

## Article

# A New Extension of the Exponentiated Weibull–Poisson Family Using the Gamma-Exponentiated Weibull Distribution: Development and Applications

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**Abstract:** This study proposes a new five-parameter distribution called the gamma-exponentiated Weibull–Poisson (GEWP) distribution. As an extension of the exponentiated Weibull–Poisson family, the GEWP distribution offers a more flexible tool for analyzing a wider variety of data due to its theoretically and practically advantageous properties. It encompasses established distributions like the exponential, Weibull, and exponentiated Weibull. The development of the GEWP distribution proposed in this paper is obtained by combining the gamma-exponentiated Weibull (GEW) and the exponentiated Weibull–Poisson (EWP) distributions. Therefore, it serves as an extension of both the GEW and EWP distributions. This makes the GEWP a viable alternative for describing the variability of occurrences, enabling analysis in situations where GEW and EWP may be limited. This paper analyzes the probability distribution functions and provides the survival and hazard rate functions, the sub-models, the moments, the quantiles, and the maximum likelihood estimation of the GEWP distribution. Then, the numerical experiments for the parameter estimation of GEWP distribution for some finite sample sizes are presented. Finally, the comparative study of GEWP distribution and its sub-models is investigated via the goodness of fit test with real datasets to illustrate its potentiality.

**Keywords:** exponentiated Weibull distribution; exponentiated Weibull–Poisson distribution; gamma-exponentiated Weibull distribution; maximum likelihood estimation; moment-generating function



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## 1. Introduction

It is a well-documented statistical reality that many real-world data cannot be accurately modeled by using existing probability distributions. This has led to the development of new continuous distributions with properties that better suit the analysis of such data. The classical Weibull distribution and its numerous modifications and extensions exemplify this ongoing effort. Researchers have continually proposed these alternative distributions for application in diverse fields of study. The exponentiated Weibull (EW) distribution, which extends the Weibull family by introducing a second shape parameter, was introduced by [1]. The gamma-exponentiated Weibull (GEW) distribution, derived by incorporating the distribution function of the EW distribution into the gamma-generated distribution, as discussed in [2], is akin to the gamma extended Weibull distribution proposed in [3]. In addition, the exponentiated Weibull–Poisson (EWP) distribution, derived from the EW [4], was later evolved into the Poisson exponentiated Weibull distribution by [5]. Furthermore, numerous other developed distributions have been applied in various fields such as engineering, medicine, lifetime data analysis, failure analysis, and reliability analysis [6–15].

Herein, we introduce the gamma-exponentiated Weibull Poisson (GEWP) distribution. This was developed by replacing the EW distribution in the EWP distribution with the GEW distribution. We provide the modified distribution obtained from the combined distributions of the EW distribution, which is proposed in [4], and the GEW distribution, which is proposed in [2]. We provide the modified distribution obtained from the combined

distributions of the EW distribution, which was proposed in [1]; the GEW distribution, which was proposed in [2]; and the EWP distribution. In other words, these three distributions are considered special sub-models, among others. We present various aspects of the GEWP distribution, including the analysis of its probability functions, survival and hazard rate functions, sub-models, moments, quantiles, and maximum likelihood estimation. The performance of the GEWP distribution is investigated through simulation studies. Finally, we demonstrate the application of the GEWP distribution to real datasets.

## 2. Preliminaries: Classical Distribution and Expansion of the EW

In this work, the probability density function (pdf) and the cumulative density function (cdf) for a continuous random variable following the distribution of interest while assuming that all of their values are positive or zero are defined in the following subsections.

### 2.1. The Exponential Distribution

The exponential distribution, a special case of the gamma distribution, is often expressed with a rate parameter denoted by  $\theta$ . Assuming that the random variable  $X$  follows the exponential distribution  $X \sim \text{Exp}(\theta)$ , then the pdf and the cdf are given by

$$f_E(x) = \theta e^{-\theta x}, \quad \theta > 0,$$

and

$$F_E(x) = 1 - e^{-\theta x}, \quad \theta > 0.$$

### 2.2. The Weibull Distribution

The Weibull distribution has two parameters: shape and scale. The pdf and cdf of a random variable following a Weibull distribution with parameters  $X \sim \text{Weibull}(\beta, \theta)$ , where  $\beta$  is the shape parameter and  $\frac{1}{\theta}$  is the scale parameter, are, respectively, given by

$$f_W(x) = \beta \theta^\beta x^{\beta-1} e^{-(\theta x)^\beta},$$

and

$$F_W(x) = 1 - e^{-(\theta x)^\beta},$$

where  $\beta, \theta > 0$ . Noteworthy, the exponential distribution is a sub-model of the Weibull distribution when  $\beta = 1$ .

### 2.3. The EW Distribution

The exponentiated Weibull (EW) distribution was initially proposed by [1] to extend the Weibull family by adding another shape parameter and elaborately arranging the classical Weibull distribution. The pdf and the cdf of a random variable following the EW distribution  $X \sim \text{EW}(\alpha, \beta, \theta)$  are, respectively, given by

$$f_{EW}(x) = \alpha \beta \theta^\beta x^{\beta-1} u(1-u)^{\alpha-1}, \quad (1)$$

where  $\alpha, \beta, \theta > 0$ ,  $u = e^{-(\theta x)^\beta}$ , and

$$F_{EW}(x) = (1 - e^{-(\theta x)^\beta})^\alpha. \quad (2)$$

### 2.4. The GEW Distribution

In [2], the gamma-exponentiated Weibull (GEW) distribution is proposed by using the alternative gamma-generated distribution proposed in [6]. The cdf and the pdf of the generated GEW distribution are, respectively, given by

$$F(x) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log G(x)} t^{\delta-1} e^{-t} dt, x, \delta > 0,$$

and

$$f(x) = \frac{1}{\Gamma(\delta)} [-\log G(x)]^{\delta-1} g(x),$$

where  $G(x)$  is any continuous distribution function with density  $g(x)$  and  $\Gamma(\cdot)$  is the gamma function. The GEW distribution is obtained by incorporating the pdf and cdf of the EW distribution shown in Equations (1) and (2) into the pdf and cdf functions of the gamma-generated distribution. Therefore, the corresponding pdf and cdf can be, respectively, derived as

$$\begin{aligned} f_{\text{GEW}}(x) &= \frac{1}{\Gamma(\delta)} [-\log F_{\text{EW}}(x)]^{\delta-1} f_{\text{EW}}(x) \\ &= \frac{\beta \theta^\beta \alpha^\delta}{\Gamma(\delta)} x^{\beta-1} u (1-u)^{\alpha-1} [-\log(1-u)]^{\delta-1}, \end{aligned} \quad (3)$$

where  $\delta, \alpha, \beta, \theta > 0$ ;  $u = e^{-(\theta x)^\beta}$ ; and

$$\begin{aligned} F_{\text{GEW}}(x) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log F_{\text{EW}}(x)} t^{\delta-1} e^{-t} dt \\ &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\alpha \log(1-u)} t^{\delta-1} e^{-t} dt \\ &= \frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)}, \end{aligned} \quad (4)$$

where  $\Gamma(\cdot, \cdot)$  is the upper incomplete gamma function.

### 2.5. The EWP Distribution

Similarly to the GEW, the EWP distribution is obtained by incorporating the Poisson distribution into the EW distribution [4]. Let  $N$  be a random variable distributed as a zero-truncated Poisson distribution, the pdf of which is

$$Pr(N = n) = \frac{e^{-\lambda} \lambda^n}{(1 - e^{-\lambda}) n!}, n = 1, 2, \dots, \lambda > 0.$$

Let  $Y_1, \dots, Y_N$  be independent and identically distributed random variables as an EW distribution, and let  $X = \max(Y_1, \dots, Y_N)$  be a random variable following the EWP distribution; then, the marginal cdf and pdf of  $X$  can be, respectively, defined as

$$F_{\text{EWP}}(x) = \frac{e^{\lambda(1-u)^\alpha} - 1}{e^\lambda - 1}$$

and

$$f_{\text{EWP}}(x) = \frac{\lambda \alpha \beta \theta^\beta}{(e^\lambda - 1)} x^{\beta-1} u (1-u)^{\alpha-1} e^{\lambda(1-u)^\alpha},$$

where  $\lambda, \alpha, \beta$ , and  $\theta > 0$ .

### 3. The GEWP Distribution

The GEWP distribution was developed by combining the EWP and GEW distributions. The probability distributions for the five-parameter GEWP distribution are analyzed and illustrated in Section 3.1; the survival and hazard rate functions are derived in Section 3.2; the association between the parameters in the GEWP corresponding to its sub-models is

covered in Section 3.3; the moments are provided in Section 3.4; the quantiles are given in Section 3.5; and, finally, the maximum likelihood estimation (MLE) method is presented in Section 3.6.

### 3.1. The cdf and pdf

For  $Y_1, Y_2, \dots, Y_N \sim \text{GEW}$ , the pdf and cdf denoted by  $f_{\text{GEW}}$  and  $F_{\text{GEW}}$ , respectively, are defined by Equations (3) and (4), respectively. The distribution of  $N$  is a zero-truncated Poisson distribution. For  $X = \max\{Y_1, Y_2, \dots, Y_N\}$ , the cdf of  $X|N = n$  is given by

$$\begin{aligned} F_{X|N=n}(x) &= [F_{\text{GEW}}(x)]^n \\ &= \left[ 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\alpha \log(1-u)} t^{\delta-1} e^{-t} dt \right]^n, \end{aligned}$$

where  $u = e^{-(\theta x)^\beta}$  and  $\alpha, \beta, \theta, \delta > 0$ . The marginal cdf of  $X$  can be written as

$$\begin{aligned} F_{\text{GEWP}}(x) &= \sum_{n=1}^{\infty} [F_{\text{GEW}}(x)]^n \frac{e^{-\lambda} \lambda^n}{(1 - e^{-\lambda}) n!} \\ &= \frac{\sum_{n=1}^{\infty} \frac{[F_{\text{GEW}}(x) \lambda]^n}{n!}}{e^\lambda - 1} \\ &= \frac{\sum_{n=1}^{\infty} \frac{[F_{\text{GEW}}(x) \lambda]^n}{n!} - 1}{e^\lambda - 1} \\ &= \frac{e^{F_{\text{GEW}}(x) \lambda} - 1}{e^\lambda - 1} \\ &= \frac{e^{\lambda \left( \frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)} \right)} - 1}{e^\lambda - 1}. \end{aligned}$$

The pdf of  $X$  when it is GEWP-distributed is given by

$$\begin{aligned} f_{\text{GEWP}}(x) &= \frac{d}{dx} \frac{e^{\lambda F_{\text{GEW}}(x)} - 1}{e^\lambda - 1} \\ &= \frac{\lambda f_{\text{GEW}}(x) e^{F_{\text{GEW}}(x) \lambda}}{e^\lambda - 1} \\ &= \frac{\lambda \beta \theta^\beta \alpha^\delta}{(e^\lambda - 1) \Gamma(\delta)} x^{\beta-1} u (1-u)^{\alpha-1} [-\log(1-u)]^{\delta-1} e^{\lambda \left( \frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)} \right)}. \end{aligned}$$

### 3.2. The Survival and Hazard Rate Functions

From the cdf derivation in the previous subsection, we can, respectively, obtain the survival and hazard rate functions of the GEWP as follows:

$$S(t) = 1 - F_{\text{GEWP}}(t) = \frac{e^\lambda - e^{\lambda \left( \frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)} \right)}}{e^\lambda - 1},$$

where  $u = e^{-(\theta t)^\beta}$ , and

$$\begin{aligned} h(t) &= -\frac{d}{dt} \log S(t) \\ &= \frac{f_{\text{GEWP}}(t)}{S(t)} \\ &= \frac{\lambda \beta \theta^\beta \alpha^\delta}{\left(e^\lambda - e^{\lambda \left(\frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)}\right)}\right) \Gamma(\delta)} t^{\beta-1} u (1-u)^{\alpha-1} [-\log(1-u)]^{\delta-1} e^{\left(\frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)}\right)}. \end{aligned}$$

The plots for the pdf and the hazard rate functions of the GEWP distribution for the values of  $\theta, \beta, \alpha, \delta$ , and  $\lambda$  are shown in Figures 1 and 2, respectively. The bathtub-shaped hazard distribution indicates that the GEWP distribution is suitable for scenarios in which the data are skewed.

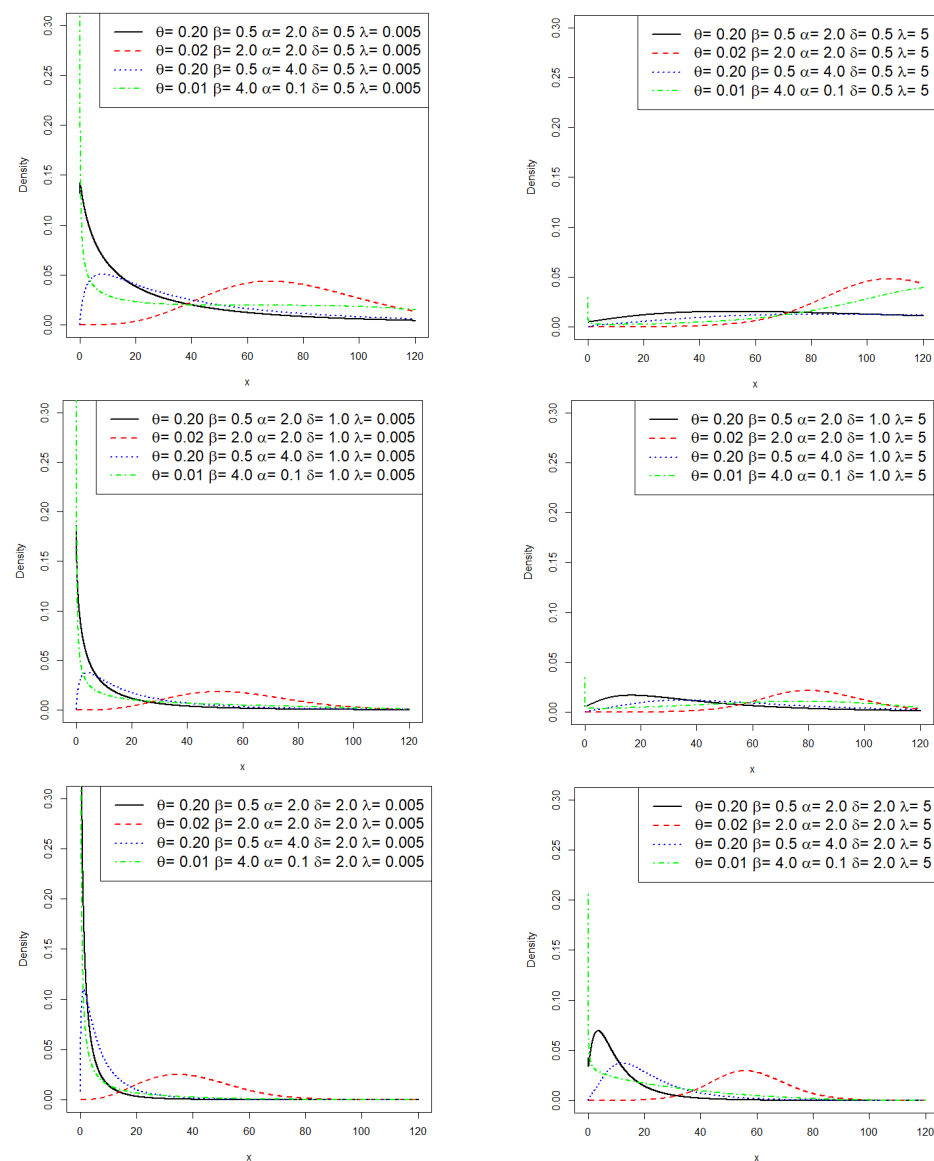
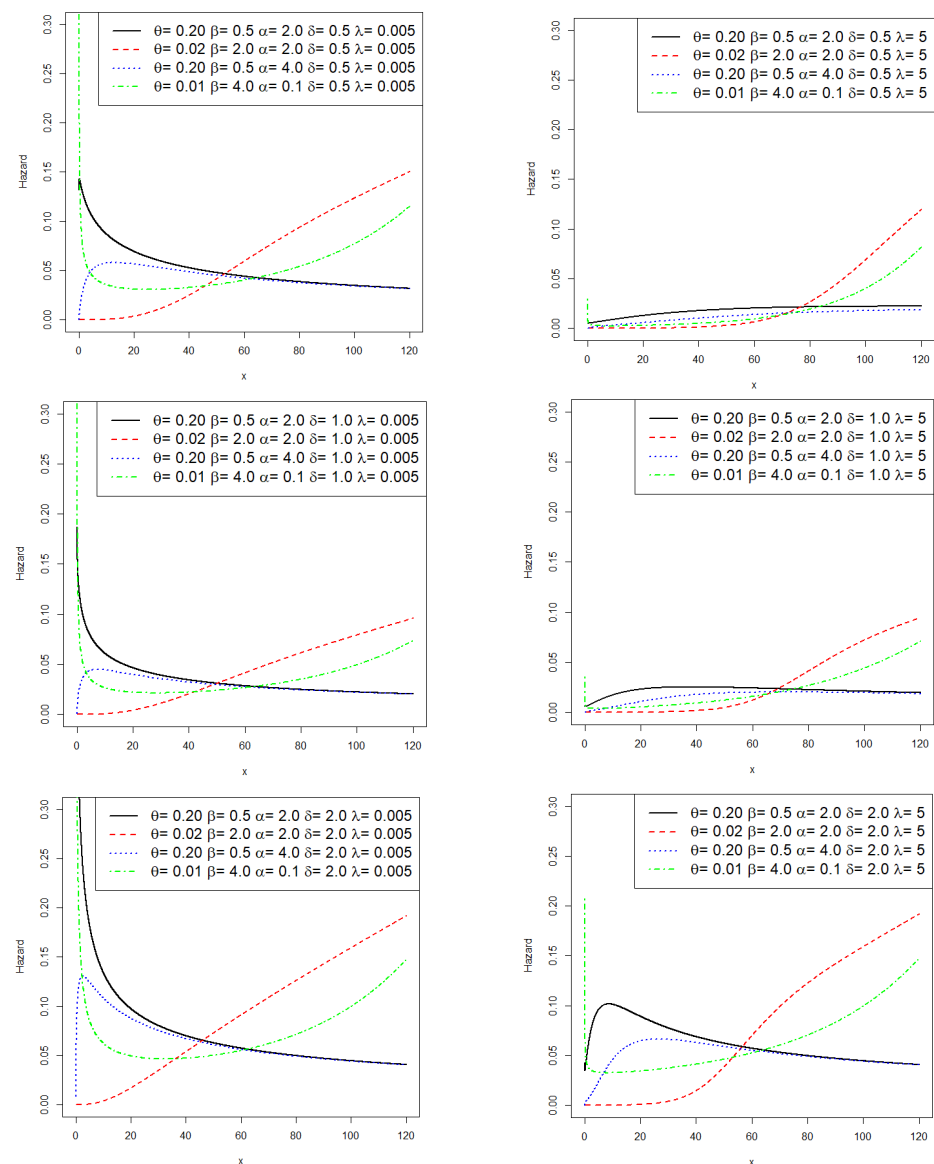


Figure 1. Density functions of the GEWP.



**Figure 2.** Hazard rate functions of the GEWP.

### 3.3. The Sub-Models

By specifying some of the parameters in the GEWP model, we can use the exponential, Weibull, EW, GEW, and EWP as its sub-models. For instance, if parameter  $\lambda$  in the cdf in Section 3.1 is close to 0, then the cdf of the GEWP distribution becomes

$$F_{\text{GEWP}}(x) = F_{\text{GEW}}(x) = \frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)}.$$

We can see that this function is, in fact, the cdf of GEW distribution. Similarly, the other sub-models of the GEWP distribution are provided in Table 1.

**Table 1.** The sub-models of GEWP distribution.

| Distribution<br>(Sub-Models) | $\theta$ | $\beta$ | Parameter<br>$\alpha$ | $\delta$ | $\lambda$               | cdf                                                        |
|------------------------------|----------|---------|-----------------------|----------|-------------------------|------------------------------------------------------------|
| EWP                          | $\theta$ | $\beta$ | $\alpha$              | 1        | $\lambda$               | $\frac{e^{\lambda(1-u)^\alpha} - 1}{e^\lambda - 1}$        |
| GEW                          | $\theta$ | $\beta$ | $\alpha$              | $\delta$ | $\lambda \rightarrow 0$ | $\frac{\Gamma(\delta, -\alpha \log(1-u))}{\Gamma(\delta)}$ |
| EW                           | $\theta$ | $\beta$ | $\alpha$              | 1        | $\lambda \rightarrow 0$ | $(1 - e^{-(\theta x)^\beta})^\alpha$                       |
| Weibull                      | $\theta$ | $\beta$ | 1                     | 1        | $\lambda \rightarrow 0$ | $1 - e^{-(\theta x)^\beta}$                                |
| Exponential                  | $\theta$ | 1       | 1                     | 1        | $\lambda \rightarrow 0$ | $1 - e^{-\theta x}$                                        |

### 3.4. The Moments of the GEWP Distribution

Suppose that  $X \sim \text{GEWP}(\theta, \beta, \alpha, \delta, \lambda)$ ;  $Y_{(n)} = \max\{Y_1, Y_2, \dots, Y_N\}$  for each  $Y_i \sim \text{GEW}(\theta, \beta, \alpha, \delta)$ ; and  $N$  is a zero-truncated Poisson distributed with parameter  $\lambda$ . Then, the moment-generating function for  $X$  can be defined as

$$\begin{aligned} M_X(t) &= \sum_{n=1}^{\infty} \Pr(N = n) M_{Y_{(n)}}(t) \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} M_{Y_{(n)}}(t). \end{aligned}$$

As a result, the  $k$ th moment of the GEWP distribution is given by

$$E(X^k) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} E(Y_{(n)}^k),$$

where  $E(Y_{(n)}^k)$  can be derived similarly to Section 5 in [2].

### 3.5. The Quantiles of the GEWP Distribution

To generate data from the  $\text{GEWP}(\theta, \beta, \alpha, \delta, \lambda)$  distribution, we can derive its  $p$ th quantiles as

$$x_p = \frac{1}{\theta} [-\log(1 - \tilde{p})]^{1/\beta} = w_{\tilde{p}},$$

where  $w_{\tilde{p}}$  is the  $\tilde{p}$  quantiles of the Weibull( $\theta, \beta$ ) and

$$\tilde{p} = e^{-\frac{\Gamma^{-1}(\delta, \Gamma(\delta)\tilde{p})}{\alpha}},$$

where  $\Gamma^{-1}(\cdot, \cdot)$  is the inverse upper incomplete gamma function and

$$\tilde{p} = \frac{1}{\lambda} \log[(e^\lambda - 1)p + 1].$$

### 3.6. Parameter Estimation

Assume that a sample of size  $n$  is drawn from a GEWP-distributed population and  $\Theta = (\theta, \beta, \alpha, \delta, \lambda)^T$  is the vector of the five parameters. Consequently, the likelihood function of the GEWP distribution for  $\underline{x} = (x_1, x_2, \dots, x_n)$  is given by

$$L(\underline{x}) = \prod_{i=1}^n \frac{\lambda \beta \theta^\beta \alpha^\delta}{(e^\lambda - 1) \Gamma(\delta)} x_i^{\beta-1} u_i (1 - u_i)^{\alpha-1} [-\log(1 - u_i)]^{\delta-1} e^{\lambda \left( \frac{\Gamma(\delta, \alpha \log(1 - u_i))}{\Gamma(\delta)} \right)},$$

where  $u_i = e^{-(\theta x_i)^\beta}$ . Following this, the log-likelihood function can be written as

$$l(\underline{x}) = \sum_{i=1}^n \left\{ \log(\lambda) + \log(\beta) + \beta \log(\lambda) + \delta \log(\alpha) - \log(\Gamma(\delta)) - \log(e^\lambda - 1) \right. \\ \left. + (\beta - 1) \log(x_i) - (\theta x_i)^\beta + (\alpha - 1) \log(1 - u_i) + (\delta - 1) \log[-\log(1 - u_i)] \right. \\ \left. + \lambda \frac{\Gamma(\delta, -\alpha \log(1 - u_i))}{\Gamma(\delta)} \right\}.$$

Therefore, by using the MLE method, the score vector

$$U = \left( \frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \delta}, \frac{\partial l}{\partial \lambda} \right)^T$$

is set to zero and

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= \frac{\lambda}{\Gamma(\delta)} \sum_{i=1}^n \frac{\alpha \beta}{\theta} (\theta x_i)^\beta \frac{u_i}{1 - u_i} [-\alpha \log(1 - u_i)]^{\delta-1} e^{\alpha \log(1 - u_i)} + \frac{n\beta}{\theta} - \beta \theta^{\beta-1} \sum_{i=1}^n x_i^\beta \\ &\quad + (\alpha - 1) \beta \theta^{\beta-1} \sum_{i=1}^n \frac{u_i}{1 - u_i} x_i^\beta + (\delta - 1) \beta \theta^{\beta-1} \sum_{i=1}^n \frac{u_i}{(1 - u_i) \log(1 - u_i)} x_i^\beta \\ \frac{\partial l}{\partial \beta} &= \frac{\lambda}{\Gamma(\delta)} \sum_{i=1}^n -\{\alpha^2 (\theta x_i)\}^\beta \log(\theta x_i) \log(1 - u_i) [-\alpha \log(1 - u_i)]^{\delta-1} + \frac{n}{\beta} \\ &\quad + n \log(\theta) + \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n (\theta x_i)^\beta \log(\theta x_i) + (\alpha - 1) \sum_{i=1}^n \frac{u_i (\theta x_i)^\beta}{1 - u_i} \log(\theta x_i), \\ \frac{\partial l}{\partial \alpha} &= \frac{-\alpha \lambda}{\Gamma(\delta)} \sum_{i=1}^n [-\alpha \log(1 - u_i)]^{\delta-1} e^{\alpha \log(1 - u_i)} + \frac{n\alpha}{\delta} + \sum_{i=1}^n \log(1 - u_i), \\ \frac{\partial l}{\partial \delta} &= -\lambda \Psi(\delta) \sum_{i=1}^n \left[ 1 + \frac{\Gamma(\delta, -\alpha \log(1 - u_i))}{\Gamma(\delta)} \right] + n \log(\alpha) - n \Psi(\delta) + \sum_{i=1}^n \log[-\log(1 - u_i)], \\ \frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - \frac{e^\lambda}{(e^\lambda - 1)} + \sum_{i=1}^n \frac{\Gamma(\delta, -\alpha \log(1 - u_i))}{\Gamma(\delta)}, \end{aligned}$$

where  $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{\delta}{\delta x} \Gamma(x)$  is a digamma function. The maximum likelihood estimator

$\hat{\Theta} = (\hat{\theta}, \hat{\beta}, \hat{\alpha}, \hat{\delta}, \hat{\lambda})^T$  is the solution of  $U = \mathbf{0}$ , which cannot be derived analytically. Therefore, numerical methods are required to obtain the parameter estimates of the GEWP. In this work, we applied the Newton–Raphson method via the optim function in the R package. We determine the initial values for more complex distributions by using the maximum likelihood estimates obtained from their sub-models. For example, for the initial values of the EW distribution, we set  $\theta$  and  $\beta$  from the MLE obtained from the Weibull distribution for  $\alpha = 1$ .

### 4. Numerical Experiments

In this section, we present numerical experiments using simulated data from the GEW, EWP, and GEWP distributions with sample sizes of  $n = 200, 500$ , and  $1000$ , respectively.

We considered three cases each for GEW, EWP, and GEWP, with the parameters specified as follows:

- (1) GEW:  $\theta = 5, \beta = 2, \alpha = 2, \delta = 10$ , that is  
GEWP:  $\theta = 5, \beta = 2, \alpha = 2, \delta = 10, \lambda \rightarrow 0$
- (2) EWP:  $\theta = 0.5, \beta = 1, \alpha = 1, \lambda = 20$ , that is  
GEWP:  $\theta = 0.5, \beta = 1, \alpha = 1, \delta = 1, \lambda = 20$ ,
- (3) GEWP:  $\theta = 5, \beta = 1, \alpha = 1, \delta = 10, \lambda = 20$ .

Although the simulated data were drawn from the GEW and EWP distributions, they can be regarded as coming from a GEWP distribution with specified values for some of the parameters. Each experiment was repeated 10,000 times. The estimated parameters were obtained from the MLE using the optim function in the R package. To determine the performance of the estimated distribution, we computed the average estimate as

$$\text{Average estimate} = \frac{\sum_{i=1}^{10,000} \hat{\tau}_i}{10,000},$$

where  $\hat{\tau}_i$  is the estimated value for the considered parameter, and the mean squared error (MSE) is

$$\text{MSE} = \frac{\sum_{i=1}^{10,000} (\hat{\tau}_i - \tau)^2}{10,000},$$

where  $\tau$  is the true value of the considered parameter. The numerical results are presented in Table 2. We can see that the parameter estimates from the MLE are close to the true parameters, especially when  $n$  is large. To illustrate how well the GEWP performs compared to its sub-models, we evaluated the performance of the GEWP using the Kolmogorov–Smirnov (K-S) goodness of fit test statistic along with the associated  $p$ -values obtained using the ks.test function in the R package to fit the simulated data where the true parameter values are known. The results are provided in Table 3.

**Table 2.** The average estimates of the parameters and the MSE for simulated data of size  $n$ .

|                     | Parameter               | Average Estimates |           |            | MSE       |           |            |
|---------------------|-------------------------|-------------------|-----------|------------|-----------|-----------|------------|
|                     |                         | $n = 200$         | $n = 500$ | $n = 1000$ | $n = 200$ | $n = 500$ | $n = 1000$ |
| <sup>1</sup> Case 1 | $\theta = 5$            | 5.9266            | 5.3048    | 4.9943     | 14.084    | 6.9176    | 4.6026     |
|                     | $\beta = 2$             | 2.2633            | 2.1996    | 2.1837     | 0.7774    | 0.5265    | 0.4088     |
|                     | $\alpha = 2$            | 2.1058            | 2.0770    | 2.0604     | 1.7098    | 0.9514    | 0.6434     |
|                     | $\delta = 10$           | 10.713            | 10.874    | 11.047     | 34.843    | 23.515    | 17.839     |
|                     | $\lambda \rightarrow 0$ | 0.0854            | 0.1008    | 0.0884     | 2.2343    | 1.2494    | 0.7587     |
| Case 2              | $\theta = 0.5$          | 0.5806            | 0.5467    | 0.5293     | 0.0960    | 0.0388    | 0.0185     |
|                     | $\beta = 1$             | 0.9946            | 0.9927    | 0.9924     | 0.0577    | 0.0254    | 0.0130     |
|                     | $\alpha = 1$            | 1.8474            | 1.4703    | 1.2894     | 8.4568    | 3.5346    | 1.4297     |
|                     | $\delta = 1$            | 1.1852            | 1.0810    | 1.0485     | 0.4343    | 0.1490    | 0.0780     |
|                     | $\lambda = 20$          | 27.662            | 24.828    | 23.320     | 281.58    | 140.23    | 96.416     |
| Case 3              | $\theta = 5$            | 8.1710            | 6.7841    | 6.1391     | 71.600    | 28.707    | 14.724     |
|                     | $\beta = 1$             | 1.0502            | 1.0192    | 1.0085     | 0.0887    | 0.0367    | 0.0193     |
|                     | $\alpha = 1$            | 0.9642            | 0.9652    | 0.9786     | 0.1446    | 0.0840    | 0.0579     |
|                     | $\delta = 10$           | 9.9564            | 9.8626    | 9.8977     | 9.9907    | 5.9698    | 4.1754     |
|                     | $\lambda = 20$          | 35.744            | 27.529    | 24.047     | 1915.7    | 485.87    | 182.84     |

<sup>1</sup> In Case 1, where  $\lambda \rightarrow 0$ , we replace  $\lambda$  with 0 to compute the MSE.

**Table 3.** The goodness of fit test for simulated data of size  $n = 1000$ .

| Generating Distribution | Fitted Model | K-S Statistic | $p$ -Value |
|-------------------------|--------------|---------------|------------|
| Case 1<br>GEW           | GEW          | 0.01633       | 0.95243    |
|                         | EWP          | 0.29357       | <0.0001    |
|                         | GEWP         | 0.01718       | 0.92934    |
| Case 2<br>EWP           | GEW          | 0.01959       | 0.83767    |
|                         | EWP          | 0.01680       | 0.94032    |
|                         | GEWP         | 0.01761       | 0.91584    |
| Case 3<br>GEWP          | GEW          | 0.03091       | 0.29483    |
|                         | EWP          | 0.03883       | 0.09798    |
|                         | GEWP         | 0.01763       | 0.91505    |

According to the results in Table 3, the K-S test values indicate that when the fitted model matches the generating distribution, we have a good fit. However, in certain cases, we may observe both good and poor fits, as exemplified by the GEW and EWP models. Thus, the results suggest that the GEWP may offer greater flexibility in fitting all cases compared to the GEW and the EWP.

### 5. Application of the GEWP Distribution to Real Data

The GEWP distribution was applied to two real datasets. Unlike the simulation study in which all of the parameter values were known, the K-S test with estimated parameter values was not appropriate for goodness of fit testing in these scenarios because it could yield a smaller Type I error value than expected. Consequently, we employed the extended Shapiro–Wilk test for assessing the goodness of fit for any continuous distribution [16], in which the  $p$ -value is used for comparison purposes. Furthermore, we used the Akaike information criterion (AIC) to compare the models (a lower AIC score indicates a better fit for the data). However, instead of using the AIC directly, we calculated the difference in AIC, which is defined as

$$AIC_{\text{diff}} = AICC - AIC_{\text{GEWP}}, \quad (5)$$

where AICC is the AIC of the considered distribution and  $AIC_{\text{GEWP}}$  is the AIC value for the GEWP.

#### 5.1. Insurance: Claim Data

The dataset comprises 251 motor insurance claims collected from a survey conducted by an insurance company in Thailand in 2013. We applied all five models to fit these data, and the MLE was employed to obtain the parameter estimates. The results for the goodness of fit testing using the Shapiro–Wilk test and the difference in AIC defined in Equation (5) are provided in Table 4.

**Table 4.** The parameter estimates, Shapiro–Wilk test, and  $AIC_{\text{diff}}$  for the claim data.

| Distribution | Parameter      |               |                |                |                 | Shapiro–Wilk   |            | $AIC_{\text{diff}}$ |
|--------------|----------------|---------------|----------------|----------------|-----------------|----------------|------------|---------------------|
|              | $\hat{\theta}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\delta}$ | $\hat{\lambda}$ | Test Statistic | $p$ -Value |                     |
| Exponential  | 0.00007        |               |                |                |                 | 0.903          | <0.001     | 178.554             |
| Weibull      | 0.00007        | 0.881         |                |                |                 | 0.916          | <0.001     | 173.217             |
| EW           | 13,469.3       | 0.109         | 1348.457       |                |                 | 0.986          | 0.012      | 146.248             |
| GEW          | 14,754.0       | 0.102         | 1410.038       | 2.182          |                 | 0.988          | 0.035      | 30.4320             |
| EWP          | 14,827.0       | 0.121         | 1328.948       |                | 7.153           | 0.991          | 0.123      | 116.085             |
| GEWP         | 14,764.4       | 0.130         | 7301.531       | 0.839          | 4.030           | 0.992          | 0.157      | 0.0000              |

### 5.2. Engineering: Strength of Glass Fibers Data

This dataset consisted of the strengths of 63 glass fibers measured at the National Physical Laboratory, England, obtained from [17], originally appearing in [18]. Unfortunately, the unit of measurement was not specified. The results of the goodness of fit tests are provided in Table 5.

**Table 5.** The parameter estimates, Shapiro–Wilk test, and AIC<sub>diff</sub> for the strength of glass fibers data.

| Distribution | Parameter      |               |                |                |                 | Shapiro–Wilk   |                 | AIC <sub>diff</sub> |
|--------------|----------------|---------------|----------------|----------------|-----------------|----------------|-----------------|---------------------|
|              | $\hat{\theta}$ | $\hat{\beta}$ | $\hat{\alpha}$ | $\hat{\delta}$ | $\hat{\lambda}$ | Test Statistic | <i>p</i> -Value |                     |
| Expo         | 0.6636         |               |                |                |                 | 0.866          | <0.001          | 358.21              |
| Weibull      | 0.6142         | 5.781         |                |                |                 | 0.953          | 0.018           | 212.96              |
| EW           | 0.5820         | 7.286         | 0.671          |                |                 | 0.958          | 0.030           | 213.90              |
| GEW          | 0.6673         | 7.083         | 0.470          | 0.408          |                 | 0.962          | 0.049           | 18.972              |
| EWP          | 0.6464         | 5.504         | 0.573          |                | 2.798           | 0.971          | 0.149           | 212.51              |
| GEWP         | 0.7946         | 5.292         | 0.367          | 0.380          | 2.556           | 0.973          | 0.183           | 0.0000              |

The results in Tables 4 and 5 indicate that the AIC<sub>diff</sub> values obtained for the sub-models were all greater than zero. Meanwhile, the smallest AIC value was achieved when using the GEWP distribution. Although the AIC value for the GEW distribution was relatively small, it did not fit the data as well as the GEWP distribution.

## 6. Conclusions

We proposed the novel GEWP distribution, which was obtained by combining the GEW and EWP distributions. It is a five-parameter probability distribution containing several sub-models. We analyzed the cdf, pdf, survival, and hazard rate functions, moments, quantiles, and parameter estimates for the GEWP distribution using the MLE approach. Since the log-likelihood function cannot be expressed in a closed form, we conducted simulation studies to investigate the performance of the parameter estimates. We found that the parameter estimates approximated the true parameter values, especially when the sample size was large. Finally, we used the parameter estimates to test the fitting of the GEWP distribution to real datasets and compared its efficacy with those of its sub-models. Both the simulation and real-dataset results indicate that the GEWP distribution performed better than the others in all of the scenarios tested. Notably, it significantly outperformed the exponential, Weibull, and EW distributions, as evidenced by its smaller *p*-value and relatively larger AIC values. While it may not be significantly superior to its sub-models (the GEW and EWP distributions), it offers greater flexibility, thereby enabling it to fit data where the GEW and EWP distributions may not be appropriate.

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