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Bayesian and E-Bayesian Estimation for a Modified Topp Leone–Chen Distribution Based on a Progressive Type-II Censoring Scheme

Zakiah I. Kalantan ¹ , Eman M. Swielum ², Neama T. AL-Sayed ², Abeer A. EL-Helbawy ^{2,*} ,
Gannat R. AL-Dayian ² and Mervat Abd Elaal ^{3,4} 

¹ Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

² Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo 11865, Egypt; aldayian@yahoo.com (G.R.A.-D.)

³ Department of Statistics, AL-Azhar University, Cairo 11751, Egypt

⁴ Canal High Institute of Engineering and Technology, Suez 43512, Egypt

* Correspondence: abeer@azhar.edu.eg

Abstract: This paper is concerned with applying the Bayesian and E-Bayesian approaches to estimating the unknown parameters of the modified Topp–Leone–Chen distribution under a progressive Type-II censored sample plan. The paper explores the complexities of different estimating methods and investigates the behavior of the estimates through some computations. The Bayes and E-Bayes estimators are obtained under two distinct loss functions, the balanced squared error loss function, as a symmetric loss function, and the balanced linear exponential loss function, as an asymmetric loss function. The estimators are derived using gamma prior and uniform hyperprior distributions. A numerical illustration is given to examine the theoretical results through using the Metropolis–Hastings algorithm of the Markov chain Monte Carlo method of simulation by the R programming language. Finally, real-life data sets are applied to prove the flexibility and applicability of the model.

Keywords: modified Topp–Leone–Chen distribution; progressive censored samples; Bayesian estimation; E-Bayesian estimation; balanced squared error loss function; balanced linear exponential loss function; Markov chain Monte Carlo method



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1. Introduction

Many statistical distributions are considered in the literature for studying and forecasting various real-life data applications in several fields, including biology, engineering, economics, finance, insurance, hydrology, electronic sciences, human mortality, and biological studies. Thus, different lifetime distribution shapes are needed to fit these kinds of lifetime data; however, many standard distributions often do not propose appropriate fits to real-life data sets. Therefore, a need for flexible family of distributions has arisen, and many generalized distributions have been suggested. Hence, several techniques for generating other distributions, generalizing and expanding classical distributions have been proposed to construct flexible distributions than the present ones. Commonly known methods are procedures of differential equations: the Pearson system and Burr system, the method of adding parameters, the method of transformation of variables and distribution functions, probability integral transforms, compound distributions, composite distributions, and finite and infinite mixture distributions.

Composite models based on the *Topp–Leone* (TL) distribution have become popular in actuarial sciences, biology, economics, ecology, and related fields. This paper deals with a composite distribution named the *modified Topp–Leone–Chen* (MTLCh) distribution. It was constructed by composing a *cumulative distribution function* (cdf) of the TL distribution with the cdf of the Chen distribution. This approach adds an additional parameter to produce a new rich and more flexible distribution that can be used for fitting and analyzing data.

The TL distribution is a continuous unimodal distribution that was proposed by [1]. The most important property of TL distribution is that it has bounded support. Hence, it is suitable for fitting a lifetime of distributions with determinate support applications, for instance, limited power supply, maintenance/repair resource, or the design life of the system. It provides closed forms of the *probability density function* (pdf) and the cdf.

The pdf and cdf of the TL distribution are, respectively, given by

$$f(y; \alpha, \theta) = \frac{2\alpha}{\theta} \left(1 - \frac{y}{\theta}\right) \left[2\frac{y}{\theta} - \left(\frac{y}{\theta}\right)^2\right]^{\alpha-1}, \quad 0 < y < \theta; \alpha, \theta > 0, \quad (1)$$

and

$$F(y; \alpha, \theta) = \begin{cases} 0, & y \leq 0, \\ \left[2\frac{y}{\theta} - \left(\frac{y}{\theta}\right)^2\right]^\alpha, & 0 < y < \theta, \\ 1, & y \geq \theta, \end{cases} \quad (2)$$

where α is a shape parameter, θ is a scale parameter, and the pdf in (1) is the J-shaped distribution. For $\theta = 1$, the standard TL distribution is obtained.

The TL distribution has some desirable reliability properties, such as the bathtub-shaped *hazard rate function* (hrf), the decreasing *reversed hazard rate function* (rhrf), the upside-down mean residual life, and the increasing expected inactivity time. This has attracted many authors to study the statistical inference of the TL distribution. For example, ref. [2] proved that the TL distribution has a bathtub failure rate function with common applications in reliability. Some reliability and hazard rate properties were presented by [3] such as the bathtub-shaped hrf, the decreasing rhrf, the upside-down mean residual life, and the increasing expected inactivity time. Also, ref. [4] considered the order statistics from the TL distribution and derived the moments of the order statistics from it. In addition, ref. [5] constructed, via the parameter's estimation, a class of goodness-of-fit tests for the TL distribution. Ref. [6] introduced the TL-generated family of distributions. The authors [7–10] concentrated on the study of the properties of Bayesian and non-Bayesian estimation in their papers.

Although the TL distribution has important reliability properties, this is not enough to analyze various real-life applications. Also, classical distributions are insufficient to cover the increasing need for flexible lifetime distributions due to the rapid development of applied statistical studies, biomedical sciences, engineering, computer sciences, reliability, econometrics, etc. Hence, scientists have extended classical distributions to more flexible distributions that can fit different failure rates and analyze real-life data.

A composition of TL and a cdf, $G(t)$, with positive support, results in a new cdf, which is given by

$$F(t) = H(G(t)) = \left[2G(t) - (G(t))^2\right]^\alpha. \quad (3)$$

The composites between the TL and other distributions have been studied in the literature. For example, ref. [11] derived a Bayesian estimation for the TL Weibull distribution under dual generalized-order statistics. Also, a finite mixture of the two-component TL Rayleigh distribution was obtained by [12]. Ref. [13] introduced the TL inverse Rayleigh distribution and studied some of its essential mathematical properties. Also, ref. [14] presented the TL compound Rayleigh distribution and derived some of its statistical properties. In addition, ref. [15] proposed the Type-II TL inverted Kumaraswamy distribution and studied the main statistical properties of this distribution. Furthermore, they showed through two real lifetime data sets that the proposed distribution fits better than some other distributions, investigating its flexibility and applicability. The TL Marshall Olkin Weibull distribution was introduced by [16].

Recently, ref. [17] proposed the Kies inverted TL distribution with applications to COVID-19 mortality rates in the United Kingdom and Canada. Also, ref. [18] derived the moments of dual generalized-order statistics from the TL-weighted Weibull, along with its description and properties. Ref. [19] obtained a new power TL distribution, with

applications to engineering and industrial data. In addition, ref. [20] introduced a new extension of the TL family of distributions and studied its statistical properties. Also, they obtained a new, extended TL exponential distribution as a special case of this family and estimated the parameters under several non-Bayesian methods. Finally, they provided two real data sets to demonstrate the importance of the family in practice; the data sets contain daily mortality due to COVID-19 in California and New Jersey, USA. Finally, ref. [21] proposed the TL-exponentiated exponential model, which is used in fitting and analyzing claim and hazard data applied in actuarial and insurance research. Furthermore, he studied the main statistical properties of this distribution and used the *maximum likelihood* (ML) estimation method to estimate the unknown parameters of the distribution.

Ref. [22] established a lifetime distribution with two parameters, which has a bathtub-shaped or increasing failure-rate function that allows it to model real lifetime data sets.

The cdf of the Chen distribution is given by

$$G(t; \lambda, b) = 1 - \exp\left[\lambda(1 - e^{tb})\right], \quad t > 0; (\lambda, b > 0), \quad (4)$$

where λ and b are shape parameters. The parameter b governs the shape of hrf for this distribution. The Chen distribution has attracted more attention among scientists due to its capability to model several shapes of the hrf. The study of distributions that have a bathtub hrf originated from various applied research studies such as those observing strength of specific materials, mortality rate, and the latency period of a deadly disease. Some references on the Chen distribution include [23–29].

Let $b = 1$ in (4), and substituting it into (3), the cdf for the *Topp–Leone–Chen* (TLCh) distribution is obtained as follows:

$$F(t) \equiv F(t; \lambda, \alpha) = [1 - \exp[2\lambda(1 - e^t)]]^\alpha, \quad t > 0; (\lambda, \alpha > 0) \quad (5)$$

The pdf corresponding to the cdf given in (5) is as follows:

$$f(t; \lambda, \alpha) = 2\lambda\alpha e^{[t+2\lambda(1-e^t)]} [1 - \exp(2\lambda(1 - e^t))]^{\alpha-1}, \quad t > 0; (\lambda, \alpha > 0). \quad (6)$$

Adding one or more parameters to a distribution yields a new rich and more flexible distribution that can be used to analyze and fit real-life data sets.

Let $X = \frac{T}{\beta}$; then, the pdf and cdf of the MTLCh distribution are, respectively, given by

$$f(x; \lambda, \alpha, \beta) = 2\lambda\alpha\beta e^{[\beta x + 2\lambda(1 - e^{\beta x})]} [1 - \exp(2\lambda(1 - e^{\beta x}))]^{\alpha-1}, \quad x > 0; (\lambda, \alpha, \beta > 0), \quad (7)$$

and

$$F(x; \lambda, \alpha, \beta) = [1 - \exp(2\lambda(1 - e^{\beta x}))]^\alpha, \quad x > 0; (\lambda, \alpha, \beta > 0), \quad (8)$$

where λ, α are shape parameters and β is a scale parameter.

The *reliability function* (rf) and hrf of the MTLCh distribution are, respectively, given by

$$R(x; \lambda, \alpha, \beta) = 1 - [1 - \exp(2\lambda(1 - e^{\beta x}))]^\alpha, \quad x > 0; (\lambda, \alpha, \beta > 0), \quad (9)$$

and

$$h(x; \lambda, \alpha, \beta) = \frac{2\lambda\alpha\beta e^{[\beta x + 2\lambda(1 - e^{\beta x})]} [1 - \exp(2\lambda(1 - e^{\beta x}))]^{\alpha-1}}{1 - [1 - \exp(2\lambda(1 - e^{\beta x}))]^\alpha}, \quad x > 0; (\lambda, \alpha, \beta > 0). \quad (10)$$

Ref. [30] introduced the MTLCh distribution as a composite distribution. They obtained some statistical properties of the proposed distribution such as rf, some stress-strength models, hrf, rhf, a quantile function, mean residual life, mean past lifetime, order statistics, and Renyi entropy. They derived the ML estimators, under progressive Type-II censored samples, for the parameters rf, and hrf. They gave a numerical example to illus-

trate the theoretical results and used two real data sets to demonstrate how the results can be used in practice.

The MTLCh distribution, as a composite distribution, presents a better fit to the data compared to other lifetime distributions, and it is appropriate to use it for modeling lifetime distributions in various fields such as engineering, reliability analysis, and finance. The MTLCh distribution is related to a wide range of well-known distributions by considering special values of the parameters $(\lambda, \alpha, \beta)'$, such as exponentiated Chen, Chen, and standard Chen distributions or through variable transformation of the variable X such as TL-beta Type-II, Kumaraswamy Weibull, Kumaraswamy exponential, Kumaraswamy Rayleigh, exponentiated Weibull, exponentiated exponential, Weibull, Burr Type X, TL-left truncated exponential, left truncated exponential, TL-exponential, Kumaraswamy power function, exponentiated power function, power function, TL-Weibull, TL-exponential, TL-Rayleigh, Rayleigh, TL-F, F, log-logistic, TL-Dagum, Dagum, TL-Burr Type-III, and Burr Type-III distributions. Ref. [31] predicted the future observations from the MTLCh distribution based on progressive Type-II censored scheme. Furthermore, they applied two-sample prediction technique to obtain the ML, Bayesian, and *expected Bayesian* (E-Bayesian) prediction (point and interval) for future order statistics. Finally, they provided a numerical example to illustrate the theoretical results and used an application using real data sets to demonstrate how the results can be used in real life.

There are many situations in life-testing and reliability studies in which the experimenter may be unable to obtain complete information on failure times of all experimental items. There are also situations wherein the removal of items prior to failure is pre-planned to reduce the cost and time associated with testing. Data obtained from such experiments are called censored data. The most common censoring schemes are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing the removal of items at points other than the terminal point of the experiment. For this reason, a more general censoring scheme called progressive Type-II censoring is considered in this paper as a generalization of the Type-II censoring scheme, and it can be briefly described as follows.

Considering n identical units are put to the test, the lifetime distribution of the n units are denoted by $X_1, X_2, \dots, X_n$. The integer $m (< n)$ is fixed at the beginning of the experiment and $R_1, R_2, \dots, R_m$ are m pre-fixed integers satisfying $R_1 + R_2 + \dots + R_m + m = n$. At the time of the first failure $X_{1:m:n}$, R_1 units are chosen randomly from the remaining $n - 1$ units and they are removed from the experiment. Similarly, at the time of the second failure $X_{2:m:n}$, R_2 of the remaining $n - R_1 - 2$ units are removed from the test and so on. Finally, when the m -th failure is observed, the experiment is terminated, and the remaining surviving units R_m with $R_m = n - R_1 - R_2 - \dots - R_{m-1} - m$ are removed. Here, $(R_1, R_2, \dots, R_m)$ is known as the censoring scheme, and it is prefixed before the experiment starts. More detail please see the Figure 1 below:

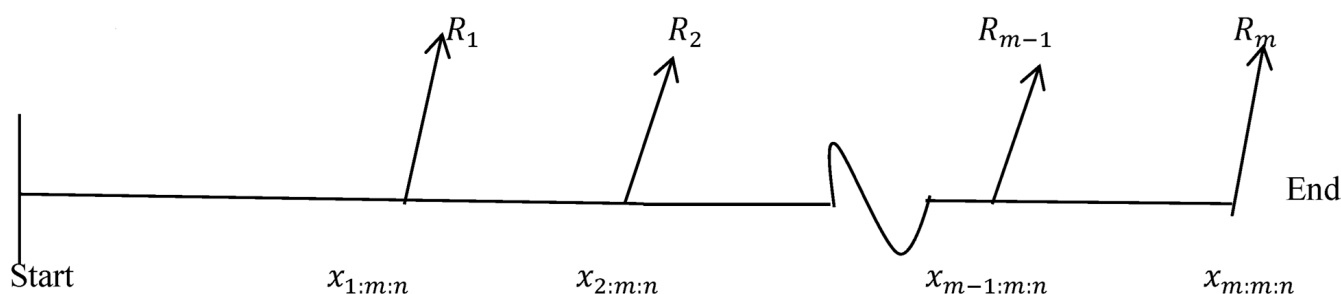


Figure 1. Number of removals.

The progressive Type-II censoring scheme has specific advantages that allow the experimenter to remove survival testing units from the experiment at different testing stages, which makes it more flexible and efficient than conventional Type-I and Type-II censoring schemes. Applying the progressive Type-II censoring scheme reduces the cost

and time for the tests and improves the efficiency of the experiment. Lifetime distributions under progressive Type-II censored scheme have been attracting great interest due to their wide applications in the fields of science, engineering, social sciences, and medicine. For more details, see [32–39].

To make the right decisions and predict a future observation, the distribution's parameters should be estimated accurately. Real-life applications often contain incomplete data. Under different types of censoring, non-Bayesian approaches of estimation like ML estimation might not perform well because they may lead to biased or inefficient estimates. Bayesian method can deal with censored data efficiently. Recently, Bayesian methods and reliability assessment have been studied by [40–42]. In Bayesian estimation approach prior information about the parameters is incorporated, which is beneficial when data are censored.

Ref. [43] introduced the E-Bayesian estimation method, which is very simple, and it is a special Bayesian method used in the area related to the life testing of products with high reliability and small sample size or censored data. E-Bayesian estimation is an extension of Bayesian method that allows one to combine more complex prior distributions with the *likelihood function* (LF) that may not have a closed form. One of the advantages of the E-Bayesian method of estimation is dealing with prior information efficiently with small samples and censored data. Thus, this flexibility can improve the performance of the estimators in certain situations. Many researchers have applied the E-Bayesian method to many distributions, such as [28,44–61].

The results of the Bayesian estimation depend on the choice of the prior distribution and loss function.

Ref. [62] suggested the use of the *balanced loss function* (BLF), which was created by [63], to be of the form

$$L^*(\theta, \theta^*) = \omega l(\theta_0, \theta^*) + (1 - \omega) l(\theta, \theta^*), \quad (11)$$

where $l(\theta, \theta^*)$ is an arbitrary loss function, θ_0 is a chosen target estimator of θ^* , and the weight is $\omega \in [0, 1]$. The BLF specializes in various choices of loss functions, such as the absolute error loss, entropy, *linear exponential* (LINEX), and *squared error loss* (SEL) functions. The Bayes estimator of a function using BLF is a mixture of the non-Bayes estimator (ML estimator, least squares estimator or any other estimator) and the Bayes estimator using any loss function. The Bayes estimator of a function, using *balanced square error loss* (BSEL), is a mixture of the ML estimator and the Bayes estimator, using SEL. This permit, weighing the importance of being close to both the true parameter and θ_0 , is a chosen target estimator.

The Bayes estimator of θ , using the BSEL function is given by

$$\tilde{\theta}_{\text{BSE}} = \omega \hat{\theta}_{\text{ML}} + (1 - \omega) \tilde{\theta}_{\text{SE}}, \quad (12)$$

where $\hat{\theta}_{\text{ML}}$ is the ML estimator of θ , $\tilde{\theta}_{\text{SE}}$ is its Bayes estimator using SEL function, and ω is between 0 and 1.

If $\omega = 0$, the BSEL function acts like the SEL function, aiming for only closeness to the true parameter.

But when $\omega = 1$, it gives more weight to focus on closeness to θ_0 , a chosen target estimator.

Hence, the BSEL function extends more control over the Bayesian estimation process through expanding a method to balance closeness to the true parameter and a chosen target estimator.

The SEL function is one of the most widespread loss functions used in literature. Although the symmetric type of SEL function gives equal weight to the over- and under-estimation of the parameters under consideration, conversely, in life-testing, over-estimation may be more important than under-estimation or vice versa.

The Bayes estimator using the *balanced LINEX loss* (BLL) function of θ is obtained as follows:

$$\tilde{\theta}_{\text{BL}} = \frac{-1}{v} \ln \{ \omega \exp(-v \hat{\theta}_{\text{ML}}) + (1 - \omega) E(\exp(-v \theta) | \underline{x}) \}, \quad (13)$$

where $v \neq 0$ is the shape parameter of BLL function. In this paper, the Bayes estimators are obtained using the BSEL and BLL functions.

The rest of this paper is organized as follows. In Section 2, the Bayes estimators of the parameters of the MTLCh distribution based on the BSEL and BLL functions are obtained. The E-Bayes estimators of the parameters based on the BSEL and BLL functions are discussed in Section 3. A Monte Carlo simulation study and two applications are given to illustrate the theoretical results in Section 4. A general conclusion is given in Section 5.

2. Bayesian Estimation

One of the key advantages of the Bayesian method is its ability to calculate uncertainty, since comprehensive information in real-life application is rare. In the Bayesian approach, the prior distribution of the parameter combines with the LF, which results in the posterior distribution reflecting the updated certainty about the parameter after combining the prior information and the observed data. The posterior distribution does not give a point estimator for the parameter only; it shows the entire range of possible values with their probabilities to quantify the uncertainty associated with the estimate. This is important for making decisions in several fields such as science, engineering, economics, and finance.

In this section, the Bayes estimators of the unknown parameters of the MTLCh distribution based on progressive Type-II censored sample are derived under two different loss functions, the BSEL function; as a symmetric loss function, and the BLL function; as an asymmetric loss function.

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ denote a progressive Type-II censored sample obtained from the MTLCh (λ, α, β) distribution. The LF is given by

$$L(\theta|\underline{x}) = C(n, m-1) \prod_{i=1}^m f(x_{(i)}; \theta) \left[1 - F(x_{(i)}; \theta)\right]^{R_i}, \quad (14)$$

where $\theta = (\lambda, \alpha, \beta)'$, $\underline{x} = (x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n})$ denotes an observed value of $\underline{X} = (X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n})$ and $C(n, m-1) = n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-\dots-R_{m-1}-m+1)$, with $C(n, 0) = n$. Then, substituting (7) and (8) into (14) yields

$$\begin{aligned} L(\theta; \underline{x}) &= C(n, m-1) \prod_{i=1}^m \left[2\lambda\alpha\beta e^{[\beta x_{(i)} + 2\lambda(1-e^{\beta x_{(i)}})]} \right] \\ &\quad \times \prod_{i=1}^m \left[1 - \exp\left(2\lambda\left(1 - e^{\beta x_{(i)}}\right)\right) \right]^{\alpha-1} \\ &\quad \times \prod_{i=1}^m \left\{ 1 - \left[1 - \exp\left(2\lambda\left(1 - e^{\beta x_{(i)}}\right)\right) \right]^\alpha \right\}^{R_i}. \end{aligned} \quad (15)$$

To be easier to obtain the Bayes estimators, the LF given by (15) can be written as follows:

$$\begin{aligned} L(\theta|\underline{x}) &\propto (\lambda\alpha\beta)^m \exp\left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)}\right] \\ &\quad \times \exp\left[(\alpha-1) \sum_{i=1}^m \ln(1-z_{(i)}) + \sum_{i=1}^m R_i \ln\{1 - [1-z_{(i)}]^\alpha\}\right], \end{aligned} \quad (16)$$

where

$$z_{(i)} = \exp\left[2\lambda\left(1 - e^{\beta x_{(i)}}\right)\right]. \quad (17)$$

The natural logarithm of LF in (16) is given by

$$\begin{aligned} \ell &\propto m \ln \lambda + m \ln \alpha + m \ln \beta + \beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} \\ &\quad + (\alpha-1) \sum_{i=1}^m \ln[1-z_{(i)}] + \sum_{i=1}^m R_i \ln\left[1 - (1-z_{(i)})^\alpha\right]. \end{aligned} \quad (18)$$

The ML estimators of the parameters $\theta = (\lambda, \alpha, \beta)'$ can be obtained by differentiating (18) with respect to λ, α , and β and then setting it to zero. Hence,

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} = & \frac{m}{\lambda} + \sum_{i=1}^m (2 - 2e^{\beta x_{(i)}}) - (\alpha - 1) \sum_{i=1}^m \frac{\dot{z}_{(i)}(\lambda)}{[1 - z_{(i)}]} \\ & + \alpha \sum_{i=1}^m R_i \frac{\dot{z}_{(i)}(\lambda) (1 - z_{(i)})^{\alpha-1}}{[1 - (1 - z_{(i)})^\alpha]}, \end{aligned} \quad (19)$$

$$\text{where } \dot{z}_{(i)}(\lambda) = \frac{\partial z_{(i)}}{\partial \lambda} = z_{(i)} (2 - 2e^{\beta x_{(i)}}),$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^m \ln[1 - z_{(i)}] - \sum_{i=1}^m R_i \frac{\ln[1 - z_{(i)}] (1 - z_{(i)})^\alpha}{[1 - (1 - z_{(i)})^\alpha]}, \quad (20)$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} = & \frac{m}{\beta} + \sum_{i=1}^m x_{(i)} - 2\lambda \sum_{i=1}^m x_{(i)} e^{\beta x_{(i)}} - (\alpha - 1) \sum_{i=1}^m \frac{\dot{z}_{(i)}(\beta)}{[1 - z_{(i)}]} \\ & + \alpha \sum_{i=1}^m R_i \frac{\dot{z}_{(i)}(\beta) (1 - z_{(i)})^{\alpha-1}}{[1 - (1 - z_{(i)})^\alpha]}, \end{aligned} \quad (21)$$

$$\text{where } \dot{z}_{(i)}(\beta) = \frac{\partial z_{(i)}}{\partial \beta} = -2\lambda x_{(i)} z_{(i)} e^{\beta x_{(i)}}.$$

The ML estimators are obtained by equating the derivatives (19)–(21) to zeros. The system of non-linear equations can be solved numerically using Newton–Raphson method, to obtain the ML estimates of the parameters λ, α , and β .

Consider the fact that the prior knowledge of the vector of parameters, $\underline{\theta} = (\lambda, \alpha, \beta)'$, is adequately represented by gamma distribution with parameters a_j and b_j and pdf as follows:

$$\pi(\theta_j; a_j, b_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j-1} \exp(-b_j \theta_j), \quad \theta_j > 0; (a_j, b_j > 0), \quad j = 1, 2, 3, \quad (22)$$

where $\theta_1 = \lambda$, $\theta_2 = \alpha$ and $\theta_3 = \beta$, a_j and b_j are the hyper-parameters of the prior distribution.

Assume that the parameters, $\underline{\theta} = (\lambda, \alpha, \beta)'$, are unknown and independent. Then, the joint prior distribution of all the unknown parameters has a joint pdf given by

$$\pi(\underline{\theta}; \underline{a}, \underline{b}) \propto \lambda^{a_1-1} \alpha^{a_2-1} \beta^{a_3-1} \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)], \quad \underline{\theta} > \underline{0}; (\underline{a}, \underline{b} > \underline{0}). \quad (23)$$

Combining the LF in (16) and the joint prior distribution given by (23), then the joint posterior distribution of the parameters, $\underline{\theta} = (\lambda, \alpha, \beta)'$ can be obtained as follows:

$$\begin{aligned} \pi(\underline{\theta} | \underline{x}) = & K L(\underline{\theta} | \underline{x}) \pi(\underline{\theta}; \underline{a}, \underline{b}) \\ = & K \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} \right] \\ & \times \exp \left[(\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) + \sum_{i=1}^m R_i \ln \left\{ 1 - [1 - z_{(i)}]^\alpha \right\} \right] \\ & \times \exp[-(b_1 \lambda + b_2 \alpha + b_3 \beta)], \end{aligned} \quad (24)$$

where K is a normalizing constant,

$$\begin{aligned} K^{-1} = & \int_{\underline{\theta}} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} \right] \\ & \times \exp \left[(\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) + \sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} \right] \\ & \times \exp\{-(b_1 \lambda + b_2 \alpha + b_3 \beta)\} d\underline{\theta} \end{aligned} \quad (25)$$

where

$$\int_{\underline{\theta}} = \int_{\lambda} \int_{\alpha} \int_{\beta} \quad \text{and} \quad d\underline{\theta} = d\beta \, d\alpha \, d\lambda. \quad (26)$$

The marginal posterior distributions of the parameters $\underline{\theta} = (\lambda, \alpha, \beta)'$ are obtained from (24) as follows:

$$\pi(\theta_\ell|\underline{x}) = \int_{\theta_j} \pi(\underline{\theta}|\underline{x}) d\theta_j, \quad \ell \neq j, \ell, j = 1, 2, 3. \quad (27)$$

2.1. Bayes Estimators under Balanced Squared Error Loss Function

The Bayes estimators of the parameters under the BSEL function can be obtained from (12) and (24) as given below:

$$\begin{aligned} \tilde{\theta}_{jBBSE} &= \omega \hat{\theta}_{jML} + (1 - \omega) \int_{\underline{\theta}} \theta_j \pi(\underline{\theta}|\underline{x}) d\theta \\ &= \omega \hat{\theta}_{jML} + (1 - \omega) K \int_{\underline{\theta}} \theta_j \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \\ &\quad \times \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) \right] \\ &\quad \times \exp \left[\sum_{i=1}^m R_i \ln \left\{ 1 - [1 - z_{(i)}]^\alpha \right\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\theta, \\ &\quad \theta_j > 0, j = 1, 2, 3, \end{aligned} \quad (28)$$

where $\theta_1 = \lambda$, $\theta_2 = \alpha$ and $\theta_3 = \beta$, $\hat{\theta}_{jML}$ is the estimator of θ_j using the ML method based on (19)–(21), $z_{(i)}$ is given by (17), K^{-1} is given by (25), and $\int_{\underline{\theta}}$ and $d\theta$ are given by (26).

The Bayes estimates of λ , α and β under BSEL function can be calculated numerically by utilizing the Metropolis–Hastings algorithm of the *Markov chain Monte Carlo* (MCMC) method of simulation using the R programming language.

2.2. Bayes Estimators under Balanced Linear Exponential Loss Function

The Bayes estimators of the parameters under the BLL function can be obtained from (13) and (24) as follows:

$$\begin{aligned} \tilde{\theta}_{jBBL} &= \frac{-1}{v} \ln \left\{ \omega \exp(-v \hat{\theta}_{jML}) + (1 - \omega) \int_{\underline{\theta}} \exp(-v \theta_j) \pi(\underline{\theta}|\underline{x}) d\theta \right\} \\ &= \frac{-1}{v} \ln \left\{ \omega \exp(-v \hat{\theta}_{jML}) + (1 - \omega) K \int_{\underline{\theta}} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \right. \\ &\quad \times \exp \left[-v \theta_j + \beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) \right] \\ &\quad \times \exp \left[\sum_{i=1}^m R_i \ln \left\{ 1 - [1 - z_{(i)}]^\alpha \right\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\theta \left. \right\}, \\ &\quad \theta_j > 0, j = 1, 2, 3, \end{aligned} \quad (29)$$

where $\theta_1 = \lambda$, $\theta_2 = \alpha$ and $\theta_3 = \beta$, $\hat{\theta}_{jML}$ is the estimator of θ_j using the ML method based on (19)–(21), $z_{(i)}$ is given by (17), K^{-1} is given by (25), and $\int_{\underline{\theta}}$ and $d\theta$ are given by (26).

The Bayes estimates of λ , α and β under the BLL function can be obtained numerically by applying the Metropolis–Hastings algorithm of the MCMC method of simulation using the R programming language.

The Bayesian method is insufficient for addressing the inference problem. Although the Bayesian approach is an effective tool to obtain the Bayes estimators, the results are sensitive to the selected data and the prior distribution, which sometimes lead to unexpected shapes for the posterior distribution. Though the advantages of the E-Bayesian method of estimation are dealing with prior information and efficiency with small samples and censored data, one should be careful when selecting the prior distribution, and there is a need to be aware when applying different numerical methods to obtain the E-Bayes estimates. E-Bayesian estimation is considered in the following section.

3. E-Bayesian Estimation

In this section, the E-Bayes estimators for the parameters of the MTLCh distribution based on progressive Type-II censored sample are derived under two different loss functions, the BSEL function; as a symmetric loss function, and the BLL function; as an asymmetric loss function.

According to ref. [43], the hyper-parameters a_j and b_j should be selected to guarantee that $\pi(\theta_j; a_j, b_j)$, given in (22), can be decreasing functions of θ_j ($j = 1, 2, 3$).

The derivative of $\pi(\theta_j; a_j, b_j)$ with respect to θ_j is given below

$$\frac{d \pi(\theta_j; a_j, b_j)}{d\theta_j} = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j-2} \exp(-b_j\theta_j) [(a_j - 1) - b_j\theta_j], \quad j = 1, 2, 3, \quad (30)$$

for $0 < a_j < 1$ and $b_j > 0$, then $\frac{d\pi(\theta_j; a_j, b_j)}{d\theta_j} < 0$, which means that $\pi(\theta_j; a_j, b_j)$ can be decreasing functions of θ_j .

The E-Bayes estimators of the parameters are obtained based on three different distributions of the hyper-parameters a_j and b_j . These distributions are used to investigate the influence of different prior distributions on the E-Bayesian estimation of θ_j .

Assuming that the hyper-parameters a_j and b_j are independent with bivariate density functions

$$\pi_h(a_j, b_j) = \pi_h(a_j) \pi_h(b_j), \quad j = 1, 2, 3, \quad h = 1, 2, \dots, 9. \quad (31)$$

Then, the bivariate uniform hyperprior distributions are

$$\pi_h(a_j, b_j) = \frac{2(c_j - b_j)}{c_j^2}, \quad 0 < a_j < 1, \quad 0 < b_j < c_j, \quad (32)$$

$$\pi_h(a_j, b_j) = \frac{1}{c_j}, \quad 0 < a_j < 1, \quad 0 < b_j < c_j, \quad (33)$$

and

$$\pi_h(a_j, b_j) = \frac{2b_j}{c_j^2}, \quad 0 < a_j < 1, \quad 0 < b_j < c_j. \quad (34)$$

The E-Bayes estimators of θ_j (expectation of the Bayes estimators of θ_j) can be derived as follows:

$$\tilde{\theta}_{jEB} = E\pi_h(\tilde{\theta}_{jB}(a_j, b_j)) = \iint_D \tilde{\theta}_{jB}(a_j, b_j) \pi_h(a_j, b_j) da_j db_j, \quad j = 1, 2, 3, \quad h = 1, 2, \dots, 9, \quad (35)$$

where $E\pi_h(h = 1, 2, \dots, 9)$ stands for the expectation of the bivariate hyperprior distributions and $\tilde{\theta}_{jB}(a_j, b_j)$ are the Bayes estimators of the parameters θ_j based on the BSEL and BLL functions.

3.1. E-Bayes Estimators under the Balanced Squared Error Loss Function

The three E-Bayes estimators of the parameter θ_j under the BSEL function can be obtained by substituting (28) and (32)–(34) into (35) as, respectively, given below:

$$\begin{aligned} \tilde{\theta}_{jEBBS1} = & \frac{2}{c_j^2} \int_0^1 \int_0^{c_j} \left\{ \omega \hat{\theta}_{jML} + (1 - \omega) K \int_{\underline{\theta}} \theta_j \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \right. \\ & \times \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) \right] \\ & \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\theta \left. \right\} (c_j - b_j) db_j da_j, \end{aligned} \quad (36)$$

$$\begin{aligned} \tilde{\theta}_{jEBBS2} = & \frac{1}{c_j} \int_0^1 \int_0^{c_j} \left\{ \omega \hat{\theta}_{jML} + (1 - \omega) K \int_{\underline{\theta}} \theta_j \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \right. \\ & \times \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln(1 - z_{(i)}) \right] \\ & \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\theta \left. \right\} db_j da_j, \end{aligned} \quad (37)$$

and

$$\begin{aligned}\tilde{\theta}_{jEBBS3} = & \frac{2}{c_j^2} \int_0^1 \int_0^{c_j} b_j \left\{ \omega \hat{\theta}_{jML} + (1 - \omega) K \int_{\underline{\theta}} \theta_j \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \right. \\ & \times \exp \left[\beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln (1 - z_{(i)}) \right] \\ & \left. \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\underline{\theta} \right\} db_j da_j,\end{aligned}\quad (38)$$

where $j = 1, 2, 3$, $\theta_1 = \lambda$, $\theta_2 = \alpha$, $\theta_3 = \beta$, $\hat{\theta}_{jML}$ is the estimator of θ_j using the ML method based on (19)–(21), $z_{(i)}$ is given by (17), K^{-1} is given by (25), and $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given by (26).

The three E-Bayes estimators of the parameter λ under the BSEL function can be obtained by substituting $j = 1$ into (36)–(38); similarly, the three E-Bayes estimators of the parameter α under the BSEL function can be derived by substituting $j = 2$ into (36)–(38). Also, the three E-Bayes estimators of the parameter β under the BSEL function can be obtained by substituting $j = 3$ in (36)–(38).

The three E-Bayes estimates of each λ , α , and β under the BSEL function can be obtained numerically by applying the Metropolis–Hastings algorithm of MCMC method of simulation through the R programming language.

3.2. E-Bayes Estimation under the Balanced Linear Exponential Loss Function

The E-Bayes estimators of the parameter θ_j under BLL function can be obtained by substituting (29) and (32)–(34) into (35) as, respectively, given below

$$\begin{aligned}\tilde{\theta}_{jEBBL1} = & \frac{2}{c_j^2} \int_0^1 \int_0^{c_j} \frac{-1}{v} \ln \{ \omega \exp(-v \hat{\theta}_{jML}) + (1 - \omega) K \int_{\underline{\theta}} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \\ & \times \exp \left[-v \theta_j + \beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln (1 - z_{(i)}) \right] \\ & \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\underline{\theta} \} (c_j - b_j) db_j da_j,\end{aligned}\quad (39)$$

$$\begin{aligned}\tilde{\theta}_{jEBBL2} = & \frac{1}{c_j} \int_0^1 \int_0^{c_j} \frac{-1}{v} \ln \{ \omega \exp(-v \hat{\theta}_{jML}) + (1 - \omega) K \int_{\underline{\theta}} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \\ & \times \exp \left[-v \theta_j + \beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln (1 - z_{(i)}) \right] \\ & \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\underline{\theta} \} db_j da_j,\end{aligned}\quad (40)$$

and

$$\begin{aligned}\tilde{\theta}_{jEBBL3} = & \frac{2}{c_j^2} \int_0^1 \int_0^{c_j} \frac{-1}{v} \ln \{ \omega \exp(-v \hat{\theta}_{jML}) + (1 - \omega) K \int_{\underline{\theta}} \lambda^{m+a_1-1} \alpha^{m+a_2-1} \beta^{m+a_3-1} \\ & \times \exp \left[-v \theta_j + \beta \sum_{i=1}^m x_{(i)} + \sum_{i=1}^m \ln z_{(i)} + (\alpha - 1) \sum_{i=1}^m \ln (1 - z_{(i)}) \right] \\ & \times \exp \left[\sum_{i=1}^m R_i \ln \{1 - [1 - z_{(i)}]^\alpha\} - (b_1 \lambda + b_2 \alpha + b_3 \beta) \right] d\underline{\theta} \} b_j db_j da_j,\end{aligned}\quad (41)$$

where $\theta_1 = \lambda$, $\theta_2 = \alpha$ and $\theta_3 = \beta$, $\hat{\theta}_{jML}$ is the estimator of θ_j using the ML method based on (19)–(21), $z_{(i)}$ is given by (17), K^{-1} is given by (25), $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given by (26).

Substituting $j = 1$ into (39)–(41), the three E-Bayes estimators of the parameter λ under the BLL function can be obtained; similarly, the three E-Bayes estimators of the parameter α under the BLL function can be obtained by substituting $j = 2$ in (39)–(41). Also, for $j = 3$ in (39)–(41), the three E-Bayes estimators of the parameter β can be obtained.

The three E-Bayes estimates of each λ , α , and β under BLL function can be computed numerically by utilizing the Metropolis–Hastings algorithm of MCMC method of simulation using the R programming language.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian and E-Bayesian estimation based on simulated and real data sets.

4.1. Simulation Study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes and E-Bayes estimates based on the data generated from the MTLCh distribution. All simulation studies are performed using R programming language.

4.2. Simulation Algorithm

Step 1: Generate m independent $U(0, 1)$ random variables $U_1, U_2, \dots, U_m$.

Step 2: Generate a_j and b_j from the bivariate uniform hyperprior distributions; $\pi_h(a_j, b_j)$, $j = 1, 2, 3$, $h = 1, 2, \dots, 9$, given in (32)–(34).

Step 3: For given values of a_j and b_j , generate λ, α and β from the gamma prior distributions.

By applying the algorithm in [64], the following steps are used to generate a progressive Type-II censored sample from the MTLCh distribution as follows:

Step 4: For given values of the progressive censoring scheme $R_1, R_2, \dots, R_m$, set

$$Y_i = U_i^{1/(i + \sum_{j=m-i+1}^m R_j)}, \quad \text{for } i = 1, 2, \dots, m.$$

Step 5: Set $UP_i = 1 - (Y_m Y_{m-1} Y_{m-2} \dots Y_{m-i+1})$, $i = 1, 2, \dots, m$. Then, $UP_1, UP_2, \dots, UP_m$ are progressive Type-II censored sample of size m from $U(0, 1)$ distribution.

Step 6: For given values of the parameters λ, α , and β , the inverse cdf method can be used to generate m progressive Type-II censored sample from the MTLCh distribution whose cdf is given by (8). Thus, by solving the nonlinear equation

$$x_i = \frac{1}{\beta} \ln \left[1 - \frac{1}{2\lambda} \ln \left(1 - (UP_i)^{\frac{1}{\alpha}} \right) \right], \quad i = 1, 2, \dots, m,$$

the resulting set, $(x_1, x_2, \dots, x_m)$, is the required progressive Type-II censored sample of size m from the MTLCh distribution, and this obtained sample is ordered.

Step 7: Calculate the Bayes and E-Bayes estimates of the parameters based on the BSEL and BLL functions.

Step 8: Repeat all the previous steps $N = 10,000$ times for the samples of size ($n = 30, 60, 100, 200$ and 500). For each sample size, there is a different ratio of effective sample sizes, $r = \frac{m}{n} = 0.2, 0.5$ and 0.8 and a set of different sample schemes, where

Scheme I: $R_1 = n - m$ and $R_2 = R_3 = \dots = R_m = 0$.

Scheme II: $R_1 = R_2 = \dots = R_{m-1} = 1$ and $R_m = n - 2m + 1$, for $r = 0.2$ and 0.5 . If $r = 0.8$, then $R_{i+1} = 1$ and $R_j = 0$; $i = 0, 4, 8, 12, \dots, m$, $i \neq j$.

Scheme III: $0, 0, \dots, 0, \frac{n-m}{3}, 0, 0, \dots, 0, \frac{n-m}{3}, 0, 0, \dots, 0, \frac{n-m}{3}$.

- The best scheme is the scheme that minimizes the *estimated risk* (ER).

Evaluating the performance of the estimates is considered through the measurement of accuracy. To study the precision and variation of the estimates, it is convenient to use the $ER = \frac{\sum_{i=1}^N (\text{estimated value} - \text{true value})^2}{N}$.

Tables A1–A6 display the Bayes and E-Bayes ERs of the unknown parameters under the BSEL and BLL functions based on progressive Type-II censoring under different sample schemes, while Tables A11 and A12 present the Bayes and E-Bayes ERs of the unknown parameters under the BSEL and BLL functions based on progressive Type-II censoring under different sample schemes and for different weights ω , where $\omega = 0, 0.4, 0.7$ and 1 .

4.3. Some Applications

This subsection aims to demonstrate how the proposed methods can be used in practice. Two real-life data sets are used for this purpose. The MTLCh distribution is fitted to the two real data sets using the Kolmogorov–Smirnov goodness-of-fit test through Mathematica 9.

Application 1:

The first application was given by [65]; the data refer to the time between the failures of a repairable item. It is important to evaluate the reliability and availability of a system or component. It indicates the average amount of operating time an item can expect to function before facing a failure that needs fixing. This helps with maintenance planning, since recognizing the average time between failures allows for maintenance to be scheduled, which can minimize failures and make sure that the system operates effectively. It is important to improve the product design and maintenance plans.

The data that refer to the time between failures for a repairable item are 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, and 1.17.

Application 2:

The second data set was presented by [66]. The experiment was fundamentally testing the dielectric strength of the insulating fluid. Dielectric strength is the maximum voltage a material can resist before it conducts electricity. This experiment concerns measuring the time it takes for the fluid to become conductive. The experiment likely measured the time it takes for the insulating fluid to fail (break down) when exposed to a constant voltage of 34 kilovolts (kV). This voltage is the electrical stress put on the fluid.

The data refer to the time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, and 72.89.

The Kolmogorov–Smirnov goodness-of-fit test is applied to check the validity of the introduced fitted model. The p values are given, respectively, as 0.9988 and 0.5379. The p value given in each case showed that the proposed model fits the data very well.

The three progressive censoring schemes considered in the simulation study were applied for the real data sets. The results of the Bayesian and E-Bayesian *standard error* (SE) of the unknown parameters for the real data sets under the BSEL and BLL functions based on the progressive Type-II censoring under different sample schemes are displayed in Tables A7–A10.

Remark 1. When the scheme $R_1 = R_2 = \dots = R_{m-1} = 0$ and $R_m = n - m$ is applied, all the results obtained in this paper based on progressive Type-II censoring under for different weights ω reduce to those of Type-II censored samples.

4.4. Concluding Remarks

From Tables A1–A12, one can observe the following.

- Tables A1–A6 show that the accuracy of the ER gets better when the sample size increases, which ensures that the estimates are consistent and approach the true parameter values as the sample size increases.
- Tables A11 and A12 show that the accuracy of the ER becomes better when the weight ω decreases.
- Tables A1–A6 show that the ERs decrease as the ratio of effective samples of size (r) increases, which is expected since increasing the ratio of effective samples of size (r) means that more information is provided, and hence the accuracy of the estimates increases.
- Since the best scheme is the scheme that minimizes the ERs, the best scheme is Scheme I when $n = 30, 60$ and 100 , but Scheme II is the best when $n = 200$ and 500 .
- The ERs of the E-Bayes estimates of the parameters are in most cases less than the ERs of the Bayes estimates; hence, the E-Bayes estimators act better than the Bayes estimators.
- The ERs of the Bayes and E-Bayes estimates under the BLL function are in most cases less than the ERs of Bayes and E-Bayes estimates under the BSEL function, so the Bayes and E-Bayes estimators under the BLL function perform better than the Bayes and E-Bayes estimators under the BSEL function.
- Tables A11 and A12 present the Bayes and E-Bayes ERs of the unknown parameters under the BSEL and BLL functions based on the progressive Type-II censoring under

different sample schemes and for different weights ω , where $\omega = 0, 0.4, 0.7$, and 1 . When $\omega = 1$, one obtains the ML estimates, while when $\omega = 0$, one obtains the Bayes estimates under the SEL or LINEX loss functions.

5. General Conclusions

In this research, the Bayes and E-Bayes estimators are compared for the parameters of the MTLCh distribution based on progressive Type-II censored samples. The estimators are considered under two different loss functions, the BSEL function, as a symmetric loss function, and the BLL function, as an asymmetric loss function. The BLF is a mixture of Bayesian and non-Bayesian estimators. The estimators are obtained based on the conjugate gamma prior and uniform hyperprior distributions. A numerical example is given to illustrate the theoretical results, and an application using real data sets is used to demonstrate how the results can be used in practice. In general, the numerical computations showed that the ERs of the E-Bayes estimates of the parameters are in most cases less than the ERs of the Bayes estimates, so the E-Bayes estimators perform better than the Bayes estimators. The ERs of the Bayes and E-Bayes estimates under the BLL function are in most cases less than the ERs of the Bayes and E-Bayes estimates under the BSEL function, so the Bayes and E-Bayes estimators under the BLL function perform better than the Bayes and E-Bayes estimators under the BSEL function. Also, the numerical computations showed that the E-Bayes estimates are more efficient than the corresponding Bayes estimates.

Some suggestions for future research that can be carried as follows: studying different Bayesian estimation methods such as a new modified artificial bee colony algorithm for an energy demand forecasting problem and an adaptive search-equation-based artificial bee colony algorithm for transportation energy demand forecasting; considering other methods of estimation such as modified ML method or modified moments; also, deriving interval estimation in the future. The Bayesian and E-Bayesian methods may also be studied based on different types of loss functions (symmetric or asymmetric), such as general entropy and precautionary loss functions, for estimating the parameters of the MTLCh distribution. Other types of progressive censoring schemes can be considered to estimate parameters such as adaptive Type-II progressive censoring and Type-II stepwise progressive censoring schemes. This study can be generalized to generalized-order statistics or dual generalized-order statistics and subsequently specialized to record statistics. Empirical Bayes methods can be utilized to study Bayesian and E-Bayesian estimation for the MTLCh distribution and compare the results with the results obtained in this paper. Optimal accelerated life testing model for the MTLCh distribution may also be derived.

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Appendix A

Table A1. Estimated risks of the MTLCh parameters under a balanced squared error loss function based on progressive Type-II censoring under Scheme I ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$ and $\omega = 0.3$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	0.8852	0.2191	0.6934	0.1548	0.3658	0.1784
			0.1664		0.6053		0.3499
			0.7955		0.0961		0.1814
	15	0.3927	0.0351	0.1001	0.0842	0.1001	0.0848
			0.0933		0.0603		0.0852
			0.137		0.0986		0.0785
60	24	0.0417	0.0191	0.0511	0.0294	0.0862	0.011
			0.0078		0.023		0.0797
			0.0264		0.0049		0.0236
	12	0.0647	0.0497	0.1849	0.0602	0.1375	0.0422
			0.0143		0.0898		0.0596
			0.0239		0.0823		0.0322
100	30	0.0331	0.0183	0.0579	0.0051	0.0996	0.0039
			0.0101		0.0196		0.0223
			0.0132		0.0134		0.0229
	48	0.0062	0.0023	0.0235	0.0024	0.0201	0.0011
			0.0031		0.0009		0.0016
			0.001		0.0021		0.0038
200	20	0.0435	0.0383	0.04	0.021	0.0407	0.0193
			0.0051		0.0254		0.0088
			0.0323		0.01		0.0158
	50	0.0277	0.0015	0.0197	0.0005	0.0116	0.0032
			0.0022		0.0079		0.0046
			0.0131		0.0024		0.003
500	80	0.0019	4.54×10^{-4}	0.0034	6.45×10^{-5}	0.0001	6.03×10^{-5}
			9.22×10^{-5}		1.91×10^{-4}		6.36×10^{-5}
			2.54×10^{-4}		8.48×10^{-4}		3.60×10^{-4}
	40	0.0194	0.0052	0.0132	0.0017	0.0262	0.0052
			0.005		0.0018		0.0011
			0.0051		0.0039		0.0037
1000	100	0.0066	0.0011	0.0074	0.0003	0.0039	0.0015
			0.001		0.0017		0.0009
			0.0011		0.0017		0.0002
	160	0.0013	3.78×10^{-5}	0.0002	1.32×10^{-5}	0.0001	2.99×10^{-5}
			1.01×10^{-5}		1.49×10^{-5}		2.63×10^{-5}
			5.65×10^{-5}		6.95×10^{-5}		1.69×10^{-5}
2000	100	0.0071	0.0031	0.006	0.0002	0.0048	0.0029
			0.0016		0.0012		0.0007
			0.0014		0.0013		0.0006
	250	0.0032	0.0006	0.0006	0.0001	0.0025	0.0004
			0.0004		0.0001		0.0003
			0.0011		0.0001		0.0002
5000	400	5.30×10^{-5}	3.15×10^{-5}	1.38×10^{-5}	2.51×10^{-6}	7.34×10^{-5}	9.53×10^{-6}
			7.05×10^{-6}		4.39×10^{-6}		3.75×10^{-6}
			3.76×10^{-5}		1.25×10^{-5}		2.96×10^{-5}

Table A2. Estimated risks of the MTLCh parameters under the balanced linear exponential loss function based on progressive Type-II censoring under Scheme I ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$, $\omega = 0.3$ and $v = -2$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	0.4982	0.1475 0.2841 0.1795	0.4359	0.2095 0.288 0.2339	1.4624	0.3616 0.3053 1.4408
	15	0.2414	0.099 0.073 0.1147	0.1088	0.0477 0.0499 0.077	0.9667	0.3132 0.1629 0.342
	24	0.0174	0.0019 0.0034 0.0143	0.0537	0.0104 0.0216 0.0042	0.0176	0.0053 0.0125 0.0156
60	12	0.3132	0.0447 0.0411 0.0518	0.2362	0.0485 0.1958 0.0853	0.9656	0.0392 0.0188 0.1234
	30	0.0196	0.0113 0.0068 0.0141	0.0241	0.0178 0.0146 0.0106	0.0778	0.0038 0.0031 0.0073
	48	0.0104	0.001 0.0013 0.0034	0.0049	0.0002 0.001 0.0021	0.0043	0.0015 0.0005 0.0004
100	20	0.0429	0.0052 0.0018 0.0146	0.0137	0.0019 0.002 0.0032	0.0206	0.002 0.0018 0.0042
	50	0.0055	0.0018 0.0018 0.0016	0.004	0.0001 0.0039 0.0018	0.0167	0.0012 0.001 0.0048
	80	0.0011	1.61×10^{-5} 1.18×10^{-4} 3.52×10^{-4}	0.0008	2.35×10^{-4} 5.10×10^{-5} 2.46×10^{-4}	0.0006	2.78×10^{-5} 4.57×10^{-5} 4.92×10^{-5}
200	40	0.0019	0.0005 0.0012 0.001	0.0014	0.0013 0.0013 0.0011	0.0021	0.0015 0.0015 0.0007
	100	0.0011	2.57×10^{-4} 9.08×10^{-4} 8.66×10^{-4}	0.0006	0.0001 0.0002 0.0004	0.0014	0.0006 0.0004 0.0004
	160	0.0003	7.44×10^{-6} 2.09×10^{-5} 6.07×10^{-5}	0.0002	2.12×10^{-5} 4.25×10^{-5} 1.34×10^{-5}	0.0001	1.65×10^{-5} 4.00×10^{-5} 4.67×10^{-5}
500	100	0.0017	0.0004 0.0001 0.0005	0.0013	0.0012 0.0002 0.0002	0.0014	2.63×10^{-4} 8.40×10^{-5} 2.10×10^{-4}
	250	0.0007	8.61×10^{-5} 7.20×10^{-5} 4.26×10^{-4}	0.0005	7.64×10^{-5} 9.19×10^{-5} 2.85×10^{-5}	0.0003	5.85×10^{-5} 1.65×10^{-5} 7.48×10^{-5}
	400	1.82×10^{-5}	1.66×10^{-6} 1.26×10^{-5} 2.73×10^{-6}	2.57×10^{-5}	2.11×10^{-6} 7.32×10^{-6} 5.86×10^{-6}	5.25×10^{-5}	4.26×10^{-6} 1.42×10^{-5} 4.61×10^{-6}

Table A3. Estimated risks of the MTLCh parameters under the balanced squared error loss function based on progressive Type-II censoring under Scheme II ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$ and $\omega = 0.3$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	1.6465	1.049 0.2886 0.1747	0.9666	0.3109 0.3572 0.0967	0.8193	0.2436 0.6947 0.143
	15	0.5443	0.0421 0.0953 0.0328	0.3286	0.067 0.0247 0.0557	0.2468	0.0534 0.0594 0.0555
	24	0.2637	0.047 0.0117 0.0238	0.1868	0.0163 0.0059 0.0053	0.0533	0.0228 0.0044 0.0283
60	12	0.116	0.0385 0.081 0.0999	0.3138	0.2853 0.1099 0.0607	0.2321	0.03 0.1182 0.0544
	30	0.0493	0.0111 0.0119 0.0158	0.0772	0.0488 0.0131 0.0464	0.0846	0.0056 0.0142 0.0134
	48	0.0186	0.006 0.0011 0.0021	0.0273	0.0073 0.003 0.0011	0.0095	0.0029 0.0039 0.0032
100	20	0.1093	0.0042 0.0011 0.0007	0.0062	0.0021 0.0019 0.0021	0.1673	0.006 0.0076 0.0082
	50	0.0314	0.0022 0.0011 0.0003	0.0027	0.0012 0.0003 0.001	0.0154	0.0023 0.0013 0.0081
	80	0.0031	0.0021 0.0005 0.0003	0.0005	3.45×10^{-4} 1.54×10^{-4} 4.96×10^{-5}	0.0018	0.0003 0.0001 0.0005
200	40	0.0043	0.0007 0.0006 0.0005	0.0049	0.0014 0.0015 0.0018	0.0087	0.0012 0.0022 0.002
	100	0.0008	1.13×10^{-4} 1.01×10^{-4} 1.75×10^{-4}	0.0018	1.65×10^{-4} 1.75×10^{-4} 2.64×10^{-4}	0.004	2.17×10^{-4} 7.42×10^{-4} 2.41×10^{-4}
	160	0.0005	2.55×10^{-5} 4.54×10^{-5} 6.77×10^{-5}	0.0001	3.93×10^{-5} 4.76×10^{-5} 1.04×10^{-5}	0.0001	8.46×10^{-5} 2.76×10^{-5} 6.44×10^{-5}
500	100	0.0023	2.12×10^{-4} 1.88×10^{-4} 1.96×10^{-4}	0.0015	6.01×10^{-4} 7.07×10^{-4} 1.12×10^{-4}	0.0008	1.14×10^{-4} 3.98×10^{-4} 1.27×10^{-4}
	250	0.0006	7.31×10^{-5} 2.05×10^{-5} 7.12×10^{-5}	0.0002	2.68×10^{-5} 2.95×10^{-5} 2.23×10^{-5}	0.0001	3.13×10^{-5} 1.64×10^{-4} 1.44×10^{-5}
	400	2.17×10^{-5}	1.23×10^{-5} 1.00×10^{-6} 8.67×10^{-6}	5.28×10^{-5}	1.13×10^{-5} 2.65×10^{-5} 9.94×10^{-6}	2.34×10^{-5}	6.12×10^{-6} 8.00×10^{-6} 5.92×10^{-6}

Table A4. Estimated risks of the MTLCh parameters under the balanced linear exponential loss function based on progressive Type-II censoring under Scheme II ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$, $\omega = 0.3$ and $v = -2$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	0.5972	0.0207 0.3606 0.0759	0.4633	0.0185 0.315 0.0274	0.1152	0.0825 0.0963 0.0218
	15	0.1763	0.0091 0.0182 0.0204	0.1872	0.0123 0.0352 0.0083	0.0451	0.0251 0.0239 0.0127
	24	0.0244	0.0053 0.0022 0.0115	0.034	0.001 0.002 0.0028	0.0309	0.0011 0.0047 0.0023
60	12	0.0641	0.0097 0.0101 0.0218	0.0237	0.0092 0.0069 0.0084	0.1091	0.0076 0.0879 0.0212
	30	0.0221	0.0031 0.0026 0.004	0.0146	0.0102 0.0049 0.0016	0.0262	0.0012 0.0019 0.0023
	48	0.0076	0.0006 0.0007 0.0006	0.0085	0.0006 0.0011 0.0008	0.003	0.0003 0.0007 0.0016
100	20	0.0135	0.0014 0.0026 0.0053	0.003	0.0013 0.0012 0.0009	0.0132	0.0028 0.0028 0.0021
	50	0.0042	9.12×10^{-4} 1.22×10^{-3} 1.41×10^{-3}	0.0021	3.22×10^{-4} 5.29×10^{-5} 1.60×10^{-4}	0.0019	6.44×10^{-4} 4.17×10^{-5} 2.19×10^{-4}
	80	0.0013	5.00×10^{-6} 1.41×10^{-5} 6.40×10^{-4}	0.0002	6.73×10^{-5} 4.29×10^{-5} 3.25×10^{-5}	0.0004	4.92×10^{-5} 3.31×10^{-5} 2.61×10^{-5}
200	40	0.0034	0.0014 0.0014 0.0006	0.0012	0.0006 0.0006 0.0005	0.0064	0.0018 0.0012 0.0011
	100	0.0009	4.30×10^{-4} 1.40×10^{-4} 4.26×10^{-4}	0.0001	1.15×10^{-4} 2.00×10^{-5} 1.11×10^{-4}	0.0005	3.87×10^{-4} 2.79×10^{-5} 6.15×10^{-5}
	160	6.22×10^{-5}	3.22×10^{-6} 9.10×10^{-6} 1.91×10^{-5}	5.62×10^{-5}	2.27×10^{-5} 1.18×10^{-5} 2.48×10^{-5}	2.74×10^{-4}	1.49×10^{-6} 1.05×10^{-5} 2.66×10^{-5}
500	100	0.0005	1.47×10^{-4} 1.75×10^{-4} 5.09×10^{-4}	0.0006	5.59×10^{-4} 7.73×10^{-5} 7.74×10^{-5}	0.0007	1.17×10^{-4} 3.73×10^{-5} 9.35×10^{-5}
	250	2.92×10^{-5}	2.67×10^{-5} 6.13×10^{-6} 2.81×10^{-5}	9.15×10^{-5}	3.60×10^{-5} 1.00×10^{-5} 3.14×10^{-5}	8.35×10^{-5}	1.31×10^{-5} 1.65×10^{-5} 3.11×10^{-5}
	400	8.10×10^{-6}	2.32×10^{-7} 1.92×10^{-7} 1.57×10^{-6}	2.84×10^{-5}	2.89×10^{-7} 2.33×10^{-7} 1.65×10^{-6}	9.53×10^{-6}	1.54×10^{-7} 2.17×10^{-6} 3.60×10^{-7}

Table A5. Estimated risks of the MTLCh parameters under the balanced squared error loss function based on progressive Type-II censoring under Scheme III ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$ and $\omega = 0.3$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	1.4995	1.1904	0.4667	0.48	0.469	0.1234
			0.127		0.4247		0.1849
			0.2904		0.2884		0.1724
	15	0.2842	0.1619	0.1642	0.0652	0.213	0.1048
			0.0794		0.1692		0.0413
			0.1638		0.0872		0.0586
60	24	0.1194	0.0693	0.0961	0.0362	0.0898	0.054
			0.0229		0.0083		0.0412
			0.0218		0.003		0.0404
	12	0.4135	0.0329	0.1831	0.067	0.0977	0.0505
			0.042		0.0838		0.08
			0.1202		0.149		0.0345
100	30	0.0858	0.0246	0.0482	0.046	0.0547	0.0337
			0.0238		0.0237		0.0102
			0.0637		0.0384		0.0188
	48	0.0087	0.0068	0.0203	0.0139	0.011	0.0032
			0.0012		0.0073		0.0069
			0.0031		0.008		0.0081
200	20	0.0592	0.0135	0.126	0.0252	0.0429	0.0288
			0.0126		0.0298		0.0131
			0.0446		0.0139		0.0335
	50	0.0095	0.0015	0.0193	0.0011	0.0113	0.0024
			0.0017		0.0086		0.0038
			0.007		0.0061		0.0031
500	80	0.0016	0.0006	0.004	0.0006	0.0007	0.0002
			0.0001		0.0027		0.0003
			0.0012		0.0002		0.0006
	40	0.0087	0.0042	0.03	0.0017	0.0131	0.011
			0.0041		0.0036		0.002
			0.0041		0.005		0.0028
1000	100	0.0023	0.0002	0.0019	0.001	0.0046	0.0004
			0.0006		0.0011		0.0001
			0.0009		0.0002		0.0005
	160	0.0005	1.22×10^{-4}	0.0008	0.0001	0.0004	1.80×10^{-4}
			1.33×10^{-4}		0.0002		6.83×10^{-5}
			1.75×10^{-4}		0.0002		3.72×10^{-4}
2000	100	0.0054	0.0006	0.0016	0.0002	0.0048	0.002
			0.0005		0.0006		0.0005
			0.0005		0.0015		0.0004
	250	0.0018	6.53×10^{-5}	0.0004	5.65×10^{-5}	0.0019	1.93×10^{-4}
			1.96×10^{-4}		4.93×10^{-5}		1.35×10^{-4}
			6.62×10^{-5}		5.86×10^{-5}		1.07×10^{-4}
5000	400	0.0003	4.93×10^{-5}	6.86×10^{-5}	7.03×10^{-6}	0.0002	1.54×10^{-5}
			3.45×10^{-5}		1.19×10^{-6}		1.86×10^{-5}
			3.26×10^{-5}		8.51×10^{-6}		2.54×10^{-5}

Table A6. Estimated risks of the MTLCh parameters under the balanced linear exponential loss function based on progressive Type-II censoring under Scheme III ($N = 10,000$, $r = 0.2, 0.5$ and 0.8 , $\lambda = 2.3$, $\alpha = 1.5$, $\beta = 1.1$, $\omega = 0.3$ and $v = -2$).

n	m	ER		ER		ER	
		$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
30	6	0.8596	0.6496 0.1263 0.1174	0.2991	0.1583 0.0978 0.1938	0.3863	0.1395 0.1202 0.3279
	15	0.2688	0.059 0.0502 0.0431	0.1603	0.0339 0.0442 0.1453	0.0904	0.0438 0.0281 0.0623
	24	0.0732	0.0351 0.0118 0.0122	0.0322	0.0259 0.0103 0.0222	0.0461	0.043 0.0119 0.0264
60	12	0.0809	0.0257 0.0166 0.0434	0.1596	0.0855 0.0436 0.117	0.1285	0.0298 0.0325 0.0115
	30	0.0385	0.0215 0.0102 0.0301	0.032	0.0227 0.0179 0.0234	0.0212	0.0296 0.0079 0.0132
	48	0.0047	0.0046 0.001 0.0031	0.0123	0.0114 0.0098 0.011	0.0066	0.006 0.0017 0.0069
100	20	0.0484	0.0141 0.011 0.0277	0.0388	0.024 0.0157 0.0353	0.0174	0.007 0.0032 0.0039
	50	0.0132	0.0034 0.0041 0.0014	0.0111	0.0041 0.0054 0.0025	0.0034	0.0014 0.0004 0.0028
	80	0.0029	6.72×10^{-4} 6.04×10^{-4} 8.14×10^{-4}	0.002	6.74×10^{-4} 1.40×10^{-4} 1.03×10^{-4}	0.0004	2.62×10^{-5} 2.98×10^{-4} 1.73×10^{-4}
200	40	0.0035	0.002 0.0015 0.003	0.007	0.002 0.0018 0.0018	0.0064	0.0024 0.0016 0.0016
	100	0.0016	0.0006 0.0005 0.0009	0.0007	0.0003 0.0004 0.0004	0.0018	0.0006 0.0001 0.0001
	160	0.0004	1.02×10^{-4} 2.55×10^{-5} 5.09×10^{-5}	0.0004	2.12×10^{-5} 4.25×10^{-5} 1.34×10^{-5}	0.0002	1.65×10^{-5} 4.00×10^{-5} 4.67×10^{-5}
500	100	0.0018	0.0006 0.0001 0.0002	0.001	0.0002 0.0001 0.0008	0.0049	0.0005 0.0005 0.0007
	250	0.001	8.61×10^{-5} 7.20×10^{-5} 1.10×10^{-4}	0.0004	4.85×10^{-5} 4.29×10^{-5} 5.86×10^{-5}	0.0005	1.62×10^{-5} 2.04×10^{-5} 3.83×10^{-5}
	400	3.26×10^{-5}	6.46×10^{-6} 1.78×10^{-6} 6.90×10^{-6}	4.93×10^{-5}	9.45×10^{-6} 1.40×10^{-6} 2.46×10^{-6}	6.10×10^{-6}	2.03×10^{-6} 2.59×10^{-6} 1.74×10^{-6}

Table A7. Standard errors of the MTLCh parameters for the real data sets (Application 1) under the balanced squared error loss function based on progressive Type-II censoring under different sample schemes.

Scheme	n	m	SE		SE		SE	
			$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
Scheme I	30	6	0.0796	0.0188	0.1107	0.0373	0.0749	0.0318
				0.0573		0.0485		0.0445
				0.0364		0.0675		0.0702
		15	0.0717	0.0160	0.0981	0.0315	0.0700	0.0274
				0.0493		0.0412		0.0398
				0.0319		0.0566		0.0596
Scheme II	30	6	0.0535	0.0109	0.0721	0.0208	0.0567	0.0184
				0.0326		0.0280		0.0270
				0.0215		0.0377		0.0381
		15	0.0551	0.0312	0.0776	0.0482	0.0598	0.0346
				0.0328		0.0431		0.0395
				0.0246		0.0347		0.0482
Scheme III	30	6	0.0418	0.0205	0.0592	0.0321	0.0525	0.0249
				0.0223		0.0295		0.0285
				0.0164		0.0232		0.0335
		15	0.0736	0.0415	0.1043	0.038	0.0966	0.0304
				0.0511		0.0842		0.0493
				0.0448		0.0675		0.0799
Scheme I	30	6	0.0544	0.0169	0.0530	0.0309	0.0824	0.0194
				0.0130		0.0178		0.0333
				0.0282		0.0346		0.0276
		15	0.0178	0.0078	0.0311	0.0051	0.0242	0.0141
				0.0035		0.0089		0.0034
				0.0089		0.0100		0.0079

Table A8. Standard errors of the MTLCh parameters for the real data sets (Application 1) under the balanced linear exponential loss function based on progressive Type-II censoring under different sample schemes.

Scheme	n	m	SE		SE		SE	
			$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
Scheme I	30	6	0.0384	0.0192	0.053	0.0404	0.0576	0.0552
				0.0267		0.0462		0.0307
				0.0316		0.0219		0.0307
		15	0.0347	0.0134	0.0487	0.0179	0.0410	0.0357
				0.0188		0.0399		0.0412
				0.0221		0.0178		0.0229
Scheme II	30	6	0.0230	0.0079	0.0278	0.0029	0.0373	0.0154
				0.0087		0.0083		0.0114
				0.0035		0.0082		0.0049
		15	0.0517	0.0500	0.0668	0.0580	0.0457	0.0307
				0.0326		0.0244		0.0423
				0.0311		0.0432		0.0302

Table A8. Cont.

Scheme	n	m	SE		SE		SE	
			$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
Scheme II	30	15	0.0446	0.0416 0.0272 0.0259	0.0501	0.0396 0.0160 0.0289	0.0412	0.0256 0.0353 0.0251
		24	0.0203	0.015 0.0199 0.0120	0.0272	0.0164 0.0109 0.0215	0.0285	0.0154 0.0214 0.0150
Scheme III	30	6	0.0677	0.0478 0.048 0.0272	0.0661	0.0589 0.0200 0.0439	0.0893	0.0542 0.0735 0.0363
		15	0.0316	0.0218 0.0135 0.0151	0.0406	0.0148 0.0122 0.0207	0.0276	0.0158 0.0210 0.0166
		24	0.0202	0.0072 0.0099 0.0059	0.0271	0.0083 0.0059 0.0106	0.0154	0.0066 0.0077 0.0087

Table A9. Standard errors of the MTLCh parameters for the real data sets (Application 2) under the balanced squared error loss function based on progressive Type-II censoring under different sample schemes.

Scheme	n	m	SE		SE		SE	
			$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
Scheme I	19	4	0.0783	0.04 0.0416 0.0578	0.0896	0.0474 0.0518 0.0534	0.0983	0.04 0.0559 0.0882
		10	0.0498	0.0171 0.0176 0.0251	0.0475	0.0345 0.0173 0.0158	0.0443	0.0402 0.0413 0.0363
		15	0.0143	0.0116 0.0111 0.0100	0.0214	0.0119 0.0147 0.0116	0.0293	0.0205 0.0205 0.0174
Scheme II	19	4	0.0755	0.0526 0.0549 0.0404	0.1091	0.0803 0.0718 0.0567	0.0734	0.0494 0.0452 0.0724
		10	0.0611	0.0324 0.0343 0.0261	0.0880	0.0505 0.0458 0.0361	0.0619	0.0353 0.0318 0.0515
		15	0.0242	0.0215 0.0160 0.0227	0.0584	0.0119 0.0255 0.0212	0.0379	0.0141 0.0126 0.0195
Scheme III	19	4	0.0682	0.0389 0.0418 0.0306	0.1237	0.0399 0.0521 0.0723	0.0900	0.0343 0.0479 0.0756
		10	0.0492	0.0198 0.0241 0.0272	0.0414	0.0227 0.0291 0.025	0.0570	0.0194 0.0160 0.0309
		15	0.0334	0.0114 0.0117 0.0167	0.0397	0.0119 0.0170 0.0244	0.0499	0.0152 0.0098 0.0137

Table A10. Standard errors of the MTLCh parameters for the real data sets (Application 2) under the balanced linear exponential loss function based on progressive Type-II censoring under different sample schemes.

Scheme	n	m	SE		SE		SE	
			$\tilde{\lambda}_B$	$\tilde{\lambda}_{EB}$	$\tilde{\alpha}_B$	$\tilde{\alpha}_{EB}$	$\tilde{\beta}_B$	$\tilde{\beta}_{EB}$
Scheme I	19	4	0.0705	0.0222	0.0544	0.0439	0.0649	0.0528
				0.0408		0.0393		0.0366
				0.0284		0.0491		0.0645
		10	0.0436	0.0084	0.0261	0.0187	0.0325	0.0257
				0.0168		0.0141		0.0111
				0.0095		0.0174		0.012
		15	0.0082	0.0044	0.0120	0.0113	0.0104	0.0074
				0.0048		0.0047		0.0087
				0.0079		0.0050		0.0100
Scheme II	19	4	0.0468	0.0365	0.0383	0.0365	0.0493	0.0261
				0.0221		0.0327		0.0364
				0.0254		0.0327		0.0458
		10	0.0194	0.0182	0.0304	0.0112	0.0263	0.0222
				0.0113		0.025		0.0261
				0.0126		0.0112		0.0141
		15	0.0144	0.0100	0.0174	0.0037	0.0144	0.0050
				0.0111		0.0103		0.0078
				0.0044		0.0104		0.0089
Scheme III	19	4	0.0397	0.0364	0.0604	0.0225	0.0539	0.0450
				0.0224		0.0495		0.0517
				0.0253		0.0221		0.0290
		10	0.0155	0.0133	0.0320	0.0084	0.0399	0.0205
				0.0151		0.0143		0.0124
				0.0104		0.0104		0.0142
		15	0.0111	0.0104	0.0250	0.0081	0.0109	0.0098
				0.0082		0.0075		0.0096
				0.0084		0.0055		0.0105

Table A11. Estimated risks of the MTLCh parameters under the balanced squared error loss function based on progressive Type-II censoring ($N = 10,000$, $n = 60$, $r = 0.5$, $\lambda = 2.3$, $\alpha = 1.5$ and $\beta = 1.1$).

Scheme	ϑ	$\omega = 0$ "BSEL"	$\omega = 0.4$ "SEL"	$\omega = 0.7$ "SEL"	$\omega = 1$ "ML"
Scheme I	λ	0.0886	0.0491	0.089	0.0269
			0.0247	0.0134	0.0224
			0.0051	0.017	0.0745
	α	0.0337	0.0299	0.0712	0.018
			0.0173	0.0075	0.1452
			0.0209	0.0464	0.0642
	β	0.0414	0.0224	0.0055	0.0895
			0.0176	0.0273	0.0841
			0.0390	0.0176	0.0188
Scheme II	λ	0.0213	0.0093	0.0614	0.1543
			0.0149	0.0076	0.0711
			0.0132	0.0793	0.0143
	α	0.0243	0.0035	0.0299	0.083
			0.0047	0.0173	0.0479
			0.0179	0.0209	0.0616

Table A11. Cont.

Scheme	$\underline{\theta}$		$\omega = 0$ "BSEL"		$\omega = 0.4$ "SEL"		$\omega = 0.7$ "SEL"		$\omega = 1$ "ML"
Scheme II	β	0.0131	0.0033	0.0667	0.0224	0.0806	0.0628	0.1592	0.0904
			0.0033		0.0176		0.0500		0.0720
			0.0065		0.0390		0.1190		0.1715
Scheme III	λ	0.0505	0.0218	0.0785	0.0381	0.1809	0.1017	0.5197	0.1077
			0.0110		0.0183		0.0465		0.0877
			0.0020		0.0182		0.0533		0.3027
	α	0.0187	0.0139	0.0581	0.0158	0.2345	0.016	0.809	0.0712
			0.0078		0.0336		0.0801		0.5858
			0.0090		0.0891		0.2126		0.2668
	β	0.0237	0.0101	0.0689	0.0108	0.3592	0.1362	1.4068	0.5148
			0.0079		0.0405		0.1265		0.4436
			0.0167		0.0170		0.0300		0.9382

Table A12. Estimated risks of the MTLCh parameters under the linear exponential loss function based on progressive Type-II censoring ($N = 10,000$, $n = 60$, $r = 0.5$, $\lambda = 2.3$, $\alpha = 1.5$ and $\beta = 1.1$).

Scheme	$\underline{\theta}$		$\omega = 0$ "LINEX"		$\omega = 0.4$ "BLL"		$\omega = 0.7$ "BLL"		$\omega = 1$ "ML"
Scheme I	λ	0.0529	0.0061	0.0822	0.0292	0.108	0.0287	0.3543	0.0780
			0.0188		0.0307		0.0590		0.1383
			0.0165		0.0730		0.1295		0.3007
	α	0.0393	0.0183	0.2151	0.0137	0.2165	0.2048	0.413	0.0279
			0.0071		0.0152		0.0347		0.0950
			0.0175		0.0246		0.0406		0.1910
Scheme II	β	0.0667	0.0393	0.0795	0.0061	0.1261	0.0853	0.5352	0.1920
			0.0359		0.0477		0.0709		0.1609
			0.0089		0.0112		0.0702		0.1587
	λ	0.0099	0.0010	0.0305	0.0126	0.0484	0.0224	0.2708	0.1145
			0.0020		0.0075		0.0127		0.1303
			0.0058		0.0845		0.1604		0.2007
Scheme III	α	0.0159	0.0030	0.0578	0.0176	0.09	0.0311	0.1741	0.0484
			0.0018		0.0307		0.0521		0.0817
			0.0047		0.0156		0.0282		0.042
	β	0.0145	0.0109	0.0712	0.0160	0.129	0.0273	0.1493	0.0111
			0.0072		0.0126		0.0237		0.0883
			0.0062		0.0039		0.0074		0.0215
Scheme III	λ	0.0368	0.0086	0.0527	0.0265	0.3355	0.0498	0.5731	0.0995
			0.0139		0.030		0.0569		0.2005
			0.0338		0.0613		0.1331		0.4459
	α	0.0348	0.0313	0.0656	0.0318	0.1495	0.0542	0.2989	0.0905
			0.0051		0.0213		0.0366		0.1663
			0.0078		0.0316		0.0563		0.0872
	β	0.0192	0.0099	0.0314	0.0224	0.2593	0.159	0.5711	0.4736
			0.0067		0.015		0.1704		0.5081
			0.0121		0.0263		0.1539		0.4875

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