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An Efficient Solution of Multiplicative Differential Equations through Laguerre Polynomials

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Abstract: The field of multiplicative analysis has recently garnered significant attention, particularly in the context of solving multiplicative differential equations (MDEs). The symmetry concept in MDEs facilitates the determination of invariant solutions and the reduction of these equations by leveraging their intrinsic symmetrical properties. This study is motivated by the need for efficient methods to address MDEs, which are critical in various applications. Our novel contribution involves leveraging the fundamental properties of orthogonal polynomials, specifically Laguerre polynomials, to derive new solutions for MDEs. We introduce the definitions of Laguerre multiplicative differential equations and multiplicative Laguerre polynomials. By applying the power series method, we construct these multiplicative Laguerre polynomials and rigorously prove their basic properties. The effectiveness of our proposed solution is validated through illustrative examples, demonstrating its practical applicability and potential for advancing the field of multiplicative analysis.

Keywords: multiplicative analysis; power series; multiplicative differential equations; multiplicative Laguerre polynomials



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1. Introduction

In the story of calculus, the classical analysis of derivative and integral was first defined by the popular mathematicians Isaac Newton and Gottfried Wilhelm Leibniz in the seventeenth century. The science community has witnessed a perpetual progress of calculus with several theoretical and practical application scenarios. Then, this development trend has become distinguished with the aid of arithmetical operations, resulting in diverse analysis approaches. Attempts have been started to develop new kinds of analysis approaching the nineteenth century, exploring the studies of geometric analysis [1]. Geometric analysis as a non-Newtonian approach was defined as multiplicative analysis, relying on division and multiplication operations instead of addition and subtraction [2]. The solid part of this approach consists of exponential operations that redefine the derivatives and integrals in an exponential way.

Multiplicative differential equations (MDEs) are a class of differential equations where the operations are defined in the context of multiplicative calculus instead of the traditional (additive) calculus. In contrast to standard differential equations that use the traditional derivative, MDEs involve the multiplicative derivative, which represents the rate of relative change. This key difference makes MDEs particularly suited for modeling processes with inherently multiplicative dynamics, such as exponential growth or decay, whereas standard differential equations are typically used for processes with additive rates of change. Understanding MDEs opens up new avenues for research and application in

various scientific and engineering disciplines, potentially offering more natural or efficient descriptions of certain phenomena.

At the time of the paper published in [3], it is claimed that multiplicative analysis holds a limited scope of applications when compared to traditional calculus, including positive functions more frequently [4]. Afterwards, there have been several definitions of multiplicative analysis problems which are simply solved with explicit rules and calculations. This has paved the way for the utilization of multiplicative calculus in various aspects of practical life. Example application areas are biomedical image analysis [5], economics [6], time-scale theory [7], chemical engineering [8], processing of real-world signals [9], fractional dynamical systems [10], and digital image interpolation [11]. A recent study developed an analytical model to predict COVID-19 spread by incorporating multiplicative calculus into machine learning [12]. The proposed prediction model generates measurements to identify the diffusion of COVID-19 in either precaution conditions or careless circumstances.

There are a large number of studies aiming at solving multiplicative calculus in terms of differential equations. In [13], three methods were used to obtain the solutions of nonhomogeneous linear-type differential equations in multiplicative analysis. The three methods, which are named operator, undetermined exponential, and variation of parameters, assume constant exponential. The numerical outputs acquired from the representative examples support the efficiency of the proposed ideas. Laplace transform and its definitions and properties have been emphasized to extract the solutions of multiplicative differential equations [14]. Another paper studied Adams Bashforth–Moulton algorithms in the form of multiplicative to achieve the solution of first-order multiplicative differential equations, in comparison to Runge–Kutta methods, yielding much better approximated findings [15]. The multiplicative differential transform method (MDTM) was utilized as a novel transform approach for the approximate solutions of first- and second-order MDEs [16]. Ordinary-oriented numerical methods were mentioned to have the potential to be more accurate when combined with multiplicative methods [17]. As a consequence, multiplicative and Volterra Halley ideas were collaborated with respect to Taylor theorems for achieving an optimum solution.

In recent years, the orthogonal polynomials have long been used to find the approximate solutions of numerous differential equations in a spectral manner. The popular orthogonal polynomials utilized in MDE solutions are Chebyshev [18], Legendre [19], and Hermite [20]. To the best of our knowledge, there has been no study in the literature that uses Laguerre polynomials in the sense of MDEs solutions. The orthogonality properties of Laguerre polynomials contribute significantly to solving multiplicative differential equations (MDEs) efficiently due to the several advantages. First, it can simplify the process of finding solutions to MDEs by providing a systematic way to represent the solutions as series expansions. Second, it can facilitate easier computation and manipulation as it can form a complete basis set for representing functions within a given interval. Third, it is well known that orthogonal polynomials minimize computational errors when approximating solutions, allowing for more straightforward and efficient analytical or numerical solutions. To sum up, the orthogonality properties of Laguerre polynomials streamline the process of solving MDEs by ensuring a powerful tool for series expansion, minimizing computational errors, and simplifying complex calculations. Taking these advantages into consideration, this article builds its contribution on the derivations of the Laguerre multiplicative differential equation and multiplicative Laguerre polynomials. The solutions of the multiplicative Laguerre polynomials are analyzed by exploiting multiplicative power series [21]. The power series method was selected to explore the multiplicative Laguerre polynomials because of its orthogonality preservation property, as it maintains the orthogonality properties of the orthogonal polynomials. On the other hand, it holds historical success when solving differential equations and constructing orthogonal polynomials, which makes it a possible method for solving MDEs. In addition to this, this method has a

flexible manipulation structure that enables us to transit from conventional polynomials to multiplicative forms. We support the proposed idea by numerical examples.

Deriving the definitions of Laguerre multiplicative differential equations and multiplicative Laguerre polynomials involves several key steps. The specific steps of the proposed solution can be summarized as follows. We start by converting the standard calculus operations into multiplicative calculus. Then, a formulation operation is applied to equalize the MDEs to the classical Laguerre differential equations. The solution is expressed as series expansion using Laguerre polynomials. Later, we modify the series for multiplicative operations to derive multiplicative Laguerre polynomials. Finally, we present orthogonality conditions and basic properties of the multiplicative Laguerre polynomials. In the coming sections, we describe the required definitions, theorems, Laguerre multiplicative differential equation, and multiplicative Laguerre polynomials in detail.

2. Overview of Multiplicative Calculus Analysis

It is worth revisiting the fundamental definitions, important theorems, and properties associated with multiplicative calculus, in order to give better insight for the scope of the article. This type of summary may not provide a deep analysis; a full coverage of the aforementioned issues can be found in [3].

Definition 1. We first start with the representations of numbers in an exponential way by the following:

- $R_{exp} = e^a, a \in \mathbb{R},$
- $Z_{exp} = e^a, a \in \mathbb{Z},$
- $N_{exp} = e^a, a \in \mathbb{N}.$

Here, R_{exp} represents the exponential real numbers, Z_{exp} is the group of exponential integer numbers, and N_{exp} indicates a set of exponential natural numbers. The values of R_{exp} can be further split into two parts: (1) R_{exp}^+ is for positive real numbers and R_{exp}^- is for negative real numbers.

Definition 2. The arithmetic operations in multiplicative analysis within the domain of R_{exp} can be defined as follows:

- Exponential sum: $a \oplus b = e^{[lna+lnb]} = e^{[ln(a.b)]} = a.b;$
- Exponential subtract: $a \ominus b = e^{[lna-lnb]} = e^{ln(a/b)} = a/b;$
- Exponential multiplication: $a \odot b = e^{[lna.lnb]} = a^{lnb} = b^{lna};$
- Exponential division: $a \div b = e^{[lna \div lnb]} = a^{lnb} = b^{lna}.$

Definition 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, and the multiplicative derivative of this function can be defined as

$$\frac{d^* f(t)}{dt} = f^*(t) = \lim_{h \rightarrow 0} \left(\frac{f(t+h)}{f(t)} \right)^{1/h} \quad (1)$$

In case of a positive function, the multiplicative derivative of $f(t)$ can be constructed as

$$\frac{d^* f(t)}{dt} = f^*(t) = \lim_{h \rightarrow 0} \left(1 + \frac{f(t+h) - f(t)}{f(t)} \right)^{1/h} \quad (2)$$

$$= f^*(t) = e^{\frac{f'(t)}{f(t)}} \quad (3)$$

$$= f^*(t) = e^{(lnof)'(t)} \quad (4)$$

This is valid for the condition of satisfying $(lnof)(t) = ln f(t)$ [19]. The multiplicative derivative, f^* , is treated as the first-order multiplicative derivative. Similar to this, the mul-

multiplicative derivative of f^* is second-order multiplicative derivative. By following this order, the n th order multiplicative derivative of a typical positive function can be defined as

$$f^{*(n)}(t) = e^{(\ln f)^n(t)} \quad (5)$$

Theorem 1. Let f be differentiable positive function by multiplicative derivative at the point x . This also makes f a classic differentiable and the correlation among classical and multiplicative derivative can be given as [3]

$$f'(t) = f(t) \ln f^*(t) \quad (6)$$

Theorem 2. We assume two functions, f and h , as multiplicative differentiable at a constant point c . When c is arbitrarily chosen, the differentiable relations between the two functions in the sense of multiplicative derivatives can be demonstrated as [3]

- $(cf)^*(t) = f^*(t)$
- $(f.h)^*(t) = f^*(t).h^*(t)$
- $(f+h)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+h(t)}} h^*(t)^{\frac{g(t)}{g(t)+h(t)}}$
- $\left(\frac{f}{h}\right)^* = \frac{f^*(t)}{h^*(t)}$
- $(f^h)^*(t) = f^*(t)^{h(t)} f(t)^{h'(t)}$

Definition 4. If a positive function f is a Riemann integrable, then its multiplicative integral is formed as

$$\int_a^b f(t) dt = e^{\int_a^b \ln f(t) dt} \quad (7)$$

When f and h exhibit multiplicative integrable property on $[a, b]$, the following features are satisfied:

- $\int_a^b f(t)^k dt = \left(\int_a^b f(t) dt \right)^k$
- $\int_a^b (f(t)h(t))^{dt} = \int_a^b f(t)^{dt} \int_a^b h(t)^{dt}$
- $\int_a^b \left(\frac{f(t)}{h(t)} \right)^{dt} = \frac{\int_a^b f(t)^{dt}}{\int_a^b h(t)^{dt}}$

The relation among multiplicative integrals and derivatives can be built by the following theorems:

Theorem 3. If f^* is multiplicatively integrable on $[a, b]$ and f yields a multiplicative derivative for any point in the range of a and b , then

$$\int_a^b f^*(t)^{dt} = \frac{f(b)}{f(a)} \quad (8)$$

Theorem 4. If f is multiplicatively integrable on $[a, b]$, then

$$F(x) = \int_a^x f(t)^{dt}, a \leq x \leq b \quad (9)$$

3. Multiplicative Differential Equations

A common linear MDE with order n can be given in the form of

$$(y^{*(n)})^{a_{(n)}(t)} \cdot (y^{*(n-1)})^{a_{(n-1)}(t)} (y^{*(n-2)})^{a_{(n-2)}(t)} \dots (y^{*(0)})^{a_{(0)}(t)} = f(t) \quad (10)$$

$f(t)$ is a certain positive function. The exponential coefficients, $a_{(n)}(t)$, are selected as constant, otherwise selecting variable coefficient values would make MDE with variable coefficients. On the other hand, the focus of this study is dedicated to homogeneous linear MDEs, so the value of $f(t)$ should be equal to 1.

A special type of MDE with second order can be explicitly defined as

$$y^{**}(t)(y^*)^{p(t)}y(t)^{q(t)} = 1 \quad (11)$$

Theorem 5. When we assume y function to be an infinite multiplicative-differentiable function derived at the t_0 neighborhood, y can be expressed in the form of multiplicative power series expansion as

$$y(t) = \prod_{i=0}^{\infty} c_i (t-t_0)^i \quad (12)$$

Therefore, the MDE has the independent solution form as $y_1(t), y_2(t), \dots, y_n(t)$.

The multiplicative Laguerre differential equation can be derived from the Sturm–Liouville equation using algebraic forms. To construct multiplicative Laguerre equations from the Sturm–Liouville equation, certain arithmetic operations are employed in an exponential manner, known as multiplicative algebraic operations. As defined in Definition 2, these algebraic equations form an algebraic structure. If $\otimes : A \otimes A \rightarrow A$ is called an operation such as $A \neq \emptyset$ and $A \in \mathbf{R}^+$ then $(A, \oplus)$ is called a multiplicative group and $(A, \oplus, \otimes)$ define a ring. By these operations, we may define the structures as [22]

$$\frac{d^*}{dt}(e^{p(t)} \odot \frac{d^* y}{dt}) \oplus (e^{q(t)} \odot y) \oplus (e^{\lambda w(t)}) = 1 \quad (13)$$

which is actually a Sturm–Liouville equation and $p(t)$, $q(t)$ and $w(t)$ are real valued functions to be selected specially. For Laguerre differential equation, we choose $p(t) = e^{-t} \cdot t$, $q(t) = 0$, and $w(t) = e^{-t}$, which transforms this equation into

$$L[y] = [(y^*(t))^{e^{-t} \cdot t}]^* \cdot y^{\lambda e^{-t}} = 1 \quad (14)$$

After performing multiplicative derivation, we consider the following second-order multiplicative Laguerre differential equation when $\lambda = n$:

$$y^{**}(t) \cdot y^{*(1-t)} \cdot y^n = 1 \quad (15)$$

where n indicates a constant real value. The solution of this equation in terms of multiplicative power series is examined as

$$y(t) = \prod_{i=0}^{\infty} c_i (t)^i \quad (16)$$

The first and second multiplicative derivatives of $y(t)$ can be acquired as

$$y^*(t) = \left(\prod_{i=1}^{\infty} c_i \right)^{it^{i-1}} \quad (17)$$

$$y^{**}(t) = \left(\prod_{i=2}^{\infty} c_i^{i(i-1)t^{i-2}} \right) \quad (18)$$

Substituting these multiplicative derivatives in Equation (15), it can be solely written that

$$\left(\prod_{i=2}^{\infty} c_i^{i(i-1)t^{i-2}} \right)^t \cdot \left(\prod_{i=1}^{\infty} c_i^{it^{i-1}} \right)^{1-t} \cdot \left(\prod_{i=0}^{\infty} c_i^{t^i} \right)^n = 1 \quad (19)$$

Reorganizing the parts of the final equation through equalizing all powers:

$$\left(\prod_{i=1}^{\infty} c_{i+1}^{(i+1)t^i} \right) \left(\prod_{i=0}^{\infty} c_{i+1}^{(i+1)t^i} \right) \left(\prod_{i=1}^{\infty} c_i^{-it^i} \right) \left(\prod_{i=0}^{\infty} c_i^{t^i} \right)^n = 1 \quad (20)$$

It is worth noting that for the terms with $i = 1$, it is seen that $i = 0$ has no effect and the equation can be rewritten as

$$\left(\prod_{i=0}^{\infty} c_{i+1}^{(i+1)i} \cdot c_{i+1}^{(i+1)c_i^{-i}} \cdot c_i^n \right)^{t^i} = 1 \quad (21)$$

Given the uniqueness of a power series, we set the coefficients to one, resulting in the following recurrence relation:

$$c_{i+1} = c_i^{\frac{(i-n)}{[(i+1) \cdot i + (i+1)]}} \quad (22)$$

Hence, the coefficients can be extracted as

$$c_1 = c_0^{-n} \quad (23)$$

$$c_2 = c_1^{-(n-1)/4} = c_0^{n(n-1)/4} \quad (24)$$

$$c_3 = c_2^{-(n-2)/9} = c_0^{-n(n-1)(n-2)/4.9} \quad (25)$$

and

$$c_4 = c_3^{-(n-3)/16} = c_0^{n(n-1)(n-2)(n-3)/4.9.16} \quad (26)$$

From the correlation of these equations, we can simply from a general definition:

$$c_i = c^{(i-1-n)/i^2} \quad (27)$$

$$= c_0^{\frac{(-1)^i n!}{(i!)^2 (n-i)!}} \quad (28)$$

This supports the definition of the general solution as

$$y_n(t) = \left(\prod_{i=0}^n c_0^{(-1)^i \frac{n!}{(n-i)!(i!)^2}} \right)^{t^i} \quad (29)$$

The multiplicative Laguerre polynomials are defined as Equation (30) and, peculiarly, if the initial conditions are chosen like $y(0) = c_0 = e^0 = 1$ and $y^*(0) = c_1 = e^1$, then the multiplicative Laguerre polynomial can be defined as

$$\tilde{L}_n(t) = e^1 \left(\prod_{i=1}^n e^{(-1)^i \frac{n!}{(n-i)!(i!)^2}} \right)^{t^i} \quad (30)$$

According to the above equation, the first multiplicative Laguerre polynomials are

$$\tilde{L}_0(t) = e^1 \quad (31)$$

$$\tilde{L}_1(t) = e^{(1-t)} \quad (32)$$

$$\tilde{L}_2(t) = e^{(t^2-4t+2)/2!} \quad (33)$$

$$\tilde{L}_3(t) = e^{(-t^3-9t^2-18t+6)/3!} \quad (34)$$

Example 1. Find the solution in terms of multiplicative Laguerre polynomials to the equation:

$$f(t) = e^{t^3-3t^2+2t}$$

It can be easily demonstrated that the positive and continuous function $f(t)$ on the interval $[0, \infty)$ can be expanded in a series of Laguerre polynomials [19].

$$f(t) = \prod_{i=1}^n c_i(t) \odot \tilde{L}_i(t) \quad (35)$$

where

$$\begin{aligned} c_i(t) &= \int_0^\infty f(t) \odot \tilde{L}_i(t) e^{-t} dt \\ &= e^{\int_0^\infty \ln f(t) \cdot \ln \tilde{L}_i(t) \ln e^{-t} dt} \end{aligned} \quad (36)$$

In this example, $f(t)$ is a third-order polynomial; thus, we explicitly calculate $c_i(t)$ for $n = 0, 1, 2, 3$

$$c_0 = 2, c_1 = -8, c_2 = 12, c_3 = -6$$

If we substitute the values of c_i in Equation (37), we confirm that

$$f(t) = e^{\int_0^\infty \left(2.1 - 8(1-t) + 12 \cdot \frac{(t^2+4t+2)}{2!} - 6 \cdot \frac{(-t^3-9t^2-18t+6)}{3!} \right) dt} = e^{t^3-3t^2+2t}$$

Example 2. We shall focus on another example:

$$f(t) = e^{t^3-3t}$$

When we apply the same steps as in the previous example, the c_i coefficients are found as

$$c_0 = 3, c_1 = -18, c_2 = 18, c_3 = -6$$

We get

$$e^{t^3-3t} = \tilde{L}_0^3 \cdot \tilde{L}_1^{18} \cdot \tilde{L}_2^{18} \cdot \tilde{L}_3^{-6} = e^{\int_0^\infty \left(3.1 - 18(1-t) + 18 \cdot \frac{(t^2+4t+2)}{2!} - 6 \cdot \frac{(-t^3-9t^2-18t+6)}{3!} \right) dt}$$

It is worth noting that while multiplicative Laguerre polynomials offer a systematic and efficient approach for solving MDEs, their computational efficiency compared to other methods depends on various factors and needs to be evaluated case by case. These factors include the specific characteristics of the problem, the accuracy requirements, and the available computational resources. For example, multiplicative Laguerre polynomials may offer advantages over other orthogonal polynomial bases, such as Legendre or Chebyshev polynomials, for certain classes of problems. The choice of polynomial basis depends on the specific properties of the problem, including the behavior of the solution and the boundary conditions.

4. Spectral Properties of Multiplicative Laguerre Differential Equation

The purpose of this section is to generalize the properties of traditional Laguerre polynomials with multiplicative Laguerre polynomials, subject to subsequent properties.

$$1. \quad \tilde{L}_i(t) = e^{L_i(t)}$$

Proof. The proof of this property can be easily obtained by examining the Laguerre polynomial of order n :

$$\tilde{L}_i(t) = \left(\prod_{i=0}^n e^{(-1)^i \binom{n}{i}} \right)^{\frac{t^i}{n!}} \quad (37)$$

If we remind the definition of the multiplicative Laguerre polynomial as derived in the previous section:

$$\tilde{L}_i(t) = e^{1 \left(\prod_{i=1}^n e^{(-1)^i \frac{n!}{(n-i)!i!2}} \right)^{t^i}} \quad (38)$$

Then, it can be clear that

$$\tilde{L}_i(t) = e^{L_i(t)} \quad (39)$$

□

$$2. \quad \frac{d^*}{dx^*} \tilde{L}_i(t) = \left(\prod_{i=1}^{n-1} \tilde{L}_i(t) \right)^{-1}$$

Proof. To prove this property, we benefit from the generating function of multiplicative Laguerre polynomials:

$$e^{\frac{-t}{(1-t)} e^{\frac{-xt}{1-t}}} = \prod_{i=0}^{\infty} (\tilde{L}_i(x))^{t^i} \quad (40)$$

The $*$ derivative of Equation (31) according to x is

$$e^{\frac{-t}{(1-t)^2} \frac{1}{1-t} e^{\frac{-xt}{1-t}}} = \prod_{i=1}^{\infty} (\tilde{L}_i(x))^{t^i}$$

Then, one can say that

$$\begin{aligned} e^{-t/(1-t)} e^{\frac{-xt}{1-t}} &= \prod_{i=1}^{\infty} (\tilde{L}_i(x))^{t^i} \\ &= \prod_{i=0}^{\infty} [\tilde{L}_i(x)]^{t^i \frac{-t}{1-t}} \end{aligned} \quad (41)$$

When $e^{\frac{1}{1-t}} = \sum_{r=0}^{\infty} (e^t)^r$ is considered,

$$\prod_{i=0}^{\infty} [\tilde{L}_i(x)]^{t^i \frac{-t}{1-t}} = \prod_{s=0}^{\infty} [L_s(x)]^{-t(s+1) \sum_{r=0}^{\infty} t^r} \quad (42)$$

If $n = r + s + 1$, then it is

$$\prod_{r=0}^{\infty} \prod_{s=0}^{n-1} (L_s)^{t^n} = \prod_{i=1}^{\infty} (\tilde{L}_i(x))^{t^i} \quad (43)$$

Comparing the coefficients of t^n , which completes the proof:

$$\tilde{L}_i(x) = \left(\prod_{s=0}^{n-1} \tilde{L}_s(x) \right)^{-1} \quad (44)$$

□

$$3. \quad \tilde{L}_i(x)^* = (\tilde{L}_{i-1}(x))^* / \tilde{L}_{i-1}(x)$$

Proof. For proving the property, the relation $e^{\frac{1}{1-t}} e^{\frac{-xt}{1-t}} = \prod_{i=0}^{\infty} (\tilde{L}_i(x))^{t^i}$ is used. The $*$ derivative according to x of both sides are taken as

$$e^{\frac{-t}{(1-t)^2} e^{\frac{-xt}{1-t}}} = \prod_{i=0}^{\infty} (\tilde{L}_i(x))^{t^i} \quad (45)$$

or

$$\left[\prod_{i=0}^{\infty} [L_i(x)]^{t^i} \right]^{-t/(1-t)} = \prod_{i=1}^{\infty} (\tilde{L}_i^*(x))^{t^i} \quad (46)$$

Taking the power $(1-t)$ of both sides, we obtain

$$\left(\prod_{i=0}^{\infty} (\tilde{L}_i(x))^{t^{i+1}} \right)^{-1} = \prod_{i=0}^{\infty} (\tilde{L}_i^*(x))^{t^i} \left(\prod_{i=0}^{\infty} (\tilde{L}_i^*(x))^{t^{i+1}} \right)^{-1} \quad (47)$$

The coefficients of t^i are considered on both sides and the proof is achieved:

$$(\tilde{L}_i)^*(x) = (\tilde{L}_{i-1}(x))^* / \tilde{L}_{i-1}(x) \quad (48)$$

□

$$4. \quad \tilde{L}_{i+1}^{i+1}(x) = \tilde{L}_i(x)^{2i+1-x} \cdot \tilde{L}_{i-1}(x)^{-i}$$

Proof. For property of recurrence relation, the generating function

$$e^{\frac{1}{1-t}} e^{\frac{-xt}{1-t}} = \left(\prod_{i=0}^{\infty} \tilde{L}_i(x) \right)^{t^i}$$

is used. By taking derivative with respect to t in this equation, it becomes

$$\begin{aligned} \frac{1}{(1-t)^2} \left(-x(1-t) \right) \frac{(1-t+t)}{(1-t)^3} e^{\frac{-xt}{1-t}} \cdot e^{\frac{-xt}{1-t}} \\ = \prod_{i=1}^{\infty} \tilde{L}_i(x)^{t^{i-1}/(i-1)!} \end{aligned} \quad (49)$$

Then, we have

$$e^{\left(\frac{-xt}{1-t}\right) \frac{1-t-x}{(1-t)^3}} = \prod_{i=1}^n \tilde{L}_i(x)^{it^{i-1}} \quad (50)$$

By taking the $(1-t)$ power of Equation 51), we obtain

$$\prod_{i=0}^n \left((\tilde{L}_i(x))^{\frac{t^i}{1-t^2}} \right)^{1-t-x} = \prod_{i=1}^n \left((\tilde{L}_i(x))^{it^{i-1}} \right) \quad (51)$$

After taking the $(1-t^2)$ th derivative of both sides and equating the coefficient t^n we obtain

$$\tilde{L}_i(x)^{1-x} \cdot \tilde{L}_{i-1}(x)^i = \tilde{L}_{i+1}(x)^{i+1} \cdot \tilde{L}_i(x)^{-2i} \cdot \tilde{L}_{i-1}(x)^{i-1} \quad (52)$$

and from this, we obtain

$$\tilde{L}_i(x)^{(1-x+2i)} \cdot \tilde{L}_{i-1}(x)^{-i} = \tilde{L}_{i+1}^{(i+1)}(x) \quad (53)$$

The recurrence relation is proved. □

$$5. \quad \tilde{L}_i(x) = \left(\frac{d^n}{dx^n} \left(e^{(e^{-x}x^{-n})} \right) \right)^{e^x/n!}$$

Proof. We use the Leibnitz's theorem for differentiation for the proof:

$$\frac{d^n}{dx^n} (e^{-x} x^n) = \sum_{r=0}^n \frac{n!}{(n-r)!r!} \left(\frac{d^{n-r}}{dx^{n-r}} x^n \right) \left(\frac{d^r}{dx^r} e^{-x} \right) \quad (54)$$

where

$$\frac{d^k}{dx^k} x^m = \frac{m!}{(m-k)!} x^{m-k}. \quad (55)$$

In Equation 55), if $k = n - r$, then

$$\frac{d^n}{dx^n} (e^{-x} x^n) = \sum_{r=0}^n (-1)^r \frac{n! x^r}{(r!)^2 (n-r)!} = \tilde{L}_n(x) \frac{n!}{e^x} \quad (56)$$

It is obvious that the multiplicative Laguerre polynomials are

$$e^{L_n(x)} = \tilde{L}_n(x)$$

Then the following is obtained:

$$\left[\frac{d^n}{dx^n} (e^{e^{-x}x^n}) \right]^{\frac{e^x}{n!}} = \tilde{L}_n(x) \quad (57)$$

It is the proof of the multiplicative Rodrigues equation. $\square$

$$6. \quad \left(e^{\frac{1}{1-t}e^{(-xt)/(1-t)}} \right) = \prod_{i=0}^n (\tilde{L}_n(t))^{t^n}$$

Proof. The generating function of multiplicative Laguerre polynomials is

$$e^{\frac{1}{1-t}e^{\frac{-xt}{1-t}}} = \prod_{i=0}^{\infty} (\tilde{L}_i(x))^{t^i} \quad (58)$$

Consider the following equations:

$$\begin{aligned} \tilde{L}(x, t) &= e^{\frac{1}{1-t}e^{\frac{-xt}{1-t}}} = \left(\prod_{i=0}^{\infty} \tilde{L}_i(x) \right)^{t^i} \\ \tilde{L}(x, t) &= e^{\frac{1}{1-t} \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{-xt}{1-t} \right)^m \right]} \end{aligned}$$

The last equation can be written as

$$\begin{aligned} (1-t)^{m+1} &= 1 + (m+1) + \frac{(m+1)(m+2)}{2!}t^2 + \dots \\ &= \sum_{k,m=0}^{\infty} \frac{(m+k)!}{k!m!} t^k \end{aligned}$$

It is substituted in Equation 59:

$$e^{\frac{1}{1-t}e^{\frac{-xt}{1-t}}} = e^{\left[\sum_{k=0}^{\infty} \frac{(m+k)!}{k!m!2} x^k t^{m+k} \right]} \quad (59)$$

Then, considering $(m+k) = i$, the coefficients of t^n become

$$(-1)^m \frac{n!}{m!2(n-m)!} x^m \quad (60)$$

Then, we can reach the proof:

$$e^{\frac{1}{1-t}e^{\frac{-xt}{1-t}}} = e^{\left[\sum_{k=0}^{\infty} \frac{(n)!}{(n-m)!m!2} \right] x^k t^n} \quad (61)$$

$$e^{\frac{1}{1-t}e^{\frac{-xt}{1-t}}} = e^{L_n(x)} \quad (62)$$

$$= \tilde{L}_n(x) \quad (63)$$

$\square$

$$7. \quad \left[\int_0^{\infty} \left(\tilde{L}_n(t) \ln(\tilde{L}_m(t)) \right) e^{-t} \right]^{dt} = 1, n \neq m$$

Proof. Since multiplicative Laguerre polynomials are the solution of Equation (16), then we can say

$$[(\tilde{L}_n^*(x))^{e^{-t} \cdot t}]^* \tilde{L}_n^{\lambda e^{-t}} = 1 \quad (64)$$

and

$$[(\tilde{L}_m^*(t))^{e^{-t} \cdot t}]^* \cdot \tilde{L}_m^{\lambda e^{-t}} = 1 \quad (65)$$

Let us take the $\lambda = n$, $\lambda = m$, respectively, and $\ln \tilde{L}_m$ and $\ln \tilde{L}_n$ power of the Equations (65) and (66), respectively, and if we take the multiplicative integral from 0 to ∞ , we obtain

$$\left[\int_0^\infty \left(\tilde{L}_n(t)^{\ln \tilde{L}_m(t)} \right)^{e^{-t}n} dt \right] = e^{\int_0^\infty e^{-t}n \ln \tilde{L}_n(t) \ln(\tilde{L}_m)(t) dt} \quad (66)$$

and

$$\left[\int_0^\infty \left(\tilde{L}_m(t)^{\ln \tilde{L}_n(t)} \right)^{e^{-t}m} dt \right] = e^{\int_0^\infty e^{-t}m \ln(\tilde{L}_m)(t) \ln(\tilde{L}_n)(t) dt} \quad (67)$$

We divide Equations (67) and (68) side by side, and then we obtain

$$\left[\int_0^\infty \left(\tilde{L}_m(t)^{\ln \tilde{L}_n(t)} \right)^{e^{-t}dt} \right]^{m-n} = \int_0^\infty \left[\frac{d^*}{dt} (W(\tilde{L}_m, \tilde{L}_n))^{e^{-t}t} \right]^{dt} = 1 \quad (68)$$

since

$$W(\tilde{L}_m, \tilde{L}_n) = (\tilde{L}_m \odot \tilde{L}_n^*) \ominus (\tilde{L}_m^* \odot \tilde{L}_n) \quad (69)$$

By using $\ln(\tilde{L}_m) = L_m(t)$ and $\ln(\tilde{L}_n) = L_n(t)$, and since for $m \neq n$

$$\int_0^\infty e^{-t} \ln(\tilde{L}_n) \ln(\tilde{L}_m) dt = 0,$$

then

$$\left[\int_0^\infty \left(\tilde{L}_n(t)^{\ln \tilde{L}_m(t)} \right)^{e^{-t}} dt \right] = e^0 = 1 \quad (70)$$

□

$$8. \quad \left[\int_0^\infty \left(\tilde{L}_n(t)^{\ln \tilde{L}_n(t)} \right)^{e^{-t}} dt \right] = e, n = m$$

Proof. Let $m = n$, then we use Equation (58):

$$\left[\frac{d^n}{dt^n} (e^{e^{-t}t^n}) \right]^{\frac{e^t}{n!}} = \tilde{L}_n(t)$$

and

$$\left[\int_0^\infty \left(\left[\frac{d^*}{dt} L_n^*(t) \right]^{\ln \frac{d^*}{dt} (t^n e^{-t})} \right)^{e^{-t}e^t/n!} dt \right]$$

After the necessary simplification

$$\left[\int_0^\infty \left(\left[\frac{d^*}{dt} \tilde{L}_n(t) \right]^{\ln \frac{d^*}{dt} (t^n e^{-t})} \right)^{1/n!} dt \right]$$

If the partial multiplicative integral is applied in this equation,

$$\begin{aligned} \int_0^\infty (\tilde{L}_n(t) \odot \tilde{L}_n(t)) &= e^{\frac{(-1)^n}{n!} \int_0^\infty \ln \tilde{L}_n \ln \frac{d^*(n-1)}{dt} (t^n e^{-t}) dt} \\ &= e^{\frac{(-1)^n}{n!} \int_0^\infty \ln \tilde{L}_n^{(n)} (t^n e^{-t}) dt} \end{aligned} \quad (71)$$

Then, by the property of the Laguerre polynomial given in Equation (38), and if we take the n th multiplicative derivative of $\tilde{L}_n(x)$, it is

$$L_n^{(n)} = \frac{d^{*(n)}}{dt^n} \binom{n}{n} \frac{(-t)^n}{n!} = (-1)^n$$

If we substitute this into Equation (72), it is

$$e^{(-1)^n \int_0^\infty \ln \tilde{L}_n^{(n)}(t^n e^{-t}) dt} = e^{\frac{1}{n!} \int_0^\infty t^n e^{-t} dt} \quad (72)$$

In the last equation,

$$\int_0^\infty t^n e^{-t} dt$$

is a special function definition called Gamma function, and it is

$$\int_0^\infty t^n e^{-t} dt = \Gamma(n+1) = n!$$

Thus, while $m = n$, the orthogonality proof of the multiplicative Laguerre polynomial is defined as

$$\int_0^\infty (L_n^*(t) \odot L_n^*(t)) = e^1 = e \quad (73)$$

□

5. Conclusions

Multiplicative analysis has recently attracted a lot of research activities within the calculus community. The research direction of recently published articles has been the effective solutions of multiplicative differential equations (MDEs). Many methods have been proposed to solve different type of MDEs. This article exploits the potential of orthogonal polynomials towards solving such MDEs. There have been some efforts with orthogonal polynomials for the solutions of MDEs. This article builds its structure on the utilization of Laguerre polynomials as there has been no similar study in the literature until now. The first part of the proposed structure relies on derivation of the definitions of Laguerre multiplicative differential equation and multiplicative Laguerre polynomials. To obtain the multiplicative Laguerre polynomials, a well-known method, power series, is applied. To give a better insight into the proposed idea, several attempts were carried out to prove the fundamental properties of multiplicative Laguerre polynomials. Future work directions will focus on alternative methods such as operational matrix method, generating functions, and recurrence relations. A particular focus will be placed on the derivation of operational matrix method to simplify the whole problem.

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