

Article

On the Inverse of the Linearization Coefficients of Bessel Polynomials

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Abstract: In this contribution, we first present a new recursion relation fulfilled by the linearization coefficients of Bessel polynomials (LCBPs), which is different than the one presented by Berg and Vignat in 2008. We will explain why this new recursion formula is as important as Berg and Vignat's. We introduce the matrix linearization coefficients of Bessel polynomials (MLCBPs), and we present some new results and some conjectures on these matrices. Second, we present the inverse of the connection coefficients with an application involving the modified Bessel function of the second kind. Finally, we introduce the inverse of the matrix of the linearization coefficients of the Bessel polynomials (IMLCBPs), and we present some open problems related to these IMLCBPs.

Keywords: Bessel polynomials; linearization coefficients; integral formula of Boros and Moll; student t-density; differential equations

1. Introduction

In 2008, Berg and Vignat presented a recursion relation fulfilled by the linearization coefficients of Bessel polynomials (LCBPs), which enabled them to prove the positivity of these coefficients. In 2013, they used the Student t-density to present an alternative evaluation of the Boros–Moll integral. Atia and Zeng presented, in 2012, an explicit single-sum formula for these LCBPs, which was missing in Berg and Vignat's paper. Atia and Zeng derived some immediate consequences from this explicit single sum and generalized a formula on the integral evaluation from Boros and Moll, from which they derived the formula presented by Berg and Vignat. BenAbdallah and Atia discussed, in 2018, a more general aspect of the LCBPs and presented an evaluation, via the convolution of Student t-densities, of a double integral. This is the expository part of this paper.

The second part of this contribution is detailed in the abstract. Let us start by recalling Bessel polynomials. Bessel polynomials q_n of degree n are defined by

$$q_n(u) = \sum_{k=0}^n \frac{(-n)_k 2^k}{(-2n)_k k!} u^k,$$

where we use the Pochhammer symbol $(z)_n := z(z+1) \dots (z+n-1)$ for $z \in \mathbb{C}$ and $n \in \mathbb{N}$. The first values are

$$q_0(u) = 1, \quad q_1(u) = 1 + u, \quad q_2(u) = 1 + u + \frac{u^2}{3}.$$

Using hypergeometric functions, we have $q_n(u) = {}_1F_1(-n; -2n; 2u)$. They are normalized according to $q_n(0) = 1$, and thus differ from the monic normalization $\theta_n(u)$ in [1]:

$$\theta_n(u) = \frac{(2n)!}{n! 2^n} q_n(u).$$



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Polynomials θ_n are sometimes called reverse Bessel polynomials, and $y_n(u) = u^n \theta_n(\frac{1}{u})$, ordinary Bessel polynomials. These Bessel polynomials are then written as

$$y_n(u) = \frac{(2n)!}{n!2^n} u^n q_n\left(\frac{1}{u}\right) = \sum_{k=0}^n \frac{(n+k)!}{2^k k! (n-k)!} u^k.$$

The so-called linearization coefficients of the Bessel polynomials LCBPs, $\beta_k^{(n,m)}$ [2], are defined by

$$q_n(au)q_m((1-a)u) = \sum_{k=\min(n,m)}^{n+m} \beta_k^{(n,m)}(a)q_k(u). \quad (1)$$

For example, we have $\beta_k^{(2,3)} = 0$ for $k = 0, 1$ and

$$\begin{aligned} \beta_2^{(2,3)}(a) &= a^5, \\ \beta_3^{(2,3)}(a) &= -(-1+a)(15a^4 - 20a^3 + 15a^2 - 6a + 1), \\ \beta_4^{(2,3)}(a) &= 7a(-1+a)^2(5a^2 - 4a + 1), \\ \beta_5^{(2,3)}(a) &= -21a^2(-1+a)^3, \end{aligned}$$

and we have

$$q_2(au)q_3((1-a)u) = \sum_{k=2}^5 \beta_k^{(2,3)}(a)q_k(u).$$

2. Berg and Vignat's Results

In 2008, Berg and Vignat [2,3] established some important results about the following:

- The connection coefficients $c_k^{(n)}(a)$. They proved their non-negativity for $a \in [0, 1]$ in the expansion

$$q_n(au) = \sum_{k=0}^n c_k^{(n)}(a)q_k(u) \quad (2)$$

with

$$c_k^{(n)}(a) = a^k(1-a) \frac{\binom{n}{k}}{\binom{2n}{2k}} \sum_{j=0}^{\min((n-j-1), k)} \binom{n+1}{k-j} \binom{n-k-1}{j} (1-a)^j \quad (3)$$

$$= a^k(1-a) \frac{\binom{n}{k} \binom{n+1}{k}}{\binom{2n}{2k}} {}_2F_1 \left(\begin{matrix} -k, & -n+k+1 \\ n-k+2 \end{matrix} ; 1-a \right). \quad (4)$$

- The coefficients $\beta_k^{(n,m)}(a)$ with applications. In particular, they proved Theorem 2.1 [2]: $\beta_n^{(n,0)}(a) = c_0^{(n)}(a)\beta_k^{(n,0)}(a) = a^n$ and, for $0 \leq k \leq n-1$,

$$\beta_k^{(n,0)}(a) = c_k^{(n)}(a).$$

In fact, for $n, m \geq 1$, Berg and Vignat [2] proved that the coefficients $\beta_k^{(n,m)}(a)$ satisfy the following recurrence relation, Lemma 3.6 [2]:

$$\frac{1}{2k+1} \beta_{k+1}^{(n,m)}(a) = \frac{a^2}{2n-1} \beta_k^{(n-1,m)}(a) + \frac{(1-a)^2}{2m-1} \beta_k^{(n,m-1)}(a) \quad (5)$$

for $k = 0, 1, \dots, m+n-1$ and $\beta_0^{(n,m)}(a) = 0$ (because $n, m \geq 1$). From (3) and (5), they derived the positivity of $\beta_k^{(n,m)}(a)$ when $0 \leq a \leq 1$, and also, they proved that $\beta_k^{(n,m)}(a) = 0$

for $k < \min(m, n)$. However, an explicit single-sum formula for $\beta_k^{(n,m)}(a)$ was missing in their paper.

Berg and Vignat [3] used the Student t-distribution and $\beta_k^{(n,n)}(\frac{1}{2})$ to present an alternative evaluation of the Boros–Moll integral [4]:

$$N_{0,4}(x; m) := \int_0^{+\infty} \frac{ds}{(1 + 2xs^2 + s^4)^{m+1}} = \frac{\pi P_m(x)}{2^{m+\frac{3}{2}}(x+1)^{m+\frac{1}{2}}},$$

where

$$P_m(x) = \sum_{j=0}^m d_{j,m} x^j \quad (6)$$

and

$$d_{j,m} = 2^{-2m} \sum_{i=j}^m 2^i \binom{2m-2i}{m-i} \binom{m+i}{m} \binom{i}{j}.$$

We present, at the end of this paper, a very short set of Maple instructions in Appendix A in order to assure readers about the correctness of this result.

3. Atia and Zeng's Results

In 2012, Atia and Zeng presented this explicit single-sum formula for $\beta_k^{(n,m)}(a)$, providing the unique solution of recurrence system (5) with the boundary condition (3) [5] summarized in the following theorem.

Theorem 1. For $k = 0, 1, \dots, n + m$, we have

$$\begin{aligned} \beta_k^{(n,m)}(a) &= a^{2n+m-k} (1-a)^{-n+k} \frac{(1/2)_{n+m-k} (1/2)_k}{(1/2)_n (1/2)_m} \\ &\quad \times \sum_{j=0}^{2(n+m-k)} (-1)^j \binom{n+m+1}{2n+2m-2k-j} \binom{-m+k+j}{j} a^{-j}. \end{aligned} \quad (7)$$

Equivalently, the coefficients $\beta_k^{(n,m)}(a)$ can be written in terms of a hypergeometric function as follows:

(i) If $k \geq \lceil (n+m-1)/2 \rceil$, then

$$\begin{aligned} \beta_k^{(n,m)}(a) &= a^{2n+m-k} (1-a)^{-n+k} \frac{(1/2)_{n+m-k} (1/2)_k}{(1/2)_n (1/2)_m} \binom{n+m+1}{2n+2m-2k} \\ &\quad \times {}_2F_1 \left(\begin{matrix} -2m-2n+2k, & -m+k+1 \\ 2k-m-n+2 \end{matrix}; \frac{1}{a} \right); \end{aligned} \quad (8)$$

(ii) If $k \leq \lfloor (n+m-1)/2 \rfloor$, then

$$\begin{aligned} \beta_k^{(n,m)}(a) &= a^{n+k+1} (1-a)^{-n+k} \frac{(1/2)_{n+m-k} (1/2)_k}{(1/2)_n (1/2)_m} \binom{n-k-1}{n+m-2k-1} \\ &\quad \times (-1)^{n+m-1} {}_2F_1 \left(\begin{matrix} -m-n-1, & n-k \\ n+m-2k \end{matrix}; \frac{1}{a} \right). \end{aligned} \quad (9)$$

Atia and Zeng derived some immediate consequences from the above Theorem 1. In fact, if we assume that $n \leq m$ and $0 \leq a \leq 1$, then the following apply:

1. If $k < n$, then the binomial coefficient $\binom{n-k-1}{n+m-2k-1}$ vanishes because $n+m-2k-1 > n-k-1$. Thus, Formula (9) implies that $\beta_k^{(n,m)}(a) = 0$ for $k < m \wedge n$.
2. If $k \geq n$, then each term in (7) is a polynomial in a . Indeed, it is clear that $(1-a)^{-n+k} \in \mathbb{Z}[a]$ alongside the following:

- If $k \geq m$, then $2n + m - k - j \geq 2n + m - k - 2(n + m - k) = k - m \geq 0$ for $0 \leq j \leq 2(n + m - k)$;
 - If $k < m$, then the binomial coefficient $\binom{-m+k+j}{j}$ does not vanish only if $-m + k + j < 0$, that is, $j < m - k$; therefore, $2n + m - k - j > 2n + m - k - (m - i) = 2n \geq 0$.
3. From definition (1), the symmetry property is immediately derived as

$$\beta_k^{(n,m)}(a) = \beta_k^{(m,n)}(1-a)$$

which follows also from (8) (resp. (9)) and Pfaff's transformation (see p. 94 [6]):

$${}_2F_1\left(\begin{matrix} A & B \\ C \end{matrix}; x\right) = (1-x)^{-A} {}_2F_1\left(\begin{matrix} A & C-B \\ C \end{matrix}; \frac{x}{x-1}\right),$$

with $A = -2n - 2m + 2k$, $B = -m + k + 1$, $C = -n - m + 2k + 2$, and $x = 1/a$ if $k \geq \lceil (n + m - 1)/2 \rceil$ (resp. $A = -n - m - 1$, $B = n - k$, $C = n + m - 2k$, and $x = 1/a$ if $k \leq \lfloor (n + m - 1)/2 \rfloor$).

For applications of this Theorem 1, Atia and Zeng derived a formula from Berg and Vignat's [2] when $m = n$, and they also proved the positivity of these coefficients, $\beta_k^{(n,m)}(a)$, $\min(n, m) \leq k \leq n + m$. In fact, Theorem 1 implies this immediately following positivity result. Suppose that $n < m$. Then, the coefficient $\beta_k^{(n,m)}(a)$ is positive for $n \leq k < m$ (for the proof, please see [5]).

As a particular case of the coefficient $\beta_k^{(n,m)}(a)$, Atia and Zeng found the coefficients $\beta_i^{(n)}(a)$ defined by

$$q_n(au)q_n((1-a)u) = \sum_{i=0}^n \beta_i^{(n)}(a)q_{n+i}(u).$$

For $0 \leq i \leq n$, we have

$$\begin{aligned} \beta_i^{(n)}(a) &= \frac{(4a(1-a))^i}{4^n} \frac{(-n)_i (n + \frac{1}{2})_i}{i! (-n + \frac{1}{2})_i} {}_2F_1\left(-n + i, -n - \frac{1}{2}; \frac{1}{2}; (2a-1)^2\right) \\ &\times \frac{(2n-2i)!(2n+2i)!}{(n-i)!(n+i)!} \sum_{j=0}^{n-i} \binom{2n+1}{2j} \binom{n-j}{i} (2a-1)^{2j}, \end{aligned}$$

(for the proof, please see [5]).

Finally, Atia and Zeng generalized a formula on an integral evaluation of Boros and Moll's integral [4]:

$$\int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^{n+1}(1+(x-y)^2)^{m+1}} = \frac{1}{2} \sum_{k=m \wedge n}^{m+n} \frac{A_{k+\frac{1}{2}}}{A_{n+\frac{1}{2}} A_{m+\frac{1}{2}}} \frac{\beta_k^{(n,m)}(\frac{1}{2})}{(1+\frac{1}{4}x^2)^{k+1}} \quad (10)$$

where

$$A_\nu = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\nu)}. \quad (11)$$

We present, at the end of this paper, a very short set of Maple instructions in Appendix B in order to assure readers about the correctness of the result.

4. BenAbdallah and Atia's Results

In 2018, BenAbdallah and Atia presented a recursion formula (in a more general case) for the linearization coefficients $(\beta_i^{(N_k)}(v_k))$ for Bessel polynomials (q_n) , mentioned by C. Berg and C. Vignat (p. 21 (14) [2]), in the expansion $q_{n_1}(a_1u)q_{n_2}(a_2u) \dots q_{n_k}(a_ku) =$

$\sum_{l=\min(n_1, n_2, \dots, n_k)}^{L_k} \beta_l^{(N_k)}(v_k) q_l(u)$, where $u \in \mathbb{R}$, $N_k = (n_1, n_2, \dots, n_k) \in \mathbb{N}^k$, $v_k = (a_1, a_2, \dots, a_k) \in \mathbb{R}_+^k$ with $\sum_{1 \leq i \leq k} a_i = 1$, and $L_k = \sum_{1 \leq i \leq k} n_i$. BenAbdallah and Atia proved that this recursion formula yields, again, the positivity of the coefficients. In addition, in the case when $k = 3$, they presented an explicit formula for $\beta_l^{(N_3)}(v_3)$, and they derived a double-integral formula through the convolution of Student t-densities. As a bonus, two reductions for this double integral were provided; the first reduction uses an integral formula given by Atia and Zeng [5], and the second uses the one given by Boros and Moll [4].

In fact, BenAbdallah and Atia studied the general case of the Berg–Vignat linearization problem $v_k = (a_1, \dots, a_k) \in \mathbb{R}^k$, $k \geq 3$, with $\sum_{i=1}^k a_i = 1$; see (p. 21 (14) [2]).

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) \dots q_{n_k}(a_k u) = \sum_{l=\min(n_1, n_2, \dots, n_k)}^{L_k} \beta_l^{(N_k)}(v_k) q_l(u). \quad (12)$$

BenAbdallah and Atia performed the following:

1. Provided the recurrence formula satisfied by the linearization coefficients $\beta_l^{(N_k)}(v_k)$ in (12) and then proved that they are non-negative and $\min(n_1, n_2, \dots, n_k) \leq l \leq L_k$, with $L_k = \sum_{1 \leq i \leq k} n_i$.
2. Provided a triple-sum formula of the linearization coefficients $\beta_l^{(N_3)}(v_3)$ in the expansion

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) q_{n_3}(a_3 u) = \sum_{l=\min(n_1, n_2, n_3)}^{L_3} \beta_l^{(N_3)}(v_3) q_l(u), \quad u \in \mathbb{R}.$$

3. Provided an evaluation through a convolution of the Student t-densities of the double integral

$$I_{n,m,p}(x) := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{A_{n+\frac{1}{2}} A_{m+\frac{1}{2}} A_{p+\frac{1}{2}} dt dy}{(1+t^2)^{n+1} (1+y^2)^{m+1} (1+(x-t-y)^2)^{p+1}}$$

where A_v is defined in (11).

In 2017, Atia and BenAbdallah generalized (1) in their paper published in the *Asian–European Journal of Mathematics*:

$$q_n(au) q_m(bu) = \sum_{l=0}^{n+m} \beta_l^{(n,m)}(a,b) q_l(u), \quad a+b \neq 1,$$

and also, they proved the positivity of these linearization coefficients $\beta_l^{(n,m)}(a,b)$ as follows:

- (i) For $0 \leq k \leq \lfloor \frac{n+m-1}{2} \rfloor$ we have

$$\begin{aligned} \beta_k^{(n,m)}(a,b) &= \frac{(2k)!(-1)^{k+1} a^n b^{2k+1-n}}{k!} \sum_{i=0}^{k+1} \frac{(-1)^i (2k+2-i)! b^{-i}}{i!(k+1-i)!} \\ &\times \sum_{j=\max(0, n+i-2k-1)}^n \frac{(-n)_{n-j} (-m)_{2k+1-i-n+j} (\frac{b}{a})^j}{(-2n)_{n-j} (-2m)_{2k+1-i-n+j} (n-j)!(2k+1-i-n+j)!}. \end{aligned}$$

(ii) For $\lceil \frac{n+m-1}{2} \rceil \leq k \leq n+m$ we have

$$\beta_k^{(n,m)}(a,b) = \frac{(2k)!(-1)^{m+n-k}a^n b^m}{k!} \sum_{i=0}^{n+m-k} \frac{(-1)^i(m+n-i+1)!b^{-i}}{(2k+1+i-n-m)!(m+n-k-i)!} \\ \times \sum_{j=0}^n \frac{(-n)_{n-j}(-m)_{m-i+j}(\frac{b}{a})^j}{(-2n)_{n-j}(-2m)_{m-i+j}(n-j)!(m-i+j)!},$$

for $0 \leq k \leq n+m$ and $a \neq 0$ and $b \neq 0$ and where the authors assumed that $n \leq m$.

The linearization coefficients in (12) satisfy the following recursion formula.

For $n_i \geq 1$, with $1 \leq i \leq k$, we have

$$\frac{1}{2l+1} \beta_{l+1}^{(N_k)}(v_k) = \sum_{i=1}^k \frac{a_i^2}{2n_i-1} \beta_l^{(\sigma_i(N_k))}(v_k)$$

where $\sigma_i(N_k) = (n_1, \dots, n_i-1, \dots, n_k)$, and $l = 0, 1, \dots, L_k-1$. Furthermore, $\beta_0^{(N_k)}(v_k) = 0$. For the proof, please see [7].

Atia and Benabdallah presented a triple-sum formula of the linearization coefficients $\beta_l^{(N_3)}(v_3)$ in the expansion

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) q_{n_3}(a_3 u) = \sum_{l=\min(n_1, n_2, n_3)}^{n_1+n_2+n_3} \beta_l^{(N_3)}(v_3) q_l(u), \quad u \in \mathbb{R}$$

where $\beta_l^{(N_3)}(v_3)$ is given by

$$\beta_l^{(N_3)}(v_3) = \sum_{k=0}^{\min(n_1+n_2+n_3-l, l+1)} \delta_l^{(k+l)} \sum_{j=\max(0, k+l-n_3)}^{\min(k+l, n_1+n_2)} \left(\sum_{i=\max(0, j-n_2)}^{\min(j, n_1)} \alpha_i^{(n_1)} \alpha_{j-i}^{(n_2)} a_1^i a_2^{j-i} \alpha_{k+l-j}^{(n_3)} a_3^{k+l-j} \right)$$

which, for $l \geq \min(n_1, n_2, n_3)$, they wrote it under the form

$$\beta_l^{(N_3)}(v_3) = \frac{(2l)!}{l!} \sum_{k=0}^{\min(L_3-l, l+1)} \frac{(-1)^k(l+k+1)!}{k!(l-k+1)!} \\ \times \sum_{j=\max(0, k+l-n_3)}^{\min(k+l, n_1+n_2)} \sum_{i=\max(0, j-n_2)}^{\min(j, n_1)} \frac{(-n_1)_i(-n_2)_{j-i}(-n_3)_{k+l-j}}{(-2n_1)_i(-2n_2)_{j-i}(-2n_3)_{k+l-j}} \frac{a_1^i a_2^{j-i} a_3^{k+l-j}}{i!(j-i)!(k+l-j)!}$$

Finally, Atia and Benabdallah presented the evaluation of the following multiple integrals:

$$\underbrace{\int_{\mathbb{R}} \dots \int_{\mathbb{R}}}_{(k-1) \text{ times}} \frac{A_{n_k+\frac{1}{2}} \prod_{j=1}^{k-1} \frac{A_{n_j+\frac{1}{2}} dx_j}{(1+x_j^2)^{n_j+1}}}{(1+(x-\sum_{j=1}^{k-1} x_j)^2)^{n_k+1}} = \frac{1}{k} \sum_{l=\min(n_1, n_2, \dots, n_k)}^{L_k} \frac{A_{l+\frac{1}{2}} \beta_l^{(N_k)}(\frac{1}{k})}{(1+(\frac{x}{k})^2)^{l+1}}.$$

This formula becomes, as a special case, a double integral when $k = 3$ and $N_3 = (n, m, p)$ and $v_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{A_{n+\frac{1}{2}} A_{m+\frac{1}{2}} A_{p+\frac{1}{2}} dt dy}{(1+t^2)^{n+1} (1+y^2)^{m+1} (1+(x-t-y)^2)^{p+1}} = \frac{1}{3} \sum_{l=\min(n, m, p)}^{n+m+p} \frac{A_{l+\frac{1}{2}} \beta_l^{(N_3)}(\frac{1}{3})}{(1+\frac{x^2}{9})^{l+1}}.$$

We present, at the end of this paper, a very short set of Maple instructions in Appendix C in order to assure readers about the correctness of the result.

5. Results

5.1. Another Recursion Formula for LCBPs

In 2008, Berg and Vignat [2] proved that the coefficients $\beta_k^{(n,m)}(a)$ satisfy the recurrence given by (5). In this section, we are going to present our first main result concerning another recurrence relation between $\beta_k^{(n,m)}(a)$, and at the end of this paper, we will show why this relation is as important as the relation given by Berg and Vignat.

Theorem 2. For any integers $n, m, n \geq 1, m \geq 1$, and for any $1 \leq k \leq n + m$, we have the following results:

$$(2m - 1)\beta_k^{(n,m)}(a) + (2n + 1)\beta_k^{(n+1,m-1)}(a) = (2k - 1)\beta_{k-1}^{(n,m-1)}(a) + (2n + 2m - 2k - 1)\beta_k^{(n,m-1)}(a), \quad (13)$$

taking into account either $\beta_{-1}^{(n,m-1)}(a) = 0$ or $\beta_{n+m}^{(n,m-1)}(a) = 0$.

Proof. Let us prove that the difference

$$Subs = LHS - RHS = 0.$$

$$\begin{aligned} Subs &= (2m - 1)\beta_k^{(n,m)}(a) + (2n + 1)\beta_k^{(n+1,m-1)}(a) - (2k - 1)\beta_{k-1}^{(n,m-1)}(a) \\ &\quad - (2n + 2m - 2k - 1)\beta_k^{(n,m-1)}(a) \\ &= \frac{(1 - 2m)a^{2n+m-k}(1 - a)^{k-n}(\frac{1}{2})_k(\frac{1}{2})_{n+m-k}}{(\frac{1}{2})_n(\frac{1}{2})_m} \sum_{j=0}^{2n+2m-2k} \frac{(-1)^j}{a^j} \binom{m+n+1}{2n+2m-2k-j} \binom{k-m+j}{j} \\ &\quad - \frac{(2n+1)a^{2n+m-k+1}(1-a)^{k-n-1}(\frac{1}{2})_k(\frac{1}{2})_{n+m-k}}{(\frac{1}{2})_{n+1}(\frac{1}{2})_{m-1}} \\ &\quad - \sum_{j=0}^{2n+2m-2k} \frac{(-1)^j}{a^j} \binom{m+n+1}{2n+2m-2k-j} \binom{k-m+j+1}{j} \\ &\quad - \frac{(2k-1)a^{2n+m-k}(1-a)^{k-n-1}(\frac{1}{2})_{k-1}(\frac{1}{2})_{n+m-k}}{(\frac{1}{2})_n(\frac{1}{2})_{m-1}} \\ &\quad - \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j} \binom{k-m+j}{j} \\ &\quad - \frac{(2n+2m-2k-1)a^{2n+m-k-1}(1-a)^{k-n}(\frac{1}{2})_k(\frac{1}{2})_{n+m-k-1}}{(\frac{1}{2})_n(\frac{1}{2})_{m-1}} \\ &\quad - \sum_{j=0}^{2n+2m-2k-2} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j}. \end{aligned}$$

Using that for any integer p , $(\frac{1}{2})_{p+1} = (p + \frac{1}{2})(\frac{1}{2})_p$ and taking away the common factor the remaining terms of the $Subs$ that we denote by $Rsubs$, we obtain

$$\begin{aligned} Rsubs &= a(1-a) \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} \\ &\quad + a^2 \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j+1}{j} \end{aligned}$$

$$\begin{aligned}
& -a \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& - (1-a) \sum_{j=0}^{2n+2m-2k-2} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j}.
\end{aligned}$$

The last summation can be expanded to $2n+2m-2k$ because $\binom{m+n}{-1} = \binom{m+n}{-2} = 0$. Then, we obtain

$$\begin{aligned}
R_{\text{subs}} &= a(1-a) \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& + a^2 \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j+1}{j} \\
& - a \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& - (1-a) \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j}.
\end{aligned}$$

Expanding the first and the last sums, we obtain

$$\begin{aligned}
R_{\text{subs}} &= a \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& - a^2 \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& + a^2 \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j+1}{j} \\
& - a \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j} \binom{-m+k+j}{j} \\
& - \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j} \\
& + a \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j},
\end{aligned}$$

Taking the sums that have the same power of a , we obtain

$$\begin{aligned}
R_{\text{subs}} &= \sum_{j=0}^{2n+2m-2k} (-a)^{2-j} \binom{m+n+1}{2n+2m-2k-j} \left(\binom{-m+k+j+1}{j} - \binom{-m+k+j}{j} \right) \\
& + \sum_{j=0}^{2n+2m-2k} (-a)^{1-j} \left(\binom{m+n}{2n+2m-2k-j} \binom{-m+k+j}{j} \right. \\
& \quad \left. - \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} - \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j} \right) \\
& - \sum_{j=0}^{2n+2m-2k} (-a)^{-j} \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j}.
\end{aligned}$$

Let us denote the above by

$$B_j^{(2)} = \binom{m+n+1}{2n+2m-2k-j} \left(\binom{-m+k+j+1}{j} - \binom{-m+k+j}{j} \right),$$

$$B_j^{(1)} = \left(\binom{m+n}{2n+2m-2k-j} \binom{-m+k+j}{j} - \binom{m+n+1}{2n+2m-2k-j} \binom{-m+k+j}{j} \right) - \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j},$$

$$B_j^{(0)} = \binom{m+n}{2n+2m-2k-j-2} \binom{-m+k+j+1}{j},$$

Then, R_{subs} becomes

$$R_{subs} = \sum_{j=0}^{2n+2m-2k} (-a)^{2-j} B_j^{(2)} + \sum_{j=0}^{2n+2m-2k} (-a)^{1-j} B_j^{(1)} - \sum_{j=0}^{2n+2m-2k} (-a)^{-j} B_j^{(0)}.$$

In these sums, we have the following:

- Only one term with a^2 with the coefficient

$$B_0^{(2)} = \left(\binom{m+n+1}{2n+2m-2k} \left(\binom{-m+k+1}{0} - \binom{-m+k}{0} \right) \right) a^2 = 0,$$

- Two terms with a with the coefficient

$$B_1^{(2)} + B_0^{(1)} = \binom{m+n+1}{2n+2m-2k-1} \left(\binom{-m+k+1+1}{1} - \binom{-m+k+1}{1} \right) + \left(\binom{m+n}{2n+2m-2k-1} \binom{-m+k+1}{1} - \binom{m+n+1}{2n+2m-2k-1} \binom{-m+k+1}{1} \right) - \binom{m+n}{2n+2m-2k-1-2} \binom{-m+k+1+1}{1} = 0,$$

- Only one term with $a^{-(2n+2m-2k)}$ with the coefficient

$$B_{2n+2m-2k}^{(0)} = \binom{m+n}{-2} \binom{-m+k+2n+2m-2k+1}{2n+2m-2k} = 0,$$

- Only two terms with $a^{-(2n+2m-2k)+1}$ with the coefficient

$$B_{2n+2m-2k-1}^{(0)} + B_{2n+2m-2k}^{(1)} = \binom{m+n}{-1} \binom{2n+m-k+1}{2n+2m-2k} + \left(\binom{m+n}{0} \binom{2n+m-k}{2n+2m-2k} - \binom{m+n+1}{0} \binom{-2n+m-k}{2n+2m-2k} - \binom{m+n}{-2} \binom{2n+m-k+1}{2n+2m-2k} \right) = 0,$$

Finally, the terms with a^{-j} , $0 \leq j \leq 2n+2m-2k-2$ are given by

$$B_{j+2}^{(2)} + B_{j+1}^{(1)} + B_j^{(0)} = 0.$$

□

5.2. The Matrix Linearization Coefficients of Bessel Polynomials

From (1), we can write

$$\begin{aligned} q_n(au)q_0((1-a)u) &= \beta_0^{(n,0)}(a)q_0(u) + \beta_1^{(n,0)}(a)q_1(u) + \cdots + \beta_n^{(n,0)}(a)q_n(u), \\ q_{n-1}(au)q_1((1-a)u) &= 0 + \beta_1^{(n-1,1)}(a)q_1(u) + \cdots + \beta_n^{(n-1,1)}(a)q_n(u), \\ q_{n-2}(au)q_2((1-a)u) &= 0 + 0 + \beta_2^{(n-2,2)}(a)q_2(u) + \cdots + \beta_n^{(n-2,2)}(a)q_n(u), \\ &\vdots \\ q_2(au)q_{n-2}((1-a)u) &= 0 + 0 + \beta_2^{(2,n-2)}(a)q_2(u) + \cdots + \beta_n^{(2,n-2)}(a)q_n(u), \\ q_1(au)q_{n-1}((1-a)u) &= 0 + \beta_1^{(1,n-1)}(a)q_1(u) + \cdots + \beta_n^{(1,n-1)}(a)q_n(u), \\ q_0(au)q_n((1-a)u) &= \beta_0^{(0,n)}(a)q_0(u) + \beta_1^{(0,n)}(a)q_1(u) + \cdots + \beta_n^{(0,n)}(a)q_n(u). \end{aligned}$$

This can be written under the following matrix representation:

$$\begin{pmatrix} q_n(au)q_0((1-a)u) \\ q_{n-1}(au)q_1((1-a)u) \\ \vdots \\ q_1(au)q_{n-1}((1-a)u) \\ q_0(au)q_n((1-a)u) \end{pmatrix} = \begin{pmatrix} \beta_0^{(n,0)}(a) & \beta_1^{(n,0)}(a) & \cdots & \beta_{n-1}^{(n,0)}(a) & \beta_n^{(n,0)}(a) \\ 0 & \beta_1^{(n-1,1)}(a) & \cdots & \beta_{n-1}^{(n-1,1)}(a) & \beta_n^{(n-1,1)}(a) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \beta_1^{(1,n-1)}(a) & \cdots & \beta_{n-1}^{(1,n-1)}(a) & \beta_n^{(1,n-1)}(a) \\ \beta_0^{(0,n)}(a) & \beta_1^{(0,n)}(a) & \cdots & \beta_{n-1}^{(0,n)}(a) & \beta_n^{(0,n)}(a) \end{pmatrix} \begin{pmatrix} q_0(u) \\ q_1(u) \\ \vdots \\ q_{n-1}(u) \\ q_n(u) \end{pmatrix},$$

which we denote by

$$\begin{pmatrix} q_n(au)q_0((1-a)u) \\ q_{n-1}(au)q_1((1-a)u) \\ \vdots \\ q_1(au)q_{n-1}((1-a)u) \\ q_0(au)q_n((1-a)u) \end{pmatrix} = M_n^\beta \begin{pmatrix} q_0(u) \\ q_1(u) \\ \vdots \\ q_{n-1}(u) \\ q_n(u) \end{pmatrix}, \quad (14)$$

Please note that M_n^β has an order of $(n+1) \times (n+1)$; here are some examples.

Example 1.

$$M_1^\beta = \begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix}, \quad M_2^\beta = \begin{pmatrix} 1-a & a(1-a) & a^2 \\ 0 & 3a^2-3a+1 & 3a(1-a) \\ a & a(1-a) & (1-a)^2 \end{pmatrix} \quad (15)$$

$$M_3^\beta = \begin{pmatrix} 1-a & -\frac{1}{5}a(1-a)(a-5) & \frac{6}{5}a^2(1-a) & a^3 \\ 0 & (1-a)^3 & a(6a^2-8a+3) & 5a^2(1-a) \\ 0 & a^3 & (1-a)(6a^2-4a+1) & 5a(1-a)^2 \\ a & \frac{1}{5}a(1-a)(a+4) & \frac{6}{5}a(1-a)^2 & (1-a)^3 \end{pmatrix} \quad (16)$$

After computing the traces of these three first matrices, we can announce our first result.

Proposition 1. For any integer n such that $n \geq 0$, the following result holds:

$$\text{Trace}(M_n^\beta) = \text{Tr}(M_n^\beta) = 1 + (1 - 2a)\text{Tr}(M_{n-1}^\beta) = \frac{1 - (1 - 2a)^{n+1}}{2a} \quad (17)$$

$$\text{Tr}(M_n^\beta) = 1 + (1 - 2a) + (1 - 2a)^2 + \cdots + (1 - 2a)^n. \quad (18)$$

Proof. Let us find the first values. Using (7), we obtain

$$\begin{aligned} \text{Tr}(M_0^\beta) &= \beta_0^{(0,0)}(a) = 1, \\ \text{Tr}(M_1^\beta) &= \beta_0^{(1,0)}(a) + \beta_1^{(0,1)}(a) = (1 - a) + (1 - a) = 2 - 2a = 1 + (1 - 2a), \\ \text{Tr}(M_2^\beta) &= \beta_0^{(2,0)}(a) + \beta_1^{(1,1)}(a) + \beta_2^{(0,2)}(a) \\ &= (1 - a) + (3a^2 - 3a + 1) + (1 - a)^2 = 4a^2 - 6a + 3 \\ &= 1 + (1 - 2a) + (1 - 2a)^2 = 1 + (1 - 2a)(2 - 2a), \\ \text{Tr}(M_3^\beta) &= \beta_0^{(3,0)}(a) + \beta_1^{(2,1)}(a) + \beta_2^{(1,2)}(a) + \beta_3^{(0,3)}(a) \\ &= (1 - a) + (1 - a)^3 + (1 - a)(6a^2 - 4a + 1) + (1 - a)^3 \\ &= 4(1 - a)(2a^2 - 2a + 1) = 1 + (1 - 2a)(4a^2 - 6a + 3). \end{aligned}$$

Here, it is a known fact that if we have

$$\forall n \in \mathbb{N}, u_{n+1} = \alpha u_n + \beta, \alpha \neq 1, \beta \neq 0$$

then when u_n converges, the limit l fulfills $l = \alpha l + \beta$. We define v_n as the auxiliary sequence defined by $\forall n \in \mathbb{N}, v_n = u_n - l$ then we obtain: $\forall n \in \mathbb{N}, v_{n+1} = \alpha v_n$. Thus, v_n is a geometric sequence with ratio α , and $v_n = \alpha^n v_0$ and $u_n = l + \alpha^n(u_0 - l)$. The sequence u_n converges if and only if $|\alpha| < 1$ or $u_0 = l$. In our case, if we have the recurrence

$$\text{Tr}(M_n^\beta) = 1 + (1 - 2a)\text{Tr}(M_{n-1}^\beta),$$

then when $a \neq \frac{1}{2}$, the limit of $\text{Tr}(M_n^\beta)$ is $l = \frac{1}{2a}$, and

$$\text{Tr}(M_n^\beta) = \frac{1 - (1 - 2a)^{n+1}}{2a}.$$

We still need to prove that for any positive integer n , we have

$$\text{Tr}(M_n^\beta) = 1 + (1 - 2a)\text{Tr}(M_{n-1}^\beta).$$

Using Theorem 1, we can write

$$\beta_k^{(n-k,k)}(a) = \frac{a^{2n-2k}}{(1-a)^{n-2k}} \sum_{j=0}^{2n-2k} (-a)^{-j} \binom{n+1}{2n-2k-j}$$

which, for $n = 2p$, yields

$$\text{Tr}(M_n^\beta) = \sum_{k=1}^p (1-a)^{2p+1-2k} + \sum_{k=0}^p (1-a)^{2k} ((1-a)^{1+2p} + \sum_{l=0}^{2k} \binom{1+2p}{2p+1-l} (-1)^l a^{2p+1-l})$$

and for $n = 2p - 1$, we have

$$\text{Tr}(M_n^\beta) = \sum_{k=1}^p (1-a)^{2p-2k-1} + \sum_{k=0}^{p-1} (1-a)^{2k+1} ((1-a)^{2p} + \sum_{l=0}^{2k+1} \binom{2p}{2p-l} (-1)^l a^{2p-l}).$$

Let us prove that for $n = 2p$, we have

$$\text{Tr}(M_n^\beta) = \frac{1 - (1 - 2a)^{n+1}}{2a} = \frac{1 - (1 - 2a)^{2p+1}}{2a}.$$

Using (19), we obtain

$$\begin{aligned} \text{Tr}(M_n^\beta) &= \sum_{k=1}^p (1-a)^{2p+1-2k} + \sum_{k=0}^p (1-a)^{2k} ((1-a)^{1+2p} + \sum_{l=0}^{2k} \binom{1+2p}{2p+1-l} (-1)^l a^{2p+1-l}) \\ &= (1-a)^{2p-1} + (1-a)^{2p-3} + \dots + (1-a)^1 \\ &\quad + (1-a)^0 \left((1-a)^{2p+1} + \binom{2p+1}{2p+1} a^{2p+1} \right) \\ &\quad + (1-a)^2 \left((1-a)^{2p+1} + \binom{2p+1}{2p+1} a^{2p+1} - \binom{2p+1}{2p} a^{2p} + \binom{2p+1}{2p-1} a^{2p-1} \right) \\ &\quad + \dots \\ &\quad + (1-a)^{2p} \end{aligned}$$

We can write this under the form

$$\begin{aligned} \text{Tr}(M_n^\beta) &= \binom{2p+1}{2p} a^{2p} - \binom{2p+1}{2p-1} a^{2p-1} \\ &\quad + \sum_{k=0}^{p-1} \left((1-a)^{2k+1} + a^{2k-1} \left(\binom{2p+1}{2k} a - \binom{2p+1}{2k-1} \right) \right) \sum_{i=0}^{p-k} (1-a)^{2i}. \end{aligned}$$

Using the following

$$\sum_{i=0}^m (1-a)^{2i} = \frac{(1-a)^{2m+2} - 1}{a(a-2)}$$

we obtain

$$\begin{aligned} \text{Tr}(M_n^\beta) &= \binom{2p+1}{2p} a^{2p} - \binom{2p+1}{2p-1} a^{2p-1} + (1-a) \frac{(1-a)^{2(p-1)+2} - 1}{a(a-2)} \\ &\quad + \sum_{k=0}^{p-1} a^{2k-1} \left(\binom{2p+1}{2k} a - \binom{2p+1}{2k-1} \right) \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)}, \end{aligned}$$

Equivalently,

$$\begin{aligned} \text{Tr}(M_n^\beta) &= \binom{2p+1}{2p} a^{2p} - \binom{2p+1}{2p-1} a^{2p-1} + (1-a) \frac{(1-a)^{2(p-1)+2} - 1}{a(a-2)} \\ &\quad + (a-1) \sum_{k=0}^{p-1} a^{2k-1} \binom{2p+1}{2k} \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)} \\ &\quad + \sum_{k=0}^{p-1} a^{2k-1} \left(\binom{2p+1}{2k} - \binom{2p+1}{2k-1} \right) \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)} \end{aligned}$$

This becomes

$$\begin{aligned} \text{Tr}(M_n^\beta) &= (2p+1)a^{2p} - p(2p+1)a^{2p-1} + \frac{(-1+a)(1-(-1+a)^{2p})}{a(a-2)} \\ &\quad + (a-1) \sum_{k=0}^{p-1} a^{2k-1} \binom{2p+1}{2k} \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)} \\ &\quad + \sum_{k=0}^{p-1} a^{2k-1} \left(\binom{2p+1}{2k} - \binom{2p+1}{2k-1} \right) \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)}. \end{aligned}$$

Using

$$(a-1) \sum_{k=0}^{p-1} a^{2k-1} \binom{2p+1}{2k} \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)} = \frac{(a-1)^2(-1+2a)^{2p+1}}{2a^2(a-2)} - \frac{(-1+a)(2p+1)a^{2p}}{a} + \frac{(a-1)^{2p+2} - (a-1)(a+1)^{2p+1}}{2a^2(a-2)} - \frac{(a-1)^2}{2a^2(a-2)}$$

and

$$\sum_{k=0}^{p-1} a^{2k-1} \left(\binom{2p+1}{2k} - \binom{2p+1}{2k-1} \right) \frac{(1-a)^{2(p-k)+2} - 1}{a(a-2)} = (2p+1)(p-1)a^{2p-1} + \frac{(1-2a) + (1-2a)^{2p+1} + (-1+a)^{2p+1}(a+1) + (a+1)^{2p+1}(-1+a)}{2a^2(a-2)}$$

and computing $2a\text{Tr}(M_n^\beta) - 1$, we find $(2a-1)^{2p+1}$. We follow the same steps to prove the result for $n = 2p$.

Please see the Maple instructions given in Appendix D at the end of this paper, which shortens the calculus presented here. $\square$

Our second result deals with the determinant. Using (18), we can state the following result.

Proposition 2. For any positive integer $n \geq 0$, the following result holds:

$$\text{Determinant}(M_n^\beta) = \text{Det}(M_n^\beta) = (1-2a)^{\frac{n(n+1)}{2}}. \quad (19)$$

Proof. Using (18), we can see that the trace of M_n^β differs from that of M_{n+1}^β in that the last term is $(1-2a)^{n+1}$. For instance, using (15) and (16), we can see that the eigenvalues appear as follows:

- $\text{Det}(M_1^\beta - \lambda I_2) = (\lambda-1)(\lambda-(1-2a))$
 $\text{Tr}(M_1^\beta) = 1 + (1-2a),$
- $\text{Det}(M_2^\beta - \lambda I_3) = -(\lambda-1)(\lambda-(1-2a))(\lambda-(1-2a)^2)$
 $\text{Tr}(M_1^\beta) = 1 + (1-2a) + (1-2a)^2,$
- $\text{Det}(M_3^\beta - \lambda I_4) = (\lambda-1)(\lambda-(1-2a))(\lambda-(1-2a)^2)(\lambda-(1-2a)^3)$
 $\text{Tr}(M_1^\beta) = 1 + (1-2a) + (1-2a)^2 + (1-2a)^3,$

So, we can deduce directly that

$$\text{Det}(M_{n+1}^\beta) = (1-2a)^{\frac{n(n+1)}{2}} \cdot (1-2a)^{n+1} = (1-2a)^{\frac{(n+1)(n+2)}{2}}.$$

It is important to remark here that in using (18), we can see that the *Spectrum* of M_{n+1}^β contains the *Spectrum* of M_n^β . $\square$

As an application on these matrices and using (10), we can write the following result:

$$\text{Application} \begin{pmatrix} A_{n+\frac{1}{2}} A_{\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^{n+1}(1+(x-y)^2)} \\ A_{n-\frac{1}{2}} A_{\frac{3}{2}} \int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^n(1+(x-y)^2)^2} \\ \vdots \\ A_{\frac{3}{2}} A_{n-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^2(1+(x-y)^2)^n} \\ A_{\frac{1}{2}} A_{n+\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^1(1+(x-y)^2)^{n+1}} \end{pmatrix} = \begin{pmatrix} \beta_0^{(n,0)}(\frac{1}{2}) & \beta_1^{(n,0)}(\frac{1}{2}) & \cdots & \beta_{n-1}^{(n,0)}(\frac{1}{2}) & \beta_n^{(n,0)}(\frac{1}{2}) \\ 0 & \beta_1^{(n-1,1)}(\frac{1}{2}) & \cdots & \beta_{n-1}^{(n-1,1)}(\frac{1}{2}) & \beta_n^{(n-1,1)}(\frac{1}{2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \beta_1^{(1,n-1)}(\frac{1}{2}) & \cdots & \beta_{n-1}^{(1,n-1)}(\frac{1}{2}) & \beta_n^{(1,n-1)}(\frac{1}{2}) \\ \beta_0^{(0,n)}(\frac{1}{2}) & \beta_1^{(0,n)}(\frac{1}{2}) & \cdots & \beta_{n-1}^{(0,n)}(\frac{1}{2}) & \beta_n^{(0,n)}(\frac{1}{2}) \end{pmatrix} \begin{pmatrix} \frac{A_{\frac{1}{2}}}{(1+\frac{x^2}{4})} \\ \frac{A_{\frac{3}{2}}}{(1+\frac{x^2}{4})^2} \\ \vdots \\ \frac{A_{n-\frac{1}{2}}}{(1+\frac{x^2}{4})^{n-1}} \\ \frac{A_{n+\frac{1}{2}}}{(1+\frac{x^2}{4})^n} \end{pmatrix}.$$

5.3. The Inverse of the Linearization Coefficients of Bessel Polynomials

5.3.1. The Inverse of the Connection Coefficients

From (2) and (3), we can talk about the inverse of the connection coefficients $c_k^n(a)$, which we denote by $ic_k^{(n)}(a)$, defined by the expansion

$$q_n(u) = \sum_{k=0}^n ic_k^{(n)}(a) q_k(au), \quad n \geq 0 \quad (20)$$

and we can announce the following theorem.

Theorem 3. The connection coefficients $ic_k^{(n)}(a)$ are given by $c_k^{(n)}(\frac{1}{a})$, i.e.,

$$q_n(u) = \sum_{k=0}^n ic_k^{(n)}(a) q_k(au) = \sum_{k=0}^n c_k^{(n)}(\frac{1}{a}) q_k(au), \quad n \geq 0. \quad (21)$$

Proof. From (2), we have

$$q_n(bu) = \sum_{k=0}^n c_k^{(n)}(b) q_k(u),$$

Then, the substitution of u by $\frac{u}{b}$, $b \neq 0$ yields

$$q_n(u) = \sum_{k=0}^n c_k^{(n)}(b) q_k(\frac{u}{b})$$

and finally, if we denote $\frac{1}{b}$ by a , we find

$$q_n(u) = \sum_{k=0}^n c_k^{(n)}(\frac{1}{b}) q_k(u).$$

By identification, we find the desired result. $\square$

5.3.2. Application

With this result, we can find a very interesting relation between the modified Bessel function of the second kind, K_ν , $\nu = i + \frac{1}{2}$, where $0 \leq i \leq n$ is an integer [8].

Proposition 3. We have the following relation involving the modified Bessel function of the second kind:

$$\frac{e^{(1-a)u}}{\Gamma(n+\frac{1}{2})} K_{n+\frac{1}{2}}(u) = \sum_{i=0}^n c_i^{(n)} \left(\frac{1}{a}\right) \frac{(\frac{2}{u})^{n-i} a^{i+\frac{1}{2}}}{\Gamma(i+\frac{1}{2})} K_{i+\frac{1}{2}}(au). \quad (22)$$

Proof. This is an immediate consequence of (21), and

$$e^{-u} q_{v-\frac{1}{2}}(u) = \frac{2^{1-v}}{\Gamma(v)} u^v K_v(u), \quad u \geq 0.$$

□

We provide, in Appendix E, a very short set of Maple instructions in order to assure readers about the correctness of the result.

5.4. The Inverse of Linearization Coefficients of Bessel Polynomials

In the previous section, we wrote (14) as follows:

$$\begin{pmatrix} q_n(au)q_0((1-a)u) \\ q_{n-1}(au)q_1((1-a)u) \\ \vdots \\ q_1(au)q_{n-1}((1-a)u) \\ q_0(au)q_n((1-a)u) \end{pmatrix} = M_n^\beta \begin{pmatrix} q_0(u) \\ q_1(u) \\ \vdots \\ q_{n-1}(u) \\ q_n(u) \end{pmatrix}$$

Then, it is natural to study the inverse of the matrix M_n^β , which we denote by M_n^γ and is defined as follows:

$$M_n^\gamma \begin{pmatrix} q_n(au)q_0((1-a)u) \\ q_{n-1}(au)q_1((1-a)u) \\ q_{n-2}(u)q_2((1-a)u) \\ \vdots \\ q_0(au)q_n((1-a)u) \end{pmatrix} = \begin{pmatrix} q_0(u) \\ q_1(u) \\ q_2(u) \\ \vdots \\ q_n(u) \end{pmatrix}$$

which we write as

$$\begin{pmatrix} \gamma_0^{(n,0)}(a) & \gamma_1^{(n,0)}(a) & \cdots & \gamma_n^{(n,0)}(a) \\ \gamma_0^{(n-1,1)}(a) & \gamma_1^{(n-1,1)}(a) & \cdots & \gamma_n^{(n-1,1)}(a) \\ \gamma_0^{(n-2,2)}(a) & \gamma_1^{(n-2,2)}(a) & \cdots & \gamma_n^{(n-2,2)}(a) \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_0^{(0,n)}(a) & \gamma_1^{(0,n)}(a) & \cdots & \gamma_n^{(0,n)}(a) \end{pmatrix} \begin{pmatrix} q_n(au)q_0((1-a)u) \\ q_{n-1}(au)q_1((1-a)u) \\ q_{n-2}(u)q_2((1-a)u) \\ \vdots \\ q_0(au)q_n((1-a)u) \end{pmatrix} = \begin{pmatrix} q_0(u) \\ q_1(u) \\ q_2(u) \\ \vdots \\ q_n(u) \end{pmatrix}$$

It is straightforward to deduce that

$$\gamma_i^{(n-k,k)}(a) = \gamma_{n-i}^{(n-k,k)}(1-a), \quad 0 \leq i, k \leq n.$$

Moreover, if we take $u = 0$, we see

$$\sum_{i=0}^n \gamma_i^{n-k,k}(a) = 1, \quad 0 \leq k \leq n.$$

Example 2.

$$M_\gamma^1 = -\frac{1}{2a-1} \begin{pmatrix} a-1 & a \\ a & a-1 \end{pmatrix},$$

$$M_{\gamma}^2 = -\frac{1}{(2a-1)^3} \begin{pmatrix} (-1+a)^2(1-3a) & -a(-1+a)(-1+2a) & -a^2(-2+3a) \\ 3(1-a)a^2 & -(a^2-a+1)(-1+2a) & -3a(-1+a)^2 \\ -(3a^2-3a+1)a & a(1-a)(2a-1) & (1-a)(3a^2-3a+1) \end{pmatrix}$$

and

$$M_{\gamma}^3 = -\frac{1}{(2a-1)^5}$$

$$\begin{pmatrix} (10a^2-6a+1)(a-1)^3 & \frac{a(a-1)^2(30a^2-26a+5)}{5} & \frac{a^2(a-1)(30a^2-34a+9)}{5} \\ 10a^3(a-1)^2 & (a-1)(6a^4-8a^3+10a^2-6a+1) & (6a^4-16a^3+22a^2-14a+3)a \\ 10a^2(a-1)^3 & (6a^3-14a^2+13a-4)a^2 & (a-1)^2(6a^3-4a^2+3a-1) \\ a(10a^4-20a^3+16a^2-6a+1) & \frac{2(15a^3-20a^2+10a-2)a(a-1)}{5} & \frac{2(15a^3-25a^2+15a-3)a(a-1)}{5} \\ & \frac{a^3(10a^2-14a+5)}{10a^2(a-1)^3} & \\ & \frac{5(a-1)^2(2a^2-2a+1)a}{(a-1)(10a^4-20a^3+16a^2-6a+1)} & \end{pmatrix}.$$

In the following, we are going to present all the results of the IMLCBPs only as conjectures.

In the abstract, we wrote that the new recursion formula that we provided is more important.

In fact, if we use the matrices M_{γ}^2 and M_{γ}^3 , we observe the following:

- If we write the recurrence formula found by Berg and Vignat written in terms of $\gamma_k^{(n,m)}$,

$$\frac{1}{2k+1}\gamma_{k+1}^{(n,m)}(a) = \frac{a^2}{2n-1}\gamma_k^{(n-1,m)}(a) + \frac{(1-a)^2}{2m-1}\gamma_k^{(n,m-1)}(a) \quad (23)$$

we can easily check that it does not work;

- if we use the matrices M_{γ}^2 and M_{γ}^3 and if we write the recurrence that we found written in terms of $\gamma_k^{(n,m)}$,

$$(2m-1)\gamma_k^{(n,m)}(a) + (2n+1)\gamma_k^{(n+1,m-1)}(a) = (2k-1)\gamma_{k-1}^{(n,m-1)}(a) + (2n+2m-2k-1)\gamma_k^{(n,m-1)}(a),$$

we easily check that this works perfectly; for instance, when $n = 0$, $m = 3$, and $k = 1$, we obtain

$$\gamma_1^{(1,2)}(a) + 5\gamma_1^{(0,3)}(a) - \gamma_0^{(0,2)}(a) - 3\gamma_1^{(0,2)}(a) = \frac{(6a^3-14a^2+13a-4)a^2}{(2a-1)^5} + 5\frac{2(15a^3-20a^2+10a-2)a(a-1)}{5(2a-1)^5} - \frac{(3a^2-3a+1)a}{(2a-1)^3} - 3\frac{a(a-1)}{(2a-1)^2} = 0.$$

and we can announce the following conjecture.

Conjecture The coefficients $\gamma_k^{(n,m)}$ of the matrix M_{γ} fulfil the same recurrence (13): For any integers n, m , $n \geq 1$, $m \geq 1$ and for any $1 \leq k \leq n+m$, we have the following results:

$$(2m-1)\gamma_k^{(n,m)}(a) + (2n+1)\gamma_k^{(n+1,m-1)}(a) = (2k-1)\gamma_{k-1}^{(n,m-1)}(a) + (2n+2m-2k-1)\gamma_k^{(n,m-1)}(a), \quad (24)$$

taking into account either $\gamma_{-1}^{(n,m-1)}(a) = 0$ or $\gamma_{n+m}^{(n,m-1)}(a) = 0$.

Conjecture All the coefficients of the matrix M_{γ} are positive for $a \notin [0, 1]$.

Conjecture The expression

$$\gamma_n^{n-k,k} := \frac{(\frac{1}{2})_n (-1+a)^{n-k+1} a^{n-k} {}_2F_1([\frac{3}{2}, 1-k], [-n+\frac{3}{2}], (-1+2a)^2) \Gamma(2n-1)}{(\frac{1}{2})_k (\frac{1}{2})_{n-k} n (-1+2a)^{2n-1} \Gamma(n) 2^{2k-2}} \quad (25)$$

fulfills

$$\sum_{k=0}^n \beta_k^{(n-l,l)}(a) \cdot \gamma_n^{n-k,k} = \frac{1}{2} \delta_{l,0}, \quad 0 \leq k, l \leq n.$$

Please use the short set of Maple instructions presented in the Appendix F below.

6. Discussion

In this paper, many conjectures, mathematical statements that have not yet been rigorously proved, are presented. These conjectures arose when I noticed a pattern that holds true for many cases. However, just because a pattern holds true for many cases does not mean that the pattern will hold true for all cases. To notice a pattern that holds true is not always easy. Conjectures must be proved for the mathematical observation to be fully accepted. When a conjecture is rigorously proved, it becomes a theorem and will have many applications, maybe not in the near future but surely in the far future.

7. Conclusions

In 2008, Berg and Vignat were the first who worked on the linearization coefficients of Bessel polynomials, and since then, Atia and co-authors continued in this direction. This is an expository paper together with some new results and conjectures.

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Appendix A

```
> restart;
> A := nu -> GAMMA(nu + 1/2) / (GAMMA(nu) * GAMMA(1/2));
> Inm := (x, n, m) -> int(1 / ((1 + (y - x)^2)^(n + 1) * (y^2 + 1)^(m + 1)),
y = -infinity..infinity);
> Jnm := (x, n, m) -> (1/2) * (sum(A(k + 1/2) * Betanmka(n, m, k, 1/2) / (A(n + 1/2) *
A(m + 1/2) * (1 + (1/4) * x^2)^(k+1)), k = min(n, m)..n + m));
> factor(simplify(Inm(x, 1, 2)));
```

$$\frac{\pi * (3 * x^4 + 56 * x^2 + 560)}{8(x^8 + 16 * x^6 + 96 * x^4 + 256 * x^2 + 256)}$$

```
> factor(simplify(Jnm(x, 1, 2)));
```

$$\frac{3\pi * Betanmka(1, 2, 1, 1/2)}{16(1 + (1/4) * x^2)^2} + \frac{\pi * Betanmka(1, 2, 1, 1/2)}{4(1 + (1/4) * x^2)^3} + \frac{3\pi * Betanmka(1, 2, 1, 1/2)}{10(1 + (1/4) * x^2)^4}$$

```
> Betanmka := (n, m, k, a) -> a^(2*n+m-k) * (1 - a)^(-n+k) * pochhammer(1/2, k)
*pochhammer(1/2, n + m - k) * (sum((-a)^j
*binomial(m + n + 1, 2 * n + 2 * m - 2 * k - j) * binomial(-m + k + j, j), j = 0.2 * m + 2 *
n - 2 * k)) / (pochhammer(1/2, n) * pochhammer(1/2, m));
```

$$> \text{factor}(\text{simplify}(\frac{3\pi * \text{Betanmka}(1,2,1,1/2)}{16(1+(1/4)*x^2)^2} + \frac{\pi * \text{Betanmka}(1,2,1,1/2)}{4(1+(1/4)*x^2)^3} + \frac{3\pi * \text{Betanmka}(1,2,1,1/2)}{10(1+(1/4)*x^2)^4}));$$

$$\frac{\pi * (3 * x^4 + 56 * x^2 + 560)}{8(x^8 + 16 * x^6 + 96 * x^4 + 256 * x^2 + 256)}$$

Appendix B

```
> restart;
> djm := (m,j) -> 2-2*m * (sum(2i * binomial(2 * m - 2 * i, m - i) * binomial(m + i, m)
* binomial(i, j), i = j..m));
> Pm := m -> sum(djm(m, j) * xj, j = 0..m) ;; Pm(m)


$$\sum_{j=0}^m (2^{-2*m} * 2^j * \text{binomial}(2 * m - 2 * j, m - j) * \text{binomial}(m + j, m) \\ * \text{hypergeom}([-m + j, m + 1 + j], [j - m + 1/2], 1/2) * x^j)$$

> expm := m -> pi * (sum(2-2*m * 2j * binomial(2 * m - 2 * j, m - j) * binomial(m + j, m)
* hypergeom([-m + j, m + 1 + j], [j + 1/2 - m], 1/2) * xj, j = 0..m)) / (2m+3/2 * (x + 1)m+1/2);
simplify(expm(2))
```

$$\frac{3\pi * (4 * x^2 + 10 * x + 7) * \sqrt{2}}{128(x + 1)^{5/2}}$$

```
> assume(x > -1);
intm := m -> int(1/(s4 + 2 * s2 * x + 1)1+m, s = 0..infinity); simplify(intm(2))
```

$$\frac{3\pi * (4 * x^2 + 10 * x + 7) * \sqrt{2}}{128(x + 1)^{5/2}}.$$

Appendix C

```
> restart;
> A := nu -> GAMMA(nu + 1/2) / (GAMMA(nu) * GAMMA(1/2));
> BetaN3third := (n,m,p) -> sum(factorial(2 * l) * (-1)k * factorial(k + l + 1) *
(sum(pochhammer(-p, k + l - j) * (sum(pochhammer(-n, i) * pochhammer(-m, j - i) /
(pochhammer(-2 * n, i) * pochhammer(-2 * m, j - i) * factorial(i) * factorial(j - i)),
i = max(0, j - m)..min(j, n))) / (pochhammer(-2 * p, k + l - j) * factorial(k + l - j)),
j = max(0, k + l - p)..min(k + l, n + m))) / (3k+l+1 * factorial(l) * factorial(k) *
factorial(l + 1 - k)), k = 0..min(n + m + p - l, l + 1));
> intdouble := (n,m,p) -> int(int(Phi(n + 1/2) * Phi(m + 1/2) * Phi(p + 1/2) /
((1 + x2)n+1 * (1 + y2)m+1 * (1 + (z - x - y)2)p+1), x = -infinity..infinity),
y = -infinity..infinity);
> summation := (n,m,p) -> sum(Phi(l + 1/2) * BetaN3third(n,m,p,l) / (1 + (1/9) *
z2)(l + 1), l = min(n,m,p)..n + m + p);
> factor(intdouble(1,2,3));
```

$$\frac{2(15 * z^{10} + 1015 * z^8 + 30942 * z^6 + 555174 * z^4 + 6341571 * z^2 + 45684243)}{15\pi * (z^2 + 9)^7}$$

```
> factor(summation(1,2,3));
```

$$\frac{2(15 * z^{10} + 1015 * z^8 + 30942 * z^6 + 555174 * z^4 + 6341571 * z^2 + 45684243)}{15\pi * (z^2 + 9)^7}$$

Appendix D

```
> restart;
> tracep := p - > (1 + 2 * p) * a2*p - p * (1 + 2 * p) * a2*p-1 + (-1 + a) * (1 - (-1 + a)2*p) / (a * (a - 2)) + (1/2) * (-1 + 2 * a)1+2*p * (-1 + a)2 / (a * a * (a - 2)) - (-1 + a) * (1 + 2 * p) * a2*p / a - (1/2) * (-1 + a)2 / (a2 * (a - 2)) + (1/2) * ((-1 + a)2*p+2 - (a + 1)1+2*p * (-1 + a)) / (a2 * (a - 2)) + (1 + 2 * p) * (p - 1) * a2*p-1 + (1/2) * ((1 - 2 * a)1+2*p + 1 - 2 * a + (-1 + a)1+2*p * (a + 1) + (a + 1)1+2*p * (-1 + a)) / (a2 * (a - 2));
> factor(simplify(2*a*tracep(3)-1));
```

$$(-1 + 2a)^7$$

Appendix E

```
> restart;
> KuKau := (n, a) -> exp((1 - a) * u) * BesselK(n + 1/2, u) / GAMMA(n + 1/2) - (sum(sqrt(a) * cnk(n, i, 1/a) * (2/u)(n-i) * ai * BesselK(i + 1/2, a * u) / GAMMA(i + 1/2), i = 0..n));
```

$$KuKau := (n, a) \rightarrow e^{(1-a)u} \frac{BesselK(n + 1/2, u)}{\Gamma(n + 1/2)} - \sum_{i=0}^n \frac{\sqrt{a} cnk(n, i, 1/a) (\frac{2}{u})^{n-i} a^i BesselK(i + 1/2, au)}{\Gamma(i + 1/2)}$$

```
> simplify(KuKau(2, a2));
```

$$\begin{aligned} & -(2/3) * \sqrt{2} * \exp(-a^2 * u) * (cnk(2, 2, 1/a^2) * a^4 * u^2 + 3 * cnk(2, 1, 1/a^2) * a^2 * u \\ & + 3 * cnk(2, 2, 1/a^2) * a^2 * u - u^2 + 3 * cnk(2, 0, 1/a^2) + 3 * cnk(2, 1, 1/a^2) \\ & + 3 * cnk(2, 2, 1/a^2) - 3 * u - 3) / u^{5/2} \end{aligned}$$

```
> cnk := (n, k, a) -> ak * binomial(n, k) / binomial(2 * n, 2 * k) * sum(binomial(n + 1, k + 1 - r) * binomial(n - k - 1, r - 1) * (1 - a)r, r = 0..min(n - k, k + 1));
```

$$cnk := (n, k, a) \rightarrow$$

$$\frac{a^k \binom{n}{k} \sum_{r=0}^{\min(n-k, k+1)} \binom{n+1}{k+1-r} \binom{n-k-1}{r-1} (1-a)^r}{\binom{2n}{2k}}$$

```
> simplify(-(2/3) * sqrt(2) * exp(-a2 * u) * (cnk(2, 2, 1/a2) * a4 * u2 + 3 * cnk(2, 1, 1/a2) * a2 * u + 3 * cnk(2, 2, 1/a2) * a2 * u - u2 + 3 * cnk(2, 0, 1/a2) + 3 * cnk(2, 1, 1/a2) + 3 * cnk(2, 2, 1/a2) - 3 * u - 3) / u5/2)
```

Appendix F

```
> beta := (m, n, k) -> a(2*n+m-k) * (1 - a)(-n+k) * pochhammer(1/2, k) * pochhammer(1/2, n + m - k) * (sum((-a)(-j) * binomial(m + n + 1, 2 * n + 2 * m - 2 * k - j) * binomial(-m + k + j, j), j = 0..2 * m + 2 * n - 2 * k)) / (pochhammer(1/2, n) * pochhammer(1/2, m))
> gamma := (n, k) -> pochhammer(1/2, n) * 2(2*n-2*k) * (-1 + a)(n-k+1) * a(n-k) * hypergeom([3/2, 1 - k], [-n + 3/2], (-1 + 2 * a)2) * GAMMA(2 * n - 1) / (pochhammer(1/2, k) * pochhammer(1/2, n - k) * n * (-1 + 2 * a)(2*n-1) * GAMMA(n)2 * 2(2*n-2))
> simplify(beta(3, 0, 0) * gamma(3, 0) + beta(3, 0, 1) * gamma(3, 1) + beta(3, 0, 2) * gamma(3, 2) + beta(3, 0, 3) * gamma(3, 3)) = 1/2.
```

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