

On Symmetries of Integrable Quadrilateral Equations

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Abstract: In the paper, we describe a method for deriving generalized symmetries for a generic discrete quadrilateral equation that allows a Lax pair. Its symmetry can be interpreted as a flow along the tangent direction of its solution evolving with a Lie group parameter t . Starting from the spectral problem of the quadrilateral equation and assuming the eigenfunction evolves with the parameter t , one can obtain a differential-difference equation hierarchy, of which the flows are proved to be commuting symmetries of the quadrilateral equation. We prove this result by using the zero-curvature representations of these flows. As an example, we apply this method to derive symmetries for the lattice potential Korteweg–de Vries equation.

Keywords: discrete integrable systems; symmetries; Lax pair; zero-curvature representation

1. Introduction

In the study of integrable partial difference equations (PΔEs), the existence of an infinite hierarchy of symmetries remains a fundamental integrability criterion. The concept of symmetries of integrable PΔEs appeared as early as in 1991 in [1] as similarity constraints for integrable lattices. There have been many advancements in the research of symmetries of PΔEs [2–14]. Various methods have been developed to construct these symmetries, such as the method via the spectral problem [2–4,6], using the recursion operator [11], using master symmetry [13], the Gardner method [12] and the Lax–Darboux scheme [9,10].

Among them, the Lax–Darboux scheme is an algebraic structure that encompasses integrable partial differential equations, differential-difference equations and PΔEs within a unified framework. In particular, within this framework, the Bianchi permutability condition for the Darboux transformations leads to a PΔE, and the compatibility of the Lax pairs and the Darboux transformations lead to an infinite hierarchy of differential-difference equations which provide commuting symmetries of the PΔE [9].

In the study of symmetries of continuous and differential-difference equations with Lax pairs, the zero-curvature representations of isospectral and non-isospectral flows have proven to be a powerful tool in deriving symmetries and their Lie algebras for the isospectral equation hierarchy as well as non-isospectral equation hierarchy, e.g., [15–26]. In this paper, we aim to extend this method to Lax-integrable PΔEs, with a particular focus on isospectral flows. Specifically, we will begin by deriving the isospectral flows and their zero-curvature representations associated with either the m -part or n -part of the Lax pair of a given PΔE. Subsequently, we will use these representations, along with the zero-curvature representation of the PΔE, to prove that these isospectral flows are symmetries of the PΔE. The proof follows a similar approach to that used in the cases of the continuous and differential-difference equations involving Lax pairs. The method can be viewed as a more general version of the Lax–Darboux scheme.

This paper is organized as follows. Section 2 contains the notations and relevant definitions essential for the subsequent discussions. Section 3 elaborates on the method for



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deriving symmetries for Lax-integrable PΔEs by zero-curvature representations. As a specific application of this method, in Section 4, we derive symmetries for the lattice potential Korteweg–de Vries (lpKdV) equation. Finally, Section 5 is devoted to concluding remarks.

2. Notations and Definitions

In this paper, our focus is solely on the following type of quadrilateral equation:

$$Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) = 0, \quad (1)$$

where $u = u(n, m)$ is a function defined on $\mathbb{Z} \times \mathbb{Z}$,

$$\tilde{u} = E_n u = u(n+1, m), \quad \hat{u} = E_m u = u(n, m+1), \quad \hat{\tilde{u}} = u(n+1, m+1),$$

E_n and E_m , respectively, serve as shift operators in the n and m directions, while p and q are spacing parameters of direction n and m , respectively.

We introduce some notations that will be used in deriving isospectral hierarchy. Assume that $u = u(n, m)$ vanishes rapidly as $|n| \rightarrow \infty$ or $|m| \rightarrow \infty$. Let $\mathcal{V}$ be a linear space consisting of all functions f , where each f is a function of u and its shifts, and $f|_{u=0} = 0$. Each f is C^∞ differentiable with respect to n and m , and C^∞ -Gâteaux differentiable with respect to u and its shifts. Here, the Gâteaux (or Fréchet) derivative of $f \in \mathcal{V}$ (or f as an operator living on $\mathcal{V}$) in direction $g \in \mathcal{V}$ is defined as

$$f'[g] = \frac{d}{d\epsilon} f(u + \epsilon g)|_{\epsilon=0}. \quad (2)$$

By means of the Gâteaux derivative, one can define the Lie product for any $f, g \in \mathcal{V}$ as

$$\llbracket f, g \rrbracket = f'[g] - g'[f]. \quad (3)$$

We define a Laurent matrix polynomials space $\mathcal{Q}_2$ composed of all 2×2 matrices $\mathcal{M} = (q_{ij}(r, u))_{2 \times 2}$, where all the $\{q_{ij}\}$ are Laurent polynomials of r . In addition, we denote $\mathcal{Q}_2^{(0)} = \{\mathcal{M} \in \mathcal{Q}_2 \mid \mathcal{M}|_{u=0} = 0\}$. We also introduce operators $\Delta_n^{(\pm)} = E_n \pm 1$, whose inverse operators can be denoted as (e.g., [4])

$$(\Delta_n^{(-)})^{-1} = - \sum_{k=0}^{\infty} E_n^k, \quad (\Delta_n^{(+)})^{-1} = \sum_{k=0}^{\infty} (-1)^k E_n^k.$$

Lie symmetries of Equation (1) are obtained by requiring the infinitesimal invariant condition

$$\text{pr}\mathbf{X} Q|_{Q=0} = 0. \quad (4a)$$

Here, $\mathbf{X}$ is the infinitesimal generator; $\text{pr}\mathbf{X}$ is the prolongation of the infinitesimal generator $\mathbf{X}$. They are defined as follows

$$\mathbf{X} = \xi \partial_u, \quad (4b)$$

$$\text{pr}\mathbf{X} = \xi \partial_u + \tilde{\xi} \partial_{\tilde{u}} + \hat{\xi} \partial_{\hat{u}} + \hat{\tilde{\xi}} \partial_{\hat{\tilde{u}}}, \quad (4c)$$

where ξ is a function of u and its shifts $u(n+i, m+j)$, $i, j \in \mathbb{Z}$. If ξ exclusively depends on u , then the symmetries are called point symmetries; otherwise, they are referred to as generalized symmetries. In this paper, we only focus on discussing the generalized symmetries of Equation (1). In terms of the Gâteaux derivative, to meet the condition (4a) is to find ξ such that

$$Q'[\xi]|_{Q=0} = 0. \quad (5)$$

Corresponding to the infinitesimal generator (4b), a group transformation between solutions of Equation (1), namely, $g_t : u(n, m) \mapsto u(n, m; t)$, can be derived in principle by solving the initial value problem

$$\frac{du(t)}{dt} = \xi(u(t), \tilde{u}(t), \hat{u}(t), \dots), \text{ with } u(0) = u, \quad (6)$$

where $t \in \mathbb{R}$ is the Lie group parameter, $u = u(n, m)$ and $u(t) \doteq u(n, m; t)$. If u is a solution of Equation (1) and ξ is its symmetry, then $u(t)$ determined by the initial problem (6) provides a solution for Equation (1) as well. Thus, (6) can be understood as that $u(t)$ evolves with the parameter t along the tangent direction ξ in the solution manifold of Equation (1) (see [27]). In this context, if $u(t)$ is a solution of (1), i.e.,

$$Q(u(t), \tilde{u}(t), \hat{u}(t), \hat{\hat{u}}(t); p, q) = 0, \quad (7)$$

we always have

$$\frac{d}{dt} Q(u(t)) = 0, \quad (8)$$

where we drop off $\tilde{u}(t)$, $\hat{u}(t)$ and $\hat{\hat{u}}(t)$ for short. If we assume $\frac{d}{dt}$ and shift operators commute, i.e., $\frac{d}{dt} E_n = E_n \frac{d}{dt}$, $\frac{d}{dt} E_m = E_m \frac{d}{dt}$, then (8) gives rise to (4a) (i.e., (5)) provided the initial problem (6) holds.

3. General Description of the Method

In this section, we elaborate on the method for deriving symmetries of PΔE (1).

3.1. Isospectral Flows $\{K^{(l)}\}$

Suppose (1) has a Lax pair:

$$\tilde{\phi} = M(u, \tilde{u}; p, r)\phi, \quad (9a)$$

$$\hat{\phi} = N(u, \hat{u}; q, r)\phi, \quad (9b)$$

where M and N are matrices in $\mathcal{Q}_2$, $\phi = (\phi_1, \phi_2)^T$ and r stands for a spectral parameter. In other words, Equation (1) functions as the compatibility condition for (9), i.e.,

$$\hat{M}N = \tilde{N}M, \quad (10)$$

if and only if u solves (1).

To construct symmetries for Equation (1), we introduce a parameter t into u . In this context, the linear problem (9a) is written as

$$\tilde{\phi} = M(u(t), \tilde{u}(t); p, r)\phi, \quad (11a)$$

and we assume that ϕ evolves with t as

$$\phi_t = U\phi, \quad (11b)$$

where $U \in \mathcal{Q}_2$ is a matrix function of $E_n^j u(t)$ and p and r . The compatibility condition between (11a) and (11b) is

$$M_t = \tilde{U}M - MU. \quad (12)$$

Next, we present some conditions (cf. [28]), under which we can construct an isospectral hierarchy

$$u_{t_l} = K^{(l)}, \quad l = 0, 1, 2, \dots. \quad (13)$$

Note that u can be considered as $u(t_0, t_1, t_2, \dots)$, while in the following, we denote it by u for short and without causing any confusion.

Condition 1. Assume that the linear problems (11a) and (11b) satisfy the following conditions:
 (1) The matrix equation

$$M'[X] = \tilde{U}M - MU \quad (14)$$

possesses a unique couple of non-zero solutions $X \in \mathcal{V}$ and $U \in \mathcal{Q}_2$ satisfying $U|_{u=0} = U_0$, where U_0 is a given matrix independent of u and meeting commuting relation $U_0 M|_{u=0} = M|_{u=0} U_0$;

(2) for any given $Y \neq 0 \in \mathcal{V}$, there exist solutions $X \in \mathcal{V}$ and $U \in \mathcal{Q}_2^{(0)}$ satisfying

$$M'[X - r^\alpha Y] = \tilde{U}M - MU, \quad (15)$$

where α is some integer related to (11a).

In fact, in light of the first assumption in Condition 1, there exists a unique $K^{(0)} \in \mathcal{V}$ and a unique $U^{(0)} \in \mathcal{Q}_2$, such that

$$M'[K^{(0)}] = \tilde{U}^{(0)}M - MU^{(0)}, \quad (16)$$

where $U^{(0)}|_{u=0} = U_0$. Thus, the first equation is given by

$$u_{t_0} = K^{(0)}. \quad (17)$$

The expression (16) is commonly referred to as the zero-curvature representation of Equation (17) or the flow $K^{(0)}$. Once $K^{(0)}$ is determined, by the second assumption in Condition 1, there exists a unique field $K^{(1)} \in \mathcal{V}$ and a unique matrix $V^{(1)} \in \mathcal{Q}_2^{(0)}$, such that

$$M'[K^{(1)} - r^\alpha K^{(0)}] = \tilde{V}^{(1)}M - MV^{(1)}. \quad (18)$$

Repeating this procedure, one can determine a unique sequence of fields $\{K^{(l)}\} \in \mathcal{V}$ and a unique sequence of matrices $\{V^{(l)}\} \in \mathcal{Q}_2^{(0)}$, for $l = 2, 3, \dots$, such that

$$M'[K^{(l)} - r^\alpha K^{(l-1)}] = \tilde{V}^{(l)}M - MV^{(l)}, \quad l = 2, 3, \dots \quad (19)$$

From Equations (16), (18) and (19), we can derive

$$M'[K^{(l)}] = \tilde{U}^{(l)}M - MU^{(l)}, \quad l = 0, 1, 2, \dots, \quad (20a)$$

where

$$U^{(l)} = \sum_{j=0}^l V^{(j)} r^{\alpha(l-j)} \in \mathcal{Q}_2, \quad V^{(0)} = U^{(0)}, \quad U^{(l)}|_{u=0} = U_0 r^{\alpha l}. \quad (20b)$$

It means that we obtain the following isospectral hierarchy

$$u_{t_l} = K^{(l)}, \quad l = 0, 1, 2, \dots, \quad (21)$$

with (20) as their zero-curvature representations. The Lax pair of the above hierarchy now is composed of (11a) and

$$\phi_{t_l} = U^{(l)}\phi. \quad (22)$$

Note that the map $Y \mapsto X$ determined by (15) defines a recursion operator L such that $X = LY$. This means the above hierarchy can be expressed alternatively as

$$u_{t_l} = K^{(l)} = L^l K^{(0)}, \quad l = 0, 1, 2, \dots \quad (23)$$

In addition, it follows from Condition 1 that the following proposition is valid.

Proposition 1. If $U_0 = 0$ in Condition 1, then the matrix Equation (14) has only zero solutions $X = 0$ and $U = 0$.

Utilizing the identity [29]

$$M'[[f, g]] = (M'[f])'g - (M'[g])'f, \quad \forall f, g \in \mathcal{V}, \quad (24)$$

from the zero-curvature representations (20), we can derive that the isospectral flows $\{K^{(l)}\}$ satisfy (cf. [22,25])

$$M'[[K^{(l_1)}, K^{(l_2)}]] = (E_n \langle U^{(l_1)}, U^{(l_2)} \rangle)M - M \langle U^{(l_1)}, U^{(l_2)} \rangle, \quad (25a)$$

where

$$\langle U^{(l_1)}, U^{(l_2)} \rangle = U^{(l_1)'}[K^{(l_2)}] - U^{(l_2)'}[K^{(l_1)}] + [U^{(l_1)}, U^{(l_2)}], \quad (25b)$$

and $[U^{(l_1)}, U^{(l_2)}] = U^{(l_1)}U^{(l_2)} - U^{(l_2)}U^{(l_1)}$. Meanwhile, from (20b), one can obtain

$$\langle U^{(l_1)}, U^{(l_2)} \rangle|_{u=0} = 0. \quad (26)$$

Consequently, considering (25) and (26), and applying Proposition 1, we can obtain the following.

Proposition 2. The isospectral flows $\{K^{(l)}\}$ satisfy

$$[[K^{(l_1)}, K^{(l_2)}]] = 0. \quad (27)$$

3.2. Symmetries

We have constructed the isospectral hierarchy (21) associated with (11a) and (11b). Next, we will prove that under certain conditions these isospectral flows $\{K^{(l)}\}$ provide commuting symmetries of Equation (1).

We assume $u(t)$ is a solution of Equation (1), i.e., (7) holds, so the Lax pair of (7) is then composed of (11a) and

$$\hat{\phi} = N(u(t), \hat{u}(t); q, r)\phi. \quad (28)$$

Note that by $u(t)$, we denote a more general $u(t_0, t_1, t_2, \dots)$. Now, we introduce a second condition.

Condition 2. For the matrices N in (28) and $U^{(l)}$ in (22), we assume

$$(N_{t_l} + NU^{(l)})N^{-1}|_{u=0} = \hat{U}^{(l)}|_{u=0}, \quad l = 0, 1, 2, \dots \quad (29)$$

From the compatibility of (11a) and (28), we have (cf. (10))

$$\hat{M} = \tilde{N}MN^{-1}. \quad (30)$$

Taking derivative w.r.t. t_l yields

$$(\hat{M})_{t_l} = \tilde{N}_{t_l}MN^{-1} + \tilde{N}M_{t_l}N^{-1} - \hat{M}N_{t_l}N^{-1}. \quad (31)$$

For the terms on the right-hand side, by using (30) and (12), we obtain

$$\begin{aligned} \tilde{N}_{t_l}MN^{-1} &= \tilde{N}_{t_l}\tilde{N}^{-1}\tilde{N}MN^{-1} = \tilde{N}_{t_l}\tilde{N}^{-1}\hat{M}, \\ \tilde{N}M_{t_l}N^{-1} &= \tilde{N}(\tilde{U}^{(l)}M - MU^{(l)})N^{-1} \\ &= \tilde{N}\tilde{U}^{(l)}\tilde{N}^{-1}\tilde{N}MN^{-1} - \tilde{N}MN^{-1}NU^{(l)}N^{-1} \\ &= \tilde{N}\tilde{U}^{(l)}\tilde{N}^{-1}\hat{M} - \hat{M}NU^{(l)}N^{-1}. \end{aligned}$$

Thus, we have

$$(\widehat{M})_{t_l} = \widehat{M}'[(\widehat{u})_{t_l}] = \{E_n[(N_{t_l} + NU^{(l)})N^{-1}]\}\widehat{M} - \widehat{M}[(N_{t_l} + NU^{(l)})N^{-1}], \quad (32)$$

where $\widehat{M}'[(\widehat{u})_{t_l}]$ represents the Gâteaux derivative of $\widehat{M}$ in the direction $(\widehat{u})_{t_l}$ with respect to $\widehat{u}$. Here, we remark that we have admitted the commutative relations $E_n E_m u = E_m E_n u$ and $E_n \frac{d}{dt_l} = \frac{d}{dt_l} E_n$, while the relation

$$(\widehat{u})_{t_l} = \widehat{(u_{t_l})} \quad (33)$$

remains open. We will see that once (33) holds, then the flows $\{K^{(l)}\}$ are symmetries of Equation (1). (For a continuous equation

$$u_t = S(u, u_x, u_{xx}, \dots),$$

the flow $K^{(l)}(u)$ defined by $u_{t_l} = K^{(l)}(u)$ is a symmetry of the above equation if $(u_t)_{t_l} = (u_{t_l})_t$. For the discrete Equation (1), if we formally write it as

$$\widehat{u} = P(E_n^j u),$$

then the commutative relation (33) can be viewed as a counterpart of $(u_t)_{t_l} = (u_{t_l})_t$ in the discrete case). Let us proceed. By recalling (20), we can have

$$\widehat{M}'[\widehat{K}^{(l)}] = \widetilde{U}^{(l)} \widehat{M} - \widehat{M} \widehat{U}^{(l)}, \quad l = 0, 1, 2, \dots, \quad (34)$$

where $\widehat{M}'[\widehat{K}^{(l)}]$ represents the Gâteaux derivative of $\widehat{M}$ in direction $\widehat{K}^{(l)}$ with respect to $\widehat{u}$. Now, comparing (32) and (34), noticing the condition (29) and following Proposition 1, we immediately obtain

$$(\widehat{u})_{t_l} = \widehat{K}^{(l)}, \quad l = 0, 1, 2, \dots \quad (35)$$

and

$$(N_{t_l} + NU^{(l)})N^{-1} = \widehat{U}^{(l)}, \quad l = 0, 1, 2, \dots \quad (36)$$

Let us focus on the first result (35). It immediately indicates the commuting relation (33) since $\widehat{K}^{(l)} = \widehat{(u_{t_l})}$. Consequently,

$$Q'[K^{(l)}]|_{Q=0} = 0,$$

i.e., $K^{(l)}$ is a symmetry of Equation (1).

We can summarize the above results in the following theorem.

Theorem 1. Assuming that the PΔE (1) has Lax pair (9), matrix M satisfies Condition 1, and Condition 2 holds for N and $U^{(l)}$, then the flows $\{K^{(l)}\}$ in (21) are commuting symmetries of the PΔE (1).

3.3. Isospectral Flows $\{H^{(l)}\}$

One can also start from (28) and in this turn, assuming ϕ evolves with the parameter t , as (we will see that with certain assumptions, there is $W = U \pmod{\text{Equation (7)}}$)

$$\phi_t = W\phi, \quad (37)$$

where $W \in \mathcal{Q}_2$. Then, a second isospectral hierarchy can be derived by assuming the following conditions.

Condition 3. Assume that the linear problems (28) and (37) satisfy the following conditions:
 (1) The matrix equation

$$N'[X] = \widehat{W}N - NW \quad (38)$$

possesses a unique couple of non-zero solutions $X \in \mathcal{V}$ and $W \in \mathcal{Q}_2$ satisfying $W|_{u=0} = W_0$, where W_0 is a matrix independent of u and meeting $W_0 N|_{u=0} = N|_{u=0} W_0$;

(2) for any given $Y \neq 0 \in \mathcal{V}$, there exist solutions $X \in \mathcal{V}$ and $W \in \mathcal{Q}_2^{(0)}$ satisfying

$$N'[X - r^\beta Y] = \widehat{W}N - NW, \quad (39)$$

where β is some integer related to (28).

With the above assumption, a second isospectral hierarchy

$$u_{t_l} = H^{(l)}, \quad l = 0, 1, 2, \dots, \quad (40)$$

with the zero-curvature representations

$$N'[H^{(l)}] = \widehat{W}^{(l)}N - NW^{(l)}, \quad W^{(l)}|_{u=0} = W_0 r^{\beta l}, \quad l = 0, 1, 2, \dots, \quad (41)$$

can be derived. At the same time, we can obtain the following two propositions.

Proposition 3. If $W_0 = 0$ in Condition 3, then the matrix Equation (38) has only zero solutions $X = 0$ and $W = 0$.

Proposition 4. The isospectral flows $\{H^{(l)}\}$ satisfy

$$[[H^{(l_1)}, H^{(l_2)}]] = 0. \quad (42)$$

Now, we have two sets of flows, namely, $\{K^{(l)}\}$ and $\{H^{(l)}\}$. Apparently, $K^{(l)}$ is a function of $E_n^j u$ and $H^{(l)}$ is a function of $E_m^j u$. However, we can unify them in light of Condition 1, 2, 3 and a further assumption:

$$U^{(l)}|_{u=0} = W^{(l)}|_{u=0}, \quad l = 0, 1, 2, \dots. \quad (43)$$

To show the uniformity, we recall the results (36) together with (33), which yields

$$N_{t_l} = N'[u_{t_l}] = N'[K^{(l)}] = \widehat{U}^{(l)}N - NU^{(l)}, \quad l = 0, 1, 2, \dots. \quad (44)$$

Comparing the above equation with (41), in light of Proposition 3 and the assumption (43), we immediately obtain (modulo Equation (1))

$$K^{(l)} = H^{(l)}, \quad l = 0, 1, 2, \dots.$$

Now that the two sets are the same, we conclude that the flows $\{H^{(l)}\}$ in (40) are also commuting symmetries of the PΔE (1).

4. Symmetries for the lpKdV Equation

In this section, we employ the method described in Section 3 to derive symmetries for the lpKdV Equation.

The lpKdV Equation is given by [30,31]

$$Q = (p - q + \widehat{u} - \widetilde{u})(p + q + u - \widehat{\widetilde{u}}) - p^2 + q^2 = 0, \quad (45)$$

where p and q serves as the spacing parameters in n -direction and m -direction, respectively. It is known as the H1 equation in the Adler–Bobenko–Suris (ABS) classification [32]. It also

appears as the nonlinear superposition formula (Bianchi identity) of the solutions of the (potential) KdV Equation [33]. The lpKdV Equation has the following Lax pair [34,35]:

$$\tilde{\phi} = M\phi = \begin{pmatrix} v & v^2 - p^2 - r \\ 1 & v \end{pmatrix} \phi, \quad (46a)$$

$$\hat{\phi} = N\phi = \begin{pmatrix} v_1 & v_1^2 - q^2 - r \\ 1 & v_1 \end{pmatrix} \phi, \quad (46b)$$

where $\phi = (\phi_1, \phi_2)^T$, r is the spectral parameter, and $v = \tilde{u} - u - p$, $v_1 = \hat{u} - u - q$. Note that the Lax pair (46) exhibits a symmetric property in the sense of switching $(p, \tilde{\cdot})$ and $(q, \hat{\cdot})$.

To obtain a differential-difference hierarchy associated with the spectral problem (46a), we introduce the parameter t and assume ϕ evolves with t as

$$\phi_t = U\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \phi, \quad U \in \mathcal{Q}_2, \quad (47)$$

where U satisfies

$$U|_{u=0} = U_0 = \begin{pmatrix} 0 & -r \\ 1 & 0 \end{pmatrix} \quad (48)$$

and r is independent of t . In the method we described in Section 3, the flows of the differential-difference hierarchy determined by (46a) and (47) will provide symmetries for the lpKdV Equation (45). The compatibility of (46a) and (47), i.e., $(E_n\phi)_t = E_n(\phi_t)$, yields the zero-curvature Equation (12). With the above matrices M and U , the zero-curvature Equation (12) gives rise to

$$v_t = v(E_n - 1)A + \tilde{B} - (-r - p^2 + v^2)C, \quad (49a)$$

$$2vv_t = (-r - p^2 + v^2)\tilde{A} + v(E_n - 1)B - (-r - p^2 + v^2)D, \quad (49b)$$

$$0 = v(E_n - 1)C + \tilde{D} - A, \quad (49c)$$

$$v_t = (-r - p^2 + v^2)\tilde{C} + v(E_n - 1)D - B. \quad (49d)$$

From (49), we can derive

$$(r + p^2)(E_n - 1)(A + D) = 0, \quad (50)$$

which implies that $(E_n - 1)(A + D)$ is zero, i.e.,

$$D = -A + a, \quad (51)$$

where a is a constant. However, considering the boundary condition (48), we need to take $a = 0$ and thus we have

$$D = -A. \quad (52)$$

The subsequent analysis indicates that the linear problems (46a) and (47) satisfies Condition 1, and simultaneously, we derive the isospectral hierarchy associated with these linear problems. We begin by substituting the matrix M from (46a) and the matrix U from (47) into Equation (15), with $\alpha = 1$, which yields (noting that the Gâteaux derivative of M in (15) is taken with respect to u not v).

$$\Delta_n^{(-)}(X - rY) = -r\Delta_n^{(-)}C + L_1C, \quad (53a)$$

$$A = (\Delta_n^{(+)})^{-1}v\Delta_n^{(-)}C, \quad (53b)$$

$$B = -rC + (\Delta_n^{(+)})^{-1}[-2v\Delta_n^{(-)}(\Delta_n^{(+)})^{-1}v\Delta_n^{(-)} + (v^2 - p^2)\Delta_n^{(+)}]C, \quad (53c)$$

where

$$L_1 = [2(\Delta_n^{(+)-1} - 1)v\Delta_n^{(-)}(\Delta_n^{(+)-1}v\Delta_n^{(-)} + [1 - (\Delta_n^{(+)-1}](v^2 - p^2)\Delta_n^{(+)} - v^2 + p^2). \quad (53d)$$

To derive the initial flow $K^{(0)}$, one should take $Y = 0$ in (53a). Then, by comparing the coefficients of the same powers of r in (53a), we can have

$$C = 1, \quad (54)$$

$$X = (\Delta_n^{(+)-1}(v^2 - p^2). \quad (55)$$

Substituting (54) into (53b) and (53c) leads to

$$A = 0, \quad (56)$$

$$B = -r - p^2 + 2(\Delta_n^{(+)-1}v^2. \quad (57)$$

Hence, there exists a unique $K^{(0)} \in \mathcal{V}$ and a unique $V^{(0)} \in \mathcal{Q}_2$, given by

$$K^{(0)} = (\Delta_n^{(+)-1}(v^2 - p^2), \quad (58a)$$

$$V^{(0)} = \begin{pmatrix} 0 & -r - p^2 + 2(\Delta_n^{(+)-1}v^2 \\ 1 & 0 \end{pmatrix}, \quad (58b)$$

which satisfies the following equation

$$M'[K^{(0)}] = \tilde{V}^{(0)}M - MV^{(0)}. \quad (59)$$

Next, to derive the flow $K^{(1)}$, we take $Y = K^{(0)}$ in (53a). Then, by comparing the coefficients of the same powers of r in (53a), we obtain

$$C = K^{(0)}, \quad (60)$$

$$X = LK^{(0)}, \quad (61)$$

where

$$L = (\Delta_n^{(-)})^{-1}L_1. \quad (62)$$

Substituting (60) into (53b) and (53c), we obtain

$$A = (\Delta_n^{(+)-1}v\Delta_n^{(-)}K^{(0)}, \quad (63a)$$

$$B = -rK^{(0)} + (\Delta_n^{(+)-1}[-2v\Delta_n^{(-)}(\Delta_n^{(+)-1}v\Delta_n^{(-)} + (v^2 - p^2)\Delta_n^{(+)}]K^{(0)}. \quad (63b)$$

Hence, there exists a unique $K^{(1)} \in \mathcal{V}$ and a unique $V^{(1)} \in \mathcal{Q}_2^{(0)}$, given by

$$K^{(1)} = LK^{(0)}, \quad (64)$$

$$V^{(1)} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (65)$$

where A , B and C are given by (63a), (63b) and (60), respectively, which satisfy the following equation

$$M'[K^{(1)} - rK^{(0)}] = \tilde{V}^{(1)}M - MV^{(1)}. \quad (66)$$

Repeating the above procedure from $K^{(0)}$ to $K^{(1)}$ allows us to obtain higher order flows $K^{(l)} = L^l K^{(0)} \in \mathcal{V}$ and $V^{(l)} \in \mathcal{Q}_2^{(0)}$, $l = 2, 3, \dots$, which satisfy

$$M'[K^{(l)} - rK^{(l-1)}] = \tilde{V}^{(l)}M - MV^{(l)}, \quad l = 2, 3, \dots \quad (67)$$

From Equations (59), (66) and (67), we can derive

$$M'[K^{(l)}] = \tilde{U}^{(l)}M - MU^{(l)}, \quad l = 0, 1, 2, \dots, \quad (68a)$$

where

$$U^{(l)} = \sum_{j=0}^l V^{(j)} r^{l-j} \in \mathcal{Q}_2, \quad U^{(l)}|_{u=0} = U_0 r^l. \quad (68b)$$

Up to this point, we have obtained the following isospectral hierarchy

$$u_{t_l} = K^{(l)} = L^l K^{(0)}, \quad l = 0, 1, 2, \dots, \quad (69)$$

where $K^{(0)}$ and L are given by (58a) and (62), with Equation (68) as their zero-curvature representations. The first equation in this hierarchy is

$$u_{t_0} = K^{(0)} = (\Delta_n^{(+)})^{-1}(v^2 - p^2), \quad (70)$$

i.e.,

$$\tilde{u}_{t_0} + u_{t_0} = (\tilde{u} - u)^2 - 2p(\tilde{u} - u). \quad (71)$$

There also exists an isospectral hierarchy associated with the spectral problem (46b). Due to the symmetric property, this isospectral hierarchy and the corresponding zero-curvature representations can be, respectively, obtained from (69) and (68) by replacing $(p, \sim)$ with $(q, \hat{\sim})$. Their expressions are

$$u_{t_l} = H^{(l)} = \underline{L}^l H^{(0)}, \quad l = 0, 1, 2, \dots, \quad (72a)$$

$$N'[H^{(l)}] = \hat{W}^{(l)}N - NW^{(l)}, \quad l = 0, 1, 2, \dots, \quad (72b)$$

where $W^{(l)}$ can be obtained from $U^{(l)}$ in (68b) by replacing $(p, \sim)$ with $(q, \hat{\sim})$, and we have

$$W^{(l)}|_{u=0} = U_0 r^l, \quad (72c)$$

$$H^{(0)} = (\Delta_m^{(+)})^{-1}(v_1^2 - q^2), \quad (72d)$$

$$\begin{aligned} \bar{L} = & (\Delta_m^{(-)})^{-1} \{ [2(\Delta_m^{(+)})^{-1} - 1] v_1 \Delta_m^{(-)} (\Delta_m^{(+)})^{-1} v_1 \Delta_m^{(-)} \\ & + [1 - (\Delta_m^{(+)})^{-1}] (v_1^2 - q^2) \Delta_m^{(+)} - v_1^2 + q^2 \}. \end{aligned} \quad (72e)$$

Through straightforward calculations, we can obtain

$$(N_{t_l} + NU^{(l)})N^{-1}|_{u=0} = \begin{pmatrix} -q & -r \\ 1 & -q \end{pmatrix} U_0 r^l \begin{pmatrix} -q & -r \\ 1 & -q \end{pmatrix}^{-1} = U_0 r^l = \hat{U}^{(l)}|_{u=0}, \quad (73)$$

$$U^{(l)}|_{u=0} = W^{(l)}|_{u=0} = U_0 r^l. \quad (74)$$

Then, by Theorem 1 and the result in Section 3.3, we conclude the following:

- Both flows $\{K^{(l)}\}$ and $\{H^{(l)}\}$ are symmetries of the lpKdV Equation (45).

- The symmetries commute:

$$[[K^{(l_1)}, K^{(l_2)}]] = 0, \quad [[H^{(l_1)}, H^{(l_2)}]] = 0, \quad l_1, l_2 = 0, 1, 2, \dots \quad (75)$$

- $\{K^{(l)}\}$ and $\{H^{(l)}\}$ are same up to modulo the lpKdV Equation (45).

5. Concluding Remarks

In this paper, we have described an approach to deriving symmetries for Lax-integrable PΔEs. The primary technique employed in this paper involves the zero-curvature representations of flows, which have been used in deriving symmetries for continuous and differential-difference Lax-integrable equations. In references [3,4,6], similar results have already been obtained for several specific equations. The difference is that they proved this point in the space of the spectral data while we employed zero-curvature representations of flows. In addition, our method can be viewed as a more general version of the Lax–Darboux scheme (cf. [9]).

For continuous and differential-difference integrable systems, non-isospectral flows can also be used to construct symmetries, and in particular, some non-isospectral flows act as master symmetries to generate more symmetries. Non-isospectral flows are usually characterized by containing explicitly independent variables. In the case of PΔEs, it has been pointed out that all the ABS equations have symmetries that depend explicitly on n (or m) [13], and are closely related to master symmetries [13]. We can also explore non-isospectral symmetries and master symmetries, provided that we can drive non-isospectral flows in our approach. However, for the lpKdV Equation, we are currently unable to derive non-isospectral flows from its Lax pair (46) (unless we change the asymptotic matrix $U^{(0)}$). This method we described in this paper can also be applied to other quadrilateral equations such as ABS equations [32] and equations from the Lax–Darboux scheme (e.g., [35]) and multi-component PΔEs, which we will explore in the future research. In addition, Toda-type lattice equations have been shown to have symmetries [13]. These equations were classified by Adler [36,37]. Their integrability is understood as 2[1,1] type ABS-coupled equations (see [38]) by eliminating one component. It would be interesting to make clear how their symmetries are related to those of the ABS equations.

In this paper, the symmetry ζ of the PΔE (1) is interpreted as a flow along the tangent direction of solution $u(t)$, which is in the solution manifold of the PΔE. This approach relies on introducing t as a parameter. Such an idea has been used in describing Hamiltonian structures as well as symmetries for the continuous and differential-difference integrable equations (e.g., [19,27,39,40]). One can introduce a parameter as a continuous independent variable by implementing continuum limits. For the lpKdV Equation (45) that we considered in the previous section, in its continuum limits, the leading term gives rise to a differential-difference equation (see Equation (5.13) in [34] or Equation (4.6) in [41]) which is nothing but the first equation $u_{t_0} = K^{(0)}$ written in the form (71). There is also a way to introduce high order parameters $\{t_l\}$ and obtain a hierarchy $u_{t_l} = S^{(l)}$ from the continuum limits of the lpKdV Equation, see [42]. The obtained flows $\{S^{(l)}\}$ should be symmetries of the lpKdV Equation. In principle, it is understood the lpKdV Equation itself includes its infinitely many commuting symmetries via its continuum limits. One can also consider the spacing parameter p (or q) as a parameter. In that case, one can have non-autonomous (or non-isospectral) equations that involve explicit independent variables (see Chapter 10.4 of [34] and cf. [13]), which can also provide symmetries. It would be interesting to understand the connections of these symmetries obtained from different ways and also to understand how these symmetries are related to the solutions and conservation laws of the considered discrete integrable equations.

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