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The Hopf Automorphism Group of Two Classes of Drinfeld Doubles

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Abstract: Let $D(R_{m,n}(q))$ be the Drinfeld double of Radford Hopf algebra $R_{m,n}(q)$ and $D(H_{s,t})$ be the Drinfeld double of generalized Taft algebra $H_{s,t}$. Both $D(R_{m,n}(q))$ and $D(H_{s,t})$ have very symmetric structures. We calculate all Hopf automorphisms of $D(R_{m,n}(q))$ and $D(H_{s,t})$, respectively. Furthermore, we prove that the Hopf automorphism group $\text{Aut}_{\text{Hopf}}(D(R_{m,n}(q)))$ is isomorphic to the direct sum $\mathbb{Z}_n \oplus \mathbb{Z}_m$ of cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$, the Hopf automorphism group $\text{Aut}_{\text{Hopf}}(D(H_{s,t}))$ is isomorphic to the semi-direct products $\mathbb{k}^* \rtimes \mathbb{Z}_d$ of multiplicative group $\mathbb{k}^*$ and cyclic group $\mathbb{Z}_d$, where $s = td$, $\mathbb{k}^* = \mathbb{k} \setminus \{0\}$, and $\mathbb{k}$ is an algebraically closed field with $\text{char}(\mathbb{k}) \nmid t$.

Keywords: Hopf algebra; Drinfeld double; automorphism group

MSC: 16T05; 16G30



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1. Introduction

In 1986, Drinfeld introduced the concept of a quasitriangular Hopf algebra and provided a way to construct a quasitriangular Hopf algebra $D(H)$ from a finite-dimensional Hopf algebra H . $D(H)$ called the Drinfeld double (or quantum double) of H . The representation category of a quasitriangular Hopf algebra is a braided monoidal category, and the braided structure can provide solutions to the Yang–Baxter equation. Drinfeld doubles of finite dimensional Hopf algebras have attracted the interests of many scholars; see [1–4] for more details.

The automorphism group of an algebra reflects the symmetry of its algebraic structure and plays an important role in understanding and mastering the algebraic structure. Automorphism groups of finite-dimensional algebra are often used on the internet to model social networks, as well as scenarios such as business transactions, recommender systems, and financial data analyses. There are many profound results concerning the automorphism groups of algebras. In [5,6], researchers studied the automorphisms of some quantum enveloping algebras. Automorphisms of many kinds of quantum polynomial algebras were obtained by researchers in [7–9]. In [10,11], researchers studied the automorphisms of some special Hopf algebras, such as path coalgebras and solvable Lie algebras.

The automorphism group has been widely focused on in the study of algebra, but there is no general way to obtain the automorphism group of any given Hopf algebra. Thus, it is essential to study the automorphisms and automorphism groups of Hopf algebras. In this paper, we will study the Hopf automorphism groups of the Drinfeld double $D(R_{m,n}(q))$, Radford Hopf algebra $R_{m,n}(q)$, and the Drinfeld double $D(H_{s,t})$ of the generalized Taft algebra $H_{s,t}$. Both $D(R_{m,n}(q))$ and $D(H_{s,t})$ are the Drinfeld doubles of pointed Hopf algebras of rank one.

This paper is organized as follows: In Section 1, we will review some basic concepts and the structure of $D(R_{m,n}(q))$. In Section 2, we will study the structure of Hopf algebra $D(H_{s,t})$. In Section 3, we will present a class of Hopf automorphisms of $D(R_{m,n}(q))$ and a class of Hopf automorphisms of $D(H_{s,t})$, respectively. In Section 4, we will present

our main results, i.e., that the Hopf automorphism group $\text{Aut}_{\text{Hopf}}(D(R_{m,n}(q)))$ is a finite group, which is isomorphic to the direct sum $\mathbb{Z}_n \oplus \mathbb{Z}_m$ and the Hopf automorphism group $\text{Aut}_{\text{Hopf}}(D(H_{s,t}))$ is an infinite group, which is isomorphic to the semi-direct products $\mathbb{K}^* \rtimes \mathbb{Z}_d$.

2. Preliminaries and Notations

Throughout the following, let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Unless otherwise stated, all algebras and Hopf algebras are defined over $\mathbb{K}$. Our references for basic concepts and notations about Hopf algebras are [12–14]. In particular, for a Hopf algebra, we will use ε , Δ , and S to denote the counit, comultiplication, and antipode, respectively.

Definition 1. Let G be a group. If N is a normal subgroup of G and H is a subgroup of G , such that
 (1) $G = NH$,
 (2) $N \cap H = \{1\}$,
 then G is called the semidirect product of N and H .

Let $0 \neq q \in \mathbb{K}$. For any nonnegative integer n , define $(n)_q$ by $(0)_q = 0$ and $(n)_q = 1 + q + \cdots + q^{n-1}$ for $n > 0$. Observe that $(n)_q = n$ when $q = 1$, and

$$(n)_q = \frac{q^n - 1}{q - 1}$$

when $q \neq 1$. Define the q -factorial of n by $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q$ for $n > 0$. Note that $(n)!_q = n!$ when $q = 1$, and

$$(n)!_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q - 1)^n}$$

when $n > 0$ and $q \neq 1$. The q -binomial coefficients $\binom{n}{i}_q$ are defined inductively as follows for $0 \leq i \leq n$:

$$\begin{aligned} \binom{n}{0}_q &= 1 = \binom{n}{n}_q \quad \text{for } n \geq 0, \\ \binom{n}{i}_q &= q^i \binom{n-1}{i}_q + \binom{n-1}{i-1}_q \quad \text{for } 0 < i < n. \end{aligned}$$

It is well-known that $\binom{n}{i}_q$ is a polynomial in q with integer coefficients and with a value of $q = 1$ equal to the usual binomial coefficient $\binom{n}{i}$, and that

$$\binom{n}{i}_q = \frac{(n)!_q}{(i)!_q(n-i)!_q}$$

when $(n-i)!_q \neq 0$ and $0 < i < n$ (see [13,15])

Let $n > 1$ and $m > 1$ be integers and let $q \in \mathbb{K}$ be a primitive n -th root of unity with $\text{char}(\mathbb{K}) \nmid mn$. Then the Radford Hopf algebra $R_{m,n}(q)$ is generated as an algebra by g and x subject to the following relations:

$$g^{mn} = 1, \quad x^n = g^n - 1, \quad xg = qgx.$$

The comultiplication Δ , counit ε , and antipode S are given by the following:

$$\begin{aligned} \Delta(x) &= x \otimes g + 1 \otimes x, & \varepsilon(x) &= 0, & S(x) &= -xg^{-1}, \\ \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1, & S(g) &= g^{-1}. \end{aligned}$$

$R_{m,n}(q)$ has a $\mathbb{k}$ -basis $\{g^i x^j | i \in \mathbb{Z}_{mn}, 0 \leq j \leq n-1\}$. For more details, one can refer to [2,16–18].

Let $\xi \in \mathbb{k}$ be a primitive mn -th root of unity with $q = \xi^m$. Then, by [2], the Drinfeld double $D(R_{m,n}(q))$ can be described as follows: $D(R_{m,n}(q))$ is generated as an algebra by a, b, c , and d subject to the following relations:

$$\begin{aligned} a^n &= b^n - 1, \quad b^{mn} = 1, \quad c^{mn} = 1, \quad d^n = 0, \\ ab &= \xi^m ba, \quad dc = \xi cd, \quad bc = cb, \quad bd = \xi^m db, \quad ad - da = b - c^m, \\ ac &= \xi^{-1} ca + \frac{\xi^{-1} - \xi^{n-1}}{(n-1)!_q} (c^{m+1} - \xi^m bc) d^{n-1}. \end{aligned}$$

The comultiplication, counit, and antipode of $D(R_{m,n}(q))$ are given by the following:

$$\begin{aligned} \Delta(a) &= a \otimes b + 1 \otimes a, \quad \Delta(b) = b \otimes b, \quad \Delta(d) = c^m \otimes d + d \otimes 1, \\ \Delta(c) &= c \otimes c + (\xi^n - 1) \sum_{t=1}^{n-1} \frac{1}{(t)!_q (n-t)!_q} c^{mt+1} d^{n-t} \otimes cd^t, \\ \varepsilon(a) &= 0, \quad \varepsilon(b) = 1, \quad \varepsilon(c) = 1, \quad \varepsilon(d) = 0, \\ S(a) &= -ab^{-1}, \quad S(b) = b^{-1}, \quad S(c) = c^{-1}, \quad S(d) = -c^{-m}d. \end{aligned}$$

$D(R_{m,n}(q))$ has a $\mathbb{k}$ -basis $\{a^i b^j c^l d^k | j, l \in \mathbb{Z}_{mn}, 0 \leq i, k \leq n-1\}$, and $\dim(D(R_{m,n}(q))) = m^2 n^4$.

3. The Construction of Drinfeld Doubles of Generalized Taft Algebra

In this section, we recall the structure of generalized Taft algebra $H_{s,t}$ in [19] and construct the Drinfeld double $D(H_{s,t})$, where $s, t \geq 2$, $d \geq 1$ are integers and $s = td$. Assume that $\text{char}(\mathbb{k}) \nmid t$.

Let $p \in \mathbb{k}$ be a primitive t -th root of unity. Then the generalized Taft algebra $H_{s,t}$ is generated as an algebra by G and X subject to the following relations:

$$G^s = 1, \quad X^t = 0, \quad XG = pGX.$$

The comultiplication, counit, and antipode of $H_{s,t}$ are given by the following:

$$\begin{aligned} \Delta(X) &= X \otimes G + 1 \otimes X, & \varepsilon(X) &= 0, & S(X) &= -XG^{-1}, \\ \Delta(G) &= G \otimes G, & \varepsilon(G) &= 1, & S(G) &= G^{-1}. \end{aligned}$$

$H_{s,t}$ has a $\mathbb{k}$ -basis $\{G^i X^j | i \in \mathbb{Z}_s, 0 \leq j \leq t-1\}$. For more details, one can refer to [1,19,20].

Definition 2 (Definition IX 2.2 in [13]). A pair (X, A) of bialgebras is considered matched if there exist linear maps $\alpha : A \otimes X \rightarrow X$ and $\beta : A \otimes X \rightarrow A$ turning X into a module coalgebra over A , and turning A into a right module-coalgebra over X , such that, if we set the following:

$$\alpha(a \otimes x) = a \cdot x \text{ and } \beta(a \otimes x) = a^x$$

the following conditions are satisfied:

$$\begin{aligned} a \cdot (xy) &= \sum_{(a)(x)} (a_1 \cdot x_1)(a_2^{x_2} \cdot y), \quad a \cdot 1 = \varepsilon(a)1, \\ (ab)^x &= \sum_{(b)(x)} a^{b_1 \cdot x_1} b_2^{x_2}, \quad 1^x = \varepsilon(x)1, \\ \sum_{(a)(x)} a_1^{x_1} \otimes a_2 \cdot x_2 &= \sum_{(a)(x)} a_2^{x_2} \otimes a_1 \cdot x_1 \end{aligned}$$

for all $a, b \in A$ and $x, y \in X$.

Lemma 1 (Theorem IX.2.3 in [13]). Let (X, A) be a matched pair of Hopf algebras. There exists a unique Hopf algebra structure on the vector space $X \otimes A$, with a unit equal to $1 \otimes 1$, such that its product is given by the following:

$$(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a_1 \cdot y_1) \otimes a_2 y_2 b,$$

its coproduct by

$$\Delta(x \otimes a) = \sum_{(a)(x)} (x_1 \otimes a_1) \otimes (x_2 \otimes a_2),$$

its counit by

$$\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a),$$

and its antipode S given by the following:

$$S(x \otimes a) = \sum_{(x)(a)} S_A(a_2) \cdot S_X(x_2) \otimes S_A(a_1)^{S_X(x_1)}$$

where S_X and S_A are antipodes of X and A , respectively. $X \otimes A$ is called a bicrossed product of X and A and denoted as $X \bowtie A$. Obviously, X and A are subalgebras of $X \bowtie A$ under the injectives $i_X(x) = x \otimes 1$ and $i_A(a) = 1 \otimes a$.

Theorem 1 (Theorem IX.3.5 in [13]). Let $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra with an invertible antipode. Consider the Hopf algebra

$$X = (H^{op})^* = (H^*)^{cop} = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*)$$

Let $\alpha : H \otimes X \rightarrow X$ and $\beta : H \otimes X \rightarrow H$ be the linear maps given by

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a_2)a_1)$$

and

$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a_3)a_1)a_2$$

where $a \in H$ and $f \in X$. Then the pair (H, X) of Hopf algebras is matched in the sense of Definition 2.

Proposition 1 (Definition IX.4.1 in [13]). The Drinfeld double $D(H_{s,t})$ of the Hopf algebra $H_{s,t}$ is the bicrossed product of $H_{s,t}$ and $(H_{s,t}^*)^{cop}$.

$$D(H_{s,t}) = (H_{s,t}^*)^{cop} \bowtie H_{s,t}$$

Proposition 2. The multiplication, comultiplication, and counit in $D(H_{s,t})$ are given by

$$(f \otimes a)(g \otimes b) = \sum_{(a)} f g(S^{-1}(a_3)a_1) \otimes a_2 b$$

$$\varepsilon_D(f \otimes a) = \varepsilon(a)f(1)$$

$$\Delta_D(f \otimes a) = \sum_{(a)(f)} (f_1 \otimes a_1) \otimes (f_2 \otimes a_2)$$

where $f, g \in (H_{s,t}^*)^{cop}$ and $a, b \in H_{s,t}$, $\sum f_1 \otimes f_2$ are determined by the following: $\sum f_1(x)f_2(y) = f(yx)$ for all $x, y \in H_{s,t}$.

Note that $H_{s,t}$ and $(H_{s,t}^*)^{cop}$ are Hopf subalgebras of $D(H_{s,t})$ via the identifications $y = \varepsilon \bowtie y, y \in H_{s,t}$ and $f = f \bowtie 1, f \in (H_{s,t}^*)^{cop}$, respectively, where ε is the unit element of $(H_{s,t}^*)^{cop}$. Let $\{\overline{G^i X^j} | 0 \leq i \leq s-1, 0 \leq j \leq t-1\}$ be the basis in $H_{s,t}^*$ dual to the basis $\{G^i X^j | 0 \leq i \leq s-1, 0 \leq j \leq t-1\}$ of $H_{s,t}$. That is, $\overline{G^i X^j}(G^i X^j) = 1$, and $\overline{G^i X^j}(G^{i'} X^{j'}) = 0$ if $(i', j') \neq (i, j)$, where $0 \leq i, i' \leq s-1, 0 \leq j, j' \leq t-1$.

Lemma 2. The multiplication of $H_{s,t}^*$ is determined by the following:

$$\overline{G^i X^j} * \overline{G^k X^l} = \begin{cases} \binom{l+j}{j}_p \overline{G^i X^{j+l}}, & \text{if } j+l < t \text{ and } k \equiv i+j \pmod{s}, \\ 0, & \text{otherwise.} \end{cases}$$

where $0 \leq i, k \leq s-1$ and $0 \leq j, l \leq t-1$.

Proof. By the coalgebra structure of $H_{s,t}$, we have $\Delta(G^a X^b) = (G^a \otimes G^a)(X \otimes G + 1 \otimes X)^b = \sum_{u=0}^b \binom{b}{u}_p G^a X^{b-u} \otimes G^{a+b-u} X^u$ for $0 \leq a \leq s-1$ and $0 \leq b \leq t-1$. If $0 \leq i, k \leq s-1, 0 \leq j, l \leq t-1$, then $(\overline{G^i X^j} * \overline{G^k X^l})(G^a X^b) = \sum_{u=0}^b \binom{b}{u}_p \overline{G^i X^j}(G^a X^{b-u}) \overline{G^k X^l}(G^{a+b-u} X^u)$. Hence, $(\overline{G^i X^j} * \overline{G^k X^l})(G^a X^b) \neq 0$ if and only if $a = i, b = l+j, j+l < t$, and $k \equiv i+j \pmod{s}$. Obviously, $(\overline{G^i X^j} * \overline{G^k X^l})(G^a X^b) = \binom{l+j}{j}_p$, and so we have the following:

$$\overline{G^i X^j} * \overline{G^k X^l} = \begin{cases} \binom{l+j}{j}_p \overline{G^i X^{j+l}}, & \text{if } j+l < t \text{ and } k \equiv i+j \pmod{s}, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. $\square$

Obviously, $\sum_{i=0}^{s-1} \overline{G^i} = \epsilon$ is the identity of the algebra $(H_{s,t})^*$. Let $\omega \in \mathbb{k}$ be a primitive s -th root of unity with $\omega^d = p$. Put $\alpha = \sum_{i=0}^{s-1} \omega^i \overline{G^i}, \beta = \sum_{i=0}^{s-1} \overline{G^i X}$.

Lemma 3. $H_{s,t}^*$ is generated, as an algebra, by α, β .

Proof. Let A be the subalgebra of $H_{s,t}^*$ generated by α, β . By Lemma 2, we have the following:

$$\begin{aligned} \alpha^0 &= \sum_{i=0}^{s-1} \overline{G^i} = \epsilon \\ \alpha &= \sum_{i=0}^{s-1} \omega^i \overline{G^i} \\ \alpha^2 &= \sum_{i=0}^{s-1} \omega^{2i} \overline{G^i} \\ \alpha^3 &= \sum_{i=0}^{s-1} \omega^{3i} \overline{G^i} \\ &\dots \dots \dots \\ \alpha^{s-1} &= \sum_{i=0}^{s-1} \omega^{(s-1)i} \overline{G^i} \end{aligned}$$

For any $0 \leq i, i' \leq s-1$, we have $\omega^i = \omega^{i'}$ if and only if $i = i'$. Hence, $\overline{G^i} \in A$ for any $0 \leq i \leq s-1$. Moreover, by $\overline{G^i} \beta = \overline{G^i X}, \overline{G^i X} \in A$ for $0 \leq i \leq s-1$. Consequently, $H_{s,t}^* = A$. $\square$

Corollary 1. The following holds in $H_{s,t}^*$.

$$\alpha^s = \epsilon, \beta^t = 0, \beta\alpha = \omega\alpha\beta.$$

Proof. It follows from Lemma 2 and the proof of Lemma 3. $\square$

Corollary 2. $H_{s,t}^*$ has a $\mathbb{k}$ -basis $\{\alpha^i \beta^j | 0 \leq i \leq s-1, 0 \leq j \leq t-1\}$.

Proof. It follows from Lemma 3 and Corollary 1. $\square$

Proposition 3. The comultiplication, counit, and antipode of $(H_{s,t}^*)^{cop}$ are given by the following:

$$\begin{aligned}\Delta(\alpha) &= \alpha \otimes \alpha, & \varepsilon(\alpha) &= 1, & S(\alpha) &= \alpha^{-1}, \\ \Delta(\beta) &= \beta \otimes \epsilon + \alpha^d \otimes \beta, & \varepsilon(\beta) &= 0, & S(\beta) &= -\alpha^{-d} \beta.\end{aligned}$$

Proof. In $H_{s,t}^*$, for any $0 \leq a \leq s-1$, we claim

$$\Delta(\overline{G^a}) = \sum_{(i+j) \equiv a \pmod{s}} \overline{G^i} \otimes \overline{G^j}.$$

In fact, we assume the following: $\Delta(\overline{G^a}) = \sum_{i,k,j,l} \theta_{i,k,j,l} \overline{G^i X^k} \otimes \overline{G^j X^l}$, where $\theta_{i,k,j,l} = \overline{G^a}(G^i X^k G^j X^l)$. By a straightforward computation, one has that $\theta_{i,j,k,l} = 1$ if and only if $k = l = 0$ and $(i+j) \equiv a \pmod{s}$. Moreover, we have $\theta_{i,j,k,l} = 0$ for other cases. Similarly, one can prove

$$\Delta(\overline{G^a X}) = \sum_{(i+j) \equiv a \pmod{s}} \overline{G^i} \otimes \overline{G^j X} + \sum_{(i+j) \equiv a \pmod{s}} p^i \overline{G^i X} \otimes \overline{G^j}.$$

Hence, by a straightforward computation, we have the following:

$$\Delta(\alpha) = \alpha \otimes \alpha, \Delta(\beta) = \epsilon \otimes \beta + \beta \otimes \alpha^d.$$

Consequently, in $(H_{s,t}^*)^{cop}$, we have the following:

$$\Delta(\alpha) = \alpha \otimes \alpha, \Delta(\beta) = \beta \otimes \epsilon + \alpha^d \otimes \beta.$$

It is easy to see that

$$\varepsilon(\overline{G^i X^j}) = \begin{cases} 1, & \text{if } i = j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leq i \leq s-1$ and $0 \leq j \leq t-1$. Consequently, $\varepsilon(\alpha) = 1$, $\varepsilon(\beta) = 0$. Moreover, a straightforward verification shows that $S(\overline{G^i}) = \overline{G^{s-i}}$ and $S(\overline{G^i X}) = -p^{1+i} \overline{G^{s-1-i} X}$, where $0 \leq i \leq s-1$. Hence, we have the following:

$$S(\alpha) = \alpha^{-1}, S(\beta) = -\alpha^{-d} \beta.$$

This completes the proof. $\square$

Proposition 4. Let Δ , ε and S be the comultiplication, counit, and antipode of $D(H_{s,t})$. Then, we have the following:

$$\begin{aligned}\Delta(\epsilon \bowtie X) &= (\epsilon \bowtie X) \otimes (\epsilon \bowtie G) + (\epsilon \bowtie 1) \otimes (\epsilon \bowtie X), \\ \Delta(\epsilon \bowtie G) &= (\epsilon \bowtie G) \otimes (\epsilon \bowtie G), \\ \Delta(\alpha \bowtie 1) &= (\alpha \bowtie 1) \otimes (\alpha \bowtie 1), \\ \Delta(\beta \bowtie 1) &= (\beta \bowtie 1) \otimes (\epsilon \bowtie 1) + (\alpha \bowtie 1)^d \otimes (\beta \bowtie 1). \\ \varepsilon(\epsilon \bowtie X) &= 0, \varepsilon(\epsilon \bowtie G) = 1, \varepsilon(\alpha \bowtie 1) = 1, \varepsilon(\beta \bowtie 1) = 0. \\ S(\epsilon \bowtie X) &= -(\epsilon \bowtie X)(\epsilon \bowtie G)^{-1}, S(\epsilon \bowtie G) = (\epsilon \bowtie G)^{-1}, \\ S(\alpha \bowtie 1) &= (\alpha \bowtie 1)^{-1}, S(\beta \bowtie 1) = -(\alpha \bowtie 1)^{-d}(\beta \bowtie 1).\end{aligned}$$

where $\epsilon \bowtie 1$ is the unit element of $D(H_{s,t})$.

Proof. We only check the rules of the antipode since other rules can be easily obtained. By the definition of the antipode given above, we have the following:

$$\begin{aligned} S(\epsilon \bowtie X) &= G^{-1} \cdot \epsilon \bowtie (-XG^{-1})^\epsilon + (-XG^{-1}) \cdot \epsilon \bowtie 1^\epsilon \\ &= \epsilon \bowtie (-XG^{-1}) = -(\epsilon \bowtie X)(\epsilon \bowtie G)^{-1}, \\ S(\epsilon \bowtie G) &= G^{-1} \cdot \epsilon \bowtie (G^{-1})^\epsilon = (\epsilon \bowtie G)^{-1}, \\ S(\alpha \bowtie 1) &= 1 \cdot \alpha^{-1} \bowtie 1^{\alpha^{-1}} = \alpha^{-1} \bowtie 1 = (\alpha \bowtie 1)^{-1}, \\ S(\beta \bowtie 1) &= 1 \cdot \beta \bowtie 1^{-\alpha^{-d}\beta} + 1 \cdot (-\alpha^{-d}\beta) \bowtie 1^{\alpha^{-d}} = (-\alpha^{-d}\beta) \bowtie 1 = -(\alpha \bowtie 1)^{-d}(\beta \bowtie 1). \end{aligned}$$

□

Let E be an algebra generated by x_1, x_2, x_3 and x_4 , subject to the following relations:

$$\begin{aligned} x_1^t &= 0, \quad x_2^s = 1, \quad x_3^s = 1, \quad x_4^t = 0, \\ x_1x_2 &= \omega^d x_2x_1, \quad x_1x_3 = \omega^{-1}x_3x_1, \quad x_2x_3 = x_3x_2, \\ x_4x_3 &= \omega x_3x_4, \quad x_1x_4 - x_4x_1 = x_2 - x_3^d, \quad x_2x_4 = \omega^d x_4x_2. \end{aligned}$$

Then E is a Hopf algebra with the comultiplication, counit, and antipode determined by the following:

$$\begin{aligned} \Delta(x_1) &= x_1 \bowtie x_2 + 1 \bowtie x_1, & \epsilon(x_1) &= 0, & S(x_1) &= -x_1x_2^{-1}, \\ \Delta(x_2) &= x_2 \bowtie x_2, & \epsilon(x_2) &= 1, & S(x_2) &= x_2^{-1}, \\ \Delta(x_3) &= x_3 \bowtie x_3, & \epsilon(x_3) &= 1, & S(x_3) &= x_3^{-1}, \\ \Delta(x_4) &= x_3^d \bowtie x_4 + x_4 \bowtie 1, & \epsilon(x_4) &= 0, & S(x_4) &= -x_3^{-d}x_4. \end{aligned}$$

One can easily check that E has a $\mathbb{k}$ -basis $\{x_1^i x_2^j x_3^l x_4^k | j, l \in \mathbb{Z}_s, 0 \leq i, k \leq t-1\}$, and $\dim(E) = s^2 t^2$.

Theorem 2. There is an algebra isomorphism φ from E to $D(H_{s,t})$ given by the following:

$$\varphi(x_1) = \epsilon \bowtie X, \quad \varphi(x_2) = \epsilon \bowtie G, \quad \varphi(x_3) = \alpha \bowtie 1, \quad \varphi(x_4) = \beta \bowtie 1.$$

Proof. Let $A = \epsilon \bowtie X$, $B = \epsilon \bowtie G$, $C = \alpha \bowtie 1$ and $D = \beta \bowtie 1$. Since $H_{s,t}$ and $(H_{s,t}^*)^{cop}$ are Hopf subalgebras of $D(H_{s,t})$ as stated before, we have $A^t = 0$, $B^s = 1$, $C^s = 1$, $D^t = 0$, $AB = \omega^d BA$ and $DC = \omega CD$. By Proposition 2 and Lemma 2, we have the following:

$$\begin{aligned} (\epsilon \bowtie X)(\beta \bowtie 1) &= (\sum_{i=0}^{s-1} \overline{G^i} \bowtie X)(\sum_{i=0}^{s-1} \overline{G^i} \bowtie 1) \\ &= \sum_{i=0}^{s-1} \overline{G^i} \bowtie G + \sum_{i=0}^{s-1} \overline{G^i} \bowtie X - \alpha^d \bowtie 1 \\ &= \epsilon \bowtie G + \beta \bowtie X - \alpha^d \bowtie 1. \end{aligned}$$

and

$$\begin{aligned} (\beta \bowtie 1)(\epsilon \bowtie X) &= (\sum_{i=0}^{s-1} \overline{G^i} \bowtie 1)(\sum_{i=0}^{s-1} \overline{G^i} \bowtie X) \\ &= \sum_{i=0}^{s-1} \overline{G^i} \bowtie X \\ &= \beta \bowtie X. \end{aligned}$$

Hence, we have $AD - DA = B - C^d$. Similarly, one can check

$$BC = CB, \quad BD = \omega^d DB, \quad AC = \omega^{-1}CA.$$

It follows that there exists a unique algebra map $\psi : E \rightarrow D(H_{s,t})$, such that $\psi(x_1) = A$, $\psi(x_2) = B$, $\psi(x_3) = C$, and $\psi(x_4) = D$. By Lemma 3 and the definition of $D(H_{s,t})$, $D(H_{s,t})$ is generated as an algebra by A, B, C , and D . Hence, ψ is surjective. By comparing their

dimensions, ψ is an isomorphism. It is easy to see that ψ is also a coalgebra morphism and, consequently, ψ is a Hopf algebra isomorphism. $\square$

4. Some Hopf Automorphisms of $D(R_{m,n}(q))$ and $D(H_{s,t})$

4.1. A Class of Hopf Automorphisms of $D(R_{m,n}(q))$

In this subsection, we provide a class of Hopf automorphisms W of $D(R_{m,n}(q))$.

Proposition 5. Let $\beta \in \mathbb{k}$ be a primitive n -th root of unity, $i \in \mathbb{Z}_n$ and $s \in \mathbb{Z}_m$. Then there is a class of Hopf automorphisms of $D(R_{m,n}(q))$, such that

$$\phi(a) = \beta^i a, \phi(b) = b, \phi(c) = b^{sn} c, \phi(d) = \beta^{-1} d.$$

Denote these Hopf automorphisms of $D(R_{m,n}(q))$ by $\phi_{i,s}$.

Proof. One can easily check that $\phi_{i,s}$ is an algebra endomorphism of $D(R_{m,n}(q))$. Obviously, we have the following:

$$(\phi(a))^n = a^n = b^n - 1 = (\phi(b))^n - 1, (\phi(b))^{mn} = b^{mn} = 1,$$

$$(\phi(c))^{mn} = c^{mn} = 1, (\phi(d))^n = d^n = 0.$$

and

$$\phi(a)\phi(b) = \beta^i ab = \beta^i \xi^m ba = \xi^m \phi(b)\phi(a),$$

$$\phi(d)\phi(c) = \beta^{-i} db^{sn} c = \beta^{-i} \xi b^{sn} cd = \xi \phi(c)\phi(d).$$

Similarly, one can check the following:

$$\phi(a)\phi(d) - \phi(d)\phi(a) = \phi(b) - (\phi(c))^m,$$

$$\phi(b)\phi(c) = \phi(c)\phi(b), \phi(b)\phi(d) = \xi^m \phi(d)\phi(b),$$

$$\phi(a)\phi(c) = \xi^{-1} \phi(c)\phi(a) + \frac{\xi^{-1} - \xi^{n-1}}{(n-1)!_q} ((\phi(c))^{m+1} - \xi^m \phi(b)\phi(c))(\phi(d))^{n-1}.$$

Since Δ is an algebra map of $D(R_{m,n}(q))$ to $D(R_{m,n}(q))$, both $\Delta \circ \phi$ and $(\phi \otimes \phi) \circ \Delta$ are algebra maps of $D(R_{m,n}(q))$ to $D(R_{m,n}(q))$. Furthermore, we have the following:

$$(\Delta \circ \phi)(a) = \Delta(\beta^i a) = \beta^i (a \otimes b + 1 \otimes a) = \phi(a) \otimes \phi(b) + 1 \otimes \phi(a),$$

and

$$((\phi \otimes \phi) \circ \Delta)(a) = (\phi \otimes \phi)(a \otimes b + 1 \otimes a) = \phi(a) \otimes \phi(b) + 1 \otimes \phi(a).$$

Hence, we have $(\Delta \circ \phi)(a) = ((\phi \otimes \phi) \circ \Delta)(a)$. Similarly, one can easily check that $(\Delta \circ \phi)(b) = ((\phi \otimes \phi) \circ \Delta)(b)$, $(\Delta \circ \phi)(c) = ((\phi \otimes \phi) \circ \Delta)(c)$ and $(\Delta \circ \phi)(d) = ((\phi \otimes \phi) \circ \Delta)(d)$. So we have $(\Delta \circ \phi)(x) = ((\phi \otimes \phi) \circ \Delta)(x)$ for any $x \in D(R_{m,n}(q))$, then $(\Delta \circ \phi) = (\phi \otimes \phi) \circ \Delta$. Similarly, one can check that $\varepsilon \circ \phi = \varepsilon$. Therefore, ϕ is a coalgebra endomorphism of $D(R_{m,n}(q))$. Thus, it can be easily checked that $\phi_{i,s}$ is a Hopf homomorphism. $\square$

Let $W := \{\phi_{i,s} | i \in \mathbb{Z}_n, s \in \mathbb{Z}_m\}$.

4.2. A Class of Hopf Automorphisms of $D(H_{s,t})$

In this subsection, we give a class of Hopf automorphisms W' of $D(H_{s,t})$.

Proposition 6. Let $\gamma \in \mathbb{k}^*$ and $r \in \mathbb{Z}_d$. Then there is a class of Hopf automorphisms of $D(H_{s,t})$ such that we have the following:

$$\psi(A) = \gamma^{-1} A, \psi(B) = B, \psi(C) = B^{rt} C, \psi(D) = \gamma D.$$

Denote these Hopf automorphisms of $D(H_{s,t})$ by $\psi_{\gamma,r}$.

Proof. It is similar to the proof of Proposition 5. $\square$

Let $W' := \{\psi_{\gamma,r} | \gamma \in \mathbb{k}^*, r \in \mathbb{Z}_d\}$.

5. The Hopf Automorphism Groups of $R_{m,n}(q)$ and $D(H_{s,t})$

5.1. The Hopf Automorphism Group of $D(R_{m,n}(q))$

In this subsection, we prove that the Hopf automorphism group of $D(R_{m,n}(q))$ is W and isomorphic to the direct sum $\mathbb{Z}_m \oplus \mathbb{Z}_n$ of cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$. Throughout the following, we denote $D(R_{m,n}(q))$ by H , and denote the Hopf automorphism group of $D(R_{m,n}(q))$ by $\text{Aut}_{\text{Hopf}}(H)$.

Lemma 4. Let $\phi : H \rightarrow H$ be a Hopf automorphism. Then,

(1) $\phi(G(H)) = G(H)$,

(2) $\phi(P_{h,g}(H)) = P_{\phi(h),\phi(g)}(H)$.

where $G(H)$ is the group consisting of all group-like elements of H , $P_{h,g}(H)$ is the group consisting of all (h,g) -primitive elements of H , and $h,g \in G(H)$.

Proof. It follows from a straightforward verification. $\square$

Lemma 5. $G(H) = \{b^j c^{ms} | j \in \mathbb{Z}_{mn}, s \in \mathbb{Z}_n\}$.

Proof. Let $x = \sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k \in G(H)$, where $j,l \in \mathbb{Z}_{mn}$, $0 \leq i,k \leq n-1$. Let $\beta_{t,l} = \frac{\zeta^n - 1}{(t)_{!q}(n-t)_{!q}} (\zeta^{(n-t)(l-1)+t(l-1)} + \zeta^{(n-t)(l-2)+t(l-2)} + \dots + \zeta^{(n-t)+t} + 1) = \frac{\zeta^{nl} - 1}{(t)_{!q}(n-t)_{!q}}$. It can be easily seen that for any $1 \leq t \leq n-1$, $\beta_{t,l} = 0$ if and only if $l = ms$ for some $s \in \mathbb{Z}_n$. Since $x \in G(H)$, we have the following:

$$\Delta(x) = \sum_{i,j,l,k} \lambda_{i,j,l,k} (a \otimes b + 1 \otimes a)^i (b^j \otimes b^j) \cdot (c^l \otimes c^l + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l} d^{n-t} \otimes c^l d^t) (c^m \otimes d + d \otimes 1)^k \quad (1)$$

and

$$\Delta(x) = x \otimes x = (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k) \otimes (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k). \quad (2)$$

By comparing (1) and (2), we have $\lambda_{i,j,l,k} = 0$ for any $0 < i \leq n-1$, $0 \leq k \leq n-1$ and $j,l \in \mathbb{Z}_{mn}$. Hence, we have the following:

$$\Delta(x) = \sum_{j,l,k} \lambda_{0,j,l,k} (b^j \otimes b^j) (c^l \otimes c^l + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l} d^{n-t} \otimes c^l d^t) (c^m \otimes d + d \otimes 1)^k, \quad (3)$$

and

$$\Delta(x) = (\sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k) \otimes (\sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k). \quad (4)$$

By comparing (3) and (4), we have the following:

$$\sum_{j,l} \lambda_{0,j,l,0} (b^j c^l \otimes b^j c^l) = (\sum_{j,l} \lambda_{0,j,l,0} b^j c^l) \otimes (\sum_{j,l} \lambda_{0,j,l,0} b^j c^l).$$

It follows that there are two cases. The first case is that there only exists one array (j_0, l_0) , such that $\lambda_{0,j_0,l_0,0} = 1$, and $\lambda_{0,j,l,0} = 0$ for any $(j,l) \neq (j_0, l_0)$. The second case is that for

any (j, l) , we have $\lambda_{0,j_0,l_0,0} = 0$. If the second case is true, then $x = \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k$, where $j, l \in \mathbb{Z}_{mn}$ and $0 < k \leq n-1$. Thus, $\varepsilon(x) = 0$, which contradicts with $x \in G(H)$. Hence, the first case must be true. For any $j, l \in \mathbb{Z}_{mn}$ and $0 < k \leq n-1$, we have the following:

$$x = b^{j_0} c^{l_0} + \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k.$$

By (3) and (4) again, we have the following:

$$\begin{aligned} \Delta(x) = & (b^{j_0} \otimes b^{j_0})(c^{l_0} \otimes c^{l_0} + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l_0} d^{n-t} \otimes c^{l_0} d^t) \\ & + (\sum_{j,l,k} \lambda_{0,j,l,k} (b^j \otimes b^j)(c^l \otimes c^l + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l} d^{n-t} \otimes c^l d^t)(c^m \otimes d + d \otimes 1)^k, \end{aligned} \quad (5)$$

and

$$\Delta(x) = (b^{j_0} c^{l_0} + \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k) \otimes (b^{j_0} c^{l_0} + \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k). \quad (6)$$

By comparing (5) and (6), it follows that

$$\sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k \otimes b^j c^l = \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k \otimes b^{j_0} c^{l_0}$$

and

$$\sum_{j,l,k} \lambda_{0,j,l,k} b^j c^{l+mk} \otimes b^j c^l d^k = b^{j_0} c^{l_0} \otimes \sum_{j,l,k} \lambda_{0,j,l,k} b^j c^l d^k.$$

Since $0 < k \leq n-1$, $l_0 \neq l_0 - mk$. Thus we have $\lambda_{0,j,l,k} = 0$ for any $j, l \in \mathbb{Z}_{mn}$ and $0 < k \leq n-1$. Consequently, $x = b^{j_0} c^{l_0}$. If $l_0 = ms$ for any $s \in \mathbb{Z}_n$. Then it is easy to check that $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. If $l_0 = ms + h$ for any $s \in \mathbb{Z}_n$, $1 \leq h \leq m-1$, then one can check that it contradicts with $\Delta(x) = x \otimes x$. Therefore, by the discussion above, we have $G(H) = \{b^j c^{ms} | j \in \mathbb{Z}_{mn}, s \in \mathbb{Z}_n\}$. $\square$

Lemma 6. Let $h = b^{j_1} c^{ms_1}$, $g = b^{j_2} c^{ms_2} \in G(H)$, and $0 \neq x \in P_{h,g}(H)$ with $j_1, j_2 \in \mathbb{Z}_{mn}$, $s_1, s_2 \in \mathbb{Z}_n$. Then $x = \alpha(h - g) + \beta g + \gamma h d$, where $\alpha, \beta, \gamma \in \mathbb{k}$ and $(j_1, s_1) \neq (j_2, s_2)$. If $(j_1, s_1) \neq (j_2 + 1, s_2)$, then $\beta = 0$; if $(j_1, s_1) \neq (j_2, s_2 - 1)$, then $\gamma = 0$; if $(j_1, s_1) \neq (j_2 + 1, s_2)$ and $(j_1, s_1) \neq (j_2, s_2 - 1)$, then $\beta = \gamma = 0$.

Proof. Let $h = b^{j_1} c^{ms_1}$, $g = b^{j_2} c^{ms_2} \in G(H)$ with $j_1, j_2 \in \mathbb{Z}_{mn}$, $s_1, s_2 \in \mathbb{Z}_n$. Let $x = \sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k \in P_{h,g}(H)$, where $j, l \in \mathbb{Z}_{mn}$, $0 \leq i, k \leq n-1$. Then we have

$$\begin{aligned} \Delta(x) = & \sum_{i,j,l,k} \lambda_{i,j,l,k} (a \otimes b + 1 \otimes a)^i (b^j \otimes b^j) \\ & (c^l \otimes c^l + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l} d^{n-t} \otimes c^l d^t)(c^m \otimes d + d \otimes 1)^k \end{aligned} \quad (7)$$

and

$$\begin{aligned} \Delta(x) = & x \otimes h + g \otimes x \\ = & (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k) \otimes b^{j_1} c^{ms_1} + b^{j_2} c^{ms_2} \otimes (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k). \end{aligned} \quad (8)$$

A straightforward computation shows that the coefficients of $b^j c^{l+m} d^{k-1} \otimes b^j c^l d$ and $a^{i-1} b^j c^l d^k \otimes a b^{i+j-1} c^l$ in (7) are $\lambda_{0,j,l,k} \frac{1-\xi^{mk}}{1-\xi^m}$ and $\lambda_{i,j,l,k} \frac{1-\xi^{-mi}}{1-\xi^{-m}}$, respectively. Then it is easy

to check that $\lambda_{0,j,l,k} = 0$ for any $j, l \in \mathbb{Z}_{mn}$, $2 \leq k \leq n-1$; $\lambda_{i,j,l,k} = 0$ for any $j, l \in \mathbb{Z}_{mn}$, $2 \leq i \leq n-1$, $0 \leq k \leq n-1$; and $\lambda_{1,j,l,k} = 0$ for any $j, l \in \mathbb{Z}_{mn}$, $1 \leq k \leq n-1$. Thus, $x = \sum_{i,j,l} \lambda_{i,j,l,0} a^i b^j c^l + \sum_{j,l} \lambda_{0,j,l,1} b^j c^l d$, where $j, l \in \mathbb{Z}_{mn}$, $0 \leq i \leq 1$. Then by comparing (7) and (8), one has $\lambda_{i,j,l,0} = \lambda_{0,j,l,1} = 0$ for any $l = ms + h$, $j \in \mathbb{Z}_{mn}$, $s \in \mathbb{Z}_n$, $0 \leq i \leq 1$ and $1 \leq h \leq m-1$. Moreover, we have the following:

$$\sum_{j,s} \lambda_{0,j,ms,0} b^j c^{ms} \otimes b^j c^{ms} = \sum_{j,s} \lambda_{0,j,ms,0} b^j c^{ms} \otimes b^{j_1} c^{ms_1} + \sum_{j,s} \lambda_{0,j,ms,0} b^{j_2} c^{ms_2} \otimes b^j c^{ms}, \quad (9)$$

$$\sum_{j,s} \lambda_{1,j,ms,0} a b^j c^{ms} \otimes b^{j+1} c^{ms} = \sum_{j,s} \lambda_{1,j,ms,0} a b^j c^{ms} \otimes b^{j_1} c^{ms_1}, \quad (10)$$

$$\sum_{j,s} \lambda_{1,j,ms,0} b^j c^{ms} \otimes a b^j c^{ms} = \sum_{j,s} \lambda_{1,j,ms,0} b^{j_2} c^{ms_2} \otimes a b^j c^{ms}, \quad (11)$$

$$\sum_{j,s} \lambda_{0,j,ms,1} b^j c^{ms+m} \otimes b^j c^{ms} d = \sum_{j,s} \lambda_{0,j,ms,1} b^{j_2} c^{ms_2} \otimes b^j c^{ms} d, \quad (12)$$

$$\sum_{j,s} \lambda_{0,j,ms,1} b^j c^{ms} d \otimes b^j c^{ms} = \sum_{j,s} \lambda_{0,j,ms,1} b^{j_1} c^{ms_1} d \otimes b^{j_1} c^{ms_1}, \quad (13)$$

where $j \in \mathbb{Z}_{mn}$, $s \in \mathbb{Z}_n$. Consider (9), if $(j_1, s_1) = (j_2, s_2)$, then $\lambda_{0,j,ms,0} = 0$. Otherwise, $\lambda_{0,j_1,ms_1,0} = -\lambda_{0,j_2,ms_2,0}$, and $\lambda_{0,j,ms,0} = 0$ for any $j \neq j_1, j_2$, or $s \neq s_1, s_2$. By (10) and (11), if $(j_1, s_1) \neq (j_2 + 1, s_2)$, then $\lambda_{1,j,ms,0} = 0$. Otherwise, $\lambda_{1,j,ms,0} = 0$ for any $j \neq j_2$ or $s \neq s_2$. By (12) and (13), if $(j_1, s_1) \neq (j_2, s_2 - 1)$, then $\lambda_{0,j,ms,1} = 0$. Otherwise, $\lambda_{0,j,ms,1} = 0$ for any $j \neq j_1$ and $s \neq s_1$. By discussion above, one can obtain Lemma 6. $\square$

Lemma 7. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$. Then $\phi(c^m) = c^m$, $\phi(d) = \alpha d$, where $\alpha \in \mathbb{k}^*$.

Proof. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$. It is easy to check that $c^m \in G(H)$, and $dc^m \in P_{c^m, c^{2m}}$. By Lemmas 4–6, one can assume

$$\phi(c^m) = b^j c^{ms}, \quad \phi(dc^m) = \alpha(b^j c^{ms} - b^{2j} c^{2ms}) + \beta a b^{2j} c^{2ms} + \gamma b^j c^{ms} d,$$

where $j \in \mathbb{Z}_{mn}$, $s \in \mathbb{Z}_n$, and α, β, γ are not all zeros. Thus,

$$\phi(d) = \alpha(1 - b^j c^{ms}) + \beta a b^j c^{ms} + \gamma \xi^{m(j-s)} d.$$

Then, we have the following:

$$\Delta(\phi(d)) = \alpha(1 \otimes 1 - b^j c^{ms} \otimes b^j c^{ms}) + \gamma \xi^{m(j-s)} (c^m \otimes d + d \otimes 1) + \beta(a b^j c^{ms} \otimes b^{j+1} c^{ms} + b^j c^{ms} \otimes a b^j c^{ms})$$

and

$$(\phi \otimes \phi)(\Delta(d)) = \alpha(1 \otimes 1 - b^j c^{ms} \otimes b^j c^{ms}) + \gamma \xi^{m(j-s)} (b^j c^{ms} \otimes d + d \otimes 1) + \beta(a b^j c^{ms} \otimes 1 + b^j c^{ms} \otimes a b^j c^{ms}).$$

Since ϕ is the Hopf automorphism of H , the above two equations are equal. It is easy to see that there are three cases. The first case is $\beta = \gamma = 0$. In this case, $\phi(d) = \alpha(1 - b^j c^{ms}) = \alpha(1 - (\phi(c))^m)$. Thus, $\phi(d)$ can be generated by $\phi(c)$, which is a contradiction. The second

case is $\gamma = s = 0$, $j = mn - 1$. In this case, $(\phi(c))^m = b^{-1}$, and $(\phi(c))^{mn} = b^{-n} \neq 1$, which is a contradiction. So the third case must be true. The third case is $\beta = j = 0$, $s = 1$. Then, $\phi(c^m) = c^m$, $\phi(d) = \alpha(1 - c^m) + \gamma\zeta^{-m}d$, where α and γ are not all zeros. By $d^n = 0$, then $(\phi(d))^n = (\alpha(1 - c^m) + \gamma\zeta^{-m}d)^n = 0$. It is easy to know that the coefficient of c^m is $-n\alpha^n$, thus $\alpha = 0$. Consequently, $\phi(d) = \gamma\zeta^{-m}d$, where $\gamma \neq 0$. $\square$

Lemma 8. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$. Then $\phi(a) = \beta a$, $\phi(b) = b$, $\phi(d) = \beta^{-1}d$, where $\beta^n = 1$.

Proof. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$. It is easy to check that $b \in G(H)$, $a \in P_{b,1}$. By Lemmas 4–6, we can assume that $\phi(b) = b^j c^{ms}$, $\phi(a) = \alpha(b^j c^{ms} - 1) + \beta a + \gamma b^j c^{ms} d$, where $j \in \mathbb{Z}_{mn}$, $s \in \mathbb{Z}_n$, and α, β, γ are not all zeros. Then, we have the following:

$$\begin{aligned} \Delta(\phi(a)) &= \alpha(b^j c^{ms} \otimes b^j c^{ms} - 1 \otimes 1) + \beta(a \otimes b + 1 \otimes a) \\ &\quad + \gamma(b^j c^{m(s+1)} \otimes b^j c^{ms} d + b^j c^{ms} d \otimes 1) \end{aligned}$$

and

$$\begin{aligned} (\phi \otimes \phi)(\Delta(a)) &= \alpha(b^j c^{ms} \otimes b^j c^{ms} - 1 \otimes 1) + \beta(a \otimes b^j c^{ms} + 1 \otimes a) \\ &\quad + \gamma(b^j c^{ms} d \otimes b^j c^{ms} + 1 \otimes b^j c^{ms} d). \end{aligned}$$

Since ϕ is the Hopf automorphism of H , the above two equations are equal. It is easy to see that there are three cases. The first case is $\beta = \gamma = 0$. In this case, $\phi(a) = \alpha(b^j c^{ms} - 1) = \alpha(\phi(b) - 1)$. Thus, $\phi(a)$ can be generated by $\phi(b)$, a contradiction. The second case is $\beta = j = 0$, $s = n - 1$. In this case, $\phi(b) = b^{mn-m}$. Thus, $(\phi(b))^n = 1$ is a contradiction. So the third case must be true. The third case is $\gamma = s = 0$, $j = 1$. Then, $\phi(b) = b$, $\phi(a) = \alpha(b - 1) + \beta a$, where α and β are not all zeros. By $ab = \zeta^m ba$, we have $\phi(a)\phi(b) = \zeta^m \phi(b)\phi(a)$. It is easy to check that $\alpha = 0$. Thus, $\phi(a) = \beta a$. In a similar way, we have $(\phi(a))^n = (\phi(b))^n - 1$ and $\phi(a)\phi(d) - \phi(d)\phi(a) = \phi(b) - (\phi(c))^m$. By Lemma 7, we have $\phi(c^m) = c^m$, $\phi(d) = \lambda d$, where $\lambda \in \mathbb{K}^*$. Then a straightforward computation shows that $\beta^n = 1$ and $\phi(d) = \beta^{-1}d$. $\square$

Lemma 9. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$. Then, $\phi \in W$.

Proof. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$, $\phi(c) = \sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k$, where $j, l \in \mathbb{Z}_{mn}$, $0 \leq i, k \leq n - 1$. We have the following:

$$\begin{aligned} \Delta(\phi(c)) &= \sum_{i,j,l,k} \lambda_{i,j,l,k} (a \otimes b + 1 \otimes a)^i (b^j \otimes b^j) \\ &\quad (c^l \otimes c^l + \sum_{t=1}^{n-1} \beta_{t,l} c^{mt+l} d^{n-t} \otimes c^l d^t) (c^m \otimes d + d \otimes 1)^k \end{aligned}$$

and

$$\begin{aligned} (\phi \otimes \phi)(\Delta(c)) &= (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k) \otimes (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^k) \\ &\quad + \sum_{t=1}^{n-1} \delta_{t,k} (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^{mt+l} d^{n-t+k}) \otimes (\sum_{i,j,l,k} \lambda_{i,j,l,k} a^i b^j c^l d^{t+k}), \end{aligned}$$

where $\delta_{t,k} = \frac{(\zeta^n - 1) \cdot \zeta^{tmk}}{(t)!_q (n-t)!_q}$. Since ϕ is the Hopf automorphism of H , the above two equations are equal. Next, it is similar to the proof of Lemma 5, we have $\phi(c) = b^{j_0} c^{l_0}$, where $j_0, l_0 \in \mathbb{Z}_{mn}$. By Lemma 7, we have $\phi(c^m) = c^m$. A straightforward computation shows that $\phi(c) = b^{sn} c^{1+tn}$, where $s, t \in \mathbb{Z}_m$. By Lemma 8, we have $\phi(a) = \beta a$, $\phi(b) = b$, $\phi(d) = \beta^{-1}d$,

where $\beta^n = 1$. Note that $\phi(d)\phi(c) = \zeta\phi(c)\phi(d)$. Then it is easy to check that $\phi(c) = b^{sn}c$. It follows that ϕ is a Hopf automorphism of H . Consequently, $\phi \in W$. $\square$

Given the cyclic group $\mathbb{Z}_m$ and the cyclic group $\mathbb{Z}_n$. Then we can construct the direct sum $\mathbb{Z}_n \oplus \mathbb{Z}_m = \{(i, s) | i \in \mathbb{Z}_n, s \in \mathbb{Z}_m\}$. The comultiplication is given by

$$(i_1, s_1)(i_2, s_2) = (i_1 + i_2, s_1 + s_2).$$

Lemma 10. $Aut_{Hopf}(H) = W \cong \mathbb{Z}_n \oplus \mathbb{Z}_m$.

Proof. By Lemma 9, $Aut_{Hopf}(H) \subseteq W$. Since $W \subseteq Aut_{Hopf}(H)$, $Aut_{Hopf}(H) = W$. Then there exists a linear map $\varphi : \mathbb{Z}_n \oplus \mathbb{Z}_m \rightarrow W$ determined by $(i, s) \mapsto f_{i,s}$. It is easy to check that φ is a group isomorphism. Therefore, we have $Aut_{Hopf}(H) = W \cong \mathbb{Z}_n \oplus \mathbb{Z}_m$. $\square$

5.2. The Hopf Automorphism Group of $D(H_{s,t})$

In this subsection, we prove that the Hopf automorphism group of $D(H_{s,t})$ is W' and isomorphic to the semidirect products $\mathbb{k}^* \rtimes \mathbb{Z}_d$. Throughout the following, denote $D(H_{s,t})$ by T , and denote the Hopf automorphism group of $D(H_{s,t})$ by $Aut_{Hopf}(T)$.

Lemma 11. Let $\psi : T \rightarrow T$ be a Hopf automorphism. Then,

$$(1) \psi(G(T)) = G(T),$$

$$(2) \psi(P_{h,g}(T)) = P_{\psi(h),\psi(g)}(T).$$

where $G(T)$ is the group consisting of all group-like elements of T , and $P_{h,g}(T)$ is the group consisting of all (h, g) -primitive elements of T , and $h, g \in G(T)$.

Proof. It follows from a straightforward verification. $\square$

Lemma 12. $G(T) = \{B^j C^l | j, l \in \mathbb{Z}_s\}$.

Proof. It is similar to the proof of Lemma 5. $\square$

Lemma 13. Let $h = B^{j_1} C^{l_1}$, $g = B^{j_2} C^{l_2} \in G(T)$, and $0 \neq x \in P_{h,g}(T)$ with $j_1, j_2, l_1, l_2 \in \mathbb{Z}_s$. Then $x = \alpha(h - g) + \beta ag + \gamma hd$, where $\alpha, \beta, \gamma \in \mathbb{k}$ and $(j_1, l_1) \neq (j_2, l_2)$. If $(j_1, l_1) \neq (j_2 + 1, l_2)$, then $\beta = 0$; if $(j_1, l_1) \neq (j_2, l_2 - d)$, then $\gamma = 0$; if $(j_1, l_1) \neq (j_2 + 1, l_2)$ and $(j_1, l_1) \neq (j_2, l_2 - d)$, then $\beta = \gamma = 0$.

Proof. It is similar to the proof of Lemma 6. $\square$

Lemma 14. Let $\psi \in Aut_{Hopf}(T)$. Then $\psi(C) = B^r C$, $\psi(D) = \gamma d$, where $\gamma \in \mathbb{k}^*$, $r \in \mathbb{Z}_d$.

Proof. It is similar to the proof of Lemmas 7 and 9. $\square$

Lemma 15. Let $\psi \in Aut_{Hopf}(T)$. Then $\psi(A) = \gamma^{-1}a$, $\psi(B) = B$, $\psi(D) = \gamma d$, where $\gamma \in \mathbb{k}^*$.

Proof. It is similar to the proof of Lemma 8. $\square$

Lemma 16. Let $\psi \in Aut_{Hopf}(T)$. Then, $\psi \in W'$.

Proof. It follows from a straightforward verification. $\square$

Define a left action of the cyclic group $\mathbb{Z}_d$ on the multiplicative group $\mathbb{k}^*$:

$$r \cdot \gamma = \gamma, r \in \mathbb{Z}_d, \gamma \in \mathbb{k}^*.$$

So we can construct the semidirect products $\mathbb{k}^* \rtimes \mathbb{Z}_d = \{(\gamma, r) | \gamma \in \mathbb{k}^*, r \in \mathbb{Z}_d\}$. The comultiplication is given by the following:

$$(\gamma_1, r_1)(\gamma_2, r_2) = (\gamma_1 \gamma_2, r_1 + r_2).$$

Lemma 17. $Aut_{Hopf}(T) = W' \cong \mathbb{k}^* \rtimes \mathbb{Z}_d$.

Proof. By Lemma 16, $Aut_{Hopf}(T) \subseteq W'$. Since $W' \subseteq Aut_{Hopf}(T)$, $Aut_{Hopf}(T) = W'$. Then there exists a linear map $\phi : \mathbb{k}^* \rtimes \mathbb{Z}_d \rightarrow W'$ determined by $(\gamma, r) \mapsto \psi_{\gamma, r}$. It is easy to check that ϕ is a group isomorphism. Therefore, we have $Aut_{Hopf}(T) = W' \cong \mathbb{k}^* \rtimes \mathbb{Z}_d$. $\square$

6. Conclusions

We found that if H is a Hopf algebra and $\phi \in Aut_{Hopf}(H)$, then $\phi(G(H)) = G(H)$ and $\phi(P_{h,g}(H)) = P_{\phi(h), \phi(g)}(H)$. Based on the above results, we cleverly calculated all Hopf automorphisms of $D(R_{m,n}(q))$ and $D(H_{s,t})$, and further obtained their Hopf automorphism groups. This article provided a new method for the study of automorphism groups of finite-dimensional Hopf algebras, which can lay a solid foundation for the study of automorphism groups of infinite-dimensional Hopf algebras. The following questions deserve further consideration:

- (a) Can this work be solved from the perspectives of eigenproblem and molecular alignment?
- (b) Which automorphism groups of infinite-dimensional Hopf algebras can be solved using our method?

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