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Numerical Solution of the Linear Fractional Delay Differential Equation Using Gauss–Hermite Quadrature

Salma Aljawi ^{1,*} , Sarah Aljohani ² , Kamran ^{3,*} , Asma Ahmed ⁴  and Nabil Mlaiki ^{2,*} 

¹ Department of Mathematical Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

² Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia; sjohani@psu.edu.sa

³ Department of Mathematics, Islamia College Peshawar, Peshawar 25120, Khyber Pakhtoonkhwa, Pakistan

⁴ Department of Computer Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia; a.amohammed@ut.edu.sa

* Correspondence: snaljawi@pnu.edu.sa (S.A.); kamran.maths@icp.edu.pk (K.); nmlaiki@psu.edu.sa (N.M.)

Abstract: Fractional order differential equations often possess inherent symmetries that play a crucial role in governing their dynamics in a variety of scientific fields. In this work, we consider numerical solutions for fractional-order linear delay differential equations. The numerical solution is obtained via the Laplace transform technique. The quadrature approximation of the Bromwich integral provides the foundation for several commonly employed strategies for inverting the Laplace transform. The key factor for quadrature approximation is the contour deformation, and numerous contours have been proposed. However, the highly convergent trapezoidal rule has always been the most common quadrature rule. In this work, the Gauss–Hermite quadrature rule is used as a substitute for the trapezoidal rule. Plotting figures of absolute error and comparing results to other methods from the literature illustrate how effectively the suggested approach works. Functional analysis was used to examine the existence of the solution and the Ulam–Hyers (UH) stability of the considered equation.

Keywords: delay differential equation; Caputo’s derivative; Laplace transform; existence; Ulam–Hyers (UH) stability; Gauss–Hermite quadrature



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1. Introduction

Fractional calculus (FC) is a branch of mathematics which generalizes the classical calculus. A close connection between FC and the dynamics of complex real-world problems exists. Despite a long history, the concept of FC has not been applied in engineering or other science fields. However, in the last two decades the FC has attracted a remarkable amount of attention from the research community. It has been pointed out that fractional order operators are very helpful in describing various real world phenomena [1], for example rheology, signal analysis, viscoelasticity, continuum mechanics, polymeric chemistry, etc [2,3]. Differential equations involving fractional order operators arising in engineering and other sciences describe nature adequately in terms of symmetry properties [4]. The authors of [5] studied hyperchaotic behavior of fractional order problems including fractional operator with two fractional orders. Tang et al. [6] modeled the dynamical behaviors of different stages of breast cancer and used the Adams–Bashforth technique for the numerical simulations. The authors of [7] analyzed the dynamics of the dengue infection by formulating a compartmental model of fractional order. In [8], the authors modeled tumor-immune cells interactions using operators from fractional calculus.

The delay or lags are encountered very often in many real-world problems, for example, immunology [9], population dynamics and epidemiology [10], drug therapy [11], respiratory systems [12], etc. Time-fractional delay differential equations (FDDEs) usually

do not have analytical solutions and can be solved only by using some numerical techniques. The development of efficient numerical methods for solving FDDEs has received great attention recently. For instance, Morgado et al. [13] discussed the existence and uniqueness of the solution to FDDEs. They utilized the fractional backward difference method to study the solution to the considered FDDEs. The authors of [14] used the finite difference method for obtaining the approximate solution of FDDEs. In [15], the authors developed a novel method for solving FDDEs and compared their results with the fractional Adams method and fractional predictor–corrector method. The authors of [16] solved FDDEs including Riemann–Liouville and Liouville–Caputo fractional operators using the shifted Jacobi polynomials. Chishti et al. [17] utilized the extended versions of the two numerical methods, the fractional finite difference method and the predictor–corrector method for the approximation of FDDEs. In [18], the authors solved a class of FDDEs including Liouville–Caputo fractional operator using the orthogonal Chelyshkov functions. Naseem et al. [19] used the differential transform method for the analytical approximation of FDDE. Rebenda and Šmarda [20] used a differential transformation approach for solving functional differential equations with multiple delays. Li et al. [21] utilized a spectral scheme to investigate the numerical solution of fractional stochastic delay differential equations with a comprehensive stability analysis. The authors of [22] established a comprehensive analysis of a general class of FDDEs with the Caputo–Fabrizio fractional derivative. Sher et al. [23] used prior estimate method to investigate the existence and uniqueness of solution for a class of evolution fractional order differential equations with a proportional delay using the Caputo derivative. They also studied different kinds of Ulam stability. The authors of [24] studied the asymptotic stability of nonlinear fractional-order differential equations with multiple delays under the Caputo’s fractional derivative with $1 < \alpha < 2$.

The Laplace transform is a powerful method for solving differential equations. However, for computational work, this technique has not become popular, as researchers have focused on discretization methods such as finite difference, finite element, and boundary elements, possibly coupled with Runge–Kutta or linear multistep methods for integration in time. This lack of interest from the research community is possibly due to two factors: (i) the Laplace transform cannot handle nonlinear differential equations; (ii) the numerical inversion of the Laplace transform is very hard to compute. Despite these drawbacks, Laplace transform methods have been utilized recently by researchers, particularly in the area of linear PDEs [25–29]. The solution of many differential equations may be found in terms of the Laplace transform which is then, however, very difficult for inversion via the techniques of complex analysis. Many strategies have been suggested for inverting the Laplace transform. However, the Laplace transform cannot be directly applied to DDEs. In [30], the authors have recently proposed an effective Laplace transform technique to solve second-order DDEs. The author derives some reliable theoretical results to transform the given DDE to an equivalent algebraic equation in the Laplace transform space and solves the transformed equation for the unknown. Finally, the exact solution of the considered DDE is obtained using the techniques of complex analysis. However, sometimes, using the Laplace transform for many DDEs, the analytical inversion of the Laplace transform becomes very laborious and difficult. Thus, the best alternative is to utilize some numerical techniques. The main advantages of the Laplace transform are as follows:

- The Laplace transform converts the complex FDDE into a simpler algebraic equation in the Laplace domain. This transformation significantly reduces the challenge of solving differential equations with fractional derivatives and delays, which leads to a more regulated and efficient procedure.
- The Laplace transform effectively incorporates the initial conditions into the transformed equation. This feature is particularly beneficial for FDDEs because it makes it possible to provide initial states easily without the need for more intricate manipulations.
- The Laplace transform method is applicable to a large class of FDDEs with different kinds of fractional derivative and delay terms. This adaptability makes it a powerful

tool in a variety of fields such as applied mathematics, physics, engineering, and control theory.

In this work, our objective was to use the method developed in [30] for solving FDDEs. We employ the Gauss–Hermite quadrature method for numerical inversion of the Laplace transform. We consider a fractional order DDE given as follows:

$$D_x^\beta y(x) = f(x) + A_1 y(x) + A_2 y(x - \chi), \quad \beta \in (0, 1], \quad x > 0, \quad (1)$$

$$y(x) = \varrho(x), \quad -\chi \leq x \leq 0, \quad y(0) = \gamma_1, \quad (2)$$

where $f: [0, T] \rightarrow \mathbb{R}$ is the linear continuous function, A_1, A_2, γ_1 are real constants, $\varrho(x)$ is the known sufficiently smooth real valued function, and $\chi \in \mathbb{R}^+$ is a finite delay.

2. Preliminaries

Let $\mathcal{J} = [0, T]$, and $\Theta = C(\mathcal{J}, \mathbb{R})$; then, for any function $y \in \Theta$, the supremum norm $\|\cdot\|$ on Θ is defined as

$$\|y\| = \sup_{x \in \mathcal{J}} |y(x)|.$$

Definition 1 ([3]). Let $\beta > 0$ and let $f(x)$ be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. Then, for $x > 0$, we call

$$I_{0+}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds,$$

the Riemann–Liouville fractional integral of $f(x)$ of order β .

Definition 2 ([3]). If $f \in C^p[0, T]$, $p \in \mathbb{N}$, and $\beta \in (p-1, p]$, then the Caputo fractional derivative of order β is defined as

$$D_x^\beta f(x) = \frac{1}{\Gamma(p-\beta)} \int_0^x \frac{\frac{d^p f(s)}{ds^p}}{(x-s)^{\beta-p+1}} ds, \quad x > 0, \quad (3)$$

where $p = [\beta] + 1$ with $[\beta]$ the integer part of β . The above derivative exists if the integral on the right side converges; further, if $0 < \beta \leq 1$, then Equation (3) can be written as follows:

$$D_x^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x \frac{\frac{df(s)}{ds}}{(x-s)^\beta} ds,$$

provided the integral on the right side converges.

Lemma 1 ([31]). Let $0 < \beta \leq 1$ with f continuous on $\mathcal{J} \times J$, where $J \subset \mathbb{R}$ is an unbounded interval. If $y(x)$ is continuous on $(0, T]$ and $y, x \rightarrow f(x, y(x))$ belongs to $L^1[0, T]$, then $y(x)$ satisfies the initial value problem

$$D_x^\beta y(x) = f(x, y(x)), \quad x \in \mathcal{J},$$

$$y(0) = y_0,$$

if and only if it satisfies the following Volterra integral equation:

$$y(x) = y_0 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s, y(s)) ds.$$

Definition 3 ([3]). Let $y(x)$ be a piecewise continuous real valued function defined for $x > 0$; then, its Laplace transform (LT) is denoted and defined as follows:

$$\widehat{y}(\mu) = \mathcal{L}\{y(x)\} = \int_0^{\infty} e^{-\mu x} y(x) dx. \quad (4)$$

Theorem 1 ([3]). The LT of $D_x^\beta y(x)$ is defined as follows:

$$\mathcal{L}\left\{D_x^\beta y(x)\right\} = \mu^\beta \mathcal{L}\{y(x)\} - \sum_{k=0}^{p-1} \mu^{\beta-k-1} y^{(k)}(0), \quad p-1 < \beta < p, \quad p \in \mathbb{N}. \quad (5)$$

Theorem 2 ([30]). Let $q(x)$, $q'(x)$ be continuous on the interval $[-\chi, 0]$; then, the LT of $y(x - \chi)$ is given as follows:

$$\mathcal{L}\{y(x - \chi)\} = \bar{q}(\mu) + \exp(-\mu\chi) \mathcal{L}\{y(x)\},$$

Also, the LT of $y'(x - \chi)$ is given as follows:

$$\mathcal{L}\{y'(x - \chi)\} = \bar{q}'(\mu) + \exp(-\mu\chi) \mathcal{L}\{y'(x)\},$$

where

$$\bar{q}(\mu) = \int_{-\chi}^0 e^{-\mu(x+\chi)} q(x) dx,$$

and

$$\bar{q}'(\mu) = \int_{-\chi}^0 e^{-\mu(x+\chi)} q'(x) dx.$$

3. Existence

In this section, we study the existence of solution to Problems (1) and (2).

Lemma 2. Let $0 < \beta \leq 1$ with f continuous on $\mathcal{J} \times J$, where $J \subset \mathbb{R}$ is an unbounded interval; then, the solution of the problem

$$D_x^\beta y(x) = f(x) + A_1 y(x) + A_2 y(x - \chi), \quad x \in \mathcal{J}, \quad (6)$$

$$y(x) = q(x), \quad -\chi \leq x \leq 0, \quad y(0) = \gamma_1,$$

can be expressed as

$$y(x) = \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [f(s) + A_1 y(s) + A_2 y(s-\chi)] ds, \quad x \in \mathcal{J}.$$

Proof. Using Lemma (1), the solution of the initial value problem (6) can be written as

$$y(x) = \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [f(s) + A_1 y(s) + A_2 y(s-\chi)] ds.$$

To prove the existence of solution to Problems (1) and (2), let us define an operator $\mathcal{S}: \Omega \rightarrow \Omega$ by

$$\mathcal{S}y(x) = \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [f(s) + A_1 y(s) + A_2 y(s-\chi)] ds, \quad x \in \mathcal{J}. \quad (7)$$

□

The following assumptions are needed for further analysis:

Hypothesis 1. For a continuous function $\mathcal{L}(y)$, and a real constant $\mathcal{N} > 0$, the following Lipschitz conditions holds:

$$|\mathcal{L}(y_1) - \mathcal{L}(y_2)| \leq \mathcal{N}|y_1 - y_2|.$$

Hypothesis 2. There exist $\eta_1 > 0$, $\eta_2 > 0$ such that

$$|f(s, y_1, y_2) - f(s, \hat{y}_1, \hat{y}_2)| \leq \eta_1|y_1 - \hat{y}_1| + \eta_2|y_2 - \hat{y}_2|.$$

Theorem 3 (Schauder's fixed point theorem [32]). Suppose that M is a nonempty, convex, compact subset of a Banach space Ω and that $\mathcal{S}: M \rightarrow M$ is a compact operator that maps M into itself. Then, $\mathcal{S}$ has a fixed point in M .

From the considered Problems (1) and (2), the equivalent integral form is obtained as

$$y(x) = \begin{cases} \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds + \int_0^x (x-s) [A_1 y(s) + A_2 y(s-\chi)] ds, & x \in \mathcal{J}, \\ \varrho(x), & -\chi \leq x \leq 0. \end{cases} \quad (8)$$

Since f is linear bounded function, we have $|f(x)| \leq k_f$, $k_f > 0$.

Theorem 4. Under the hypothesis $\mathbf{H}_1$, the considered problem has a solution.

Proof. Let us define the Banach space Ω under the norm described by $\|y\| = \sup_{x \in \mathcal{J}} |y(x)|$. Consider a nonempty, convex, and compact subset M of Ω defined by $M = \{y: y \in B_r, \|y(x_1) - y(x_2)\| \leq N\|x_1 - x_2\|, x_1, x_2 \in \mathcal{J}\}$.

Define the operator $\mathcal{S}: M \rightarrow M$ by

$$\mathcal{S}[y(x)] = \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [A_1 y(s) + A_2 y(s-\chi)] ds, \quad x \in \mathcal{J}. \quad (9)$$

Let $y \in M$; then, to show that M is bounded, we use (9)

$$\begin{aligned} |\mathcal{S}y(x)| &= \left| \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [A_1 y(s) + A_2 y(s-\chi)] ds \right| \\ &\leq |\gamma_1| + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} |f(s)| ds + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [A_1 |y(s)| + A_2 |y(s-\chi)|] ds \\ &\leq |\gamma_1| + \frac{k_f T^\beta}{\Gamma(\beta+1)} + \frac{NA_1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} + \frac{NA_2}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ &\leq |\gamma_1| + \frac{k_f T^\beta}{\Gamma(\beta+1)} + \frac{NA_1 T^\beta}{\Gamma(\beta+1)} + \frac{NA_2 T^\beta}{\Gamma(\beta+1)} \\ &= \rho. \end{aligned}$$

which implies that

$$\|\mathcal{S}y\| \leq \rho.$$

This shows that $\mathcal{S}$ is bounded. Obviously, we can claim that $\mathcal{S}$ maps bounded set to bounded sets into bounded sets.

To show that $\mathcal{S}$ is continuous, let y_n in M since M is compact and contains all of its limit points. Therefore, $y_n \rightarrow y$, as $n \rightarrow \infty$. Therefore, we take

$$\begin{aligned}
|\mathcal{S}y_n - \mathcal{S}y| &= \frac{1}{\Gamma(\beta)} \left| \int_0^x (x-s)^{\beta-1} [A_1(y_n(s) - y(s)) + A_2(y_n(s-\chi) - y(s-\chi))] ds \right| \\
&\leq \frac{A_1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} |y_n(s) - y(s)| ds + \frac{A_2}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} |y_n(s-\chi) - y(s-\chi)| ds \\
&\leq \frac{A_1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} N|x-s| ds + \frac{A_2}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} N|x-s| ds.
\end{aligned} \tag{10}$$

let $|x-s| < \delta$, then

$$\begin{aligned}
|\mathcal{S}y_n - \mathcal{S}y| &< \frac{(A_1 + A_2)N}{\Gamma(\beta)} \int_0^x \delta^{\beta-1} \delta ds \\
&= \frac{(A_1 + A_2)NT\delta^\beta}{\Gamma(\beta)} \\
&= \varepsilon,
\end{aligned}$$

which implies $\|\mathcal{S}y_n - \mathcal{S}y\| < \varepsilon$, where $\varepsilon = \frac{(A_1 + A_2)NT\delta^\beta}{\Gamma(\beta)}$ or $\delta = \left\{ \frac{\Gamma(\beta)\varepsilon}{(A_1 + A_2)NT} \right\}^{\frac{1}{\beta}}$. Since y is continuous, we have $\|\mathcal{S}y_n - \mathcal{S}y\| \rightarrow 0$ as $n \rightarrow \infty$. Hence, the operator $\mathcal{S}$ is continuous.

Next, let $x_1, x_2 \in \mathcal{J}$; then, consider

$$\begin{aligned}
|\mathcal{S}y(x_2) - \mathcal{S}y(x_1)| &= \left| \frac{1}{\Gamma(\beta)} \int_0^{x_2} (x_2-s)^{\beta-1} f(s) ds + \frac{1}{\Gamma(\beta)} \int_0^{x_2} (x_2-s)^{\beta-1} [A_1y(s) + A_2y(s-\chi)] ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1-s)^{\beta-1} f(s) ds - \frac{1}{\Gamma(\beta)} \int_0^{x_1} (x_1-s)^{\beta-1} [A_1y(s) + A_2y(s-\chi)] ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \left[\int_0^{x_1} [(x_1-s)^{\beta-1} - (x_2-s)^{\beta-1}] |f(s)| ds + \int_{x_1}^{x_2} (x_2-s)^{\beta-1} |f(s)| ds \right] \\
&\quad + \frac{1}{\Gamma(\beta)} \left[\int_0^{x_1} [(x_1-s)^{\beta-1} - (x_2-s)^{\beta-1}] |A_1y(s) + A_2y(s-\chi)| ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_2} (x_2-s)^{\beta-1} |A_1y(s) + A_2y(s-\chi)| ds \right] \\
&\leq \frac{k_f}{\Gamma(\beta)} \left(\int_0^{x_1} [(x_1-s)^{\beta-1} - (x_2-s)^{\beta-1}] ds + \int_{x_1}^{x_2} (x_2-s)^{\beta-1} ds \right) \\
&\quad + \frac{1}{\Gamma(\beta)} \left(\int_0^{x_1} [(x_1-s)^{\beta-1} - (x_2-s)^{\beta-1}] ds + \int_{x_1}^{x_2} (x_2-s)^{\beta-1} ds \right) (A_1 + A_2)N.
\end{aligned}$$

Thus

$$\begin{aligned}
|\mathcal{S}y(x_2) - \mathcal{S}y(x_1)| &\leq \frac{k_f}{\Gamma(\beta+1)} \left(x_1^\beta - x_2^\beta + (x_2 - x_1)^\beta - (x_2 - x_1)^\beta \right) \\
&\quad + \frac{1}{\Gamma(\beta+1)} \left(x_1^\beta - x_2^\beta + (x_2 - x_1)^\beta - (x_2 - x_1)^\beta \right) (A_1 + A_2)N \\
&= \frac{2(k_f + (A_1 + A_2)N)}{\Gamma(\beta+1)} (x_1^\beta - x_2^\beta).
\end{aligned}$$

We see that if $x_1 \rightarrow x_2$, then $x_1^\beta - x_2^\beta \rightarrow 0$; obviously, the right of the above equation goes to zero. Therefore, $|\mathcal{S}y(x_2) - \mathcal{S}y(x_1)| \rightarrow 0$ as $x_1 \rightarrow x_2$. As $\mathcal{S}$ is bounded and continuous, it is uniformly continuous. Therefore, $\|\mathcal{S}y(x_2) - \mathcal{S}y(x_1)\| \rightarrow 0$ as $x_1 \rightarrow x_2$. Hence, $\mathcal{S}$ is equi-continuous, so by Arzela–Ascoli theorem, $\mathcal{S}$ is relatively compact. Thus, according to Schauder’s fixed point theorem, the operator $\mathcal{S}$ has at least one fixed point. Consequently, the problem under consideration has a solution. $\square$

4. Stability

Consider the problem

$$D_x^\beta y(x) = f(x) + A_1 y(x) + A_2 y(x - \chi) + g(x), \quad x \in \mathcal{J}, \quad (11)$$

$$y(x) = \varrho(x), \quad -\chi \leq x \leq 0, \quad y(0) = \gamma_1,$$

here, $g(x) \in \Omega$ is a function such that $|g(x)| < \varepsilon$, for $\varepsilon > 0$. Then, (11) has a solution

$$y(x) = \gamma_1 + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [f(s) + A_1 \bar{y}(s) + A_2 \bar{y}(s-\chi) + g(s)] ds, \quad (12)$$

Using Theorem (4), Equation (12) can be written as

$$\mathcal{T}y(x) = y(x) + g(x), \quad x \in \mathcal{J}. \quad (13)$$

From Equation (12), one can use (11)

$$\begin{aligned} |\mathcal{T}y(x) - y(x)| &\leq \frac{\varepsilon}{\Gamma(\beta+1)}, \\ &\leq \mathbb{B}\varepsilon, \end{aligned} \quad (14)$$

where $\mathbb{B} = \frac{1}{\Gamma(\beta+1)}$.

Theorem 5. Problem (6) is UH and generalized to be UH stable if

$$Y = \left(\frac{1}{\Gamma(\beta+1)} (|A_1| + |A_2|) \right) < 1$$

holds.

Proof. Let $y, \bar{y} \in \Omega$ and be unique solution, respectively, of (6); then,

$$\begin{aligned} \|y - \bar{y}\| &= \sup_{x \in \mathcal{J}} |y(x) - \mathcal{T}\bar{y}(x)|, \\ &\leq \sup_{x \in \mathcal{J}} |y(x) - \mathcal{T}y(x)| + \sup_{x \in \mathcal{J}} |\mathcal{T}y(x) - \mathcal{T}\bar{y}(x)|, \\ &\leq \mathbb{B}\varepsilon + Y\|y - \bar{y}\|, \\ &\leq \frac{\mathbb{B}\varepsilon}{1 - Y}. \end{aligned}$$

□

5. Numerical Method

In this section, we describe the proposed numerical method for obtaining the solution of FDDEs. The main steps in our numerical method are as follows: (i) Using the Laplace transform, we reduce the considered FDDE to an algebraic equation; (ii) we solve the reduced equation in the Laplace domain; and (iii) finally, we utilize the inverse Laplace transform to recover the original problem's solution. Since, the analytical inversion is difficult to compute, we instead use the numerical inversion technique. For the numerical inversion of the Laplace transform, we use the Gauss–Hermite quadrature method. The flowchart of the proposed numerical method is shown in Figure 1.

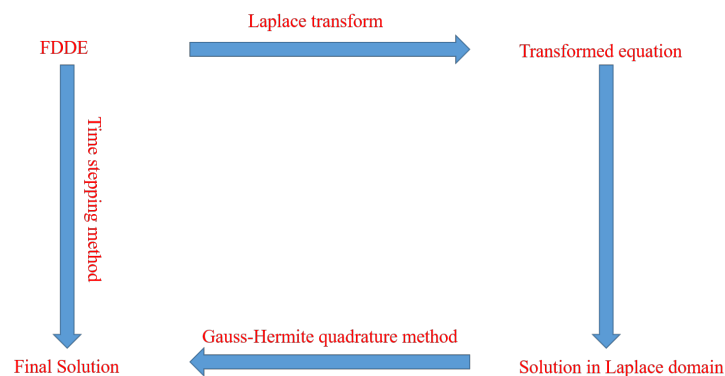


Figure 1. The flowchart of the proposed numerical method.

By applying the LT to (1)–(2), we obtain the following:

$$\begin{aligned}\mathcal{L}\{D_x^\beta y(x)\} &= \mathcal{L}\{f(x) + A_1 y(x) + A_2 y(x - \chi)\}, \\ \Rightarrow \mu^\beta \mathcal{L}\{y(x)\} - \mu^{\beta-1} y(0) &= \mathcal{L}\{f(x)\} + A_1 \mathcal{L}\{y(x)\} + A_2 \mathcal{L}\{y(x - \chi)\},\end{aligned}$$

Then, using Theorem (2), we have

$$\Rightarrow \mu^\beta \hat{y}(\mu) - \mu^{\beta-1} \gamma_1 = \hat{f}(\mu) + A_1 \hat{y}(\mu) + A_2 \bar{q}(\mu) + A_2 \exp(-\mu\chi) \hat{y}(\mu),$$

or

$$\Rightarrow \left[\mu^\beta - A_1 - A_2 \exp(-\mu\chi) \right] \hat{y}(\mu) = \mu^{\beta-1} \gamma_1 + \hat{f}(\mu) + A_2 \bar{q}(\mu). \quad (15)$$

Equation (15) can be simplified as

$$\hat{y}(\mu) = \frac{\mu^{\beta-1} \gamma_1 + \hat{f}(\mu) + A_2 \bar{q}(\mu)}{\mu^\beta - A_1 - A_2 \exp(-\mu\chi)}. \quad (16)$$

Taking the inverse LT of (16), we have

$$y(x) = \mathcal{L}^{-1} \left\{ \frac{\mu^{\beta-1} \gamma_1 + \hat{f}(\mu) + A_2 \bar{q}(\mu)}{\mu^\beta - A_1 - A_2 \exp(-\mu\chi)} \right\}. \quad (17)$$

For numerous functions, the inverse LT in (17) cannot be inverted analytically using the techniques of complex analysis; therefore, numerical inversion methods are then used. The inverse LT in (17) is expressed as

$$y(x) = \frac{1}{2\pi i} \int_{\vartheta-i\infty}^{\vartheta+i\infty} e^{\mu x} \hat{y}(\mu) d\mu = \frac{1}{2\pi i} \int_{\Gamma} e^{\mu x} \hat{y}(\mu) d\mu. \quad (18)$$

Many numerical schemes for the evaluation of the integral (18) use quadrature rule. The function $\hat{y}(\mu)$ is assumed to be analytic in the plane $\Re(\mu) > \vartheta_0$, with ϑ_0 being the convergence abscissa. Typically, Γ is selected to be the line $\Re(\mu) = \vartheta$ with $\vartheta > \vartheta_0$. However, using Cauchy's theorem, it is possible to deform this contour into a better one for computation. These deformations techniques provide efficient methods for analytical and numerical computation of (18). In numerical computation, we use the deformation, as the integral (18) is not suitable for quadrature on the line $\Re(\mu) = \vartheta$ due to the slow decaying transform function $\hat{y}(\mu)$ as $\Im(\mu) \rightarrow \pm\infty$ and the highly oscillatory exponential factor $e^{\mu x}$. The line $\Re(\mu) = \vartheta$ can be deformed to a contour which begins at $-\infty$ in the third quadrant and turns around all the singularities of $\hat{y}(\mu)$ and ends at $+\infty$ in the second quadrant. On such a contour, the integrand decays rapidly due to the factor $\hat{y}(\mu)$; furthermore, if the contour is smooth, this turns the problem in hand into a classical setting in which the trapezoidal converges very quickly. Fewer quadrature nodes and consequently less function

evaluations are needed to achieve the desired accuracy. Such deformation is valid if the singularities of $\hat{y}(\mu)$ and $|\hat{y}(\mu)| \rightarrow 0$ as $\mu \rightarrow \infty$ [33] are all enclosed by the contour. For an effective numerical scheme, the implementation of two key factors must be considered: (i) integration contour and (ii) the quadrature-rule. The most famous methods in this area on the selection of optimal contour, the optimal values of parameters that define the contour and quadrature rule, are reported in [33–37]. In this article, the Gauss–Hermite quadrature (GHQ) rule is utilized as an alternative to the mid-point/trapezoidal rule.

Gauss–Hermite Quadrature Rule

The GHQ rule for a real valued function $h(t)$ defined on real line is given as [38]

$$\int_{-\infty}^{\infty} e^{-t^2} h(t) dt \approx \sum_{\xi=1}^q \eta_{\xi} h(t_{\xi}). \quad (19)$$

The points t_{ξ} , are the roots of $H_{\nu}(t)$, the Hermite polynomial of degree ν . The weights η_{ξ} , are determined uniquely by the property that if $H_{\nu}(t)$ is a polynomial not greater than $\nu - 1$, then the approximation sign can be replaced by an equal sign. For smooth functions $H_{\nu}(t)$, the convergence of Gauss–Hermite can be very fast. The following theorem, which was proven in [39,40], provides clarification for this statement.

Theorem 6. The error in (19) can be expressed as

$$R_{\nu}(h) = \frac{1}{2\pi i} \int_{\Gamma} \phi_{\nu}(\mu) h(\mu) d\mu, \quad (20)$$

where

$$\phi_{\nu}(\mu) = \frac{Q_{\nu}(\mu)}{H_{\nu}(\mu)}, \quad Q_{\nu}(\mu) = \int_{-\infty}^{\infty} e^{-t^2} \frac{H_{\nu}(t)}{\mu - t} dt. \quad (21)$$

The contour Γ in (20) starts at $\mu = -\infty$, turns around the zeros of H_{ν} , and ends at $\mu = \infty$, with $h(\mu)$ assumed to be analytic inside of it. The function $Q_{\nu}(\mu)$ in (21), known as a Hermite function of the second kind, decreases rapidly as $|\mu| \rightarrow \infty$. Similarly, $H_{\nu}(\mu)$ rapidly increases, so the function $\phi_{\nu}(\mu)$ in (20) can be fairly small, provided that the contour Γ is significantly restricted by the growth or singularities of $h(\mu)$. High accuracy corresponds to singularities further from the real axis, and this fact can be used as follows. If we select $t = \zeta x$, $\zeta \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} e^{-t^2} h(t) dt = \zeta \int_{-\infty}^{\infty} e^{-\zeta^2 x^2} h(\zeta x) dx = \zeta \int_{-\infty}^{\infty} e^{-x^2} e^{x^2} e^{-\zeta^2 x^2} h(\zeta x) dx, \quad (22)$$

Now, we can employ the rule (19) to the function $y(x) = e^{x^2(1-\zeta^2)} h(\zeta x)$. Then, $\hat{y}(\mu)$ will have singularities farther from the x -axis as compared to the singularities of $h(\mu)$, and this may improve the accuracy. For evaluation of (18), the following contour is considered:

$$\Gamma: \mu = \vartheta(1 + i\zeta)^2, \quad -\infty < \zeta < \infty, \quad \vartheta > 0. \quad (23)$$

The contour (23) intersects the x -axis at $\mu = \vartheta$ and y -axis at $\mu = \pm 2\vartheta i$. Using (23), in (18), we obtain

$$\int_{\Gamma} \hat{y}(\mu) e^{\mu x} d\mu = \int_{-\infty}^{\infty} e^{\mu(\zeta)x} \hat{y}(\mu(\zeta)) \mu'(\zeta) d\zeta,$$

The scaling technique in (22) can be used via $\zeta = \zeta\sigma$ as follows:

$$y(x) = \frac{\zeta}{2\pi i} \int_{-\infty}^{\infty} e^{-\sigma^2} e^{\sigma^2 + \mu(\zeta\sigma)x} \hat{y}(\mu(\zeta\sigma)) \mu'(\zeta\sigma) d\sigma = \int_{-\infty}^{\infty} e^{-\sigma^2} h(\sigma) d\sigma,$$

or

$$y(x) = \int_{-\infty}^{\infty} e^{-\sigma^2} h(\sigma) d\sigma, \quad (24)$$

where

$$h(\sigma) = \frac{\zeta}{2\pi i} e^{\sigma^2 + \mu(\zeta\sigma)x} \hat{y}(\mu(\zeta\sigma)) \mu'(\zeta\sigma).$$

Now, employing the Gauss–Hermite quadrature to the right side of (24), we have

$$y_{App}(x) \approx 2Re \left\{ \sum_{\xi=1}^p \eta_{\xi} h(\sigma_{\xi}) \right\},$$

where the σ_{ξ} are the positive roots of H_{ν} , and for even ν , we have $p = \nu/2$, and for odd ν , we have $p = (\nu + 1)/2$.

Theorem 7. Suppose $\mathcal{C}: \mathcal{B} \rightarrow \mathcal{B}$ is a contraction mapping, $\mathcal{B}$ is the Banach space, and $0 < \mathcal{S} < 1$ is a constant; then, via the inverse LT, the solution can be expressed as follows:

$$y_k = \mathcal{C}(y_{k-1}), \quad y_{k-1} = \sum_{j=1}^{k-1} y_j, \quad k = 1, 2, 3, \dots$$

and

$$y_k \in M_{\delta}(y) = \{\bar{y} \in \mathcal{B}: \|y - \bar{y}\| < \delta\},$$

$$\lim_{k \rightarrow \infty} y_k = y, \quad \text{where } y_0 = y(0).$$

Proof. We use mathematical induction to obtain the desired result. For $k = 1$, we obtain

$$\|y_1 - y\| = \|\mathcal{C}y_0 - \mathcal{C}y\| \leq \mathcal{S}\|y_0 - y\|.$$

Assuming the result is true for $k - 1$, we have

$$\|y_{k-1} - y\| \leq \mathcal{S}^{k-1} \|y_0 - y\|. \quad (25)$$

Now

$$\|y_k - y\| = \|\mathcal{C}(y_{k-1}) - \mathcal{C}(y)\| \leq \mathcal{S}\|y_{k-1} - y\|. \quad (26)$$

Using (25) and (26), we obtain

$$\|y_k - y\| \leq \mathcal{S}\mathcal{S}^{k-1} \|y_0 - y\| \leq \mathcal{S}^k \delta < \delta.$$

which implies

$$y_k \in M_{\delta}(y). \quad \text{Also } k \rightarrow \infty, \quad \mathcal{S}^k \rightarrow 0.$$

Therefore,

$$\lim_{k \rightarrow \infty} y_k = y.$$

which is the desired result. $\square$

6. Application

In this section, we present the simulation results of the discussed numerical technique for fractional order DDEs. We have presented some numerical examples to show the efficiency of the proposed numerical method. Numerical experiments were performed with $\nu = 20$ using MATLAB. The maximum absolute error err_{∞} and root-mean-squared error rms_{err} were calculated for the considered problems. The error norms are defined as follows :

$$err_{\infty} = \max_{1 \leq j \leq n} |y(x_j) - y_{App}(x_j)|,$$

and

$$rms_{err} = \sqrt{\frac{\sum_{j=1}^n (y(x_j) - y_{App}(x_j))^2}{n}},$$

where $y(x)$ and $y_{App}(x)$ denotes the exact and numerical solutions, respectively.

6.1. Example 1

Consider a FDDE

$$D_x^\beta y(x) = y(x-1) - y(x) + 2x - 1 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}}, \quad x > 0,$$

with delay and initial conditions

$$y(x) = x^2, \quad -1 \leq x \leq 0,$$

and

$$y(0) = 0.$$

The analytic solution is $y(x) = x^2$. The numerical solution of example 1 is obtained by employing GHQ method. The two error norms err_∞ and rms_{err} computed for example 1 are presented in Tables 1 and 2. Numerical and exact solutions are plotted in Figure 2a. In Figure 2b, the err_∞ and rms_{err} are plotted for $x \in [0, 1]$. In Table 1, we present the comparison of the errors of the proposed method with that of the fractional backward difference method [13]. Table 1 indicates that the solutions obtained by using the proposed method are closer to the exact solution when compared with the method of [13].

Table 1. The tabulated results of the err_∞ and rms_{err} computed by the suggested method for problem 1.

x	err_∞	rms_{err}	CPU(s)
0.1	1.4014×10^{-13}	3.1337×10^{-14}	0.110374
0.3	1.2613×10^{-12}	2.8203×10^{-13}	0.130806
0.5	3.5035×10^{-12}	7.8340×10^{-13}	0.155005
0.7	6.8680×10^{-12}	1.5357×10^{-12}	0.116574
0.9	1.1351×10^{-11}	2.5382×10^{-12}	0.117173
1	1.4015×10^{-11}	3.1338×10^{-12}	0.111987
[13]	9.8085×10^{-4}		

Table 2. Numerical solution of example 1.

x	Exact Solution	Proposed Method	[16]
0.2	0.04	0.04	0.04
0.4	0.16	0.16	0.16
0.6	0.36	0.36	0.36
0.8	0.64	0.64	0.64
1.0	1	1	1.028
1.2	1.44	1.44	1.4692
1.4	1.96	1.96	2.0175
1.6	2.56	2.56	2.6861
1.8	3.24	3.24	3.44
2	4	4	4.2456

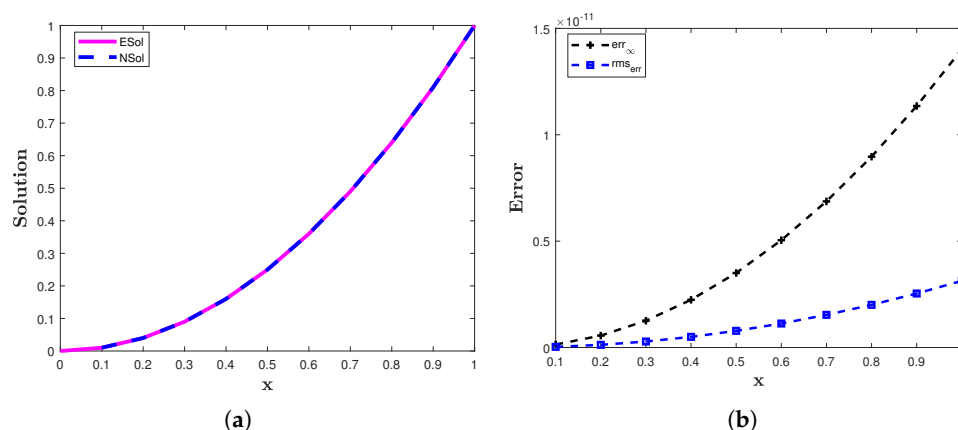


Figure 2. (a) The approximate (NSol) and analytic (ESol) solutions of example 1. (b) The err_{∞} and rms_{err} of the proposed method for example 1 with $x \in [0, 1]$ and $\beta = 0.5$.

6.2. Example 2

Consider a FDDE

$$D_x^{\beta} y(x) = y(x-1) - y(x) + 1 - 3x + 3x^2 + \frac{20t^{3-\beta}}{9\Gamma(3-\beta)}, \quad x > 0,$$

with delay and initial conditions

$$y(x) = x^3, \quad -1 \leq x \leq 0,$$

and

$$y(0) = 0.$$

The analytic solution is $y(x) = x^3$. The two error norms err_{∞} and rms_{err} computed for example 2 are presented in Table 3. In Figure 3a, we illustrate a comparison between the analytic solution and the approximate solution obtained by the proposed method. In Figure 3b, we graphically illustrate the comparison of the err_{∞} and rms_{err} for $x \in [0, 1]$. In Table 3, we present a comparison of the errors of the proposed method and the fractional backward difference method [13]. For this problem, the proposed method also produces accurate results.

Table 3. The tabulated results of the err_{∞} and rms_{err} computed by the suggested method for example 2.

x	err_{∞}	rms_{err}	$CPU(s)$
0.1	3.3054×10^{-13}	7.3911×10^{-14}	0.147631
0.3	8.9247×10^{-12}	1.9956×10^{-12}	0.118002
0.5	4.1318×10^{-11}	9.2389×10^{-12}	0.118522
0.7	1.1338×10^{-10}	2.5352×10^{-11}	0.112424
0.9	2.4097×10^{-10}	5.3881×10^{-11}	0.105683
1	3.3054×10^{-10}	7.3912×10^{-11}	0.101965
[13]	1.4487×10^{-3}		

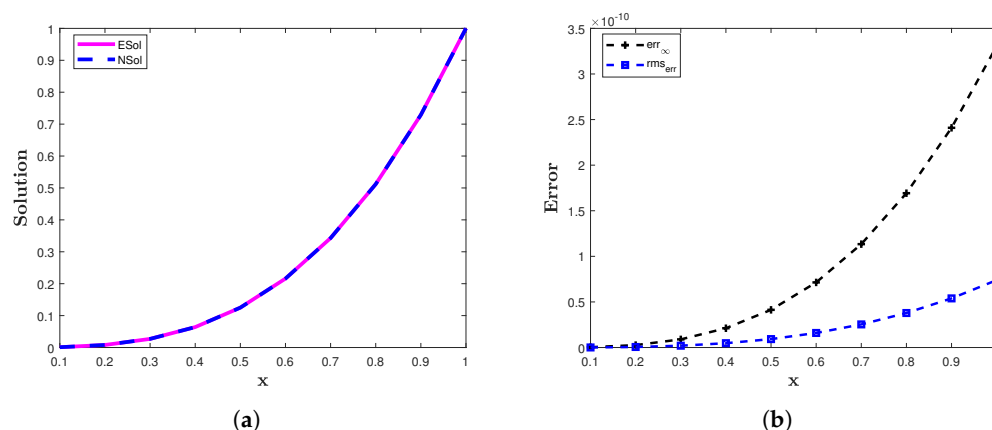


Figure 3. (a) The approximate (NSol) and analytic (ESol) solutions of example 2. (b) The err_{∞} and rms_{err} of the proposed method for example 2 with $x \in [0, 1]$ and $\beta = 0.3$.

6.3. Example 3

Consider a FDDE

$$D_x^{\beta} y(x) = \frac{2}{\Gamma(3-\beta)} x^{2-\beta} - \frac{1}{\Gamma(2-\beta)} x^{1-\beta} + 2\chi x - \chi^2 - \chi - y(x) + y(x-\chi), \quad 0 < \beta \leq 1,$$

with delay and initial conditions

$$y(x) = 0, \quad -1 \leq x \leq 0,$$

and

$$y(0) = 0.$$

The analytic solution is $y(x) = x^2 - x$. This example is solved using the suggested scheme with constant and time-varying delay. The two error norms err_{∞} and rms_{err} computed for example 3 are presented in Table 4. In Table 4, we compare the errors obtained by using our suggested numerical scheme with those results obtained using the Adams–Bashforth–Moulton method [41]. The results show that the suggested scheme produces accurate results. In Figure 4a, a comparison between the analytic solution and the approximate solution obtained by the proposed method is illustrated. In Figure 4b, a comparison between the err_{∞} and rms_{err} is graphically illustrated for $x \in [0, 1]$. The results presented in graphs and Table 4 suggest that this technique can be used in the simulation of high order FDDEs.

Table 4. The tabulated results of the err_{∞} and rms_{err} computed by the suggested method for example 3 with different χ and β .

	x	err_{∞}	rms_{err}	CPU(s)	[41]
$\chi = 0.1$ $\beta = 0.2$	5	3.3402×10^{-04}	7.4690×10^{-05}	0.130427	6.20×10^{-03}
	10	1.8837×10^{-04}	4.2120×10^{-05}	0.102818	1.34×10^{-02}
	50	5.1148×10^{-05}	1.1437×10^{-05}	0.122529	9.90×10^{-02}
$\chi = 0.1$ $\beta = 0.9$	5	3.9423×10^{-03}	8.8153×10^{-04}	0.106599	3.30×10^{-03}
	10	3.6955×10^{-03}	8.2633×10^{-04}	0.109374	3.10×10^{-03}
	50	3.1777×10^{-03}	7.1056×10^{-04}	0.124888	2.30×10^{-03}
$\chi = 0.01e^{-x}$ $\beta = 0.2$	5	4.8500×10^{-10}	1.0845×10^{-10}	0.104304	7.40×10^{-03}
	10	1.3979×10^{-09}	3.1258×10^{-10}	0.100498	8.20×10^{-03}
	50	3.5020×10^{-08}	7.8307×10^{-09}	0.120393	5.20×10^{-03}
$\chi = 0.01e^{-x}$ $\beta = 0.9$	5	2.1570×10^{-09}	4.8231×10^{-10}	0.103760	1.0×10^{-03}
	10	1.3979×10^{-09}	3.1259×10^{-10}	0.104422	4.711×10^{-04}
	50	3.5019×10^{-08}	7.8305×10^{-09}	0.105549	7.030×10^{-04}

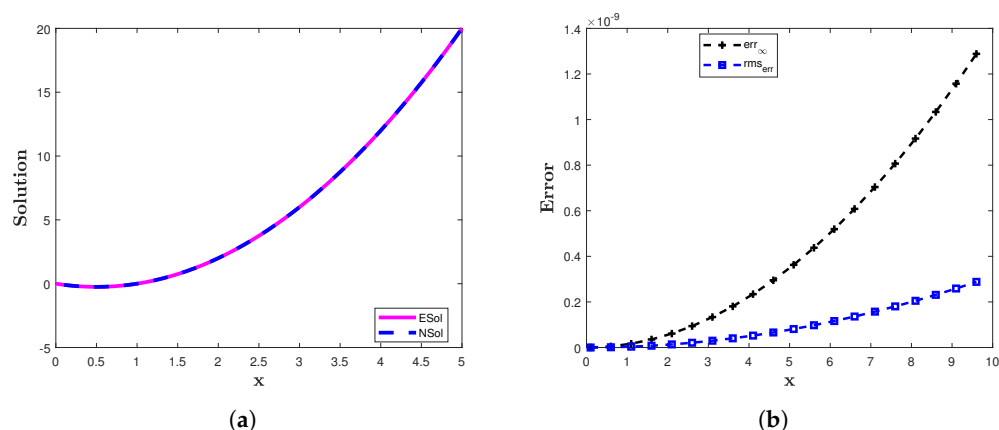


Figure 4. (a) The approximate (NSol) and analytic (ESol) solutions of example 3. (b) The err_{∞} and rms_{err} of the proposed method for example 3 with $x \in [0, 10]$ and $\beta = 0.9$.

6.4. Example 4

Consider a FDDE

$$D_x^{\beta} y(x) = y(x-1) - x, \quad 0 < \beta \leq 1,$$

with delay and initial conditions

$$y(x) = x, \quad -1 \leq x \leq 0,$$

and

$$y(0) = 0.$$

The analytic solution is

$$y(x) = \frac{-2}{\Gamma(0.5)} \sqrt{x}, \quad \text{if } 0 \leq x < 1,$$

and

$$y(x) = \frac{2}{\sqrt{\pi}} \left(\sqrt{x-1} - \sqrt{x} \right) + 1 - x - \frac{2\sqrt{x-1}(1+2x)}{3\sqrt{\pi}}, \quad \text{if } 1 \leq x < 2.$$

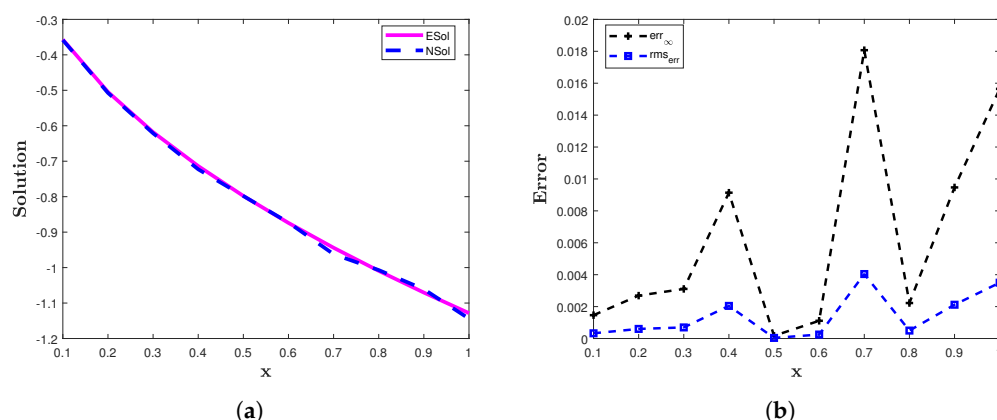
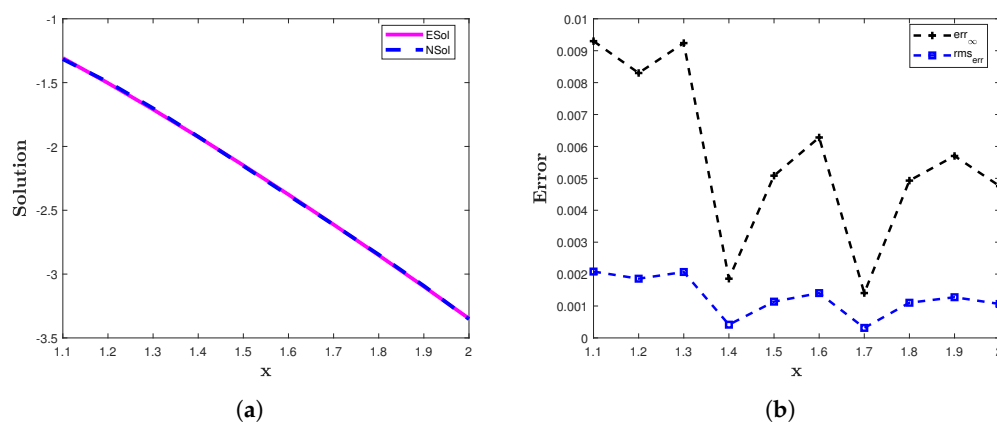
The two error norms err_{∞} and rms_{err} computed for example 4 are presented in Table 5. The graphical demonstration of the analytic solution and the approximate solution obtained by the proposed numerical technique for $0 \leq x < 1$ are presented in Figure 5a and for $1 \leq x < 2$ in Figure 6a. The err_{∞} and rms_{err} are plotted for $x \in [0, 1]$ in Figure 5b and for $x \in [1, 2]$ in Figure 6b. The computed solution of the proposed scheme is compared with the solution of [13] and [16] in Tables 5 and 6. From these results, we can see that the proposed numerical method provides accurate results for wide intervals of x .

Table 5. The tabulated results of the err_{∞} and rms_{err} computed by the suggested method for example 4.

x	err_{∞}	rms_{err}	CPU(s)
0.1	1.4630×10^{-3}	3.2714×10^{-4}	0.124060
0.3	3.1078×10^{-3}	6.9493×10^{-4}	0.125002
0.5	1.6985×10^{-4}	3.7980×10^{-5}	0.126084
0.7	1.8069×10^{-2}	4.0404×10^{-3}	0.104953
0.9	9.4689×10^{-3}	2.1173×10^{-3}	0.130307
1	1.5646×10^{-2}	3.4985×10^{-3}	0.118086
1.1	9.2996×10^{-3}	2.0794×10^{-3}	0.125332
1.3	9.2384×10^{-3}	2.0658×10^{-3}	0.105014
1.5	5.0839×10^{-3}	1.1368×10^{-3}	0.104054
1.7	1.4055×10^{-3}	3.1429×10^{-4}	0.103424
1.9	5.7012×10^{-3}	1.2748×10^{-3}	0.111321
2	4.7573×10^{-3}	1.0638×10^{-3}	0.106843
[13]	9.626×10^{-3}		

Table 6. Numerical solution of example 4.

x	Exact Solution	Proposed Method	[16]
0.20	−0.5046	−0.5073	−0.5044
0.40	−0.7136	−0.7228	−0.7142
0.60	−0.8740	−0.8729	−0.8736
0.80	−1.0093	−1.0070	−1.0096
1.00	−1.1284	−1.1440	−1.1427
1.20	−1.5034	−1.4951	−1.4637
1.40	−1.9254	−1.9236	−1.8815
1.60	−2.3769	−2.3832	−2.4091
1.80	−2.8521	−2.8472	−3.0060
2.00	−3.3480	−3.3528	−3.6385

**Figure 5.** (a) The approximate(NSol) and analytic (ESol) solutions of example 4 for $x \in [0, 1]$. (b) The err_{∞} and rms_{err} of the proposed method for example 4 with $x \in [0, 1]$ and $\beta = 0.5$.**Figure 6.** (a) The numerical (NSol) and exact (ESol) solutions of example 4 for $x \in [1, 2]$. (b) The err_{∞} and rms_{err} of the proposed method for example 4 with $x \in [1, 2]$ and $\beta = 0.5$.

7. Conclusions

In this article, we first examine the existence of solutions to linear FDDEs. Then, we discuss the UH stability of the considered FDDEs. We examine the numerical solution of the FDDEs using the Laplace transform and Gauss–Hermite quadrature method. Our main objective was to develop an efficient numerical method for solving this class of equations, which frequently arise in numerous fields such as signal processing, control theory, and biological systems. To find out if the suggested numerical technique supports the stability of the problem considered, we considered numerical examples with constant and time-varying delay. The following conclusions can be drawn from the results presented in Tables and Figures.

- The error is very small in our approximations.
- The proposed method produces accurate results in a very short computation time.
- The results demonstrate that the proposed method can solve FDDEs efficiently.

Future research could focus on enhancing the stability through numerical methods and investigating the adaptability of this method to more complex and higher-dimensional problems.

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