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# On Quasi-Subordination for Bi-Univalency Involving Generalized Distribution Series

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**Abstract:** Various researchers have considered different forms of bi-univalent functions in recent times, and this has continued to gain more attention in Geometric Function Theory (GFT), but not much study has been conducted in the area of application of the certain probability concept in geometric functions. In this manuscript, our motivation is the application of analytic and bi-univalent functions. In particular, the researchers examine bi-univalency of a generalized distribution series related to Bell numbers as a family of Caratheodory functions. Some coefficients of the class of the function are obtained. The results are new as far work on bi-univalency is concerned.

**Keywords:** quasi-subordination; bi-univalent functions; Bell numbers; caratheodory function

**MSC:** 30C45

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## 1. Introduction and Preliminaries

Let $A$ denote the class of functions defined by the following equation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) \tag{1}$$

which are analytic in the open disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Given $f$ and $g$ as two functions which are analytic in the open unit disk $E$, we find definition and notations of function $f(z)$ subordinate to $g(z)$ in [1,2] and many other papers and elementary studies in GFT.

Ma and Minda [3] studied and introduced the following class

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

which serves as a major motivation for many other studies. See [4–12] for more information.

A function $j(z)$ is said to belong to class $P$, Caratheodory function if for $j$ in $E$, $Rej(z) > 0$ and the function is of the form

$$j(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2}$$

for $z \in E$. Then, $|c_n| \leq 2$ for each $n$ [13].

Furthermore, given a function $\psi(z)$ of the for

$$\psi(z) = B_0 + B_1 z + B_2 z^2 + \ldots \quad (|\psi| \leq 1, z \in E). \tag{3}$$

We introduce the Bell numbers $B_n$ which satisfy the following recurrence relation in the work [14–17]. The function

$$Q(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6} z^3 + \frac{5}{8} z^4 + \ldots \tag{4}$$

was firstly introduced by Kumar et al. [18], which is starlike with respect to 1, and its coefficients generate the Bell numbers. Researchers have used Bell numbers to work on various subclasses of analytic functions, and their results are seen everywhere. For the benefit of doubt, readers can see [9,19].

Let $T$ be defined as

$$T = a_0 + a_1 + a_2 + a_3 + \ldots,$$

where $a_n \geq 0$ for all $n \in N$. For the generalized discrete probability distribution whose probability mass function is given as

$$p(n) = \frac{a_n}{T}, \quad n = 0, 1, 2, 3, \ldots,$$

$p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p(n) = 1$.

In addition, suppose

$$\gamma(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then from $S = \sum_{n=0}^{\infty} a_n$ series, $\gamma$ is convergent for both $|x| < 1$ and $x = 1$.

The following probability concepts exist in the literature, but Porwal [20] was the first to apply them in GFT:

If $X$ is a discrete random variable that takes values $x_1, x_2, \ldots$ associated with probabilities $p_1, p_2, \ldots,$ then the expected $X$ denoted by $E(X)$ is defined as

$$E(X) = \sum_{n=1}^{\infty} p_n x_n.$$

The moment of a discrete probability distribution ($r$th) about $x = 0$ is defined by $\mu(x)$, where $\mu'_1$ is the mean of the distribution and the variance is given as

$$\mu'_2 - (\mu'_1)^2.$$

The moment about the origin is given as

$$Mean = \mu'_1 = \frac{\gamma'}{T}$$

$$Variance = \mu'_2 - (\mu'_1)^2 = \frac{1}{S}\left[\gamma''(1) + \gamma'(1) - \frac{(\gamma'(1))^2}{T}\right].$$

The moment generating function of a random variable $X$ is represented by $M_x(t)$ and defined by

$$M_x(t) = \frac{\gamma(e^t)}{T}.$$

The coefficient $a_n$ reduces to various well-known discrete probability distributions for specific values, for example, the Bernoulli distribution, Zeta distribution, Logarithmic distribution, Poisson distribution, and many more special cases, as $a_n$ varies.

We now define a series whose coefficients are probabilities of the generalized distribution as

$$f_{\phi}(z) = z + \sum_{n=2}^{\infty} \frac{a_n - 1}{T} z^n \tag{5}$$

which is analytic in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f_{\phi}(0)$ and $f_{\phi}(0)' - 1 = 0$. A fair number of publications have gone in this direction. For the benefit of doubt, the reader can check [9,19–22].

According to the Koebe $\frac{1}{4}$ theorem, a disc with a radius of one-quarter is in the image of $E$ under every function $f \in S$. Thus, every function $f \in S$ owns an inverse function, and this inverse function can be defined on a disc with a radius of one-quarter. The inverse function of $f$ can be expressed by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots. \tag{6}$$

If function $f$ and its inverse $f^{-1}$ are univalent in $E$, then the function $f \in A$ is bi-univalent in $E$. There have been some studies that considered quite a number of bi-univalent functions. See [1,4,10,23–25] for more information.

Consequently, we define the inverse function as

$$g_{\phi}(\omega) = f_{\phi}^{-1}(\omega) = \omega - \frac{a_1}{T}\omega^2 + (2\frac{a_1^2}{T^2} - \frac{a_2}{T})\omega^3 - (5\frac{a_1^3}{T^3} - 5\frac{a_1}{T}\frac{a_2}{T} + \frac{a_3}{T})\omega^4 + \dots \tag{7}$$

which is of the particular interest for this investigation.

*Significance of Studying Bi-Univalent Functions in Geometric Function Theory*

In Geometric Function Theory, the study of bi-univalent functions holds significance due to its connection with the theory of univalent functions, which focuses on functions that are injective and holomorphic (analytic) in a given domain. Bi-univalent functions are a subclass of univalent functions that possess additional symmetries, specifically involving both the function and its inverse. A function $f(z)$ is said to be bi-univalent in a domain $D$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $D$. The significance of studying bi-univalent functions can be understood through several aspects: Applications in Conformal Mapping, Symmetry Properties, Coefficient Inequalities, Connections with other Classes of Functions, Mapping Properties, problem-solving in Complex Analysis, and lots more.

Generally, the study of bi-univalent functions in Geometric Function Theory is significant because it extends the theory of univalent functions, providing a richer class of functions with additional symmetries. This extension opens up new avenues for understanding and exploring conformal mappings, coefficient estimates, and various geometric and analytic properties in the complex plane.

In order to achieve the aim of the study, the linear combination

$$\beta p(z) + (1 - \beta)Q(z), \quad (0 \leq \beta \leq 1) \tag{8}$$

is considered.

Distribution series has a lot of applications in mathematics and comes in different forms. In particular, we use a distribution series in the study in order to extend some existing work and introduce new forms of functions, guided and motivated by earlier work from the aforementioned references. In this work, the bi-univalency of a generalized distribution series in terms of quasi-subordination involving Bell numbers was investigated. The results obtained provide a new base of available information.

For the purpose of this work, the following definitions are necessary. Readers can see some of the advantages of studying generalized distribution series associated with Bell numbers in [18,19].

The investigation of bi-univalent functions in GFT is evolving everyday as authors have attempted to apply its applications to a wide range of special functions in mathematics and engineering. The bi-univalency of a generalized probability series in geometric functions theory was first introduced by Olatunji et al. (2023) in [26] but in relation to remodeled s-sigmoid function as a Charateodory function. The current work is an attempt to improve on the previous work by investigating bi-univalency of generalized probability series relating to the Bell numbers as a Charateodory function. This creates a basis for a comparison of results. The results obtained in both studies may be uniquely different but can be compared to each other. Readers can see [1] for details. In a similar manner, the authors in [27] provided the bounds on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the class of functions in the class investigated, while [28] used a convolution operator involving Hurwitz–Lerch Zeta and the generalized derived operator that satisfies quasi subordination conditions. We now define the two classes of functions to be investigated in this study as follows.

**Definition 1.** *Let $f_\phi \in \sum$ defined by (5) be in class $S^*_\sum(\alpha,\beta)$ for $\alpha,\beta \geq 0$, then if*

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 \prec_q (\mu(z)-1) \tag{9}$$

*and*

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 \prec_q (\mu(\omega)-1) \tag{10}$$

*are satisfied for $z,\omega \in E$ and $g_\phi = f_\phi^{-1}$ given by (7).*

**Definition 2.** *Let $f_\phi \in \sum$ defined by (5) be in class $M_\sum(\alpha,\beta)$ for $\alpha,\beta \geq 0$, then if*

$$(1-\alpha)\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\left(1+\frac{zf''_\phi(z)}{f'_\phi(z)}\right) - 1 \prec_q (\mu(z)-1) \tag{11}$$

*and*

$$(1-\alpha)\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\left(1+\frac{\omega g''_\phi(\omega)}{g'_\phi(\omega)}\right) - 1 \prec_q (\mu(\omega)-1) \tag{12}$$

*are satisfied for $z,\omega \in E$ and $g_\phi = f_\phi^{-1}$ given by (7). Interestingly, authors in [2,3,26,29,30] also investigated different classes of functions of interest. See also [31–36].*

Motivated by [20–22,24], we consider the following results.

## 2. Main Results

**Theorem 1.** *Let $f_\phi(z)$ be given by (5). If $f_\phi \in S^*_\sum(\alpha,\beta)$ for $\alpha,\beta \geq 0$, then*

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|(1+\beta)}{(1+2\alpha)}, \tag{13}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|\sqrt{5\beta^2-2\beta+1}}{1+2\alpha}, \tag{14}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{\sqrt{|B_0|(1+\beta)+2|B_1|(1-\beta)}}{\sqrt{1+4\alpha}}, \tag{15}$$

$$\left|\frac{a_2}{T}\right| \leq \frac{\beta[|B_0|+|B_1|]}{1+3\alpha} + \frac{B_0^2(5\beta^2-2\beta+1)}{(1+2\alpha)^2} \tag{16}$$

*and*

$$\left|\frac{a_2}{T}\right| \leq \frac{\beta[|B_0|+|B_1|]}{1+3\alpha} + \frac{|B_0|(1+\beta)+|B_1|(1-\beta)}{1+4\alpha}. \tag{17}$$

**Proof.** Let $f_\phi \in \sum$ defined by (5) be in class $S^*_\Sigma(\alpha, \beta)$ for $\alpha, \beta \geq 0$. From (2), we have two functions $\mu(z)$ and $\mu(\omega) \in P$ such that

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 = \psi(z)(\mu(u(z)) - 1) = \psi(z)(\beta p(z) + (1-\beta)Q(z) - 1) \tag{18}$$

and

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 = \psi(\omega)(\mu(v(\omega)) - 1) = \psi(\omega)(\beta q(\omega) + (1-\beta)R(\omega) - 1). \tag{19}$$

From the left hand sides (LHS) of (18) and (19), we have

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 = (1+2\alpha)\frac{a_1}{T}z + \left(2(1+3\alpha)\frac{a_2}{T} - (1+2\alpha)\frac{a_1^2}{T^2}\right)z^2 \tag{20}$$

and

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 = -(1+2\alpha)\frac{a_1}{T}\omega + \left(-2(1+3\alpha)\frac{a_2}{T} + (3+10\alpha)\frac{a_1^2}{T^2}\right)\omega^2. \tag{21}$$

Let the functions $p, q, Q$ and $R \in P$, the class of Caratheodory functions of the form (2). Then,

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots, \quad q(\omega) = 1 + q_1\omega + q_2\omega^2 + \ldots, \quad R(\omega) = 1 + \omega + \omega^2 + \ldots \quad (z, \omega \in E), \tag{22}$$

where $Q(z)$ is earlier defined in (4).

Equivalently, from (2), the right hand sides of (18) and (19) give

$$(\beta p(z) + (1-\beta)Q(z) - 1) = (\beta p_1 + 1 - \beta)z + (\beta p_2 + 1 - \beta)z^2 + \ldots \tag{23}$$

and

$$(\beta q(\omega) + (1-\beta)R(\omega) - 1) = (\beta q_1 + 1 - \beta)\omega + (\beta q_2 + 1 - \beta)\omega^2 + \ldots. \tag{24}$$

Then, from (23), (24) and (3), we see that

$$\psi(z)(\beta p(z) + (1-\beta)Q(z) - 1) = B_0(\beta p_1 + 1 - \beta)z + ((\beta p_2 + 1 - \beta)B_0 + (\beta p_1 + 1 - \beta)B_1)z^2 + \ldots \tag{25}$$

and

$$\psi(\omega)(\beta q(\omega) + (1-\beta)R(\omega) - 1) = B_0(\beta q_1 + 1 - \beta)\omega + ((\beta q_2 + 1 - \beta)B_0 + (\beta q_1 + 1 - \beta)B_1)\omega^2 + \ldots. \tag{26}$$

Now, using Equations (20) and (21) with (25) and (26), we obtain

$$(1+2\alpha)\frac{a_1}{T} = B_0(\beta p_1 + 1 - \beta), \tag{27}$$

$$2(1+3\alpha)\frac{a_2}{T} - (1+2\alpha)\frac{a_1^2}{T^2} = (\beta p_2 + 1 - \beta)B_0 + (\beta p_1 + 1 - \beta)B_1, \tag{28}$$

$$-(1+2\alpha)\frac{a_1}{T} = B_0(\beta q_1 + 1 - \beta) \tag{29}$$

and

$$-2(1+3\alpha)\frac{a_2}{T} + (3+10\alpha)\frac{a_1^2}{T^2} = (\beta q_2 + 1 - \beta)B_0 + (\beta q_1 + 1 - \beta)B_1. \tag{30}$$

It is therefore clear that from (27) and (29), we can express

$$\frac{a_1}{T} = \frac{B_0(\beta p_1 + 1 - \beta)}{(1+2\alpha)} = -\frac{B_0(\beta q_1 + 1 - \beta)}{(1+2\alpha)}. \tag{31}$$

The parameters $p_1$ and $q_1$ are both additive inverse of each other with $p_1 + q_1 = 0$ such that

$$p_1 = -q_1 \tag{32}$$

and

$$B_0^2(\beta^2(p_1^2 + q_1^2) - 4\beta + 2\beta^2 + 2) = 2(1+2\alpha)^2 \frac{a_1^2}{T^2}. \tag{33}$$

Comparing (28) and (30), we have that

$$(\beta(p_2 + q_2) + 2(1-\beta))B_0 + 2B_1(1-\beta) = 2(1+4\alpha)\frac{a_1^2}{T^2}. \tag{34}$$

Using (31) and (33) in (34) that

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|(1+\beta)}{(1+2\alpha)},$$

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|\sqrt{5\beta^2 - 2\beta + 1}}{1+2\alpha}$$

and

$$\left|\frac{a_1}{T}\right| \leq \frac{\sqrt{|B_0|(1+\beta) + 2|B_1|(1-\beta)}}{\sqrt{1+4\alpha}},$$

Thus, (13)–(15) holds. Similarly, from (28) and (29), it implies that

$$(\beta(p_2 - q_2))B_0 + (\beta(p_1 - q_1))B_1 = 4(1+3\alpha)\left(\frac{a_2}{T} - \frac{a_1^2}{T^2}\right). \tag{35}$$

From (33) and (35), we have

$$\frac{a_2}{T} = \frac{\beta((p_2 - q_2)B_0 + (p_1 - q_1)B_1)}{4(1+3\alpha)} + \frac{(\beta(p_1^2 + q_1^2) - 4\beta + 2\beta^2 + 2)B_0^2}{2(1+2\alpha)^2}. \tag{36}$$

Following Equation (36), we remark that

$$\left|\frac{a_2}{T}\right| \leq \frac{\beta(|B_0| + |B_1|)}{(1+3\alpha)} + \frac{B_0^2(5\beta^2 - 2\beta + 1)}{(1+2\alpha)^2}. \tag{37}$$

On the other hand, by (34) and (35), we obtain

$$\frac{a_2}{T} = \frac{\beta((p_2 - q_2)B_0 + (p_1 - q_1)B_1)}{(1+3\alpha)} + \frac{B_0(\beta(p_2 + q_2) + 2(1-\beta)) + 2B_1(1-\beta)}{2(1+4\alpha)}. \tag{38}$$

So, from (38), we see that

$$\left|\frac{a_2}{T}\right| \leq \frac{\beta(|B_0| + |B_1|)}{(1+3\alpha)} + \frac{|B_0|(1+\beta) + |B_1|(1-\beta)}{1+4\alpha}.$$

Now we conclude that from (35) and (36), we know (16) and (17) are true. □

**Theorem 2.** *Let $f_\phi(z)$ be given by (5). If $f_\phi \in M^*_{\Sigma,q}(\alpha, \beta_s)$ for $\alpha, \beta \geq 0$, then*

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|(1+\beta)}{(1+\alpha)}, \tag{39}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|\sqrt{5\beta^2 - 2\beta + 1}}{1+\alpha}, \tag{40}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{\sqrt{|B_0|(1+\beta)+|B_1|(1-\beta)}}{\sqrt{2(1+2\alpha)}}, \tag{41}$$

$$\frac{a_2}{T} \leq \frac{\beta[|B_0|+|B_1|]}{1+2\alpha} + \frac{B_0^2(5\beta^2-2\beta+1)}{(1+\alpha)^2} \tag{42}$$

*and*

$$\left|\frac{a_2}{T}\right| \leq \frac{\beta[|B_0|+|B_1|]}{1+2\alpha} + \frac{|B_0|(1+\beta)+|B_1|(1-\beta)}{1+\alpha}. \tag{43}$$

**Proof.** Let $f_\phi \in \sum$ be defined by (5) be in class $M_\sum(\alpha,\beta)$ for $\alpha,\beta \geq 0$. From (2), we have functions $u(z)$ and $v(\omega) \in P$ such that

$$(1-\alpha)\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\left(1+\frac{zf''_\phi(z)}{f'_\phi(z)}\right) - 1 = \psi(z)(\mu(u(z))-1) = \psi(z)(\beta p(z)+(1-\beta)Q(z)-1) \tag{44}$$

and

$$(1-\alpha)\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\left(1+\frac{\omega g''_\phi(\omega)}{g'_\phi(\omega)}\right) - 1 = \psi(\omega)(\mu(v(\omega))-1) = \psi(\omega)(\beta q(\omega)+(1-\beta)R(\omega)-1). \tag{45}$$

From the LHS of (44) and (45), we have

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 = (1+\alpha)\frac{a_1}{T}z + \left(2(1+2\alpha)\frac{a_2}{T} - (1+3\alpha)\frac{a_1^2}{T^2}\right)z^2 \tag{46}$$

and

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 = -(1+\alpha)\frac{a_1}{T}\omega + \left(-2(1+2\alpha)\frac{a_2}{T} + (3+5\alpha)\frac{a_1^2}{T^2}\right)\omega^2. \tag{47}$$

Therefore, from (46) to (47) and (25) to (26), we obtain

$$(1+\alpha)\frac{a_1}{T} = B_0(\beta p_1 + 1 - \beta), \tag{48}$$

$$2(1+2\alpha)\frac{a_2}{T} - (1+3\alpha)\frac{a_1^2}{T^2} = (\beta p_2 + 1 - \beta)B_0 + (\beta p_1 + 1 - \beta)B_1, \tag{49}$$

$$-(1+\alpha)\frac{a_1}{T} = B_0(\beta q_1 + 1 - \beta) \tag{50}$$

and

$$-2(1+2\alpha)\frac{a_2}{T} + (3+5\alpha)\frac{a_1^2}{T^2} = (\beta q_2 + 1 - \beta)B_0 + (\beta q_1 + 1 - \beta)B_1. \tag{51}$$

From (48) and (50), we can express

$$\frac{a_1}{T} = \frac{B_0(\beta p_1 + 1 - \beta)}{(1+\alpha)} = -\frac{B_0(\beta q_1 + 1 - \beta)}{(1+\alpha)}. \tag{52}$$

The parameters $p_1$ and $q_1$ are both additive inverse of each other with $p_1 + q_1 = 0$ such that

$$p_1 = -q_1 \tag{53}$$

and

$$B_0^2(\beta^2(p_1^2+q_1^2) - 4\beta + 2\beta^2 + 2) = 2(1+\alpha)^2\frac{a_1^2}{T^2}. \tag{54}$$

From (49) and (51), we obtain

$$(\beta(p_2+q_2)+2(1-\beta))B_0+2B_1(1-\beta)=2(1+\alpha)\frac{a_1^2}{T^2}. \tag{55}$$

Consequently, from (52), (54) in (55) that

$$\left|\frac{a_1}{T}\right|\leq\frac{|B_0|(1+\beta)}{(1+\alpha)},$$

$$\left|\frac{a_1}{T}\right|\leq\frac{|B_0|\sqrt{5\beta^2-2\beta+1}}{1+\alpha},$$

and

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{|B_0|(1+\beta)+2|B_1|(1-\beta)}}{\sqrt{2(1+2\alpha)}},$$

Thus, (39)–(41) holds. Similarly, from (50) and (51), it implies that

$$(\beta(p_2-q_2))B_0+(\beta(p_1-q_1))B_1=4(1+2\alpha)\left(\frac{a_2}{T}-\frac{a_1^2}{T^2}\right). \tag{56}$$

Now from (54) and (56), we have

$$\frac{a_2}{T}=\frac{\beta((p_2-q_2)B_0+(p_1-q_1)B_1)}{4(1+2\alpha)}+\frac{\left(\beta(p_1^2+q_1^2)-4\beta+2\beta^2+2\right)B_0^2}{2(1+\alpha)^2}. \tag{57}$$

Also from (57), we remark that

$$\left|\frac{a_2}{T}\right|\leq\frac{\beta(|B_0|+|B_1|)}{(1+2\alpha)}+\frac{B_0^2(5\beta^2-2\beta+1)}{(1+\alpha)^2}.$$

On the other hand, using (55) and (56), we obtain

$$\frac{a_2}{T}=\frac{\beta((p_2-q_2)B_0+(p_1-q_1)B_1)}{4(1+2\alpha)}+\frac{B_0(\beta(p_2+q_2)+2(1-\beta))+2B_1(1-\beta)}{2(1+\alpha)}. \tag{58}$$

Therefore, from (58), we see that

$$\left|\frac{a_2}{T}\right|\leq\frac{\beta(|B_0|+|B_1|)}{(1+2\alpha)}+\frac{|B_0|(1+\beta)+|B_1|(1-\beta)}{1+\alpha}.$$

Then, from (55) and (56), we know that (42) and (43) are true. □

Bi-univalent functions provide a powerful tool for studying complex analytic functions and their geometric properties, with applications across various areas of mathematics and physics. Carathéodory functions provide a rich framework for studying the properties of conformal mappings and analytic functions in the complex plane, with applications ranging from classical problems in geometric function theory to modern developments in complex analysis. The results in this current study which involves bi-univalent functions defined in relation with Bell numbers and generalized discrete probability distribution as a Charateodory function can be compared with the existing results in [1].

## 3. Conclusions

Probability distribution is a statistical concept that was first introduced in geometric functions theory by Porwal in [20]. Several authors have also worked in that directions such as Oladipo in [22] and Olatunji et al. in [1]. There are different forms of probabilities but the work considered a generalized probability distribution as introduced and investigated by the authors mentioned earlier. Motivated by the existence of inverse of univalent functions involving the generalized distribution, the authors investigated and obtained coefficient results for the bi-univalency of a generalized probability distribution associated with the Bell numbers as a Charateodory function. The results obtained are new but similar to associated with a remodeled s-sigmoid function as a Charateodory function. The bi-univalent class of functions with the generalized probability series was first defined and studied in [1] but in relation to remodeled s-sigmoid function as a Charateodory function. The current study is however defined in relation to Bell numbers as another Charateodory function. Thus, the classes exist and are not empty. The results obtained are uniquely different from the previous studies that motivated the paper. The classes of functions can be extended and investigated in relation to other established Charateodory functions such as pseudo $\lambda$ functions and their convolution as future directions. Furthermore, the $m$ fold symmetric functions recently considered by Olatunji et al. in [10] and the Three Leaf Domain are other directions that can be considered for future investigation. In particular, two univalent functions $S^*_{\sum}(\alpha, \beta)$ and $M_{\sum}(\alpha, \beta)$ were introduced and investigated in the study providing a new direction in the literature for rs in this field of study.

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