

A Factory of Fractional Derivatives

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Abstract: This paper aims to demonstrate that, beyond the small world of Riemann–Liouville and Caputo derivatives, there is a vast and rich world with many derivatives suitable for specific problems and various theoretical frameworks to develop, corresponding to different paths taken. The notions of time and scale sequences are introduced, and general associated basic derivatives, namely, right/stretching and left/shrinking, are defined. A general framework for fractional derivative definitions is reviewed and applied to obtain both known and new fractional-order derivatives. Several fractional derivatives are considered, mainly Liouville, Hadamard, Euler, bilinear, tempered, q-derivative, and Hahn.

Keywords: shift-invariant; scale-invariant; nabla derivative; fractional q-derivative; fractional Hahn derivative

MSC: 26A33

1. Introduction

During this century, fractional calculus has been increasingly adopted as a model for many natural and artificial phenomena [1–3]. However, four centuries after the first reference to the possibility of non-integer-order derivatives [4], we have arrived at a very confusing situation, in which we find the following:

- Multiple definitions of derivatives without clear application domains;
- Great confusion between the notions of the derivative and the system;
- Mixtures of different types of derivatives without stating reasons and validity;
- The incorrect use of initial conditions;
- False derivatives.

This situation causes enormous problems that make life difficult for those who just want to find applications in science and engineering.

Since the first formal introduction of the concept of the fractional derivative (FD) by Liouville (1832) [5,6], a long distance has been covered by mathematicians, physicists, and engineers. Despite Liouville’s inability to impose his vision, the main definitions we find today are based on the formulas he presented, mainly the Riemann–Liouville (RL) [7], Dzherbashian–Caputo (C) [8,9], and Grünwald–Letnikov (GL) [7] definitions. However, based on these derivatives, new ones have been proposed, such as that of Hadamard [9] or Marchaud [7]. The success of the designation “fractional derivative” motivated and gave rise to the proposal of “new fractional derivatives”. Over the past 20 years, many modifications or combinations of the above derivatives have been introduced, along with other operators claiming to be “fractional derivatives”. Some are mere high-pass filters, and others are derivatives of order 1 in disguise. If we also consider pseudo-derivatives, we conclude that the situation is really messy and confusing, as shown first by Oliveira and Machado and more recently by Teodoro et al. [10,11], who listed such operators and introduced a classification according to some specified criteria. Since 2015, there has been a lot of discussion in forums, conferences, and articles, where we found that most people are



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unaware of the distinction between the concepts of a system and a derivative. This situation was the main reason for the attempt, described in [12], to provide a methodology for testing whether a given operator is suitable to be a fractional derivative. The pursuit of this goal was continued with the formalizations introduced in [13,14]. However, new generalizations have recently appeared that use Abel's algorithm as a base for the tautochrone problem. This was described first by Kochubei [15] and then by Luchko [16], Tarasov [17], and Fernandez [18]. Hanyga [19] used the Abel approach in the discussion of the concept of the fractional derivative.

Nevertheless, it is important to note that these developments only deal with shift-invariant operators in continuous time, without considering discrete-time and scale-invariant operators. On the other hand, we see the existence of what we can call the RL/C dictatorship, which tries to impose the formats of RL and C everywhere, even in discrete-time cases. In fact, Liouville found that the definition of a fractional anti-derivative could not be used directly to define a derivative, since the involved integral had become singular. He cleverly decided to transfer the singularity to a derivative of integer order, giving rise to two procedures that originated the RL and C derivatives. Consequently, such procedures are only necessary in the presence of a singularity and are unnecessary and sometimes useless when there is no singularity, as occurs in discrete time, since they involve an increase in the number of operations, which can be very important in practical applications. What we can observe are attempts to formulate RL and C derivatives in many situations, mainly discrete, where a single formula serves for all derivative orders, positive and negative. This work shows how we should proceed and allows us to easily detect when a singularity transfer is necessary.

Here, we recover the framework introduced in [20] and apply it to fractionalize various derivatives. We start from the general nabla and delta derivatives introduced by Aulbach and Hilger [21,22]. These are based on the concept of a time scale, which we revise and call a "time sequence" to distinguish it from the novel "scale sequence", allowing us to formally introduce two derivatives similar to the nabla and delta: the "stretching" and "shrinking" derivatives. These derivatives are simple linear operators that deserve to be generalized. To accomplish this, we calculate their eigenfunctions (generalized exponentials) and their eigenvalues (transfer functions) that are used to define generalized Laplace transforms. From these, we introduce the corresponding higher-order and fractional-order derivatives. We do not intend to present a detailed study of the different fractionalizations that exist for each derivative but to show that the path introduced is coherent and leads to general and simple solutions. This is a practical point of view that does not require dismissing other pure mathematical approaches. In particular, we do not treat existence problems.

Order-1 and, in general, positive-integer derivatives have a short memory (range). In contrast, fractional derivatives have a long memory. The tempered fractional derivatives that we will generalize have a medium range [23]. Here, we start from the Hilger approach to propose new tempered nabla/stretching and delta/shrinking derivatives. The following scheme shows that we define eight basic derivatives suitable for fractionalization:

$$\left\{ \begin{array}{l} \text{time derivatives} \left\{ \begin{array}{l} \text{nabla} \\ \text{delta} \end{array} \right\} \\ \text{scale derivatives} \left\{ \begin{array}{l} \text{stretching} \\ \text{shrinking} \end{array} \right\} \end{array} \right\} \text{tempered}$$

The (q, h) Hahn derivative that is suggested in Section 6.4 is a mixture of time and scale derivatives. We can go further and introduce another class of derivatives involving these ones. For this, we consider the traditional bilinear transformation used to obtain the discrete-time equivalent for continuous-time shift-invariant systems [24]. Such a transformation was reinterpreted as a derivative [25] and will be generalized here through combinations of the above derivatives, expanding the number of possible fields of application. It is important to stress that it is also a long-range derivative. However, it can be tempered, too.

This paper is organized as follows. A revision of the notion of a “time sequence” and the introduction of the concept of a “scale sequence” are presented in Section 2. In Section 3, we recall the discussion about the concept of the FD that we introduced in [12]. We generalize Hilger’s definitions of nabla/delta derivatives [21,22] by introducing the corresponding scale, tempered, and bilinear derivatives (Section 3.2). The general framework that we proposed in [20] is revised in Appendix B. With this framework and the derivatives introduced in Section 3.2, a general form of fractional derivatives is proposed in Section 4. For this, we use the notion of generalized exponentials that are the eigenfunctions of the derivatives (Section 4.1) and serve to define generalized Laplace transforms (Section 4.2). In Section 5, we describe the derivatives defined in time sequences, namely, continuous-time Liouville (Section 5.2), discrete-time Euler (Section 5.4), and discrete-time bilinear (Section 5.5) derivatives. The corresponding tempered derivatives are also considered. The main derivatives on scale sequences, chiefly the scale-invariant derivatives, are described in Section 6. In particular, the Hadamard (Section 6.1), tempered Hadamard (Section 6.2), and bilinear (Section 6.3) derivatives are introduced. The q -derivative and Hahn derivative are suggested in Section 6.4. Bilateral derivatives are considered in Section 7. Finally, some conclusions are presented in Section 8. Some mathematical tools are listed in Appendix A.

2. Time and Scale Sequences

Hilger’s approach to continuous/discrete unification was based on the concept of the *time scale* [21,22]. This refers to any non-empty closed subset, $\mathbb{T}$, of the real line, $\mathbb{R}$. The designation “time scale” can be misleading, since the word “scale” is associated with the notion of stretching or shrinking, frequently having a relation to a speed or a rate. We prefer the designation *time sequence*.

Let $t \in \mathbb{T}$ be the current instant. The previous next instant is denoted by $\rho(t)$. Similarly, the next following point in the time sequence $\mathbb{T}$ is denoted by $\sigma(t)$. One has

$$\rho(t) = t - \nu(t), \quad \sigma(t) = t + \mu(t),$$

where $\nu(t)$ and $\mu(t)$ are called *graininess* functions. These functions can be used to construct any time sequence. Let us define

$$\nu^0(t) := 0, \quad \nu^n(t) := \nu^{n-1}(t) + \nu(t - \nu^{n-1}(t)), \quad n \in \mathbb{N},$$

and $\rho^0(t) := t$, $\rho^n(t) := \rho(\rho^{n-1}(t))$, $n \in \mathbb{N}$. Note that $\nu^1(t) = \nu(t)$ and $\rho^1(t) = \rho(t)$. When moving into the past, we have

$$\begin{aligned} \rho^0(t) &= t = t - \nu^0(t), \\ \rho^1(t) &= \rho(t) = t - \nu(t) = t - \nu^1(t), \\ \rho^2(t) &= \rho(\rho(t)) = \rho(t) - \nu(\rho(t)) = t - \nu(t) - \nu(t - \nu(t)) = t - \nu^2(t), \\ &\vdots \\ \rho^n(t) &= t - \nu^n(t). \end{aligned}$$

Moving into the future, the definitions and results are similar:

$$\begin{aligned} \mu^0(t) &:= 0, \quad \mu^n(t) := \mu^{n-1}(t) + \mu(t + \mu^{n-1}(t)), \\ \sigma^0(t) &:= t, \quad \sigma^n(t) := \sigma(\sigma^{n-1}(t)), \end{aligned}$$

$n \in \mathbb{N}$, and we have $\sigma^n(t) = t + \mu^n(t)$.

The time sequences $\mathbb{T}$ defined by a set of discrete instants $t_n, n \in \mathbb{Z}$, and by the corresponding *graininess* functions are very interesting. We define direct graininess [26,27] by

$$t_n = t_{n-1} + v_n, \quad n \in \mathbb{Z},$$

and reverse graininess by

$$t_n = t_{n+1} - \mu_n, \quad n \in \mathbb{Z},$$

where we avoid representing any reference instant t_0 . These definitions of “irregular” sequences allow us to introduce the most interesting time sequences that we find in practice. However, we have some difficulties in performing some kinds of manipulations that are also very common. Let us consider a time sequence defined on $\mathbb{R}$ and unbounded when $t \rightarrow \pm\infty$. For example, we consider a time sequence defined by

$$t_n = nT + \vartheta_n, \quad n \in \mathbb{Z}, \quad T > 0, \quad |\vartheta_n| < \frac{T}{2},$$

which we can call an “almost linear sequence” [28]. However, in the most interesting engineering applications, we consider regular (uniform) sequences having constant graininess, h ,

$$\mathbb{T} = h\mathbb{Z} = \{\dots, -3h, -2h, -h, 0, h, 2h, 3h, \dots\},$$

with $h \in \mathbb{R}^+$.

This way of constructing sequences, which we can call additive, is not a unique way of obtaining working domains. There is another way, multiplicative, that can be obtained by introducing log-graininesses, $\theta, \zeta > 1$. To understand this idea, assume that we have

$$\sigma(t) = t\zeta(t), \quad \rho(t) = t/\theta(t),$$

so that the logarithm of the sequence is a time sequence with the graininesses $\log \theta_n$ and $\log \zeta_n$, giving rise to

$$\log \tau_n = \log \tau_{n-1} + \log \theta_n, \quad n \in \mathbb{Z},$$

and

$$\log \tau_n = \log \tau_{n+1} - \log \zeta_n, \quad n \in \mathbb{Z}.$$

Therefore, we obtain what we can call “scale sequences,” Θ , that are *progressive*

$$\tau_n = \tau_{n-1}\theta_n, \quad n \in \mathbb{Z},$$

and *regressive*

$$\tau_n = \tau_{n+1}/\zeta_n, \quad n \in \mathbb{Z}.$$

A simple example is

$$\tau_n = \tau_0 q^n \in \Theta, \quad n \in \mathbb{Z}, \tau_0 \in \mathbb{R}.$$

We will use these sequences when dealing with scale-invariant derivatives.

3. What is a Derivative?

3.1. The Classic Derivatives

The notion of the *derivative*, in the classic sense, comes from the works of Leibniz, Newton, and Lagrange and is defined for functions of a continuous variable (often time), usually in an interval on $\mathbb{R}$, using a presentation that comes from the Cauchy formulation. Simply put, we can say that a derivative is the instantaneous rate of change in a function that can be stated, for $t \in \mathbb{R}$, as

$$Df(t) = \frac{df}{dt} = f'(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t-h)}{h}. \quad (1)$$

This is nowadays referred to as a causal derivative formulation, although mathematicians usually prefer the anti-causal formula,

$$Df(t) = \frac{df}{dt} = f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}. \quad (2)$$

There are other alternative approaches, such as that of Carathéodory [29]. However, in practical applications, we deal with functions defined in domains that are not necessarily continuous, requiring more general approaches.

3.2. Nabla/Stretching and Delta/Shrinking Derivatives on General Time/Scale Sequences

The modern notions of derivatives came from the works of Hilger in an attempt to realize a discrete/continuous unification [21,22,30].

Definition 1. Consider a given time sequence, $\mathbb{T}$. According to this approach, we define the nabla derivative by

$$D_{\nabla}f(t) = \begin{cases} \frac{f(t) - f(\rho(t))}{v(t)} & \text{if } v(t) \neq 0 \\ \lim_{h \rightarrow 0} \frac{f(t) - f(t-h)}{h} & \text{if } v(t) = 0 \end{cases} \quad (3)$$

and the delta derivative by

$$D_{\Delta}f(t) = \begin{cases} \frac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if } \mu(t) \neq 0 \\ \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} & \text{if } \mu(t) = 0. \end{cases} \quad (4)$$

We can give these derivatives another form, in agreement with our comments made earlier about time sequences. According to this, the nabla derivative is given by

$$D_{\nabla}f(t_n) = \begin{cases} \frac{f(t_n) - f(t_{n-1})}{v_n} & \text{for any } t_n \text{ that is not left dense,} \\ \lim_{h \rightarrow 0^+} \frac{f(t_n) - f(t_n-h)}{h} & \text{for } t_n \text{ left dense,} \end{cases} \quad (5)$$

where $v_n = t_n - t_{n-1}$, and the delta derivative is

$$D_{\Delta}f(t_n) = \begin{cases} \frac{f(t_{n+1}) - f(t_n)}{\mu_n} & \text{for any } t_n \text{ that is not right dense,} \\ \lim_{h \rightarrow 0^+} \frac{f(t_n+h) - f(t_n)}{h} & \text{for } t_n \text{ right dense,} \end{cases} \quad (6)$$

where $\mu_n = t_{n+1} - t_n$ is the graininess.

From these basic derivatives, we can formulate others that we consider in the nabla version (the delta is similar). Let $a, b \in \mathbb{R}^+$. We define the following:

1. Tempered nabla derivative

$$D_{\nabla}f(t) = \begin{cases} \frac{f(t) - a^{-b}f(\rho(t))}{v(t)} & \text{if } v(t) \neq 0 \\ \lim_{h \rightarrow 0} \frac{f(t) - a^{-b}f(t-h)}{h} & \text{if } v(t) = 0; \end{cases} \quad (7)$$

2. Bilinear nabla derivative

$$\frac{D_{\nabla}f(t) + D_{\nabla}f(\rho(t))}{2} = \begin{cases} \frac{f(t) - f(\rho(t))}{v(t)} & \text{if } v(t) \neq 0 \\ \lim_{h \rightarrow 0} \frac{f(t) - f(t-h)}{h} & \text{if } v(t) = 0; \end{cases} \quad (8)$$

3. Tempered bilinear nabla derivative

$$\frac{D_{\nabla}f(t) + a^{-b}D_{\nabla}f(\rho(t))}{2} = \begin{cases} \frac{f(t)-a^{-b}f(\rho(t))}{v(t)} & \text{if } v(t) \neq 0 \\ \lim_{h \rightarrow 0} \frac{f(t)-a^{-b}f(t-h)}{h} & \text{if } v(t) = 0. \end{cases} \quad (9)$$

Besides these derivatives, we can define others on scale sequences.

Definition 2. Let Θ be a scale sequence. For any $t \in \Theta$, we define the stretching derivative as

$$D_{\nabla}f(t) = \begin{cases} \frac{f(t)-f(\rho(t))}{\log \theta(t)} & \text{if } \theta(t) \neq 1 \\ \lim_{q \rightarrow 1} \frac{f(t)-f(t/q)}{\log q} & \text{if } \theta(t) = 1, \end{cases} \quad (10)$$

where $\rho(t) = t\theta(t)$ and $q > 1$. Similarly, we define the shrinking derivative as

$$D_{\nabla}f(t_n) = \begin{cases} \frac{f(t_n)-f(t_{n-1})}{\log \theta_n} & \text{for any } t_n \text{ that is not left dense,} \\ \lim_{q \rightarrow 1^+} \frac{f(t_n)-f(t_n/q)}{\log q} & \text{for } t_n \text{ left dense,} \end{cases} \quad (11)$$

where $\theta_n = t_n/t_{n-1}$.

We do not present the other versions corresponding to the delta, tempered, and bilinear cases. They are obtained in a similar way.

In any time/scale sequence, we define the Heaviside unit step by

$$\varepsilon(t) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0. \end{cases}$$

Any derivative of this function

$$\delta_f(t - t_0) = D\varepsilon(t - t_0), \quad t, t_0 \in \mathbb{T}, \quad (12)$$

will be called an *impulse*.

3.3. What is a Fractional Derivative?

Some years ago, we provided answers to this question based on two criteria [12]. These criteria expressed the characteristics we desired for fractional continuous-time shift-invariant derivatives. The *wide-sense criterion* reads as follows. An operator is considered an FD in this criterion if it enjoys the properties ¹P defined as follows:

¹P1 Linearity

The operator is linear.

¹P2 Identity

The zero-order derivative of a function returns the function itself.

¹P3 Backward compatibility

When the order is an integer, the FD gives the same result as the ordinary derivative.

¹P4 The index law holds

$$D^{\alpha}D^{\beta}f(t) = D^{\alpha+\beta}f(t) \quad (13)$$

for $\alpha < 0$ and $\beta < 0$.

¹P5 Generalized Leibniz rule

$$D^{\alpha}[f(t)g(t)] = \sum_{i=0}^{\infty} \binom{\alpha}{i} D^i f(t) D^{\alpha-i} g(t) \quad (14)$$

The index law property can be modified to include positive orders. This leads to the *strict-sense criterion*. Therefore, criterion ²P has five conditions, where ¹P4 is modified to

²P4 The index law

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \quad (15)$$

for any α and β .

Remark 1.

- The strict-sense criterion in the form ²P4, presented by (15), opens a discussion: is a given property a requirement for the operator itself (fractional derivative) or for the operator domain, i.e., the space of functions? The “philosophical controversy” on whether the properties belong to operators or functions is very important in applications. If we consider that we should not use relation (15), then an electrical circuit RLC is different from an RCL. But, this is not considered in practice: they constitute the same circuit. The situation becomes more complicated if we have two coils associated in series. Therefore, commutativity must be assumed. Likewise, additivity is important in practice. Without it, the realization of a shift-invariant system would become shift-variant.
- Another important question concerns the so-called “starting point”. Should a given definition include such a point? If such a point is included, we have a definition “à la carte”, one for each function, when we expected to have one definition for all the functions (this may pose existence problems, but that is another issue). Nobody questions the definition of the Fourier transform: it is unique for all the functions, with no starting point.

4. General Fractional Derivatives

4.1. Generalized Exponentials

In Section 3.2, we introduced a general definition of order-1 derivatives. Applying the framework described in Appendix B leads us to generalized fractional derivatives. This procedure was applied in [26] for Euler-type derivatives and in [25] for bilinear cases. Here, we recall these results and present a general expression for the fractional nabla derivative (the delta case is treated similarly).

Theorem 1. Consider that our domain is a time sequence, $\mathbb{T}$, and take $t_0 \in \mathbb{T}$ as the reference instant (as before, this parameter will be removed when it is not important). The generalized nabla exponential is given by

$$e_{\nabla}(t, t_0; s) = \begin{cases} \prod_{k=1}^n [1 - s\mu_k(t_0)]^{-1} & \text{if } t = t_n > t_0 \\ 1 & \text{if } t = t_0 \\ \prod_{k=1}^m [1 - s\nu_k(t_0)] & \text{if } t = t_m < t_0 \end{cases} \quad (16)$$

and the delta exponential by

$$e_{\Delta}(t, t_0; s) = \begin{cases} \prod_{k=1}^n [1 + s\mu_k(t_0)] & \text{if } t = t_n > t_0 \\ 1 & \text{if } t = t_0 \\ \prod_{k=1}^m [1 + s\nu_k(t_0)]^{-1} & \text{if } t = t_m < t_0. \end{cases} \quad (17)$$

For proof, see [26,27]. These exponentials have several interesting properties that we will not consider here [26]. Those that we are interested in here read as follows:

1. *Relation between nabla and delta exponentials.*

The function defined in (17) with the substitution of $-s$ for s is the inverse of (16):

$$e_{\Delta}(t, t_0; s) = 1/e_{\nabla}(t, t_0; -s). \quad (18)$$

2. Let $h_M = \max(\nu_k, \mu_l)$ and $h_m = \min(\nu_k, \mu_l)$, $k, l \in \mathbb{Z}$. The nabla exponential $e_{\nabla}(t, t_0; s)$ is characterized as follows [26]:
- It is a real number for any real s ;
 - It is positive for any real number s such that $s < \frac{1}{h_M}$;
 - It oscillates for any real number s such that $s > \frac{1}{h_m}$;
 - It is bounded for values of s inside the inner Hilger circle $|1 - sh_m| = 1$;
 - It has an absolute value that increases as $|s|$ increases outside the outer Hilger circle $|1 - sh_M| = 1$, going to infinity as $|s| \rightarrow \infty$.

This means that each graininess defines a circle centered at its inverse and passing at $s = 0$. In general, we have infinite circles that reduce to one when the graininess is constant. When it is null, it degenerates into the imaginary axis.

3. *Product of exponentials.*
The following relations hold:

$$\begin{aligned} e_{\nabla}(t, t_0; s) \cdot e_{\Delta}(\tau, t_0; -s) &= e_{\nabla}(t, \tau; s) \\ &= e_{\Delta}(\tau, t; -s). \end{aligned} \quad (19)$$

Now, let $t_n > t_m + \tau$. We have

$$e_{\nabla}(t_n, t_0; s) = e_{\nabla}(t_n - \tau, t_0; s) \cdot e_{\nabla}(t_n, t_n - \tau; s) \quad (20)$$

and

$$e_{\nabla}(t_n - \tau, t_0; s) = e_{\nabla}(t_n, t_0; s) \cdot e_{\nabla}(t_n, t_n - \tau; -s). \quad (21)$$

Relation (21) states the *translation or shift property* of an exponential.

4. *Scale change.*
Let a be a positive real number. We have

$$e_{\nabla}(at, t_0; s) = e_{\nabla}(t, t_0; as).$$

If $a < 0$, a similar relation is obtained but involving the delta exponential.

Above, we introduced the nabla and delta exponentials. Now, we consider the stretching/shrinking exponentials defined on scale sequences. In agreement with our study carried out in Section 3.2, we obtain such exponentials through simple substitutions of graininesses. We obtain the stretching exponential

$$e_{\nabla}(t, t_0; s) = \begin{cases} \prod_{k=1}^n [1 - s \log(\theta_k(t_0))]^{-1} & \text{if } t = t_n > t_0 \\ 1 & \text{if } t = t_0 \\ \prod_{k=1}^m [1 - s \log(\zeta_k(t_0))] & \text{if } t = t_m < t_0 \end{cases} \quad (22)$$

and the shrinking exponential

$$e_{\Delta}(t, t_0; s) = \begin{cases} \prod_{k=1}^n [1 + s \log(\theta_k(t_0))] & \text{if } t = t_n > t_0 \\ 1 & \text{if } t = t_0 \\ \prod_{k=1}^m [1 + s \log(\zeta_k(t_0))]^{-1} & \text{if } t = t_m < t_0. \end{cases} \quad (23)$$

The examples that we will treat below will help illustrate the roles of the different parameters.

4.2. Suitable General Laplace Transforms and Corresponding Derivatives

4.2.1. Nabla case

To define a general Laplace transform, we express a given function in terms of the exponentials deduced above. This defines the inverse transform.

Definition 3. Suppose that we have a function, $f(t)$, defined in a given time sequence, having a transform $F_{\nabla}(s)$ (undefined for now; defined later by (26)). The inverse transform will serve to synthesize it from a continuous set of elementary exponentials:

$$f(t) = -\frac{1}{2\pi i} \oint_{\gamma} F_{\nabla}(s) e_{\nabla}(t + \mu(t), t_0; s) ds, \quad (24)$$

where the integration path, γ , is a simple closed contour in a region of analyticity of $F_{\nabla}(s)$ and surrounding the poles introduced by the exponential.

Consider the Hilger circle with the radius $R > \frac{1}{h_m} = \frac{1}{\min(v_k, \mu_l)}$, centered at $\frac{1}{R}$ and described in the counterclockwise direction. We call this region the *fundamental region*, and we represent it by R_{γ} . In the continuous-time case, the circle degenerates into a vertical straight line. Expression (24) states our definition of the inverse nabla LT. It is a simple task to show that the inverse nabla LT of $F_{\nabla}(s) \equiv 1$ is the impulse $\delta(t_n - t_0) = \frac{1}{v_1(t_0)}$. With this result, we can prove one of the most important properties of the nabla LT:

$$\mathcal{L}_{\nabla}[D_{\nabla}f(t)] = sF_{\nabla}(s), \quad (25)$$

where $F_{\nabla}(s)$ is the nabla LT of $f(t)$, i.e., $F_{\nabla}(s) = \mathcal{L}_{\nabla}[f(t)]$. We deduce several properties of the nabla LT from the inversion integral. Precisely, the following properties hold (we assume s to be inside the region of convergence of the nabla LT):

- *Linearity*

$$\mathcal{L}_{\nabla}[f(t) + g(t)] = \mathcal{L}_{\nabla}[f(t)] + \mathcal{L}_{\nabla}[g(t)].$$

- *The transform of the derivative*

Given equality (25), we easily deduce, by repeated application of (25), that if $N \in \mathbb{N}_0$, then

$$\mathcal{L}_{\nabla}[D_{\nabla}^N f(t)] = s^N F_{\nabla}(s),$$

restating a well-known result in the context of the LT.

- *Time scaling*

Let a be a positive real number. (It could be negative, but such a case is not interesting.) We have

$$\mathcal{L}_{\nabla}[f(at)] = \frac{1}{a} F_{\nabla}\left(\frac{s}{a}\right).$$

The shift property (21) provides a suggestion of how to define the nabla LT. In fact, it must be defined in terms of the delta exponential due to property (19) particularized to $t \rightarrow t + \mu(t)$ and $\tau = t$.

Definition 4. We define the generalized nabla LT by

$$F_{\nabla}(s) = \sum_{n=-\infty}^{+\infty} v_n f(t_n) e_{\Delta}(t_n, t_0; -s) = \sum_{n=-\infty}^{+\infty} v_n f(t_n) e_{\nabla}^{-1}(t_n, t_0; s) \quad (26)$$

with a suitable “region of convergence”.

Example 1. Unit steps. The nabla LT of the unit step is $1/s$:

$$\mathcal{L}_{\nabla}[\epsilon(t)] = \frac{1}{s}$$

for s inside R_γ and

$$\mathcal{L}_\nabla[-\epsilon(-t - \mu(t))] = \frac{1}{s}$$

for s outside R_γ .

According to our framework, we define the fractional nabla derivative by

$$D_\nabla^\alpha f(t) = \mathcal{L}^{-1}[s^\alpha F_\nabla(s)]. \quad (27)$$

Using (24),

$$D_\nabla^\alpha f(t) = \begin{cases} -\frac{1}{2\pi i} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{n+1} [1 - s\mu_k(t_0)]^{-1} ds = \\ \frac{(-1)^n}{\prod_{k=1}^{n+1} \mu_k(t_0)} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{n+1} \left[s - \frac{1}{\mu_k(t_0)}\right]^{-1} ds & t = t_n \geq t_0 \\ -\frac{1}{2\pi i} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{m+1} [1 - s\nu_k(t_0)] ds = \\ \frac{(-1)^n}{\prod_{k=1}^{m+1} \nu_k(t_0)} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{m+1} \left[s - \frac{1}{\nu_k(t_0)}\right] ds & t = t_m < t_0. \end{cases} \quad (28)$$

To express this relation directly in terms of the values of $f(t)$, we need to introduce the convolution, which is a somewhat involved process [26]. We will do so in each particular case that we will treat.

4.2.2. Stretching Case

The treatment of this case is similar to the previous one. Just remember the need to use log-graininesses. The rest of the procedure is the same, as we will see in the examples.

5. Main Derivatives on Time Sequences

5.1. Liouville Derivatives

Consider that the time sequence is the real line, $\mathbb{T} = \mathbb{R}$. Therefore, the graininess is null, so the basic derivative introduced above applies:

$$D_f f(t) = \frac{df(t)}{dt} = \lim_{h \rightarrow 0^+} \frac{f(t) - f(t-h)}{h}. \quad (29)$$

The application of the above framework gives the following:

1. The eigenfunction is the usual exponential:

$$e_f(t) = e^{st}, \quad t \in \mathbb{R}, s \in \mathbb{C}. \quad (30)$$

2. The associated transform is the classic (bilateral) LT (A1).
3. The FD of order $\alpha \in \mathbb{R}$ was introduced by Liouville (1832) as [6]

$$\frac{d^\alpha e^{st}}{dt^\alpha} = s^\alpha e^{st}, \quad (31)$$

with a suitable ROC. Therefore, from (A2), we have

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^\alpha X(s) e^{st} ds, \quad (32)$$

with $t \in \mathbb{R}$ and $a > 0$.

4. The (causal) differintegrator has TF $H(s) = s^\alpha$, being valid for $\text{Re}(s) > 0$ or $s = \pm j\omega, \omega \in \mathbb{R}^+$. The anti-causal differintegrator has the left complex half-plane as the ROC [31].
5. The Liouville–Grünwald–Letnikov derivative reads [32]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{h \rightarrow 0^+} h^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} x(t - kh). \quad (33)$$

6. Impulse response
The impulse response is given by [32]

$$\mathcal{L}^{-1}s^\alpha = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(t); \quad (34)$$

The gamma function is defined in (A13).

7. Liouville (anti-)derivative
The impulse response and the convolution give another representation for the fractional anti-derivative:

$$\frac{d^\alpha x(t)}{dt^\alpha} = \int_0^\infty x(t - \tau) \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau, \quad (35)$$

which is the causal version of Liouville's first integral formula. He noted that this formula must not be used for positive values of α (derivative case), since the integral can be singular. He introduced two procedures, stated as $s^\alpha = s^{\alpha-N}s^N = s^N s^{\alpha-N}, \alpha < N \in \mathbb{N}$. Alternatively, (35) can be regularized [32].

5.2. Tempered Shift-Invariant Derivatives

Definition 5. Let $\lambda > 0$. In agreement with (7), we define the tempered shift-invariant derivative by [23]

$$D_{\lambda,f}^1 f(t) = \lim_{h \rightarrow 0^+} \frac{f(t) - e^{-\lambda h} f(t - h)}{h}. \quad (36)$$

For this derivative, the following steps result from the framework.

1. The eigenfunction of this derivative is

$$e_{\nabla,\lambda}(t, s) = e^{(s-\lambda)t}, \quad t \in \mathbb{R}, s \in \mathbb{C}. \quad (37)$$

We could define a modified LT. However, this result expresses only a translation, suggesting that we can use the simple exponential e^{st} , modifying the eigenvalue

$$D_{\lambda,f}^1 e^{st} = (s + \lambda)e^{st}.$$

Therefore, the usual LT is suitable for our objectives.

2. We define the tempered Liouville FD by

$$D_{\nabla}^\alpha e^{st} = (s + \lambda)^\alpha e^{st}, \quad t \in \mathbb{R}, s \in \mathbb{C}^+; \quad (38)$$

3. The differintegrator, $H(s) = (s + \lambda)^\alpha$, has $\text{Re}(s) > -\lambda$ as the ROC. As

$$(s + \lambda)^\alpha = \lim_{h \rightarrow 0^+} \left(\frac{1 - e^{-(s+\lambda)h}}{h} \right)^\alpha, \quad \text{Re}(s) > -\lambda,$$

then

$$D_{\lambda,f}^\alpha f(t) = \lim_{h \rightarrow 0^+} h^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-n\lambda h} f(t - nh). \quad (39)$$

4. The impulse response corresponding to $H(s)$ is

$$h(t) = e^{-\lambda t} \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \varepsilon(t), \quad (40)$$

from which it is not difficult to obtain the integral versions of the tempered Liouville derivatives. The tempered Liouville anti-derivative reads

$$D_{\lambda, f}^{\alpha} f(t) = \int_0^{\infty} f(t-\tau) e^{-\lambda \tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau = e^{-\lambda t} \int_{-\infty}^t f(\tau) e^{\lambda \tau} \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} d\tau. \quad (41)$$

From this relation, it is possible to obtain the regularized, Riemann–Liouville, or Liouville–Caputo derivatives [23].

5.3. Discrete-Time Euler Derivatives

Assume that our domain is the discrete-time sequence $\mathbb{T} = h\mathbb{Z}$, where h is the graininess or sampling interval of the time sequence.

Set $t = nh$, $n \in \mathbb{Z}$. From (3), we obtain the Euler nabla derivative by

$$D_{\nabla} f(t) = \frac{f(t) - f(t-h)}{h}. \quad (42)$$

For this, we have the following considerations:

1. The nabla exponential is given by (A8)

$$e_{\nabla}(nh, s) = (1 - sh)^{-n}, \quad n \in \mathbb{Z}, s \in \mathbb{C}.$$

The corresponding complex sinusoid is obtained when s is over the circle $[1 - sh] = 1$. This defines the so-called right Hilger circle [22,30].

2. The nabla LT is introduced in Appendix A and is given by [33]

$$\mathcal{L}_{\nabla}[f(nh)] = F_{\nabla}(s) = h \sum_{n=-\infty}^{+\infty} f(nh) e_{\nabla}(-nh, s),$$

while its inverse transform is

$$f(nh) = -\frac{1}{2\pi j} \oint_{\gamma} F_{\nabla}(s) e_{\nabla}((n+1)h, s) ds,$$

where the integration path, γ , is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with a center at $s = \frac{1}{h}$.

With the substitution $s = \frac{1-z^{-1}}{h}$, we recover the usual Z transform. Therefore, the associated convolution is (A7).

3. We define the nabla FD as

$$D_{\nabla}^{\alpha} e_{\nabla}(nh, s) = s^{\alpha} e_{\nabla}(nh, s). \quad (43)$$

For non-integer orders, we have to consider a branch-cut line starting at $s = 0$ and lying in the left complex half-plane.

4. Nabla fractional derivative

If $|z| > 1$, then that implies $|1 - hs| < 1$ (s inside the Hilger circle) from (43) and (A7).

$$D_{\nabla}^{\alpha} x(nh) = h^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} x(nh - kh), \quad (44)$$

which is the discrete-time nabla FD, valid for any real (or complex) order.

5. The differintegrator, $H(s) = s^\alpha$, has the Hilger disk as the ROC.

5.4. Tempered Euler Derivatives

These derivatives result from Euler's derivatives by introducing a tempering factor $(\lambda, h \in \mathbb{R}^+)$. For the nabla case, we have

$$D_{\nabla, \lambda}^1 f(nh) = \frac{f(nh) - e^{-\lambda h} f(nh - h)}{h}, \quad n \in \mathbb{Z}. \quad (45)$$

It is a simple matter to see that the eigenfunction reads

$$e_{\nabla, \lambda}(nh, s) = e^{-n\lambda h} (1 - sh)^{-n}, \quad n \in \mathbb{Z}, s \in \mathbb{C}. \quad (46)$$

From this, the procedure for the corresponding transform is easy. It is interesting to state the expression for the FD, which is (from (39))

$$D_{\nabla, \lambda}^\alpha f(nh) = h^{-\alpha} \sum_{m=0}^{\infty} \frac{(-\alpha)_m}{m!} e^{-m\lambda h} f(nh - mh). \quad (47)$$

5.5. Bilinear Shift-Invariant Derivatives

Definition 6. Let us assume again that our domain is the time scale $t \in \mathbb{T} = h\mathbb{Z}$. We define the order-1 forward (nabla) bilinear derivative $\nabla_b x(nh)$ of $x(nh)$ as the solution of the difference equation [25,34]

$$\nabla_b f(nh) + \nabla_b f(nh - h) = \frac{2}{h} [f(nh) - f(nh - h)]. \quad (48)$$

With this definition, we can deduce the following results:

1. The bilinear exponential is [25]

$$e_b(nh, s) = \left(\frac{2 + hs}{2 - hs} \right)^n, \quad n \in \mathbb{Z}, s \in \mathbb{C}. \quad (49)$$

This relation suggests that we consider the discrete-time exponential function, z^n , $n \in \mathbb{Z}$. We define the forward bilinear derivative (∇_b) as an elementary discrete-time system such that

$$\nabla_b z^n = \frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} z^n. \quad (50)$$

Remark 2. This change shows that, with this formulation, it is immaterial whether we work in the plane of the variable s or in the usual "discrete-time" plane of the variable z .

2. Fractional derivative

In agreement with our scheme, the bilinear FD is defined by

$$\nabla_b^\alpha e_b(nh, s) = s^\alpha e_b(nh, s), \quad (51)$$

with a suitable ROC. In terms of the variable z , we can state

$$\nabla_b^\alpha z^n = \left(\frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^\alpha z^n, \quad |z| > 1, \quad (52)$$

a differintegrator with the TF

$$H_b(z) = \left(\frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^\alpha, \quad |z| > 1. \quad (53)$$

For non-integer orders, we again have to consider a branch-cut line. In this case, it is any line that lies inside the unit disk, joining the points $-1, +1$.

3. Z transform

As $z = \frac{2+hs}{2-hs}$ leads to the ZT and such transformation sets the unit circle $|z| = 1$ as the image of the imaginary axis in s , independently of the value of h , we will use the ZT. Once we have defined the derivative of an exponential, we can obtain the derivative of any function that has a ZT. From (51) and (53), we conclude that, if $x(n)$ is a function with the ZT $X(z)$ and analytic in the ROC defined by $z \in \mathbb{C} : |z| > a, a < 1$, then

$$\nabla_b^\alpha x(n) = \frac{1}{2\pi i j} \oint_\gamma \left(\frac{2}{h} \frac{1-z^{-1}}{1+z^{-1}} \right)^\alpha X(z) z^{n-1} dz, \quad (54)$$

with the integration path outside the unit disk.

4. The bilinear differintegrator is given by

$$H_b(z) = \left(\frac{2}{h} \frac{1-z^{-1}}{1+z^{-1}} \right)^\alpha.$$

5. GL-type derivative

It can be shown [25] that, letting $\psi_k^\alpha, k = 0, 1, \dots$, be the inverse ZT of $\left(\frac{1-z^{-1}}{1+z^{-1}} \right)^\alpha$, we define the α -order bilinear nabla derivative by

$$\nabla_b^\alpha x(n) = \left(\frac{2}{h} \right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(n-k), \quad (55)$$

an analog to the GL derivative. The impulse response is obtained as follows. If $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{Z}^+$, then

$$\psi_k^\alpha = (-1)^k \frac{(\alpha)_k}{k!} \sum_{m=0}^k \frac{(-\alpha)_m (-k)_m}{(-\alpha-k+1)_m} \frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+, \quad (56)$$

and

$$\psi_k^{-N} = \frac{(N)_k}{k!} \sum_{m=0}^{\min(k,N)} \frac{(-N)_m (-k)_m}{(-N-k+1)_m} \frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+, \quad (57)$$

when $\alpha = -N, N \in \mathbb{Z}^+$.

5.6. Tempered Bilinear Shift-Invariant Derivatives

Definition 7. In the conditions stated in the previous subsection, we define the order-1 tempered nabla bilinear derivative $\nabla_{\lambda,b} x(nh)$ of $x(nh)$ as the solution of the difference equation [25,34]

$$\nabla_{\lambda,b} f(nh) + e^{-\lambda h} \nabla_{\lambda,b} f(nh-h) = \frac{2}{h} [f(nh) - e^{-\lambda h} f(nh-h)]. \quad (58)$$

The application of our framework is similar to the previous one. We will not describe it here.

6. Main Derivatives on Scale Sequences

6.1. Hadamard Derivatives

Another interesting example of our framework is given by the scale-invariant derivatives. Recall the time sequence from the previous subsection. Let $q \in \mathbb{R}^+$. As seen in Section 2, for any $t \in \mathbb{R}$, if we set $\tau = q^t \in \mathbb{R}^+$, we obtain a scale sequence Θ .

Definition 8. We define a scale-invariant derivative by

$$\mathfrak{D}_s x(\tau) = \lim_{q \rightarrow 1} \frac{x(\tau) - x(\tau q^{-1})}{\log(q)}. \quad (59)$$

If $q > 1$, we call it a stretching derivative, and if $q < 1$, it is a shrinking derivative.

This definition agrees with the general formulae introduced in Section 4 with $\rho(\tau) = \tau q^{-1}$ and $\theta(\tau) = q$.

Our framework gives the following:

1. The eigenfunction is the usual power function [35]

$$e_s(\tau, s) = \tau^s, \quad s \in \mathbb{C}. \quad (60)$$

The condition $\pm \operatorname{Re}(s) > 0$ defines the stretching (+) and shrinking (-) derivatives.

2. The scale FD was introduced by Hadamard (1892) as [7,35]

$$\mathfrak{D}_s^\alpha \tau^s = s^\alpha \tau^s, \quad (61)$$

with a suitable ROC.

3. The transform implied by relation (60) is the usual Mellin transform (MT) (A3). Then, by (A4),

$$\mathfrak{D}_s^\alpha x(\tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^\alpha X(s) \tau^s ds, \quad (62)$$

with $\tau \in \mathbb{R}^+$ and $a > 0$.

4. Differintegrator

Again, the differintegrator has the TF $H(s) = s^\alpha$. We will consider the stretching case in what follows. Therefore, $\operatorname{Re}(s) > 0$ or $s = \pm j\omega, \omega \in \mathbb{R}^+$ (we must note the similarity to the Liouville case).

5. The stretching GL-type derivative [35] is given by

$$\mathfrak{D}_s^\alpha x(\tau) = \lim_{h \rightarrow 1} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(\tau q^{-n}). \quad (63)$$

6. Impulse response [35]

The impulse response is the derivative of an impulse at $\tau = 1, \delta(\tau - 1)$. If $\operatorname{Re}(s) > 0$, we have

$$\mathcal{M}^{-1} s^\alpha = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} u(\tau - 1). \quad (64)$$

7. Hadamard (anti-)derivatives

The impulse response and the Mellin convolution give another representation for the fractional anti-derivative. Let $\alpha < 0$ and $\tau \in \mathbb{R}^+$. The relation stated in (64) used in the Mellin convolution leads to the scale anti-derivative given by

$$\begin{aligned} \mathfrak{D}_s^\alpha x(\tau) &= \frac{1}{\Gamma(-\alpha)} \int_1^\infty x(\tau/\eta) \ln^{-\alpha-1}(\eta) \frac{d\eta}{\eta} \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^\tau x(\eta) \ln^{-\alpha-1}(\tau/\eta) \frac{d\eta}{\eta}. \end{aligned} \quad (65)$$

These relations express what are usually called Hadamard integrals [7,9]. As in the Liouville derivative case, these integrals become singular when the derivative order is positive. To avoid this problem, we can regularize them or adopt Liouville procedures based on the relations $(s^\alpha = s^{\alpha-N} s^N = s^N s^{\alpha-N}, \alpha < N \in \mathbb{N})$ [35].

6.2. Tempered Scale-Invariant Derivatives

Definition 9. Let $\mu > 0$. We define a tempered stretching derivative by [35,36]

$$\mathfrak{D}_{\mu,s}x(\tau) = \lim_{q \rightarrow 1^+} \frac{x(\tau) - q^{-\mu}x(\tau q^{-1})}{\ln q}. \quad (66)$$

This case is analogous to the previous one. With a similar procedure, we obtain the eigenfunction corresponding to the eigenvalue v :

$$e_{\lambda}(\tau, v) = \tau^{-\mu} \tau^v.$$

However, with a translation, we can use τ^v by moving the eigenvalue to $v + \mu$ and using the MT [35]. Therefore, the results obtained in Section 6.1 are easily adaptable. The situation is analogous to the tempered shift-invariant case [35].

6.3. Bilinear Scale-Invariant Derivatives

Definition 10. We can follow the development in Section 5.5 by introducing a bilinear scale derivative as the solution of the equation

$$\mathfrak{D}_{bi,s}x(\tau) + \mathfrak{D}_{bi,s}x(\tau/q) = \frac{2}{\ln(q)}[x(\tau) - x(\tau/q)]. \quad (67)$$

It is not very difficult to show that the bilinear exponential is

$$e_{b,s}(q^n, v) = \left[\frac{2 + v \log(q)}{2 - v \log(q)} \right]^n, \quad n \in \mathbb{Z}, \quad (68)$$

where v is again the eigenvalue. From here, we go on as in the case treated previously (Section 5.5). The larger difference lies in the use of exponential sampling instead of linear sampling. We can go further and introduce the tempered bilinear derivatives.

Definition 11. We define a tempered bilinear scale derivative as the solution of

$$\mathfrak{D}_{bi,\mu,s}x(\tau) + q^{-\mu}\mathfrak{D}_{bi,\mu,s}x(\tau/q) = \frac{2}{\ln(q)}[x(\tau) - q^{-\mu}x(\tau/q)]. \quad (69)$$

The procedure for obtaining the corresponding FD is the same as above.

6.4. Other Fractional Derivatives

6.4.1. q-Derivatives

Besides the anti-causal/shrinking cases, not treated here, the above framework can be used to introduce other fractional derivatives, provided that suitable order-1 derivatives are defined. These can be obtained by mixing different concepts. For instance, the approach above used to introduce the scale-invariant derivatives can give rise to other alternatives, such as the following:

- The use of (59) with the removal of the limit;
- The substitution of $\ln(q)$ by $(1 - q^{-1})\tau$ with or without a limit (quantum derivative) [37,38];
- The introduction of a tempering term, as in (Section 6.2).

Here, we will treat the second case, which is known as a quantum derivative (q-derivative) [37,39].

Definition 12. Let $q > 1$. The nabla q -derivative reads

$$\mathfrak{D}_{q^{-1}}x(t) = \frac{x(t) - x(tq^{-1})}{(1 - q^{-1})t}. \quad (70)$$

Substituting $q \rightarrow q^{-1}$, we obtain the delta q -derivative:

$$\mathfrak{D}_qx(t) = \frac{x(qt) - x(t)}{(q - 1)t}. \quad (71)$$

According to Section 2, we set $\rho(t) = t/q$, $\sigma(t) = qt$, $\mu(t) = (q - 1)t$, $\nu(t) = (1 - q^{-1})t$. These functions originate the following recursions:

1. (a) $\nu_0(t) = 0$;
 (b) $\nu_1(t) = (1 - q^{-1})t$;
 (c) $\nu_n(t) = \nu_{n-1}(t) + \nu(t - \nu_{n-1}(t)) = q^{-1}\nu_{n-1}(t) + (1 - q^{-1})t$, $n = 2, 3, \dots$
 - $\nu_2(t) = (1 - q^{-2})t$;
 - $\nu_3(t) = (1 - q^{-3})t$;
 - $\dots$
 - $\nu_n(t) = (1 - q^{-n})t$, $n = 0, 1, 2, 3, \dots$
2. (a) $\mu_0(t) = 0$;
 (b) $\mu_1(t) = (q - 1)t$;
 (c) $\mu_n(t) = \mu_{n-1}(t) + \mu(t + \mu_{n-1}(t)) = q\mu_{n-1}(t) + (q - 1)t$, $n = 2, 3, \dots$
 - $\mu_2(t) = (q^2 - 1)t$;
 - $\mu_3(t) = (q^3 - 1)t$;
 - $\dots$
 - $\mu_n(t) = (q^n - 1)t$, $n = 0, 1, 2, 3, \dots$

Using these granularity functions and the exponentials and transforms defined in Section 4, we obtain the fractional q -derivatives.

Analogously, we can introduce other scale q -derivatives. We again set $\rho(t) = t/q$ and $\sigma(t) = qt$, but now, $\mu(t) = \nu(t) = \log(q)$.

Definition 13. Let $q > 1$. The stretching q -derivative reads

$$\mathfrak{D}_{q^{-1}}x(t) = \frac{x(t) - x(tq^{-1})}{\log(q)}, \quad (72)$$

while the shrinking q -derivative is

$$\mathfrak{D}_qx(t) = \frac{x(qt) - x(t)}{\log(q)}. \quad (73)$$

Therefore, $\mu_n(t) = \nu_n(t) = \log^n(q)$. We must note the relation with the development introduced in Section 6.1.

6.4.2. (q,h)-Derivatives

Definition 14. Let $h \geq 0$, $q \geq 1$, and $t \in \mathbb{R}$. We define the nabla and delta Hahn derivatives [40,41] by

$$\nabla_{q,h}f(t) = \frac{f(t) - f(t/q - h)}{(1 - q^{-1})t + h}, \quad (1 - q^{-1})t + h \neq 0 \quad (74)$$

and

$$\Delta_{q,h}f(t) = \frac{f(qt + h) - f(t)}{(q - 1)t + h}, \quad (q - 1)t + h \neq 0. \quad (75)$$

According to Section 4, we set $\rho(t) = t/q - h$, $\sigma(t) = qt + h$, $\mu(t) = (q - 1)t + h$, $\nu(t) = (1 - q^{-1})t + h$.

As above, we obtain the following recursions:

1. (a) $\nu_0(t) = 0$;
 (b) $\nu_1(t) = (1 - q^{-1})t + h$;
 (c) $\nu_n(t) = \nu_{n-1}(t) + \nu(t - \nu_{n-1}(t)) = q^{-1}\nu_{n-1}(t) + (1 - q^{-1})t$, $n = 2, 3, \dots$
 - $\nu_2(t) = (1 - q^{-2})t + q^{-1}h$;
 - $\nu_3(t) = (1 - q^{-3})t + q^{-2}h$;
 - $\dots$
 - $\nu_n(t) = (1 - q^{-n})t + q^{-n+1}h$, $n = 1, 2, 3, \dots$
2. (a) $\mu_0(t) = 0$;
 (b) $\mu_1(t) = (q - 1)t + h$;
 (c) $\mu_n(t) = \mu_{n-1}(t) + \mu(t + \mu_{n-1}(t)) = q\mu_{n-1}(t) + (q - 1)t$, $n = 1, 2, 3, \dots$
 - $\mu_2(t) = (q^2 - 1)t + qh$;
 - $\mu_3(t) = (q^3 - 1)t + q^2h$;
 - $\dots$
 - $\mu_n(t) = (q^n - 1)t + q^{n-1}h$, $n = 1, 2, 3, \dots$

Again, using these granularity functions and the exponentials and transforms defined in Section 4, we obtain the Hahn fractional derivatives.

Remark 3. We must note the following:

- With $q = 1$, we obtain the usual Euler nabla and delta derivatives introduced above [34];
- Letting $h = 0$, we obtain the quantum derivatives considered in the previous subsection;
- $\mu(t) = 0$ implies that $q = 1$ and $h = 0$ simultaneously. In such a case, we compute the corresponding limits. We will not do so here since it is similar to the Liouville-type derivative.
- We can define scale (q, h) -derivatives by setting $\mu(t) = \log((q - 1) + h/t)$ and $\nu(t) = \log((1 - q^{-1}) + h/h)$.

7. Fractional Two-Sided Derivatives

In the previous sections, we dealt with one-sided derivatives. In particular, we studied the causal and stretching derivatives. However, two-sided derivatives are useful in many applications [42–45]. The shift-invariant bilateral derivatives have already been studied with generality [46,47]. This has not happened with their scale counterparts. We will not study this here, but the following definition has broad validity.

Definition 15. Let $\alpha, \beta \in \mathbb{R}$. As shown in [20] for the shift-invariant case, a two-sided derivative D_θ^γ can be defined by

$$D_{\alpha-\beta}^{\alpha+\beta} f(t) = \mathcal{L}^{-1} \left[s^\alpha (-s)^\beta F(s) \right], \quad (76)$$

where $\gamma = \alpha + \beta$ is the derivative order, and $\theta = \alpha - \beta$ is an asymmetry parameter and states a unified formulation that includes one-sided derivatives.

An interesting particular case is the following. Let $f(t)$, $t \in \mathbb{R}$, be a real function and $\gamma, \theta \in \mathbb{R}$ be two real parameters. The two-sided Grünwald–Letnikov-type bilateral derivative of $f(t)$ is given by

$$D_\theta^\gamma f(t) = \lim_{h \rightarrow 0^+} h^{-\gamma} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma + 1) f(t - nh)}{\Gamma(\frac{\gamma+\theta}{2} - n + 1) \Gamma(\frac{\gamma-\theta}{2} + n + 1)}. \quad (77)$$

For the scale-invariant case, it is a simple task to define a similar derivative by

$$D_{\theta}^{\gamma} f(t) = \lim_{q \rightarrow 1} \log^{-\gamma}(q) \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma + 1) f(t/q^n)}{\Gamma(\frac{\gamma+\theta}{2} - n + 1) \Gamma(\frac{\gamma-\theta}{2} + n + 1)}. \quad (78)$$

The two-sided derivatives are useful in representing the space variation in partial differential equations.

8. Conclusions

The basic principles underlying the construction of fractional derivatives were introduced and examined. The fundamental idea was the formalization and development of schemes based on the eigenfunctions of simple order-1 derivatives. We showed that the proposed scheme allows us to obtain a unified framework for dealing with continuous/discrete-time shift-invariant or scale-invariant derivatives. This opens many doors for applications. On the other hand, our previous criteria for defining an FD appeared as a consequence of the scheme.

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Abbreviations

The following abbreviations are used in this manuscript:

C	Caputo
FD	Fractional derivative
GL	Grünwald–Letnikov
LT	Laplace transform
MT	Mellin transform
RL	Riemann–Liouville
ROC	Region of convergence

Appendix A. Mathematical Tools

Some interesting mathematical tools are described below:

1. Usual (D'Alembert's or Duhamel's) convolution [48]

$$y(t) = x(t) * g(t) = \int_{-\infty}^{+\infty} x(t - \eta) g(\eta) d\eta.$$

This is suitable for expressing the input/output relation in shift-invariant systems [49].

2. Mellin convolution [35,50]

$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta) \frac{d\eta}{\eta},$$

which is used to express the input/output relation in scale-invariant systems [35].

3. The (bilateral) Laplace transform (LT) is given by [25]

$$\mathcal{L}[h(t)] = H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt, \quad s \in \mathbb{C}, \quad (A1)$$

which is assumed to converge in some non-void region (region of convergence—ROC), which may degenerate into the imaginary axis, giving rise to the Fourier transform (with $s = i\omega$).

4. We define the inverse LT by the Bromwich integral:

$$h(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(s)e^{st} ds, \quad t \in \mathbb{R}, \quad (\text{A2})$$

where $a \in \mathbb{R}$ is called the abscissa of convergence. Frequently, we denote the integration path by γ .

5. In a similar way, we define the Mellin transform (MT) by [35]

$$\mathcal{M}[h(t)] = H(v) = \int_0^\infty h(t)t^{-v-1} dt, \quad v \in \mathbb{C}, \quad (\text{A3})$$

with an inverse similar to (A2)

$$h(t) = \mathcal{M}^{-1}H(v) = \frac{1}{2\pi i} \int_{\gamma} H(v)t^v dv, \quad t \in \mathbb{R}^+. \quad (\text{A4})$$

The MT in (A3) has a parameter sign change $-v \rightarrow v$ relative to the usual MT [50]. However, this definition establishes a better parallelism with the LT concerning the region of convergence.

6. We define the Z transform [31,51] by

$$\mathcal{Z}[f(n)] = F(z) = \sum_{n=-\infty}^{+\infty} f(n)z^{-n}, \quad z \in \mathbb{C}, \quad (\text{A5})$$

with the inverse given by the Cauchy integral

$$f(n) = \frac{1}{2\pi i} \oint_c F(z)z^{n-1} dz, \quad (\text{A6})$$

where c is the unit circle. With the change in the variable $z = e^{i\omega}$, $-\pi < \omega \leq \pi$, we obtain the discrete-time Fourier transform.

7. The convolution associated with the Z transform reads

$$x(n) * y(n) = h \sum_{k=-\infty}^{\infty} x(kh)y(nh - kh). \quad (\text{A7})$$

8. Nabla exponential
This exponential reads

$$e_{\nabla}(nh, s) = (1 - sh)^{-n}, \quad n \in \mathbb{Z}, h \in \mathbb{R}^+, s \in \mathbb{C}. \quad (\text{A8})$$

9. Nabla LT (NLT)
The analysis equation for the NLT is given by

$$\mathcal{N}[f(nh)] = F_{\nabla}(s) = h \sum_{n=-\infty}^{+\infty} f(nh)e_{\nabla}(-nh, s) \quad h \in \mathbb{R}^+, s \in \mathbb{C}. \quad (\text{A9})$$

Its inverse transform (synthesis equation) is given by

$$f(nh) = -\frac{1}{2\pi i} \oint_{\gamma} F_{\nabla}(s)e_{\nabla}((n+1)h, s) ds, \quad n \in \mathbb{Z}, h \in \mathbb{R}^+. \quad (\text{A10})$$

where the integration path, γ , is any simple closed contour in a region of analyticity of $F_{\nabla}(s)$ that includes the point $s = \frac{1}{h}$. The simplest path is a circle with a center at $s = \frac{1}{h}$.

With the substitution $s = \frac{1-z^{-1}}{h}$, we recover the usual Z transform. Therefore, the associated convolution is (A7).

10. Binomial theorem

Let $\alpha \in \mathbb{R} - \mathbb{Z}^-$. Then,

$$(1-z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k, \quad |z| < 1. \quad (\text{A11})$$

We can extend it to negative integer values of α through the Pochhammer symbol for the rising factorial

$$(a)_n = \prod_{k=0}^{n-1} (a+k) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad \text{with } (a)_0 = 1,$$

obtaining

$$(1-z)^\alpha = \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} z^k, \quad |z| < 1. \quad (\text{A12})$$

11. Gamma function

Consider a real number $\alpha \notin \mathbb{Z}_0^-$ and let $N \in \mathbb{N}$ be the greatest integer less than or equal to $-\alpha$. The gamma function is given by [52,53]

$$\Gamma(\alpha) = \int_0^{\infty} \left[e^{-\tau} - \sum_{n=0}^N \frac{(-1)^n}{n!} \tau^n \right] \tau^{\alpha-1} d\tau, \quad (\text{A13})$$

where the summation is null when $N < 0$.

Appendix B. Framework for Fractionalizing Derivatives

In [20], a framework for the fractionalization of derivatives was proposed, intended to deduce derivatives that are suitable for signal processing, namely, to define linear systems. It consists of the following steps:

1. Let $\mathbb{T}$ denote any time or scale sequence and define a generically called *derivative* (see Section 3.2), $\mathcal{D}_f$. We may consider three types: right or stretching, left or shrinking, and bilateral. The treatment of the left/shrinking is similar to the right/stretching. Therefore, we will not consider them unless there is a particular interest. The bilateral will have a different approach that will be described later.
2. For such a derivative, compute the pair eigenvalue/eigenfunction from

$$\mathcal{D}_f e_f(t, s) = s e_f(t, s), \quad (\text{A14})$$

where $t \in \mathbb{T}$ and $s \in \mathbb{C}$. The eigenfunction $e_f(t, s)$ will be called the *generalized exponential*.

3. Define the corresponding fractional derivative, $\mathcal{D}_f^\alpha$, by

$$\mathcal{D}_f^\alpha e_f(t, s) = s^\alpha e_f(t, s), \quad (\text{A15})$$

where $\alpha \in \mathbb{R}$ is the derivative order. The elementary system characterized by the transfer function (TF), $H(s) = s^\alpha$, with a suitable region of convergence, is called the *differintegrator*.

4. With the generalized exponential, introduced above, define a (generalized Laplace) transform,

$$X(s) = \mathcal{L}_f[x(t)],$$

so that the following hold:

- (a) Any function, $x(t)$, can be expressed in terms of the eigenfunction (synthesis equation) by

$$x(t) = \mathcal{L}_f^{-1}X(s) = \frac{1}{2\pi j} \int_{\gamma} X(s)e_f(t,s)ds, \quad (\text{A16})$$

where γ is, in general, a closed simple integration path in a region of the complex plane where $X(s)$ is analytic. We may have to make an adjustment in (A16) so that

$$\mathcal{L}_f[\delta_f(t)] = 1,$$

in the whole complex plane. Relation (A16) defines the inverse transform.

- (b) From (A16) and using (A15), the classic relation

$$\mathcal{L}_f[\mathcal{D}^\alpha x(t)] = s^\alpha X(s) \quad (\text{A17})$$

emerges and is valid in a suitable ROC.

- (c) Let $g(t) = \mathcal{L}_f^{-1}G(s)$ and $x(t) = \mathcal{L}_f^{-1}X(s)$. From (A16), we define a convolution operation by

$$y(t) = g(t) * x(t) = \mathcal{L}_f^{-1}[G(s)X(s)]. \quad (\text{A18})$$

- (d) From (A18), we conclude that the convolution is commutative and, as $s^\alpha G(s)X(s) = G(s)s^\alpha X(s)$, then

$$D_f^\alpha[g(t) * x(t)] = D_f^\alpha g(t) * x(t) = g(t) * D_f^\alpha x(t). \quad (\text{A19})$$

- (e) The analysis equation (direct transform) is defined, for each case, in agreement with the properties of the eigenfunction.

Remark A1. The values $w \in \mathbb{R}$ for which $s = \phi(w)$ leads to $|e_f(t,s)| = 1$ are called frequencies, and $H(w)$ is the frequency response of the differintegrator. In such cases, the generalized LT degenerates into a Fourier transform, and the FD of a sinusoid is a sinusoid. This is very important in applications.

The above definition of the FD operator as an elementary system having a TF equal to $H(s) = s^\alpha$ leads to some properties that are fundamental in applications and are easily deduced from (A16) [32]:

1. Associativity of the orders

$$\mathcal{D}_f^\alpha \mathcal{D}_f^\beta x(t) = \mathcal{D}_f^{\alpha+\beta} x(t); \quad (\text{A20})$$

2. Identity

$$\mathcal{D}_f^0 x(t) = x(t); \quad (\text{A21})$$

3. Inverse

$$\mathcal{D}_f^\alpha \mathcal{D}_f^{-\alpha} x(t) = \mathcal{D}_f^{-\alpha} \mathcal{D}_f^\alpha x(t) = x(t). \quad (\text{A22})$$

Remark A2. These properties, together with the generalized Leibniz rule, constitute the SSC for deciding whether a given elementary system can be considered a fractional derivative, as seen above (Section 3.3) [12]. The main difference lies in the fact that the criterion emerges here as a consequence, without being axiomatically imposed. With the transform introduced above, we can obtain the generalized Leibniz rule, but we will not do so here.

We are going to apply the framework for several useful systems.

Remark A3. It is important to highlight some interesting facts:

1. Although, from a purely mathematical point of view, the derivative order can be any complex number, this cannot be the case from an application point of view, since a complex order leads to non-Hermitian derivatives and systems [31,54].
2. For a noninteger order, s^α represents a multivalued expression. To define a function, we need to introduce a branch-cut line, starting at $s = 0$. The complete location is specified in accordance with the ROC.

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