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Revisit Birkhoff's Theorem: The Post-Newtonian Metric of a Self-Gravitating and Collapsing Thin Spherical Shell

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Abstract: We calculate the metric of a self-gravitating and collapsing infinitely thin spherical shell in the weak-field and slow-motion limits, and we demonstrate that Birkhoff's theorem is not consistent with the theory of the post-Newtonian approximation. More importantly, it is illustrated that performing a coordinate transformation in solving Einstein field equations may change the matter energy-momentum tensor, making the resultant solution not correspond to the original problem.

Keywords: Birkhoff's theorem; coordinate transformation; reference frame; post-Newtonian approximation; second gravitational potential; gravitomagnetic field

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1. Introduction

Birkhoff derives the metric for the spacetime of any spherically symmetric system and arrives at the conclusion that spacetime is static and is characterized by the Schwarzschild metric, regardless of whether the system is dynamic or not [1]. This result is called Birkhoff's theorem. The corollary of Birkhoff's theorem also leads to "the metric inside an empty spherical cavity at the center of a spherically symmetric system must be equivalent to the flat-space Minkowski metric $\eta_{\mu\nu}$", even when the spherically symmetric body is not static [2]. This is similar to the well-known result that the gravitational field in the empty space inside a spherically symmetric body is zero in the Newton theory, following the Gauss theorem. However, Birkhoff's theorem and its corollary are counter-intuitive to general relativity, in which the gravitational source's motion induces a gravitomagnetic field and other relativistic effects in general.

In a previous work [3], we have shown that similar to the rotational motion, the radial motion of a spherically symmetric system may also produce a gravitomagnetic field. In order to fully demonstrate the incorrectness of Birkhoff's theorem, we consider the system of a massive and pressure-less infinitely thin spherical shell, which collapses under self-gravity. In fact, the spacetime of the thin spherical shell has been explored by several authors [4–7]. For example, in *Problem Book in Relativity and Gravitation* by Lightman et al. [5], Ex. 16.3 and Ex. 21.11 are dedicated to the thin spherical shell. However, all the starting points in the literature are either Birkhoff's theorem or the specific form of the spherically symmetric metric, such as [6]

$$ds^2 = -\frac{dt^2}{\tilde{F}^2(t,r)} + \tilde{X}^2(t,r)dr^2 + \tilde{Y}^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

which does not have the time-space cross component and leads to Birkhoff's theorem. In contrast, the most general form of the spherically symmetric metric should be [2]

$$ds^2 = -\frac{dt^2}{F^2(t,r)} + H(t,r)drdt + X^2(t,r)dr^2 + Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2}$$

Here, $\tilde{F}(t,r)$, $\tilde{X}(t,r)$, $\tilde{Y}(t,r)$, $F(t,r)$, $H(t,r)$, $X(t,r)$, and $Y(t,r)$ are the to-be-determined functions of coordinates t and r.

In this work, we will calculate Equation (2) directly under the theory of the post-Newtonian (PN) approximation, which is the classical method used to solve Einstein field equations in the weak-field and slow-motion limits [2,8,9]. We derive the shell's external and internal metric to the 1PN order. Our results demonstrate that the radial motion of a spherical-symmetric system can not only produce a gravitomagnetic field but also contributes to the second gravitational potential. The gravitational field outside the collapsing thin spherical shell cannot be described by the Schwarzschild metric, and the inside one is non-zero and time-varying beyond the Newtonian order.

This paper is organized as follows. Section 2 reviews the theory of the post-Newtonian approximation. In Section 3, we derive the 1PN metric of the self-gravitating and collapsing thin spherical shell. In Section 4, we give the dynamic equations of the test particles outside and inside the shell, respectively. Section 5 discusses the incompatibility between Birkhoff's theorem and the theory of the post-Newtonian approximation. Conclusions are provided in Section 6.

2. Theory of the Post-Newtonian Approximation

Einstein field equations can be written as [2]

$$\mathcal{R}_{\mu\nu} = -8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right), \tag{3}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, and $T_{\mu\nu}$ is the matter energy-momentum tensor. We use the natural units in which $G = c = 1$. Greek indices run from 0 to 3. Einstein summation notation is used.

In the weak-field and slow-motion limits, Einstein field equations can be solved via the method of the post-Newtonian approximation, which has been well established and applied to solve numerous gravitational problems [2,8]. Here, we give the basic contents for the theory of the post-Newtonian approximation, and adopt the same notations as in Weinberg's textbook [2].

Let $\bar{M}$, $\bar{v}$, and $\bar{r}$ represent the typical values of mass, velocity, and distance in a non-relativistic system, respectively. In the post-Newtonian approximation, the metric is expanded in the powers of $\bar{v}^2$, which is roughly of the same order of a typical Newtonian potential $\bar{\phi} \equiv -\bar{M}/\bar{r}$, as follows:

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \cdots, \tag{4}$$

$$g_{0i} = \overset{3}{g}_{0i} + \cdots, \tag{5}$$

$$g_{ij} = \delta_{ij} + \overset{2}{g}_{ij} + \cdots, \tag{6}$$

where Latin indices run from 1 to 3. δ_{ij} is Kronecker's delta. $\overset{N}{g}_{\mu\nu}$ denotes the term in $g_{\mu\nu}$ of order $\bar{v}^N$. The corresponding matter energy-momentum tensor is expanded as

$$T^{00} = \overset{1}{T}{}^{00} + \overset{2}{T}{}^{00} + \cdots, \tag{7}$$

$$T^{0i} = \overset{1}{T}{}^{0i} + \cdots, \tag{8}$$

$$T^{ij} = \overset{2}{T}{}^{ij} + \cdots, \tag{9}$$

where $\overset{N}{T}{}^{\mu\nu}$ denotes the term in $T^{\mu\nu}$ of order $(\bar{M}/\bar{r}^3)\bar{v}^N$.

Substituting Equations (4)–(9) into Equation (3) and making use of the harmonic coordinate conditions, we can simplify Einstein field equations to the 1PN order as

$$\nabla^2 \overset{2}{g}_{00} = -8\pi \overset{0}{T}^{00} , \tag{10}$$

$$\nabla^2 \overset{2}{g}_{ij} = -8\pi \delta_{ij} \overset{0}{T}^{00} , \tag{11}$$

$$\nabla^2 \overset{3}{g}_{0i} = 16\pi \overset{1}{T}^{0i} , \tag{12}$$

$$\nabla^2 \overset{4}{g}_{00} = \frac{\partial^2 \overset{2}{g}_{00}}{\partial t^2} + \overset{2}{g}_{ij} \frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} - \frac{\partial \overset{2}{g}_{00}}{\partial x^i} \frac{\partial \overset{2}{g}_{00}}{\partial x^i} - 8\pi \left(\overset{2}{T}^{00} - 2 \overset{2}{g}_{00} \overset{0}{T}^{00} + \overset{2}{T}^{ii} \right) , \tag{13}$$

where ∇^2 is the Laplace operator.

Imposing the infinity boundary conditions that all fields should vanish, we can obtain the 1PN metric, as follows [2]:

$$\overset{2}{g}_{00} = -2\phi , \tag{14}$$

$$\overset{2}{g}_{ij} = -2\phi \delta_{ij} , \tag{15}$$

$$\overset{3}{g}_{0i} = \zeta_i , \tag{16}$$

$$\overset{4}{g}_{00} = -2\phi^2 - 2\psi , \tag{17}$$

with

$$\phi(t, \boldsymbol{x}) = -\int \frac{\overset{0}{T}^{00}(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3 x' , \tag{18}$$

$$\zeta_i(t, \boldsymbol{x}) = -4 \int \frac{\overset{1}{T}^{0i}(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3 x' , \tag{19}$$

$$\psi(t, \boldsymbol{x}) = -\int \left[\frac{1}{4\pi} \frac{\partial^2 \phi(t, \boldsymbol{x}')}{\partial t^2} + \overset{2}{T}^{00}(t, \boldsymbol{x}') + \overset{2}{T}^{ii}(t, \boldsymbol{x}') \right] \frac{d^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} , \tag{20}$$

where ϕ denotes the Newtonian potential caused by the gravitational source's rest mass. ζ_i is the gravitational vector potential generated by the mass current, and it is also called the gravitomagnetic field [10–14]. ψ denotes the second gravitational potential contributed by the source's kinetic and Newtonian potential energy as well as the second time derivative of the Newtonian potential in the whole space.

Finally, the 1PN metric can be written as

$$ds^2 = -[1 + 2(\phi + \psi) + 2\phi^2] dt^2 + 2\zeta_i dx^i dt + (1 - 2\phi) dx_i dx^i . \tag{21}$$

For a non-relativistic test particle, its 1PN dynamic equation can be written as [2]

$$\frac{d\boldsymbol{v}}{dt} = -\nabla(\phi + 2\phi^2 + \psi) + 3\boldsymbol{v} \frac{\partial \phi}{\partial t} + 4\boldsymbol{v}(\boldsymbol{v} \cdot \nabla)\phi - \boldsymbol{v}^2 \nabla \phi - \frac{\partial \boldsymbol{\zeta}}{\partial t} + \boldsymbol{v} \times (\nabla \times \boldsymbol{\zeta}) , \tag{22}$$

where $\boldsymbol{v}$ denotes the velocity of the test particle.

All the above contents can be found in Weinberg's textbook [2]. The reason we repeat the derivation details of the post-Newtonian approximation theory is to show that Equations (18)–(20) hold in *any* harmonic coordinates. We will make use of these important facts in the later discussions.

For a relativistic test particle, the metric given in Equation (21) can only provide its partial 1PN dynamic equation. For simplicity, we only write down the complete 0.5PN dynamic equation of the relativistic test particle including a photon, as follows:

$$\frac{dv}{dt} = -(1+v^2)\nabla\phi + (3-v^2)v\frac{\partial\phi}{\partial t} + 4v(v\cdot\nabla)\phi + v\times(\nabla\times\zeta) - v(v\cdot\nabla)(v\cdot\zeta)\,. \tag{23}$$

where v denotes the velocity of the relativistic test particle, with its magnitude being close to 1.

3. The 1PN Metric of a Self-Gravitating and Collapsing Infinitely Thin Spherical Shell

We consider a massive and pressure-less infinitely thin spherical shell, which has a rest mass of M and a radius of R. The shell collapses under its self-gravity. The collapsing velocity is un_i, with n_i being the radial unit vector, i.e., $\frac{dR}{dt} = u < 0$, and $n_i = \frac{x_i}{r}$, with $r \equiv |\boldsymbol{x}|$. We calculate the metric of the self-gravitating and collapsing spherical shell to the 1PN order, so we only need the equation of motions for the shell at the Newtonian order, as follows:

$$\frac{du}{dt} = \frac{\Phi}{2R}\,, \tag{24}$$

where $\Phi \equiv -\frac{M}{R}$ is the Newtonian potential induced by the other parts of the shell. Equation (24) satisfies energy conservation in the Newton theory

$$E_N \equiv \frac{1}{2}u^2 - \frac{1}{2}\frac{M}{R} = constant\,, \tag{25}$$

where E_N denotes the sum of Newtonian kinetic and potential energies of the shell per unit mass.

Similarly, we only need the energy-momentum tensor of the self-gravitating and collapsing thin spherical shell at the Newtonian order, which can be written as

$$\overset{0}{T}{}^{00}(t,\boldsymbol{x}) = \frac{M}{4\pi R^2}\delta(r-R)\,, \tag{26}$$

$$\overset{1}{T}{}^{0i}(t,\boldsymbol{x}) = \frac{M}{4\pi R^2}un_i\delta(r-R)\,, \tag{27}$$

$$\overset{2}{T}{}^{00}(t,\boldsymbol{x}) = \frac{M}{4\pi R^2}\Big(\frac{1}{2}u^2+\Phi\Big)\delta(r-R)\,, \tag{28}$$

$$\overset{2}{T}{}^{ij}(t,\boldsymbol{x}) = \frac{M}{4\pi R^2}u^2n_in_j\delta(r-R)\,, \tag{29}$$

where $\delta(s)$ is the Dirac delta function of s.

Substituting Equations (26) and (27) into Equations (18) and (19), it is easy to obtain the Newtonian potential and the gravitomagnetic field induced by the shell, as follows:

$$\phi(t,\boldsymbol{x}) = -\frac{M}{r}\theta(r-R) - \frac{M}{R}[1-\theta(r-R)]\,, \tag{30}$$

$$\zeta_i(t,\boldsymbol{x}) = -\frac{4}{3}\frac{MRux_i}{r^3}\theta(r-R) - \frac{4}{3}\frac{Mux_i}{R^2}[1-\theta(r-R)]\,, \tag{31}$$

where $\theta(s)$ is the step function of s, which equals 1 for $s>0$ and 0 otherwise.

From Equation (30), we can observe that the Newtonian potential inside the shell is dependent on time only, and its gradient with respect to space is zero. Therefore, the internal gravitational field is zero at the Newtonian order, and this is in agreement with the prediction of Newton's theory, in which the Gauss theorem can be applied. However, beyond the Newtonian order, the dynamic equations of the test particles will involve more potentials, such as the second gravitational potential and the gravitomagnetic field, as well as the time derivative of the Newtonian potential, as given by Equations (22) and (23). Later, we will explicitly give the 1PN dynamic equation for the non-relativistic test particle and the 0.5PN dynamic equation for the relativistic one, outside and inside the collapsing shell.

For the calculation of ψ, we need to obtain $\frac{\partial^2\phi}{\partial t^2}$ first. Taking the time derivative of Equation (30), we have

$$\frac{\partial\phi(t,\boldsymbol{x})}{\partial t} = \frac{M}{R^2}\dot{R}\left[1-\theta(r-R)\right], \tag{32}$$

where "dot" denotes the time derivative, and $\dot{R}$ is just u. Taking the time derivative of Equation (32), we obtain

$$\frac{\partial^2\phi(t,\boldsymbol{x})}{\partial t^2} = \left(-2\frac{M}{R^3}\dot{R}^2+\frac{M}{R^2}\ddot{R}\right)\left[1-\theta(r-R)\right]+\frac{M}{R^2}\dot{R}^2\delta(r-R), \tag{33}$$

where we have made use of $\frac{d\theta(s)}{ds}=\delta(s)$.

Substituting Equations (28), (29) and (33) into Equation (20), we can obtain the second gravitational potential induced by the shell after integration, as follows:

$$\psi(t,\boldsymbol{x})=-\frac{M}{r}\left(\frac{11}{6}u^2+\frac{7}{6}\Phi\right)\theta(r-R)-\frac{M}{R}\left[\left(\frac{3}{2}+\frac{1}{3}\frac{r^2}{R^2}\right)u^2+\left(\frac{5}{4}-\frac{1}{12}\frac{r^2}{R^2}\right)\Phi\right]\left[1-\theta(r-R)\right], \tag{34}$$

where we have replaced $\dot{R}$ with u and made use of $\ddot{R}=\frac{\Phi}{2R}$.

Substituting Equations (30), (31) and (34) into Equation (21), to the 1PN order, we can write down the metric of the self-gravitating and collapsing thin spherical shell as follows:

$$ds^2 = -\left[1-\frac{2M}{r}\left(1+\frac{11}{6}u^2-\frac{7}{6}\frac{M}{R}\right)+\frac{2M^2}{r^2}\right]dt^2-\frac{8}{3}\frac{MRu\boldsymbol{x}\cdot d\boldsymbol{x}}{r^3}dt+\left(1+\frac{2M}{r}\right)d\boldsymbol{x}^2, \quad \text{for } r>R, \tag{35}$$

and

$$ds^2=-\left\{1-\frac{2M}{R}\left[1+\frac{3}{2}u^2-\frac{9}{4}\frac{M}{R}+\frac{1}{3}\frac{r^2}{R^2}\left(u^2+\frac{1}{4}\frac{M}{R}\right)\right]\right\}dt^2-\frac{8}{3}\frac{Mu\boldsymbol{x}\cdot d\boldsymbol{x}}{R^2}dt+\left(1+\frac{2M}{R}\right)d\boldsymbol{x}^2, \quad \text{for } r\leq R. \tag{36}$$

It can be seen that the metric is continuous at the shell's position ($r=R$). Since u and R are time-dependent, the internal and external spacetime of the shell is not static at the 1PN order.

4. The Dynamic Equations of the Test Particles

Based on the 1PN metric, we can obtain the dynamic equations of the test particles in the spacetime of the self-gravitating and collapsing thin spherical shell.

4.1. Outside the Shell

Substituting Equations (30), (31) and (34) into Equation (22), we can write down the 1PN dynamic equation of the non-relativistic test particle outside the shell as follows:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{M\boldsymbol{x}}{r^3}\left(1+E_N+\boldsymbol{v}^2-4\frac{M}{r}\right)+\frac{4M(\boldsymbol{v}\cdot\boldsymbol{x})\boldsymbol{v}}{r^3}, \tag{37}$$

here, we have assumed that $|\boldsymbol{v}|$ has the same order as $|u|$. It follows this equation that the gravitational field outside the collapsing spherical shell seems static to the 1PN order, since E_N which is constant at the Newtonian order can be absorbed into a newly-defined mass.

For the case of the relativistic test particle with $|\boldsymbol{v}|\approx 1$, the situation is different. Substituting Equations (30), (31) and (34) into Equation (23), we have the complete 0.5PN dynamic equation of the relativistic test particle, as follows:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{M\boldsymbol{x}}{r^3}(1+\boldsymbol{v}^2)+\frac{4M(\boldsymbol{v}\cdot\boldsymbol{x})\boldsymbol{v}}{r^3}+\frac{4MR\,u}{3r^3}\left[\boldsymbol{v}^2-3\frac{(\boldsymbol{v}\cdot\boldsymbol{x})^2}{r^2}\right]\boldsymbol{v}. \tag{38}$$

It can be observed that the relativistic test particle does experience the time-varying effect induced by the radial motion, since the last term in Equation (38) contains R and u. The time derivative of $R\,u$, which can be written as

$$\frac{d(R\,u)}{dt} = E_N + \frac{1}{2}u^2\,, \tag{39}$$

does not equal zero in general. In fact, this term also exists in the dynamic equation of the non-relativistic test particle, but it is counted as a 2PN term there, so it does not appear in Equation (37).

4.2. Inside the Shell

Similarly, from the gravitational potentials and gravitomagnetic field inside the shell, we can write down the 1PN dynamic equation of the non-relativistic test particle as follows:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{M\boldsymbol{x}}{R^3}\left(2u^2 + \frac{1}{2}\frac{M}{R}\right) + \frac{3Mu}{R^2}\boldsymbol{v}\,, \tag{40}$$

where we have assumed that $|\boldsymbol{v}|$ has the same order as $|\boldsymbol{u}|$.

For the relativistic test particle with $|\boldsymbol{v}| \approx 1$, we can write down the complete 0.5PN dynamic equation inside the shell as follows:

$$\frac{d\boldsymbol{v}}{dt} = \frac{3Mu}{R^2}\left(1 + \frac{1}{9}\boldsymbol{v}^2\right)\boldsymbol{v}\,. \tag{41}$$

In can be observed that the test particles' acceleration inside the shell is zero at the Newtonian order. However, the test particles will experience a time-varying gravitational field beyond the Newtonian order. It is interesting to see that at the 0.5PN order, the relativistic test particle will keep moving straight and will decelerate since the value of u is negative.

5. Discussion

In accordance with the theory of the post-Newtonian approximation, the spacetime of the self-gravitating and collapsing thin spherical shell is time-varying. This is different from Birkhoff's theorem, which states that the spacetime of any spherically symmetric system is static.

As pointed out in our previous work [3], the derivation of Birkhoff's theorem starts with the specific form of a spherically symmetric metric with $g_{0i} = 0$, which is obtained via a coordinate transformation on the most general form of the spherically symmetric metric. However, making such a coordinate transformation will change the matter energy-momentum tensor. For example, the metric of a uniformly moving point mass can be achieved by applying a Lorentz boost to the Schwarzschild metric [12,15]. This implies that the same coordinate transformation has been applied on the energy-momentum tensor of a static point mass to obtain that of the uniformly moving point mass, i.e., the reference frame is changed. We can also see this clearly in the example of the collapsing thin shell, as follows.

In the theory of the post-Newtonian approximation, Equation (19) is the 1PN solution to Einstein field equations, revealing the relation between the time-space cross component of metric and that of matter energy-momentum tensor under the harmonic-coordinate conditions. For the external space of the collapsing thin spherical shell, Equation (19) can be written in the following form:

$$\underbrace{-\frac{4}{3}\frac{MR\,u\,x_i}{r^3}}_{\overset{3}{g}_{0i}(t,\boldsymbol{x})} = -4\int_0^\infty \underbrace{\left(\frac{4\pi r' x_i}{3r^3}\right)}_{\int_0^{2\pi}\int_0^{\pi}\frac{n_i}{|\boldsymbol{x}-\boldsymbol{x}'|}\sin\theta d\theta d\varphi} \underbrace{\left[\frac{M}{4\pi R^2}\,u\,\delta(r'-R)\right]}_{\overset{1}{T}^{0i}(t,\boldsymbol{x}')/n_i} r'^2 dr'\,. \tag{42}$$

It can be observed that any coordinate transformation which makes the metric's time-space cross component become zero ($\overset{3}{g}_{0i} = 0$) will have to make u become zero if $\overset{1}{T}^{0i}$ still keeps the same form as $\frac{M}{4\pi R^2} u n_i \delta(r' - R)$, since M and R are always positive and finite. Therefore, setting $g_{0i} = 0$ in the spherically symmetric metric may limit the solution of Einstein field equations with the harmonic-coordinate conditions to the static problem.

Finally, it is worth mentioning that Birkhoff's theorem in other gravity theories, such as in the quadratic gravity and the Einstein-Aether theory, has been explored [16,17]. However, unfortunately, similarly to the derivation of Birkhoff's theorem in general relativity, these works study spacetime properties under a specific form of the spherically symmetric metric, which does not have a time-space cross component, such as Equation (1).

6. Conclusions

We have studied the metric of the self-gravitating and collapsing thin spherical shell under the theory of the post-Newtonian approximation. It was found that the external field of the shell cannot be described by the Schwarzschild metric, and the internal metric is not Minkowskian. We derived the 1PN dynamic equation for the non-relativistic test particle and the 0.5PN dynamic equation for the relativistic one, respectively, to further show that the non-static spherically symmetric system can produce time-dependent gravitational effects.

More importanly, we have shown that, according to the theory of the post-Newtonian approximation, any coordinate transformation used to eliminate the time-space cross component of a metric will make the same component of the thin spherical shell's energy-momentum tensor become zero, changing the dynamic gravitational problem into a static one. Therefore, this work not only proves that Birkhoff's theorem is not correct, but also illustrates that performing a coordinate transformation in solving Einstein field equations may change the matter energy-momentum tensor, i.e., a coordinate transformation may change the reference frame, making the achieved solution not be the answer for the original problem.

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