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Semi-Symmetric Metric Connections and Homology of CR-Warped Product Submanifolds in a Complex Space Form Admitting a Concurrent Vector Field

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Abstract: In this paper, we conduct a thorough study of CR-warped product submanifolds in a Kaehler manifold, utilizing a semi-symmetric metric connection within the framework of warped product geometry. Our analysis yields fundamental and noteworthy results that illuminate the characteristics of these submanifolds. Additionally, we investigate the implications of our findings on the homology of these submanifolds, offering insights into their topological properties. Notably, we present a compelling proof demonstrating that, under a specific condition, stable currents cannot exist for these warped product submanifolds. Our research outcomes contribute significant knowledge concerning the stability and behavior of CR-warped product submanifolds equipped with a semi-symmetric metric connection. Furthermore, this work establishes a robust groundwork for future explorations and advancements in this particular field of study.

Keywords: CR-submanifolds; warped product manifolds; semi-symmetric; Kaehler manifolds; stable currents

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1. Introduction

The geometry of warped product manifolds has been widely acknowledged as a remarkable framework for modeling the spacetime surrounding black holes and objects with significant gravitational fields. Bishop and O'Neill initially introduced the concept of warped product manifolds [1] to explore manifolds with negative curvature. These manifolds expand upon the idea of Riemannian product manifolds by incorporating warping functions. Specifically, a warped product $B \times_b F$ combines two pseudo-Riemannian manifolds: the base manifold (B, g_B) and the fiber (F, g_F). This combination is achieved through a smooth function b defined on the base manifold B. The resulting metric tensor is denoted as $g = g_B \oplus b^2 g_F$, where the direct sum symbol $\oplus$ represents the direct sum of metric tensors. In this context, the base manifold (B, g_B) represents the underlying space on which the warped product is defined, while the fiber (F, g_F) represents an additional space that is warped or scaled by the warping function b. The warping function b is a smooth function that assigns a positive value to each point in the base manifold B.

B.-Y. Chen [2] made notable advancements in the field of submanifold theory through his examination of warped products. In the context of almost Hermitian manifolds, Chen introduced the notion of CR-warped product submanifolds. He offered valuable insights into the warping function and derived an approximation for the norm of the second fundamental form within the expressions of the warping function. Chen's contributions significantly enriched our understanding of submanifolds and their relationship to warped products in the realm of almost Hermitian manifolds.

Building upon Chen's research, Hesigawa and Mihai [3] delved deeper into these submanifolds within the realm of contact geometry. They specifically examined the contact form associated with CR-warped product submanifolds and derived a similar approximation for the second fundamental form of a contact CR-warped product submanifold embedded in a Kaehler space form.

In an independent investigation [4], it was determined that the homology groups of contact CR-warped product submanifolds immersed in odd-dimensional spheres were trivial. This finding was based on the absence of stable integral currents and the vanishing of homology, indicating the lack of stable currents within such submanifolds.

F. Sahin [5,6] made significant progress in the field by showing that CR-warped product submanifolds exhibited consistent outcomes in both R^n and S^6. This discovery emphasized the similarities in the topological and differentiable characteristics of CR-warped product submanifolds within these two spaces. However, it is worth noting that different scholars have obtained diverse results concerning the topological and differentiable properties of submanifolds when imposing specific constraints on the second fundamental form [4,7–10].

Homology groups play a crucial role in characterizing the topological properties of manifolds, providing an algebraic description that captures essential information about their components, voids, tunnels, and overall structure. Homology theory is a powerful tool with diverse applications, finding relevance in fields such as root construction, molecular docking, image segmentation, and genetic expression analysis. The study of submanifolds and homological theory are intricately intertwined.

Federer and Fleming [11] established a significant connection by demonstrating that non-trivial integral homological groups $H_p(M, \mathbb{Z})$ are linked to the existence of stable currents. This result emphasized the relationship between homology groups and the presence of stable currents within manifolds. Building upon this foundation, Lawson and Simon [12] extended the investigation to submanifolds of spheres and proved that, under a pinching condition on the second fundamental form, integral currents do not exist. This finding provided insights into the absence of integral currents in specific submanifold scenarios.

Leung [13] and Xin [9] further advanced this line of research by extending the results from spheres to Euclidean spaces. Their studies explored the connection between submanifolds in Euclidean spaces and the existence of stable integral currents. In a related investigation, Zhang [14] examined the homology of tori, expanding our understanding of homological properties in this specific context.

Additionally, Liu and Zhang [8] made a significant contribution by proving the non-existence of stable integral currents for certain types of hypersurfaces in Euclidean spaces. This result shed light on the limitations and constraints associated with the presence of stable integral currents in particular scenarios.

The notion of a semi-symmetric linear connection on a Riemannian manifold was initially introduced by Friedmann and Schouten [15]. Subsequently, Hayden [16] provided a definition for a semi-symmetric connection as a linear connection ∇ defined on an n-dimensional Riemannian manifold (M, g), with a torsion tensor T satisfying $T(\varkappa_1, \varkappa_2) = \pi(\varkappa_2)\varkappa_1 - \pi(\varkappa_1)\varkappa_2$, where π is a 1-form and $\varkappa_1, \varkappa_2 \in TM$. K. Yano [17] further investigated the properties of semi-symmetric metric connections, analyzing their characteristics. Notably, Yano demonstrated that a conformally flat Riemannian manifold equipped with a semi-symmetric connection has a curvature tensor that identically vanishes.

Expanding upon these foundational works, Sular and Özgür [18] delved into the exploration of warped product manifolds with a semi-symmetric metric connection, with a specific focus on Einstein warped product manifolds possessing such a connection. They extensively investigated various aspects and properties of these manifolds. Furthermore, in [19], they obtained additional results concerning warped product manifolds with a semi-symmetric metric connection.

Building on the foundations laid by previous studies, this current research seeks to investigate the impact of a semi-symmetric metric connection on CR-warped product submanifolds and their homology within a Kaehler manifold. The primary objective is to comprehend how the presence of a semi-symmetric metric connection influences the properties and topological characteristics of these submanifolds when constructed as warped products. By exploring this relationship, we aim to gain deeper insights into the behavior and structural features of CR-warped product submanifolds in the context of warped product constructions.

2. Preliminaries

Let $(\bar{M}, g)$ denote an even-dimensional Riemannian manifold. An almost Hermitian manifold is defined as a manifold $\bar{M}$ where there exists a tensor field J of type $(1,1)$ on $\bar{M}$ such that the following conditions hold

$$J^2\varkappa_1 = -\varkappa_1$$

$$g(J\varkappa_1, J\varkappa_2) = g(\varkappa_1, \varkappa_2),$$

for $\varkappa_1, \varkappa_2 \in T\bar{M}$. There is a well-known fact that states that an almost Hermitian manifold is classified as a Kaehler manifold if and only if the following condition is satisfied

$$(\bar{\bar{\nabla}}_{\varkappa_1} J)\varkappa_2 = 0, \tag{1}$$

where $\varkappa_1, \varkappa_2 \in T\bar{M}$ and $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g.
Now, defining a connection $\bar{\nabla}$ as

$$\bar{\nabla}_{\varkappa_1}\varkappa_2 = \bar{\bar{\nabla}}_{\varkappa_1}\varkappa_2 + \pi(\varkappa_2)\varkappa_1 - g(\varkappa_1, \varkappa_2)P, \tag{2}$$

such that $\bar{\nabla} g = 0$ for any $\varkappa_1, \varkappa_2 \in T\bar{M}$. The connection $\bar{\nabla}$ is semi-symmetric because $T(\varkappa_1, \varkappa_2) = \pi(\varkappa_2)\varkappa_1 - \pi(\varkappa_1)\varkappa_2$. Using (2) in (1), we have

$$(\bar{\nabla}_{\varkappa_1} J)\varkappa_2 = \pi(J\varkappa_2)\varkappa_1 - g(\varkappa_1, J\varkappa_2)P - \pi(\varkappa_2)J\varkappa_1 + g(\varkappa_1, \varkappa_2)JP. \tag{3}$$

In the case where the holonomy group of the Riemannian manifold $\bar{M}$ leaves a point invariant, it has been established in [20] that a vector field z can be found on $\bar{M}$ which satisfies the equation:

$$\bar{\nabla}_{\varkappa_1} z = \varkappa_1.$$

Such a vector field is commonly referred to as a concurrent vector field. The presence of concurrent vector fields in Riemannian manifolds and other relevant research has been a subject of study by numerous researchers, as evidenced by the works cited in [20–22].

Suppose that the associated vector field P is concurrent [23]; that means

$$\bar{\nabla}_{\varkappa_1} P = \varkappa_1. \tag{4}$$

We define a Kaehler manifold $\bar{M}$ as a complex space form if it possesses a constant J-holomorphic sectional curvature denoted by c, and it is represented as $\bar{M}(c)$.

The curvature tensor $\bar{R}$ associated with the semi-symmetric metric connection $\bar{\nabla}$ is given by

$$\bar{R}(\varkappa_1, \varkappa_2)\varkappa_3 = \bar{\nabla}_{\varkappa_1}\bar{\nabla}_{\varkappa_2}\varkappa_3 - \bar{\nabla}_{\varkappa_2}\bar{\nabla}_{\varkappa_1}\varkappa_3 - \bar{\nabla}_{[\varkappa_1, \varkappa_2]}\varkappa_3. \tag{5}$$

Similarly, we can define the curvature tensor $\bar{\bar{R}}$ for the Riemannian connection $\bar{\bar{\nabla}}$.

Suppose

$$\beta(\varkappa_1, \varkappa_2) = (\bar{\nabla}_{\varkappa_1}\pi)\varkappa_2 - \pi(\varkappa_1)\pi(\varkappa_2) + \frac{1}{2}g(\varkappa_1, \varkappa_2)\pi(P). \tag{6}$$

Now, by the application of (2), (5), and (6), we obtain

$$\begin{aligned} \bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = \bar{\bar{R}}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) + \beta(\varkappa_1, \varkappa_3) g(\varkappa_2, \varkappa_4) \\ - \beta(\varkappa_2, \varkappa_3) g(\varkappa_1, \varkappa_4) + \beta(\varkappa_2, \varkappa_4) g(\varkappa_1, \varkappa_3) - \beta(\varkappa_1, \varkappa_4) g(\varkappa_2, \varkappa_3). \end{aligned} \tag{7}$$

When utilizing the value of $\bar{\bar{R}}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)$, the expression for the curvature tensor $\bar{R}$ of a Kaehler space form $\bar{M}(c)$ equipped with a semi-symmetric metric connection is provided in detail in [23]. This result is further discussed in [24].

$$\begin{aligned} \bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = & \frac{c}{4}\{g(\varkappa_2, \varkappa_3)\varkappa_1 - g(\varkappa_1, \varkappa_3) g(\varkappa_2, \varkappa_4) \\ & + g(\varkappa_1, J\varkappa_3) g(J\varkappa_2, \varkappa_4) - g(\varkappa_2, J\varkappa_3) g(J\varkappa_1, \varkappa_4) \\ & + 2g(\varkappa_1, J\varkappa_2) g(J\varkappa_3, \varkappa_4)\} + \beta(\varkappa_1, \varkappa_3) g(\varkappa_2, \varkappa_4) \\ & - \beta(\varkappa_2, \varkappa_3) g(\varkappa_1, \varkappa_4) + \beta(\varkappa_2, \varkappa_4) g(\varkappa_1, \varkappa_3) \\ & - \beta(\varkappa_1, \varkappa_4) g(\varkappa_2, \varkappa_3) \end{aligned} \tag{8}$$

for all $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in T\bar{M}$.

In the case of a submanifold M isometrically immersed in a differentiable manifold $\bar{M}$, the Gauss and Weingarten formulas for a semi-symmetric metric connection can be derived through a routine calculation. These formulas are given by $\bar{\nabla}_{\varkappa_1}\varkappa_2 = \nabla_{\varkappa_1}\varkappa_2 + h(\varkappa_1, \varkappa_2)$ and $\bar{\nabla}_{\varkappa_1} N = -A_N\varkappa_1 + \nabla^{\perp}_{\varkappa_1} N + \pi(N)\varkappa_1$, where ∇ represents the induced semi-symmetric metric connection on M, N belongs to the normal bundle $T^{\perp}M$, h denotes the second fundamental form of M, $\nabla^{\perp}$ represents the normal connection on $T^{\perp}M$, and A_N is the shape operator. The relationship between the second fundamental form h and the shape operator is given by the following formula:

$$g(h(\varkappa_1, \varkappa_2), N) = g(A_N\varkappa_1, \varkappa_2).$$

For vector fields $\varkappa_1 \in TM$ and $\varkappa_3 \in T^{\perp}M$, we can decompose their relationship as follows:

$$J\varkappa_1 = T\varkappa_1 + F\varkappa_1 \tag{9}$$

and

$$J\varkappa_3 = t\varkappa_3 + f\varkappa_3, \tag{10}$$

where $T\varkappa_1$ (and $t\varkappa_3$), $F\varkappa_1$ (and $f\varkappa_3$) are the tangential and normal parts of $J\varkappa_1$ (and $J\varkappa_3$), respectively.

The Gauss equation for a semi-symmetric connection can be expressed in terms of the Riemannian curvature tensor R as follows:

$$\bar{R}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) = R(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) - g(h(\varkappa_1, \varkappa_4), h(\varkappa_2, \varkappa_3)) + g(h(\varkappa_2, \varkappa_4), h(\varkappa_1, \varkappa_3)) \tag{11}$$

for $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in TM$.

Sular and Özgür (16) conducted a study on warped products denoted as $M_1 \times_f M_2$, where M_1 and M_2 are Riemannian manifolds and f is a positive differentiable function on M_1 referred to as the warping function. Their investigation focused on these warped products in the context of a semi-symmetric metric connection associated with a vector field P on $M_1 \times_f M_2$. The key findings from their work, summarized as a lemma, provide a crucial foundation for subsequent research.

Lemma 1. *Let $M_1 \times_f M_2$ be a warped product manifold with semi-symmetric metric connection $\bar{\nabla}$; we have the following:*

1. *If $P \in TM_1$, then*

$$\bar{\nabla}_{\varkappa_1}\varkappa_3 = \frac{\varkappa_1 f}{f}\varkappa_3 \quad \text{and} \quad \bar{\nabla}_{\varkappa_3}\varkappa_1 = \frac{\varkappa_1 f}{f}\varkappa_3 + \pi(\varkappa_1)\varkappa_3;$$

2. *If $P \in TM_2$, then*

$$\bar{\nabla}_{\varkappa_1}\varkappa_3 = \frac{\varkappa_1 f}{f}\varkappa_3 \quad \text{and} \quad \bar{\nabla}_{\varkappa_3}\varkappa_1 = \frac{\varkappa_1 f}{f}\varkappa_3,$$

where $\varkappa_1 \in TM_1$, $\varkappa_3 \in TM_2$, and π is the 1-form associated with the vector field P.

Consider the warped product submanifold $M = M_1 \times_f M_2$ of a Kaehler manifold $\bar{M}$. In this scenario, we have the curvature tensors R and $\tilde{R}$ associated with the submanifold M and its induced semi-symmetric metric connection ∇ and induced Riemannian connection $\tilde{\nabla}$, respectively. We can express the relationship between these curvature tensors as follows:

$$\begin{aligned} \bar{R}(\varkappa_1,\varkappa_2)\varkappa_3 =& \tilde{R}(\varkappa_1,\varkappa_2)\varkappa_3 + g(\varkappa_3,\bar{\nabla}_{\varkappa_1}P)\varkappa_2 - g(\varkappa_3,\bar{\nabla}_{\varkappa_2}P)\varkappa_1 \\ &+ g(\varkappa_1,\varkappa_3)\bar{\nabla}_{\varkappa_2}P - g(\varkappa_2,\varkappa_3)\bar{\nabla}_{\varkappa_1}P \\ &+ \pi(P)[g(\varkappa_1,\varkappa_3)\varkappa_2 - g(\varkappa_2,\varkappa_3)\varkappa_1] \\ &+ [g(\varkappa_2,\varkappa_3)\pi(\varkappa_1) - g(\varkappa_1,\varkappa_3)\pi(\varkappa_2)]P \\ &+ \pi(\varkappa_3)[\pi(\varkappa_2)\varkappa_1 - \pi(\varkappa_1)\varkappa_2] \end{aligned} \tag{12}$$

for any vector field $\varkappa_1, \varkappa_2, \varkappa_3$ on M [18].

According to part (ii) of Lemma 3.2 in [18], for the warped product submanifold $M = M_1 \times_f M_2$, we obtain the following result:

$$\tilde{R}(\varkappa_1,\varkappa_3)\varkappa_2 = \frac{H^f(\varkappa_1,\varkappa_2)}{f}\varkappa_3, \tag{13}$$

where $\varkappa_1, \varkappa_2 \in TM_1$, $\varkappa_3 \in TM_2$, respectively, and H^f is the Hessian of the warping function.

By considering Equations (12) and (13), we can deduce the following:

$$\begin{aligned} R(\varkappa_1,\varkappa_3)\varkappa_2 = \frac{H^f(\varkappa_1,\varkappa_2)}{f}\varkappa_3 + \frac{Pf}{f}g(\varkappa_1,\varkappa_2)\varkappa_3 + \pi(P)g(\varkappa_1,\varkappa_2)\varkappa_3 + g(\varkappa_2,\nabla_{\varkappa_1}P)\varkappa_3 \\ - \pi(\varkappa_1)\pi(\varkappa_2)\varkappa_3, \end{aligned} \tag{14}$$

where $\varkappa_1, \varkappa_2 \in TM_1, \varkappa_3 \in TM_2, P \in TM_1$, and H^f is the Hessian of the warping function f.

Utilizing part (1) of Lemma 1 and referring to Equation (4), we can deduce that $\frac{Pf}{f} = 0$. By substituting this result into the previous equation, we obtain the following expression:

$$R(\varkappa_1,\varkappa_3)\varkappa_2 = \frac{H^f(\varkappa_1,\varkappa_2)}{f}\varkappa_3 + 2g(\varkappa_1,\varkappa_2)\varkappa_3 - \pi(\varkappa_1)\pi(\varkappa_2)\varkappa_3. \tag{15}$$

For the warped product submanifold $M = M_1 \times_f M_2$ of a Riemanian manifold $\bar{M}$, by employing part (i) of Lemma 2.1, we are able to deduce the following relationship:

$$\nabla_{\varkappa_1}\varkappa_3 = \varkappa_1(\ln f)\varkappa_3 \tag{16}$$

and

$$\nabla_{\varkappa_3}\varkappa_1 = \varkappa_1(\ln f)\varkappa_3 + \pi(\varkappa_1)\varkappa_3, \tag{17}$$

where $\varkappa_1, P \in TM_1$ and $\varkappa_3 \in TM_2$.

It is straightforward to derive the following expression for the Laplacian Δf of the warping function:

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2. \tag{18}$$

3. CR-Warped Product Submanifolds and Their Homology

In [2], Chen proved the existence and non-existence of the warped products of the type $N_T \times_f N_\perp$ and $N_\perp \times_f N_T$ in the Kaehler manifolds, where N_T and $N_\perp$ are the holomorphic and totally real submanifolds, respectively. Basically, he called the warped product of the type $N_T \times_f N_\perp$ CR-warped product submanifolds; however, he named the warped products of the type $N_\perp \times_f N_T$ warped product CR-submanifolds. In this section, we study the impact of semi-symmetric connection on these types of warped product submanifolds in a Kaehler manifold.

Our analysis begins by considering a specific type of submanifold known as warped product CR-submanifolds. These submanifolds take the form of $N_\perp \times_f N_T$ in a Kaehler manifold equipped with a semi-symmetric metric connection and a concurrent vector field P. Here, $N_\perp$ denotes a totally real submanifold, and N_T represents a holomorphic submanifold satisfying $P \in TN_T$. Our investigation results in the following finding:

Theorem 1. *Let $(\bar{M}, g)$ be a Kaehler manifold with an ssm connection. Then there does not exist a WPCR-submanifold of the type $N_\perp \times_f N_T$, such that P is a concurrent vector field tangent to N_T.*

Proof. For any $\varkappa_1, \varkappa_2 \in TN_T$ and $\varkappa_3 \in TN_\perp$, using part (ii) of Lemma 2.1, the Gauss formula, and Equations (7) and (4), we can derive the following expression:

$$\begin{aligned}\varkappa_3(\ln f)g(\varkappa_1,\varkappa_2) + \pi(\varkappa_3)g(\varkappa_1,\varkappa_2) = g(J\bar{\nabla}_{\varkappa_1}\varkappa_3, J\varkappa_2) = g(\bar{\nabla}_{\varkappa_1}J\varkappa_3, J\varkappa_2) - g(\pi(J\varkappa_3)\varkappa_1 \\ - g(\varkappa_1, J\varkappa_3)P - \pi(\varkappa_3)J\varkappa_1 + g(\varkappa_1,\varkappa_3)JP, J\varkappa_2).\end{aligned} \tag{19}$$

Upon further simplification, we obtain

$$\varkappa_3(lnf)g(\varkappa_1,\varkappa_2) = -g(h(\varkappa_1, J\varkappa_2), F\varkappa_3) - \pi(J\varkappa_3)g(\varkappa_1, J\varkappa_2). \tag{20}$$

By exchanging $\varkappa_1$ and $\varkappa_2$, we have

$$\varkappa_3(lnf)g(\varkappa_1,\varkappa_2) = -g(h(\varkappa_2, J\varkappa_1), F\varkappa_3) - \pi(J\varkappa_3)g(\varkappa_2, J\varkappa_1).$$

On the other hand, by substituting $J\varkappa_1$ for $\varkappa_1$, and $J\varkappa_2$ for $\varkappa_2$, we then have

$$\varkappa_3(lnf)g(\varkappa_1,\varkappa_2) = g(h(\varkappa_1, J\varkappa_2), F\varkappa_3) + \pi(J\varkappa_3)g(J\varkappa_1, \varkappa_2).$$

Hence, we have $\varkappa_3(lnf)g(\varkappa_1,\varkappa_2) = 0$, which implies that $\varkappa_3(lnf) = 0$. □

In this study, our focus will be on CRWP-submanifolds of the form $N_T \times_f N_\perp$ that possess an ssm connection, where $P \in TN_T$ is a concurrent vector field. With this objective in mind, we will now introduce the initial results as follows:

Lemma 2. *Let $M = N_T \times_f N_\perp$ be a non-trivial CRWP-submanifold of a Kaehler manifold with an ssm connection and a concurrent vector field P then*

$$g(A_{F\varkappa_3}\varkappa_4, \varkappa_1) = g(A_{F\varkappa_4}\varkappa_3, \varkappa_1), \tag{21}$$

for $\xi, \varkappa_1 \in TN_T$ and $\varkappa_3, \varkappa_4 \in TN_\perp$.

Proof. Making use of Weingarten formula along with (9), we have

$$g(A_{F\varkappa_3}\varkappa_4, \varkappa_1) = -g(\bar{\nabla}_{\varkappa_1}J\varkappa_3, \varkappa_4). \tag{22}$$

Now, using (3) and (16), we obtain the required result. □

Lemma 3. *Let $M = N_T \times_f N_\perp$ be a non-trivial CRWP-submanifold of a Kaehler manifold admitting an ssm connection and a concurrent vector field P, then*

$$g(h(J\varkappa_1,\varkappa_3), J\varkappa_3) = \varkappa_1(\ln f)\|\varkappa_3\|^2 + 2\pi(\varkappa_1)\|\varkappa_3\|^2, \tag{23}$$

for $\varkappa_1 \in TN_T$ and $\varkappa_3 \in TN_\perp$.

Proof. By employing the Gauss formula in conjunction with Equation (7), we can derive the following expression:

$$g(h(J\varkappa_1,\varkappa_3),J\varkappa_3) = g(\bar{\nabla}_{\varkappa_3}J\varkappa_1,J\varkappa_3). \tag{24}$$

On applying the Gauss formula and Equation (17), we obtain

$$g(h(J\varkappa_1,\varkappa_3),J\varkappa_3) = g((\bar{\nabla}_{\varkappa_3}J)\varkappa_1,J\varkappa_3) + g(J\bar{\nabla}_{\varkappa_3}\varkappa_1,J\varkappa_3).$$

Further using Equations (3) and (17), we obtain the required result. □

In this study, we focus on investigating stable currents on CRWP-submanifolds. Our main objective is to prove that under certain specific conditions, the existence of stable currents is ruled out. Furthermore, we highlight the notable results established by Simons, Xin, and Lang, which hold significant recognition in the field.

Lemma 4 ([8,12])**.** *Let M^n be a compact submanifold of dimension n in a space form $\bar{M}(c)$ with positive curvature c. If the second fundamental form satisfies the inequality*

$$\sum_{i=1}^{p}\sum_{s=p+1}^{n}(2|h(x_i,x_j)|^2 - g(h(x_i,x_i),h(x_i,x_s))) < pqc, \tag{25}$$

then there are no stable currents in M^n. Here, $p,q \in \mathbb{Z}^+$ with $p+q=n$, $\{x_1,\ldots,x_n\}$ is an orthonormal basis in T_xM, and $x \in M$. Furthermore, we have $\tilde{H}_p(M^n,Z)=0$ and $\tilde{H}_q(M^n,Z)=0$, where $H_j(M,Z)$ denotes the j-th homology of M with integer coefficients.

Theorem 2. *Let $M^{p+q} = N_T^p \times_f N_\perp^q$ be a compact CRWP-submanifold of complex space form $\bar{M}(4)^{p+2q}$ with an ssm connection and a concurrent vector field P. If the following inequality holds*

$$\Delta f + \sum_{i=1}^{p}\beta(x_i,x_i) + \frac{p}{q}\sum_{j=p+1}^{q}\beta(x_j,x_j) > (2-q)\|\nabla(\ln f)\|^2 - \frac{q}{f}\pi(\nabla f) - 3p - \pi(P) + 2\sum_{i=1}^{p}\pi(x_i), \tag{26}$$

then there are no p-stable currents present in M^{p+q}. Furthermore, the homology groups $H_p(M^n,Z)=0$ and $H_q(M^n,Z)=0$ are satisfied, where $H_j(M,Z)$ represents the j-th homology group of M. Here, p and q denote the dimensions of the holomorphic submanifold N_T^p and the totally real submanifold $N_\perp^q$, respectively.

Proof. Suppose dim $N_T^p = p = 2\alpha$ and dim $N_\perp^q = q$, where N_T and $N_\perp$ are the integral manifolds of holomorphic distribution D_T and the totally real distribution $D_\perp$. Let $\{x_1,x_2,\ldots,x_\alpha,x_{\alpha+1} = Jx_1,\ldots,x_{2\alpha} = Jx_\alpha\}$ and $\{x_{2\alpha+1} = x_1^*,\ldots,x_{p+q} = x_q^*\}$ be the orthonormal basis of TN_T^p and $TN_\perp^q$, respectively. Therefore, an orthonormal basis for the normal subbundle $JD_\perp$ is $\{x_{n+1} = \bar{x}_1 = Jx_1^*,\ldots,x_{n+q} = \bar{x}_q = Jx_q^*\}$.

Thus, we can establish the following relationship:

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2 - g(h(x_i,x_i),h(x_j,x_j))\} &= \sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ &+ \sum_{i=1}^{p}\sum_{j=p+1}^{n}\{\|h(x_i,x_j)\|^2 - g(h(x_i,x_i),h(x_j,x_j))\}.\end{aligned} \tag{27}$$

Applying the Gauss Equation (11):

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 + \sum_{i=1}^{p}\sum_{j=p+1}^{q}g(R(x_i,x_j)x_i,x_j)-\sum_{i=1}^{p}\sum_{j=p+1}^{q}g(\bar{R}(x_i,x_j)x_i,x_j). \tag{28}$$

On making use of Formula (8) for a complex space form $\bar{M}(4)^{p+2q}$:

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=&\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2\\ &-pq-p\sum_{j=p+1}^{q}\beta(x_j,x_j)-q\sum_{i=1}^{p}\beta(x_i,x_i)\\ &+\sum_{i=1}^{p}\sum_{j=p+1}^{q}g(R(x_i,x_j)x_i,x_j).\end{aligned} \tag{29}$$

By considering Equation (15), we can express the relationship for the submanifold $N_T^p\times_f N_\perp^q$ of $\bar{M}^{p+2q}(4)$ as follows:

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{q}g(R(x_i,x_j)x_i,x_j)=&\sum_{i=1}^{p}\sum_{j=p+1}^{q}\frac{H^f(x_i,x_i)}{f}g(x_j,x_j)\\ &+\sum_{i=1}^{p}\sum_{j=p+1}^{q}\{2g(x_i,x_i)g(x_j,x_j)-\pi(x_i)\pi(x_i)g(x_j,x_j).\end{aligned} \tag{30}$$

Ultimately, the subsequent equation is obtained.

$$\sum_{i=1}^{p}\sum_{j=p+1}^{q}g(R(x_i,x_j)x_i,x_j)=\frac{q}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)+2pq-q\pi(P). \tag{31}$$

We first compute the term Δf, which represents the Laplacian of f. The derivation is as follows:

$$\Delta f=-\sum_{k=1}^{n}g(\nabla_{x_k}\nabla f,x_k)=-\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)-\sum_{j=p+1}^{q}g(\nabla_{x_j^*}\nabla f,x_j^*). \tag{32}$$

$$\frac{\Delta f}{f}=-q\|\nabla(\ln f)\|^2-\frac{1}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)-\frac{q}{f}\pi(\nabla f). \tag{33}$$

By utilizing Equation (18), we can determine that

$$\frac{1}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)=-\Delta(\ln f)+(1-q)\|\nabla(\ln f)\|^2-\frac{q}{f}\pi(\nabla f) \tag{34}$$

or

$$\frac{q}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)=-q\Delta(\ln f)+q(1-q)\|\nabla(\ln f)\|^2-\frac{q^2}{f}\pi(\nabla f). \tag{35}$$

By plugging in the aforementioned value into Equation (31), we obtain

$$\sum_{i=1}^{p}\sum_{j=p+1}^{q}R((x_i,x_j)x_i,x_j)=-q\Delta(\ln f)+q(1-q)\|\nabla(\ln f)\|^2+2pq-q\pi(P)-\frac{q^2}{f}\pi(\nabla f). \tag{36}$$

Therefore, by using Equation (29):

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2$$
$$-pq-p\sum_{j=p+1}^{q}\beta(x_j,x_j)-q\sum_{i=1}^{p}\beta(x_i,x_i) \tag{37}$$
$$-q\Delta(\ln f)+q(1-q)\|\nabla(\ln f)\|^2+2pq-q\pi(P)-\frac{q^2}{f}\pi(\nabla f),$$

or

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2$$
$$-(\pi(P)-p)q-p\sum_{j=p+1}^{q}\beta(x_j,x_j)-q\sum_{i=1}^{p}\beta(x_i,x_i) \tag{38}$$
$$-q\Delta(\ln f)+q(1-q)\|\nabla(\ln f)\|^2-\frac{q^2}{f}\pi(\nabla f).$$

Now, let $\varkappa_1=x_\alpha(1\leq\alpha\leq p)$ and $\varkappa_3=x_\beta^*(1\leq\beta\leq q)$:

$$\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2=\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}g(h(x_i,x_j^*),\bar{x}_r^*)^2$$
$$=\sum_{i=1}^{p}\sum_{j,r=1}^{q}\{g(h(x_i,x_j^*),Jx_r^*)^2. \tag{39}$$

Applying Lemma 3 to the equation above, we obtain

$$\sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2=\sum_{i=1}^{\alpha}\sum_{j=p+1}^{q}(x_i(\ln f))^2g(x_j^*,x_j^*)^2+2\sum_{i=1}^{\alpha}\pi(x_i)\|x_j^*\|^2$$
$$+\sum_{i=1}^{\alpha}\sum_{j=p+1}^{q}(Jx_i(\ln f))^2g(x_j^*,x_j^*)^2+2\sum_{i=1}^{\alpha}\pi(Jx_i)\|x_j^*\|^2, \tag{40}$$

or equivalently,

$$\sum_{r=n+1}^{q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2=q\|\nabla(\ln f)\|^2+2\sum_{i=1}^{p}\pi(x_i)\|x_j^*\|^2. \tag{41}$$

By employing Equations (38) and (41), we observe that

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}-4pq=q(2-q)\|\nabla(\ln f)\|^2$$
$$-q\Delta(\ln f)-3pq-p\sum_{j=p+1}^{q}\beta(x_j,x_j)-q\sum_{i=1}^{p}\beta(x_i,x_i)-\frac{q^2}{f}\pi(\nabla f) \tag{42}$$
$$-\pi(P)q+2q\sum_{i=1}^{p}\pi(x_i).$$

Under the assumption that condition (26) is satisfied, we can deduce the following inequality:

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2 - g(h(x_i,x_i),h(x_j,x_j))\} < 4pq. \tag{43}$$

Applying Lemma 4 to the complex space form with $c = 4$ brings us to the final conclusion of our theorem. □

4. Conclusions

In the field of Riemannian manifolds, there are two notable types of smooth connections that have attracted considerable attention: the Levi-Civita connection and the semi-symmetric metric connection. These connections exhibit distinct properties, leading to extensive efforts aimed at comparing and contrasting the geometric attributes of submanifolds associated with each type of connection. While significant research has been conducted on the homology of warped product submanifolds with respect to the Levi-Civita connection, the homology of such submanifolds in the presence of semi-symmetric metric connections remains unexplored. Recognizing this gap in knowledge, our paper sets out to investigate the homology and stable currents of CR-warped product submanifolds within Kaehler manifolds, utilizing a semi-symmetric connection. By focusing on this specific context, our goal is to uncover insights into the topological properties and behavior of generalized warped product submanifolds. We anticipate that the findings of our study will not only contribute to the understanding of homology and stable currents in the realm of CR-warped product submanifolds but also inspire further research on generalized warped product submanifolds and their associated topological characteristics.

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