha_1)} \left[\|g\|^{\alpha_1 - 1} e^{-\mu \|g\|} + \zeta \right] = \zeta' + \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_1,\mu} f(\tau, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\tau), y(\tau)) d\tau,$$

where

$$\begin{aligned}\zeta &:= \int_a^b e^{-\mu g(\tau)} \psi(\tau) (g(\tau))^{\alpha_1 - 1} d\tau, \\ \zeta' &:= \ell - \frac{y(a) \|g\|^{\alpha_1} e^{-\mu \|g\|}}{\Gamma(1 + \alpha_1)} - \frac{y(a)}{\Gamma(1 + \alpha_1)} \int_a^b e^{-\mu g(\tau)} \psi(\tau) (g(\tau))^{\alpha_1} d\tau.\end{aligned}$$

Then,

$$C_1 = \frac{\Gamma(\alpha_1)}{(\|g\|^{\alpha_1 - 1} e^{-\mu \|g\|} + \zeta)} \left[\zeta' + \int_a^b h(s) f(s, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(s), y(s)) g'(s) ds \right],$$

where

$$h(s) := \frac{1}{\Gamma(\alpha_1)} \int_s^b e^{-\mu(g(\tau) - g(s))} (g(\tau) - g(s))^{\alpha_1 - 1} \psi(\tau) d\tau = \mathfrak{S}_{a,g}^{\alpha_1,\mu} \left[\frac{\psi(s)}{g'(s)} \right].$$

Substituting C_1 into (32) results in the following integral form of the problem being studied:

$$\begin{aligned}y(t) &= -\mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(t), y(t)) \\ &\quad + \frac{e^{-\mu g(t)}(g(t))^{\alpha_1 - \alpha_2 - 1}}{(\|g\|^{\alpha_1 - 1} e^{-\mu \|g\|} + \zeta)} \left[\zeta' + \int_a^b h(s) f(s, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(s), y(s)) g'(s) ds \right].\end{aligned}\quad (33)$$

Remark 4. Since $f(\cdot, x(\cdot), y(\cdot)) \in L_p[a, b]$ and $\psi \in L_q[a, b]$, it follows, by Lemma 1, $h \in L_q[a, b]$. Therefore, the integral on the right-hand side of (33) makes sense.

In view of Assumptions (A1)–(A4), we have

$$\begin{aligned} |y(t)| &\leq \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \left(M_0 + M_1 \mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(t)| + M_2 |y(t)| \right) \\ &\quad + M \left[\zeta' + \int_a^b |h(s)| \left(M_0 + M_1 \mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(s)| + M_2 |y(s)| \right) g'(s) ds \right] \\ &= \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} M_0 + M_1 \mathfrak{S}_{a,g}^{\alpha_1,\mu} |y(t)| \\ &\quad + M_2 \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| + M \left[\ell + \int_a^b |h(s)| \left(M_0 + M_1 \mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(s)| + M_2 |y(s)| \right) g'(s) ds \right], \end{aligned}$$

where

$$M := \frac{(\|g\|)^{\alpha_1-1}}{\Gamma(\alpha_1 - \alpha_2) [(\|g\|)^{\alpha_1-1} e^{-\mu\|g\|} + \zeta]}.$$

Note that

$$(g(\tau) - g(s))^{\lambda_1-1} > (g(\tau) - g(s))^{\lambda_2-1}, \quad t, s \in [a, b], \quad t > s, \quad 0 < \lambda_1 < \lambda_2.$$

Therefore,

$$\begin{aligned} \mathfrak{S}_{a,g}^{\lambda_2,\mu} |y(t)| &= \frac{1}{\Gamma(\lambda_2)} \int_a^t (g(t) - g(s))^{\lambda_2-1} e^{-\mu(g(t)-g(s))} |y(s)| g'(s) ds \\ &\leq \frac{1}{\Gamma(\lambda)} \int_a^t (g(t) - g(s))^{\lambda_1-1} e^{-\mu(g(t)-g(s))} |y(s)| g'(s) ds \\ &= \frac{\Gamma(\lambda_1)}{\Gamma(\lambda)} \mathfrak{S}_{a,g}^{\lambda_1,\mu} |y(t)|, \quad \Gamma(\lambda) := \min\{\Gamma(\lambda_1), \Gamma(\lambda_2)\}. \end{aligned}$$

Hence,

$$|y(t)| \leq M^{**} + M^* \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| + \left[\int_a^b |h(s)| \left(M_1 \mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(s)| + M_2 |y(s)| \right) g'(s) ds \right],$$

where

$$\begin{aligned} M^* &:= \frac{M_1 \Gamma(\alpha_1 - \alpha_2)}{\min\{\Gamma(\alpha_1 - \alpha_2), \Gamma(\alpha_1)\}} + M_2, \\ M^{**} &:= \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} M_0 + M\ell + MM_0 \int_a^b |h(s)| g'(s) ds. \end{aligned} \quad (34)$$

This paragraph will show how our considerations and the demonstrated relationships between differential and integral problems are applicable to boundary /initial value research (cf. [5,42] for the basis in the case of weak topology). Let us begin by examining integral Equation (32) in the space of continuous functions. This paper examines fractional-order integral operators. In the study of differential and integral equations, it is also necessary to control superposition operators acting between corresponding function spaces.

Our analysis starts with functions that satisfy assumption (A2) because the superposition operator is well-studied in this case, and it is straightforward to obtain sufficient conditions to satisfy (A2). This fragment discusses the need for further study of the superposition operator on other spaces, particularly Hölder spaces. The issue of obtaining analogous results for Hölder spaces is still not fully resolved. Paper [43] provides generally sufficient results (cf. [44] for a detailed study of autonomous operators).

In the introduction of this section, we recalled some special cases of the issue under study. We will also analyze the applicability of various fixed-point theorems for the existence of solutions by studying the considered problems and integral equations.

However, it is important to provide a brief explanation for studying continuous solutions. In Hölder spaces, the behavior of the superposition operator is known to be quite

pathological. The acting condition does not imply the continuity or boundedness of this operator in the norm of the Hölder space. Interestingly, the Hölder condition for solutions is not excluded even in the case of discontinuous functions f ! The only straightforward scenario is the previously mentioned situation involving autonomous operators [44].

In the proof of the next theorem, as we are looking for continuous solutions and continuous functions on a compact interval are L_q integrable for any $q \geq 1$, the following well-known lemma will be useful.

Lemma 9 ([45]). Assume that f satisfies Carathéodory conditions. Then, the operator $F(x)(\cdot) = f(\cdot, x(\cdot))$ maps $L_p([a, b])$ into $L_q([a, b])$ if and only if there exist a function $b \in L_q(I)$ and constants $d \geq 0, r > 0$ such that

$$|f(t, x)| \leq b(t) + d|x(t)|^{p/q},$$

for all x from some ball $B(0, r) \subset L_p([a, b])$. Moreover, F is a continuous and bounded operator.

As previously stated, it is possible to study BVPs using equivalent integral forms. As with BVPs of integer order, these forms are useful for proofs. The results will be presented in a general form, and the use of arbitrary constants will allow for the derivation of conclusions for differential problems, as discussed in the previous paragraph. We will commence with a case study where $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, and $\alpha_1 - \alpha_2 < 1$.

Theorem 2. Let $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, $\alpha_1 - \alpha_2 < 1$, $\beta \in [0, 1]$, and $\mu \in \mathbb{R}^+$ be arbitrary, and let $g \in C^1[a, b]$ be a positive increasing function with $g'(t) \neq 0$. Under Assumptions (A1)–(A4), integral Equation (31) admits a solution $y \in C[a, b]$.

Proof. As the presented result is intended to illustrate the applications of the studied fractional-order operators to BVP, we will provide only a brief overview of the proof. The focus of this section is to demonstrate how the properties of the integral and fractional derivative that have been previously established allow us to utilize the steps of the classical proofs. It is important to note that, while the aforementioned properties allow us to apply the classical proofs, there are differences and difficulties that must be considered. Furthermore, we will complete the remaining elements of the proof, including the definition of the operator in question, the selection of an appropriate function space, and the properties of the superposition operator within the aforementioned space.

To begin, we will define an operator whose fixed point will provide the solution to the equation. The operator T is defined as the right-hand side of Equation (32):

$$T(y)(t) = x(a)e^{-\mu g(t)} - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)).$$

We will demonstrate that it is well defined, the domain will be specified, and an invariant set B will be constructed.

By Lemma 1, we see that $\mathfrak{S}_{a,g}^{\alpha_2,\mu} y \in C[a, b]$. Then, (A2) implies that $f(\cdot, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\cdot), y(\cdot)) \in L_p[a, b]$ for some $p > \max\{1, \frac{1}{\alpha_1-\alpha_2}\}$. Thus, our assumption (A3) together with Lemma 9, again by Lemma 1, we obtain $\mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f(\cdot, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\cdot), y(\cdot)) \in C[a, b]$. It was found that T maps $C[a, b]$ into itself.

In considering the functions as elements of $C[a, b]$ (not in Hölder spaces), it is sufficient to investigate pointwise estimation and then the sup-norm, as these are relevant functions for this context. Let us suppose that y is a solution of (32). It is possible to demonstrate the following “a priori” estimation.

In view of our assumptions, and by the semi-group property (cf. Proposition 1), we have

$$\begin{aligned} |y(t)| &\leq \Gamma(\alpha_1) \|g\|_\infty^{\alpha_1-1} + \left| \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \left(M_0(t) + M_1 \mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(t)| + M_2 |y(t)| \right) \right| \\ &\leq M + M_1 \left| \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} \left(\mathfrak{S}_{a,g}^{\alpha_2,\mu} |y(t)| \right) \right| + M_2 \left| \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| \right|, \\ &\leq M + M_1 \left| \mathfrak{S}_{a,g}^{\alpha_1,\mu} |y(t)| \right| + M_2 \left| \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| \right|, \end{aligned}$$

where

$$M = \Gamma(\alpha_1) \|g\|_\infty^{\alpha_1-1} + \|\mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} M_0\|_\infty.$$

As previously stated in Remark 4, the fractional integral operators under consideration are monotonic with respect to their exponent.

Consequently, $\mathfrak{S}_{a,g}^{\alpha_1,\mu} |y(t)| \leq \frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(\lambda)} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)|$, where $\Gamma(\lambda) = \min\{\Gamma(\alpha_2), \Gamma(\alpha_1 - \alpha_2)\}$. Then,

$$|y(t)| \leq M + M_1 \cdot \frac{\Gamma(\alpha_1 - \alpha_2)}{\Gamma(\lambda)} \cdot \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| + M_2 \cdot \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)|.$$

Accordingly, by Theorem 1, we arrive at the following conclusion:

$$|y(t)| \leq Me^{\mu g(t)} \mathcal{E}_{\alpha_1-\alpha_2} \left(M_1 \cdot \frac{\Gamma(\alpha_1 - \alpha_2)}{\Gamma(\lambda)} + M_2 \right) := r.$$

For $0 < \gamma < \gamma_0 = 1 - (\alpha_1 - \alpha_2)$, define the set

$$B = \{x \in \mathbb{H}_0^\gamma[a, b] \subset C[a, b] : \|x\|_\infty \leq r\}.$$

Now, we will discuss the continuity property of the operator T on B . Recall that, by Lemma 1, $T(B) \subset \mathbb{H}_0^{\gamma_0}[a, b]$. This is a superposition of several operators. The following comments will examine them one by one.

Firstly, as $C[a, b] \subset L_p[a, b]$ for any $p > \max\{1, \frac{1}{\alpha_1-\alpha_2}\}$, the operator $F_1 = \mathfrak{S}_{a,g}^{\alpha_2,\mu}$ is continuous (and compact) when acting on $C[a, b]$ by Lemma 1. By (A2) and (A3), and since $\mathfrak{S}_{a,g}^{\alpha_2,\mu} y \in C[a, b]$ for any $y \in C[a, b]$, the superposition operator $F(x, y)(\cdot) = f(\cdot, x(\cdot), y(\cdot))$ maps $C[a, b] \times C[a, b]$ into $L_p[a, b]$ and is continuous. Thus, $\mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f(\cdot, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\cdot), y(\cdot))$ is continuous as an operator from $C[a, b]$ into $C[a, b]$.

Finally, we present is a compactness argument. The criterion for compactness in $C[a, b]$ is the Arzelà–Ascoli theorem, which implies that T must map bounded sets into bounded equicontinuous sets.

It is necessary to observe that the tempered-Riemann–Liouville fractional integral maps bounded subsets of $B \subset \mathbb{H}_0^\gamma[a, b]$ into bounded (Lemma 1) and equicontinuous subsets of this space. This follows easily from the fact that T is compact as acting between $\mathbb{H}_0^\gamma[a, b]$ and $\mathbb{H}_0^{\gamma_0}[a, b]$. Consequently, it maps bounded subsets into relatively compact subsets of the latter space, which are then relatively compact in $C[a, b]$. This implies that the images are equicontinuous in $C[a, b]$. For further details, please see also [7] or (proof of Theorem 5, [22]).

It remains to note that $T(B)$, for $B \subset C[a, b] = \mathbb{H}^0[a, b]$, is a subset of Hölder space $\mathbb{H}_0^{\gamma_0}[a, b]$. But Hölder spaces are compactly embedded in $C[a, b]$ (cf. (Lemma 1, [7])), so $T(B)$ is compact in $C[a, b]$.

The fixed point for T can be obtained by applying the classical Schauder fixed-point theorem. $\square$

We must now examine the case where $\alpha_1 - \alpha_2 \geq 1$. This equation will be investigated for an arbitrary constant C_1 . However, if we wish to solve the differential problem (referring to Equation (29)), we can substitute a specific constant and obtain the result for integral

Equation (33). In the context of equivalent differential problems, the value of this constant is determined by the assumed boundary values.

Theorem 3. Let $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, $\alpha_1 - \alpha_2 \geq 1$, $\beta \in [0, 1]$, and $\mu \in \mathbb{R}^+$ be arbitrary, and let $g \in C^1[a, b]$ be a positive increasing function with $g'(t) \neq 0$. Under Assumptions (A1)–(A4), integral Equation (32) admits a solution $y \in C[a, b]$ for any constant C_1 .

Proof. As with the previous theorem, we begin by defining the operator T . The existence of a fixed point that satisfies the equation will be demonstrated:

$$T(y)(t) = y(a)e^{-\mu g(t)} + C_1 \frac{e^{-\mu g(t)}(g(t))^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} - \mathfrak{I}_{a,g}^{\alpha_1 - \alpha_2, \mu} f(t, \mathfrak{I}_{a,g}^{\alpha_2, \mu} y(t), y(t)),$$

where C_1 is an arbitrary constant.

Once more, we will demonstrate that the operator is well defined by specifying its domain and constructing an invariant set. It should be noted that the present study is conducted under assumptions analogous to those of the preceding theorem. Consequently, the continuity of the operator T is guaranteed by the same proof as previously employed.

In contrast to the previous case, it is not possible to immerse the domain B of T in a Hölder space. For any considered space of order $\gamma \in [0, 1]$, it follows that $\mathfrak{I}_{a,g}^{\alpha_1 - \alpha_2, \mu}$ should be in the space of order $\gamma + (\alpha_1 - \alpha_2) > 1$, which is impossible. We will now verify the hypothesis from the Schäfer fixed-point theorem. Let us recall: Let $F : X \rightarrow X$ be completely continuous. Then, the following alternative holds: Either $x - tF(x) = 0$ has a solution for every $t \in [0, 1]$ or $S = \{x : x = tF(x) \text{ for some } t \in (0, 1)\}$ is unbounded. Our attention must be limited to the properties of T on $C[a, b]$, so the only known property is the boundedness of T (Lemma 1).

Although T is not compact from $C[a, b]$ into $C[a, b]$, it is an “improving” operator. That is, it maps bounded subsets into equicontinuous subsets of $C[a, b]$ (see [1]).

Indeed, let $t, t + h \in [a, b]$, and $x \in C[a, b]$. Then, following the idea from (Lemma 2, [7]), for any $\alpha > 0$, we obtain that

$$|\mathfrak{I}_{a,g}^{\alpha, \mu} x(t + h) - \mathfrak{I}_{a,g}^{\alpha, \mu} x(t)| \leq K_\alpha \cdot \|x\|_\infty h^\alpha$$

for some constant K_α (see (Lemma 2, [7]) for the detailed calculation of this constant) and, thus,

$$\begin{aligned} |T(x)(t + h) - T(x)(t)| &\leq \frac{M_1}{\Gamma(\alpha_1)} \cdot K_{\alpha_1} \cdot \|x\|_\infty h_1^{\alpha_1} + \frac{M_2}{\Gamma(\alpha_1 - \alpha_2)} \cdot K_{\alpha_1 - \alpha_2} \cdot \|x\|_\infty h^{\alpha_1 - \alpha_2} \\ &\leq K \cdot \|x\|_\infty \cdot h^{\alpha_1 - \alpha_2} \end{aligned}$$

(cf. also another direct estimation for $\mathfrak{I}_{a,g}^{\alpha, 0}$ in (Theorem 9, [24])). If we restrict our attention to bounded subsets of $C[a, b]$ we see that their images are bounded and equicontinuous subsets of $C[a, b]$.

We just proved that $T : C[a, b] \rightarrow C[a, b]$ is completely continuous. Put

$$H = \{y \in C[a, b] : y = \zeta T(y) \text{ for some } \mu \in (0, 1)\}.$$

We may now apply the Schäfer fixed-point theorem. In order to demonstrate that the set of fixed points of T is not empty, it is necessary to prove that H is bounded. Let $y \in H$.

In this case, we can use the semi-group property (see Proposition 1)) to obtain

$$\begin{aligned} |y(t)| &= \zeta |T(y)(t)| \leq \mu |y(a)e^{-\mu g(a)}| + \zeta |C_1| \frac{e^{-\mu g(a)} \|g\|_{\infty}^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \\ &\quad + \zeta \left| \mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} \left(M_0(t) + M_1 \mathfrak{S}_{a,g}^{\alpha_2, \mu} |y(t)| + M_2 |y(t)| \right) \right| \\ &\leq \zeta M + \zeta M_1 \left| \mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} \left(\mathfrak{S}_{a,g}^{\alpha_2, \mu} |y(t)| \right) \right| + \zeta M_2 \left| \mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} |y(t)| \right| \\ &\leq \zeta M + \zeta M_1 \left| \mathfrak{S}_{a,g}^{\alpha_1, \mu} |y(t)| \right| + \zeta M_2 \left| \mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} |y(t)| \right|, \end{aligned}$$

where

$$M = \zeta |y(a)e^{-\mu g(a)}| + \zeta |C_1| \frac{e^{-\mu g(a)} \|g\|_{\infty}^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)}.$$

Now, as $\zeta \in (0, 1)$, we can make the following estimation (cf. [39]):

$$|y(t)| \leq M + \frac{M_1}{\Gamma(\alpha_1)} \cdot \mathfrak{S}_{a,g}^{1, \mu} |y(t)| + \frac{M_2}{\Gamma(\alpha_1 - \alpha_2)} \cdot \mathfrak{S}_{a,g}^{1, \mu} |y(t)|.$$

Therefore, we are able to apply the Gronwall-type result and, again by Theorem 1, we obtain

$$|y(t)| \leq M e^{\mu g(t)} \mathcal{E}_1 \left(M_1 \cdot \frac{1}{\Gamma(\alpha_1)} + M_2 \cdot \frac{1}{\Gamma(\alpha_1 - \alpha_2)} \right) := K < \infty.$$

Consequently, the set H of functions $\{y \in C[a, b] : y = \mu T(y) \text{ for some } \mu \in (0, 1)\}$ is bounded and then, by the Schäfer fixed-point theorem, T has a fixed point. $\square$

It is crucial to highlight that the outcome of integral Equation (32) is applicable to the investigation of a diverse range of local boundary value problems. This is due to the arbitrary nature of constant C_1 . As previously stated (in Remark 4), in nonlocal problems of the form of (29), expression C_1 is not only a function but can also depend on the unknown function y . In such cases, the selection of the fixed-point theorem is of paramount importance. The proof of the corollary will be adapted to align with the assumptions of this theorem. Consider Equation (33):

$$\begin{aligned} y(t) &= -\mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(t), y(t)) \\ &\quad + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1 - \alpha_2 - 1}}{(\|g\|)^{\alpha_1 - 1} e^{-\mu \|g\|} + \zeta} \left[\frac{\zeta' + \int_a^b h(s) f(s, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(s), y(s)) g'(s) ds}{\Gamma(\alpha_1 - \alpha_2)} \right]. \end{aligned}$$

The functions and constants presented in this equation have already been previously introduced in Remark 4. It should be noted, however, that these constants will be recalled in the course of the proof. Constants M^* and M^{**} are defined in (34).

Corollary 2. Let $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$, $\alpha_1 - \alpha_2 \geq 1$, $\beta \in [0, 1]$, and $\mu \in \mathbb{R}^+$ be arbitrary, and let $g \in C^1[a, b]$ be a positive increasing function with $g'(t) \neq 0$. Under Assumptions (A1)–(A4), integral Equation (32) admits a solution (33) provided that $\psi \in L_1[a, b]$ and

$$M^* \mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} 1 + \int_a^b |h(s)| \left(M_1 \mathfrak{S}_{a,g}^{\alpha_2, \mu} 1 + M_2 \right) g'(s) ds < 1. \quad (35)$$

Proof. The reasons supporting the significance of acting and the continuity conditions of the operator remains valid in the current form of the operator:

$$\begin{aligned} T(y)(t) &= -\mathfrak{S}_{a,g}^{\alpha_1 - \alpha_2, \mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(t), y(t)) \\ &\quad + \frac{e^{-\mu g(t)} (g(t))^{\alpha_1 - \alpha_2 - 1}}{(\|g\|)^{\alpha_1 - 1} e^{-\mu \|g\|} + \zeta} \left[\frac{\zeta' + \int_a^b h(s) f(s, \mathfrak{S}_{a,g}^{\alpha_2, \mu} y(s), y(s)) g'(s) ds}{\Gamma(\alpha_1 - \alpha_2)} \right], \end{aligned}$$

where

$$\zeta := \int_a^b e^{-\mu g(\tau)} \psi(\tau) (g(\tau))^{\alpha_1-1} d\tau,$$

$$\begin{aligned} \zeta' &= \mathfrak{I}_{a,g}^{\alpha_2,\mu} y(b) + \int_a^b \psi(\tau) \mathfrak{I}_{a,g}^{\alpha_2,\mu} y(\tau) d\tau - \frac{y(a) \|g\|^{\alpha_1} e^{-\mu \|g\|}}{\Gamma(1+\alpha_1)} \\ &\quad - \frac{y(a)}{\Gamma(1+\alpha_1)} \int_a^b e^{-\mu g(\tau)} \psi(\tau) (g(\tau))^{\alpha_1} d\tau \end{aligned}$$

and

$$h(s) = \frac{1}{\Gamma(\alpha_1)} \int_s^b e^{-\mu(g(\tau)-g(s))} (g(\tau) - g(s))^{\alpha_1-1} \psi(\tau) d\tau = \mathfrak{I}_{a,g}^{\alpha_1,\mu} \left[\frac{\psi(s)}{g'(s)} \right].$$

Nevertheless, the boundedness condition necessitates an essential modification. As previously stated in Remark 4, we have the following estimation:

$$|T(y)(t)| \leq M^{**} + M^* \mathfrak{I}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| + \left[\int_a^b |h(s)| \left(M_1 \mathfrak{I}_{a,g}^{\alpha_2,\mu} |y(s)| + M_2 |y(s)| \right) g'(s) ds \right],$$

where

$$M^* = \frac{M_1 \Gamma(\alpha_1 - \alpha_2)}{\min\{\Gamma(\alpha_1 - \alpha_2), \Gamma(\alpha_1)\}} + M_2,$$

$$M^{**} = \mathfrak{I}_{a,g}^{\alpha_1-\alpha_2,\mu} M_0 + M \left(\mathfrak{I}_{a,g}^{\alpha_2,\mu} y(b) + \int_a^b \psi(\tau) \mathfrak{I}_{a,g}^{\alpha_2,\mu} y(\tau) d\tau \right) + M M_0 \int_a^b |h(s)| g'(s) ds.$$

As illustrated, this operator maps bounded sets to bounded sets. It can be demonstrated, as in the proof of the theorem, that the images of bounded sets are also completely continuous sets in $C[a, b]$. Consequently, it is completely continuous.

In this case, instead of the Schäfer fixed-point theorem, the Leray–Schauder alternative for completely continuous operators will be useful.

Denote

$$K_1 = M^* \mathfrak{I}_{a,g}^{\alpha_1-\alpha_2,\mu} 1 + \int_a^b |h(s)| \left(M_1 \mathfrak{I}_{a,g}^{\alpha_2,\mu} 1 + M_2 \right) g'(s) ds.$$

Put $R = \frac{M^{**}}{1-K_1}$. Since $M^{**} > 0$ and from our assumptions $K_1 \in (0, 1)$, $R > 0$. Consider the eigenvalue problem

$$y(t) = \zeta T(y)(t)$$

with $\|y\|_\infty = R$ and $\zeta \in (0, 1)$. Then, a solution $y \in C[a, b]$ of this problem satisfies the following estimation

$$|y(t)| \leq \zeta \left[M^{**} + M^* \mathfrak{I}_{a,g}^{\alpha_1-\alpha_2,\mu} |y(t)| + \left[\int_a^b |h(s)| \left(M_1 \mathfrak{I}_{a,g}^{\alpha_2,\mu} |y(s)| + M_2 |y(s)| \right) g'(s) ds \right] \right],$$

and, consequently,

$$\|y\|_\infty \leq \zeta M^{**} + K_1 \cdot \|y\|_\infty.$$

Therefore,

$$R = \|y\|_\infty < M^{**} + R \cdot \left[M^* \mathfrak{I}_{a,g}^{\alpha_1-\alpha_2,\mu} 1 + \int_a^b |h(s)| \left(M_1 \mathfrak{I}_{a,g}^{\alpha_2,\mu} 1 + M_2 \right) g'(s) ds \right].$$

Based on assumption (35), $R - K_1 R < M^{**}$; then, $R < \frac{M^{**}}{1-K_1}$, when combined with the preceding equation, leads to a contradiction. Consequently, in accordance with the Leray–Schauder alternative, there exists a fixed point y of T in $B_R \subset C[a, b]$. $\square$

Corollary 3. Under Assumptions (A1)–(A4), problem (29) with $\alpha_1 - \alpha_2 \geq 1, \beta \in [0, 1]$ admits a solution $x \in AC[a, b]$.

Proof. Let $y \in C[a, b]$ be a solution of (32). Since $\mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f = \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2-1,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2-1,\mu} f$, we infer that $y \in AC[a, b]$ and $y(a) = y_0$. Consequently, by Lemma 8, we have

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} y(t) &= 0 - {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2-1,\mu} \mathfrak{S}_{a,g}^{1,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) \\ &= -\mathfrak{d}\left({}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2-1,\mu}\right) \mathfrak{S}_{a,g}^{1,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) \\ &= -\mathfrak{d}\mathfrak{S}_{a,g}^{1,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) = -f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)). \end{aligned}$$

Also, a trivial calculation shows that

$$\mathfrak{S}_{a,g}^{\alpha_2,\mu} y(b) + \int_a^b \psi(\tau) \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(\tau) d\tau = \ell.$$

So, $y \in AC[a, b]$ is a solution of problem (30).

Now, define $x := \mathfrak{S}_{a,g}^{\alpha_2,\mu} y \in AC[a, b]$. Obviously, $x(a) = \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(a) = 0$. Also, in view of Lemma 8, ${}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} y = y$. Moreover,

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} x(t) &= \left({}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} y\right)(t) = \mathfrak{d}\left({}^H\mathbb{D}_{a,g}^{\alpha_1-1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} y\right)(t) \\ &= \mathfrak{d}\left({}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2-1,\beta,\mu} y\right)(t) = {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} y. \end{aligned}$$

It can therefore be concluded that $x := \mathfrak{S}_{a,g}^{\alpha_2,\mu} y$ is a solution for problem (29) that is absolutely continuous. $\square$

Corollary 4. If Assumptions (A1)–(A3) are satisfied, along with

(B2) For each $x, y \in C[a, b]$, $f(\cdot, x(\cdot), y(\cdot)) \in \mathbb{H}_0^{\zeta+\alpha_1-\alpha_2+\beta(1-\alpha_1+\alpha_2)}[a, b]$, for some $\zeta \in (0, 1)$ such that $\zeta + \alpha_1 - \alpha_2 + \beta(1 - \alpha_1 + \alpha_2) < 1$,

then problem (29) with $\alpha_1 - \alpha_2 < 1, \beta \in [0, 1]$ admits a solution $x \in C[a, b]$.

Proof. Let $y \in C[a, b]$ be a solution of (32). By Lemma 1,

$\mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f \in \mathbb{H}_0^{\zeta+\alpha_1-\alpha_2+\beta(1-\alpha_1+\alpha_2)}[a, b]$. Hence, $y - y_0 e^{-\mu g(\cdot)} \in \mathbb{H}_0^{\zeta+\alpha_1-\alpha_2+\beta(1-\alpha_1+\alpha_2)}[a, b]$ and

$$y(t) = y(a) e^{-\mu g(t)} - \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)). \quad (36)$$

Consequently, we have $y(a) = y_0$, and by Lemma 7 we have

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} \left(y(t) - y(a) e^{-\mu g(t)}\right) &= -{}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_1-\alpha_2,\mu} f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)) \\ &= -f(t, \mathfrak{S}_{a,g}^{\alpha_2,\mu} y(t), y(t)). \end{aligned}$$

Hence, we arrive at problem (30).

Now, define $x := \mathfrak{S}_{a,g}^{\alpha_2,\mu} \left(y - y(a) e^{-\mu g(\cdot)}\right) \in \mathbb{H}_0^{\zeta+\alpha_1-\alpha_2+\beta(1-\alpha_1+\alpha_2)}[a, b]$. Obviously, $x(a) = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x(a) = 0$ and, in view of Lemma 8, we obtain

$${}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} \left(y - y(a) e^{-\mu g(\cdot)}\right) = y - y(a) e^{-\mu g(\cdot)}.$$

Since ${}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} e^{-\mu g(\cdot)} = e^{-\mu g(\cdot)}$, it follows $y = {}^H\mathbb{D}_{a,g}^{\alpha_2,\beta,\mu} x$. Moreover,

$$\begin{aligned} {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} x(t) &= {}^H\mathbb{D}_{a,g}^{\alpha_1,\beta,\mu} \mathfrak{S}_{a,g}^{\alpha_2,\mu} \left(y(t) - y(a) e^{-\mu g(t)}\right) \\ &= {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_1,\beta,\mu} \left(y(t) - y(a) e^{-\mu g(t)}\right) = {}^H\mathbb{D}_{a,g}^{\alpha_1-\alpha_1,\beta,\mu} y(t). \end{aligned}$$

Hence, $x = \mathfrak{S}_{a,g}^{\alpha_2,\mu} \left(y - y(a)e^{-\mu g(\cdot)} \right)$ solves problem (29). $\square$

Finally, we can discuss problem (29) when $\beta = 1$, namely

$$\left\{ \begin{array}{l} \frac{d_{a,g}^{\alpha_1,\mu}}{dt^{\alpha_1}} x(t) + f(t, x(t), \frac{d_{a,g}^{\alpha_2,\mu}}{dt^{\alpha_2}} x(t)) = 0, \quad t \in [a, b], \quad \beta \in [0, 1], \quad \alpha_1 \in (1, 2), \quad \alpha_2 \in (0, 1), \\ \left\{ \begin{array}{ll} x(a) = \frac{d_{a,g}^{\alpha_2,\mu}}{dt^{\alpha_2}} x(a) = 0, & \text{when } \alpha_1 - \alpha_2 < 1, \text{ or} \\ x(b) + \int_a^b \psi(\tau) x(\tau) d\tau = \ell \in \mathbb{R}, \quad x(a) = 0, & \text{when } \alpha_1 - \alpha_2 \geq 1. \end{array} \right. \end{array} \right. \quad (37)$$

By applying (12) to all occurrences of the Caputo derivative, in problem (37), we can reformulate the latter as an equivalent problem that only involves the Riemann–Liouville derivative. This new problem can also be expressed as (29) with a slightly modified function f , where $\beta = 0$. By continuing to argue as previously outlined, it can be demonstrated that a solution is Hölder continuous, i.e., $x \in \mathbb{H}_0^\gamma[a, b]$ exists.

5. Concluding Remarks and Future Work

In this paper, we present a comprehensive generalized tempered calculus, examining the interdependencies between integral and differential operators. The results obtained not only encompass and extend the previously known theorems but also permit the avoidance of difficulties encountered in techniques based on fixed-point theorems by the correct definition of equivalent problems. The problem is of significant importance in both its applications and its impact on the general theory of function spaces and operators.

The study placed significant emphasis on the functional spaces required for the study of fractional-order problems. The study conducted a comprehensive examination of these spaces and identified the constraints associated with operating these operators in absolutely continuous function spaces. The appropriate Hölder spaces are the optimal choice, as they demonstrate the equivalence of differential and integral problems. In this study, we have examined typical Hölder spaces, which allows for a comparison with existing results. It remains an open problem to obtain analogous results for spaces dependent on the g function.

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