

Article

Solutions of Second-Order Nonlinear Implicit ψ -Conformable Fractional Integro-Differential Equations with Nonlocal Fractional Integral Boundary Conditions in Banach Algebra

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Abstract: In this paper, we introduce and thoroughly examine new generalized ψ -conformable fractional integral and derivative operators associated with the auxiliary function $\psi(t)$. We rigorously analyze and confirm the essential properties of these operators, including their semigroup behavior, linearity, boundedness, and specific symmetry characteristics, particularly their invariance under time reversal. These operators not only encompass the well-established Riemann–Liouville and Hadamard operators but also extend their applicability. Our primary focus is on addressing complex fractional boundary value problems, specifically second-order nonlinear implicit ψ -conformable fractional integro-differential equations with nonlocal fractional integral boundary conditions within Banach algebra. We assess the effectiveness of these operators in solving such problems and investigate the existence, uniqueness, and Ulam–Hyers stability of their solutions. A numerical example is presented to demonstrate the theoretical advancements and practical implications of our approach. Through this work, we aim to contribute to the development of fractional calculus methodologies and their applications.

Keywords: generalized conformable fractional operators; ψ -conformable fractional integro-differential equations; nonlocal fractional integral boundary conditions; Banach algebra; semigroup behavior and boundedness; fractional boundary value problems

MSC: 34B15; 26A33; 47H10; 47G10; 34D20



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1. Introduction

The field of fractional calculus has gained substantial attention over the past few decades, captivating the interest of numerous researchers. This branch of mathematics extends the concepts of differentiation and integration to non-integer orders, offering a powerful framework for modeling and analyzing complex phenomena in various scientific and engineering disciplines. Remarkable contributions to this field have been made by eminent mathematicians, including trailblazers such as Joseph Liouville, who pioneered the theory of fractional derivatives in the 19th century [1], and Samuel H. Lenchner, whose work in the mid-20th century significantly advanced the understanding of fractional integrals [2]. Moreover, contemporary luminaries like Anatoly A. Kilbas and Hari M. Srivastava have continued to reshape the landscape of fractional calculus. This article delves into the foundational aspects of fractional calculus, exploring its historical evolution and shedding light on its modern significance through notable references like Podlubny [3] and Miller and Ross [4].

Fractional boundary value problems (FBVPs) have garnered substantial interest due to their diverse applications spanning various fields. FBVPs offer a robust framework for modeling intricate phenomena and extracting invaluable insights into the behaviors exhibited

by fractional systems. These challenges find utility across a spectrum of domains, including but not limited to heat conduction, control systems, and population dynamics. Extensive investigations have been conducted to explore the existence of positive solutions for fractional differential equations featuring integral boundary conditions. These explorations employ methodologies such as fixed-point theory, alongside upper- and lower-solution techniques. For more information, one can consult [5–13], and the references therein.

In the realm of the function ψ , fractional derivatives emerge as generalizations of the classical Riemann–Liouville derivatives. The ψ -Caputo fractional derivative diverges from its classical counterpart due to the incorporation of kernel terms. Recent work by Almeida has re-evaluated this derivative, introducing a Caputo-type regularization that showcases intriguing properties. For further insights into additional properties and applications of the ψ -Caputo fractional derivatives, references such as Abbas [14], Abdel Jawad [15], Almeida [16], Awad et al. [17–20], and ElSayed [21] provide comprehensive coverage, along with the references cited therein.

Many scholars have introduced various definitions of fractional derivatives, including notable contributions from Riemann–Liouville, Hadamard, Caputo, Grunwald–Letnikov, Katugampola, Marchaud, Erdelyi–Kober, and Riesz, among others ([22–27]), each contributing to the foundation of this field. These foundational scholars established the notion of fractional integrals, a conceptual cornerstone that laid the groundwork for subsequent advancements, including the formulation of associated fractional derivatives. It is worth noting that many of these fractional derivatives are established through the medium of fractional integrals [27]. This shared origin introduces nonlocal features into these derivative forms, which supports their varied applications, including phenomena like memory effects and future dependencies. For additional details, refer to [28–34].

In 2019, Khalil et al. [35] introduced a novel concept for conformable fractional derivatives and integral operators. This innovative idea extends the conventional limit definition of a function's derivative. As a result, they demonstrated that the fractional-order derivative follows the product and quotient rules, and it produces results analogous to Rolle's theorem and the mean value theorem from classical calculus. Additionally, Katugampola [23] expanded upon the findings in [22] and [26] by proposing a new fractional derivative that serves as a natural extension of the traditional derivative definition at a point a . Later in 2019, Tahir U. Khan and Muhammad A. Khan [36] formulated new generalized conformable fractional integral and derivative operators (both left-sided and right-sided). They established several fundamental properties of these operators, such as the semigroup and linearity properties. These operators are viewed as generalizations of Katugampola's, Riemann–Liouville's, and Hadamard's fractional operators.

In this study, we introduce novel generalized conformable fractional integral and derivative operators (both right-sided and left-sided) for a function denoted as $f(t)$ with respect to an increasing function $\psi(t) \in C^1[p, q]$. We thoroughly investigate the fundamental properties of these operators, demonstrating that they exhibit semigroup behavior, linearity, boundedness, and symmetry. Furthermore, we show that these operators generalize the new conformable fractional integral and derivative operators, as well as the Riemann–Liouville, Hadamard, Katugampola, Khan, and other fractional operators.

The use of Banach algebra is chosen for its comprehensive structure, combining the properties of both a Banach space and algebra. This structure provides a robust foundation for analyzing nonlinear operators and ensuring the convergence of iterative methods. The completeness of Banach algebra under a norm facilitates the application of fixed-point theorems and other analytical techniques crucial for establishing the existence and uniqueness of solutions to complex integro-differential equations. Moreover, Banach algebra enables the manipulation of functions and their compositions within an algebraic context, essential for addressing the complexities of fractional integro-differential equations with nonlocal boundary conditions. Thus, this study is centered on solving complex fractional boundary value problems, specifically the following second-order nonlinear

implicit ψ -conformable fractional integro-differential equations (ICFDE) with a pair of nonlocal boundary conditions within Banach algebra:

$$\begin{cases} \frac{d^2}{dt^2} y(t) = f\left(t, y(t), {}^{\tau}\mathfrak{D}_{0+}^{\beta, \psi} y(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathfrak{D}_{0+}^{\theta, \psi} y(s)\right)(t)\right), t \in [0, 1], \\ y(\eta) = {}^{\tau}\mathcal{I}_{0+}^{\eta, \psi} G_1(t, y(t)), \eta \in (0, 1], \\ y(\delta) = {}^{\tau}\mathcal{I}_{0+}^{\delta, \psi} G_2(t, y(t)), \delta \in (0, 1], \end{cases} \quad (1)$$

where $y(t) \in C^2[0, 1]$, the function $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is considered as α -differentiable with respect to $t \in [0, 1]$ and is absolutely continuous, $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, $\psi(t) \in C^1[0, 1]$ is strictly increasing on $[0, 1]$, and $G_i : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ for $i \in \{1, 2\}$ are functions with conformable integrability. Additionally, ${}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi}$, ${}^{\tau}\mathcal{I}_{0+}^{\eta, \psi}$, and ${}^{\tau}\mathcal{I}_{0+}^{\delta, \psi}$ are generalized ψ -conformable fractional integral operators of respective orders $\alpha, \sigma, \delta, \eta \in (0, 1]$ with $\eta < \delta$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$. Moreover, ${}^{\tau}\mathfrak{D}_{0+}^{\beta, \psi}$ and ${}^{\tau}\mathfrak{D}_{0+}^{\theta, \psi}$ are generalized ψ -conformable fractional derivative operators of respective orders $\beta, \theta \in (0, 1]$ with $\beta < \theta$.

The manuscript's structure is organized as follows: Section 1 serves as an introduction, where we delineate the research objective. In Section 2, crucial foundational information for this study is provided. Initially, we introduce our innovative generalized conformable fractional integral ${}^{\tau}\mathcal{I}_{0+}^{\sigma, \psi}$ as well as the derivative ${}^{\tau}\mathfrak{D}_{0+}^{\beta, \psi}$ operators (both right-sided and left-sided) concerning another function $\psi(t)$. Subsequently, we delve into the core attributes of these operators and mathematically establish their compliance with semigroup properties, linearity, and boundedness. In Section 3, an investigation into the existence, uniqueness, and stability of the boundary value problem ICFDE (1), as previously defined, is conducted. Finally, to illuminate the implications of our findings, a numerical example is presented in Section 3. This example not only illustrates the theoretical advancements but also underscores the practical utility of our approach. Ultimately, Section 4 provides a concluding summary of the study's outcomes.

2. Preliminaries

The most significant characterization was provided by Riemann and Liouville. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ on the right-hand side is formulated in [26] as follows:

$$\mathcal{I}_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \text{ with } t > a, \quad (2)$$

which is based on the iteration of the Riemann integral operator $\int_a^t f(s) ds$.

The right-sided Hadamard fractional integral introduced by J. Hadamard [22] for $\alpha > 0$ is given by

$$H_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s} \text{ with } t > a, \quad (3)$$

which is based on iterating the integral operator $\int_a^t f(s) \frac{ds}{s}$.

Katugampola [23] presented a generalized integral operator as follows:

$${}^{\rho}\mathcal{I}_{a+}^{\alpha} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^{\rho} - s^{\rho})^{\alpha-1} f(s) s^{\rho-1} ds \text{ with } t > a, \operatorname{Re}(\alpha) > 0, \text{ and } \rho > 0. \quad (4)$$

Katugampola's approach is considered as a generalization of both the Riemann–Liouville and Hadamard operators. In addition, it is based on the iteration of the integral operator $\int_a^t f(s) s^{\tau} ds$. Concurrently, the corresponding left-sided versions of the mentioned fractional integral operators were also determined, and the associated fractional derivative operators were also defined in [23,36–38].

Recently, Khalil et al. [35] presented a novel approach for new definitions of conformable fractional derivatives and integral operators defined as follows.

Definition 1. According to [35], let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The conformable fractional derivative of f of order α is defined as

$$\mathfrak{D}_\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad (5)$$

for all $t > 0$ and $\alpha \in (0, 1]$.

Remark 1. Whenever the limit exists, we say that f is conformable fractional α -differentiable; we write $f^{(\alpha)}(t)$ for $\mathfrak{D}_\alpha f(t)$, to denote the conformable fractional derivatives of f of order α . In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α -differentiable. Moreover, if f is also ordinary α -differentiable, we can establish a relationship between the conformable fractional derivative and the ordinary derivative for $t > 0$ as follows: $\mathfrak{D}_\alpha f(t) = t^{1-\alpha} f'(t)$, where $f'(t)$ represents the ordinary derivative of f at the point t . Furthermore, it can be demonstrated that a function may be α -differentiable at a specific point while not being ordinary differentiable. Detailed information can be found in reference [35]. This new definition is straightforward and encompasses nearly all fundamental properties of the ordinary derivative given in the theorems below, which are proved in [35].

Theorem 1. According to [35], let $\alpha \in (0, 1]$ and f_1 and f_2 be α -differentiable functions at a point $t > 0$. Then, for any $k_1, k_2 \in \mathbb{R}$, we have

- (1) $\mathfrak{D}_\alpha(k_1 f_1 + k_2 f_2) = k_1 \mathfrak{D}_\alpha(f_1) + k_2 \mathfrak{D}_\alpha(f_2)$.
- (2) $\mathfrak{D}_\alpha(t^n) = n t^{n-\alpha}$, for every $n \in \mathbb{R}$.
- (3) $\mathfrak{D}_\alpha(c) = 0$ for every $c \in \mathbb{R}$.
- (4) $\mathfrak{D}_\alpha(f_1 f_2) = f_2 \mathfrak{D}_\alpha(f_1) + f_1 \mathfrak{D}_\alpha(f_2)$.
- (5) $\mathfrak{D}_\alpha\left(\frac{f_1}{f_2}\right) = \frac{f_2 \mathfrak{D}_\alpha(f_1) - f_1 \mathfrak{D}_\alpha(f_2)}{f_2^2}$.

Definition 2. According to [35], the conformable fractional integral of the continuous function $f : [a, b] \subseteq [0, \infty) \rightarrow \mathbb{R}$ of the fractional order $\alpha \in (0, 1]$ is defined as

$$\mathcal{I}_\alpha(f)(u) = \int_a^t f(u) d_\alpha u = \int_a^t f(u) u^{\alpha-1} du, \quad (6)$$

where the integral $\int_a^t du$, on the right side, represents the classical Riemann improper integral.

Theorem 2. According to [35], if f is any continuous function in the domain of $\mathcal{I}_\alpha$, then $\mathfrak{D}_\alpha \mathcal{I}_\alpha f(t) = f(t)$.

In [36], Khan et al. defined new generalized conformable fractional integral and derivative operators (right-sided and left-sided), by iterating a conformable integral of order $\alpha \in (0, 1]$ as follows.

Definition 3. According to [36], let f be an α -differentiable function on the interval $[p, q] \subseteq [0, \infty)$. The right-sided and left-sided generalized conformable fractional integral operators ${}^\tau \mathcal{I}_{p+}^\beta$ and ${}^\tau \mathcal{I}_{q-}^\beta$ of order $\beta > 0$ with $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$ are defined by

$${}^\tau \mathcal{I}_{p+}^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_p^t \left(\frac{t^{\tau+\alpha} - s^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta-1} f(s) s^\tau d_\alpha s \text{ with } t > p, \quad (7)$$

and

$${}^{\tau}\mathcal{I}_{q-}^{\beta}f(t) = \frac{1}{\Gamma(\beta)} \int_t^q \left(\frac{s^{\tau+\alpha} - t^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta-1} f(s) s^{\tau} d_{\alpha}s \text{ with } q > t, \quad (8)$$

respectively, and ${}^{\tau}\mathcal{I}_{p+}^0 f(t) = {}^{\tau}\mathcal{I}_{q-}^0 f(t) = f(t)$, where Γ denotes the classical gamma function, and $d_{\alpha}s = s^{\alpha-1}ds$.

Definition 4. According to [36], let f be an α -differentiable function on the interval $[p, q] \subseteq [0, \infty)$. The right-sided and left-sided generalized conformable fractional derivative operators ${}^{\tau}\mathcal{D}_{p+}^{\beta}$ and ${}^{\tau}\mathcal{D}_{q-}^{\beta}$ of order $\beta \in (0, 1]$ with $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$ are defined by

$${}^{\tau}\mathcal{D}_{p+}^{\beta}f(t) = \frac{t^{-\tau}}{\Gamma(1-\beta)} \mathcal{D}_{\alpha} \left(\int_p^t \left(\frac{t^{\tau+\alpha} - s^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) s^{\tau} d_{\alpha}s \right) \text{ with } t > p, \quad (9)$$

and

$${}^{\tau}\mathcal{D}_{q-}^{\beta}f(t) = \frac{t^{-\tau}}{\Gamma(1-\beta)} \mathcal{D}_{\alpha} \left(\int_t^q \left(\frac{s^{\tau+\alpha} - t^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) s^{\tau} d_{\alpha}s \right) \text{ with } q > t, \quad (10)$$

respectively, and ${}^{\tau}\mathcal{D}_{p+}^0 f(t) = {}^{\tau}\mathcal{D}_{q-}^0 f(t) = f(t)$, where Γ denotes the classical gamma function and $\mathcal{D}_{\alpha}$ denotes the conformable derivative of order α .

Furthermore, the operators under consideration were demonstrated to possess semi-group and linearity properties, as well as boundedness. Furthermore, these operators were utilized to establish Riemann–Liouville-type conformable fractional operators. These newly defined fractional operators can be seen as extensions of the Katugampola fractional operators, Riemann–Liouville fractional operators, and Hadamard fractional integral operators.

Here, we introduce our innovative generalized conformable operators for fractional integration and differentiation (both on the right and left) of a function $f(t)$ with respect to another function $\psi(t)$.

Definition 5. We consider an α -differentiable and absolutely continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, and a function $\psi(t) \in C^1[0, 1]$ that is strictly increasing on the interval $[0, 1]$. The conformable fractional derivative of f with respect to ψ and of order α is defined by

$$\mathcal{D}_{\alpha, \psi}(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(\psi(t) + \epsilon \psi(t)^{1-\alpha}) - f(\psi(t))}{\epsilon}, \quad (11)$$

for all $t > 0$ and $\alpha \in (0, 1]$.

Remark 2. If the limit exists, we say that f is α -differentiable with respect to ψ and of order α , such that $\mathcal{D}_{\alpha, \psi}f(t) = \psi'(t)\psi(t)^{1-\alpha}f'(\psi(t))$, where $f'(t)$ represents the ordinary derivative of f at the point t .

Proof. We assume that f is α -differentiable with respect to ψ . Then, by definition, the conformable fractional derivative of f with respect to ψ of order α is

$$\mathcal{D}_{\alpha, \psi}(f)(t) = \lim_{\epsilon \rightarrow 0^+} \frac{f(\psi(t) + \epsilon \psi(t)^{1-\alpha}) - f(\psi(t))}{\epsilon}.$$

To find an explicit form, we consider the Taylor expansion of f around $\psi(t)$:

$$f(\psi(t) + \epsilon \psi(t)^{1-\alpha}) \approx f(\psi(t)) + f'(\psi(t)) \cdot \epsilon \psi(t)^{1-\alpha} + o(\epsilon).$$

Substituting this expansion into the definition of the conformable fractional derivative, we obtain

$$\begin{aligned}\mathfrak{D}_{\alpha,\psi}(f)(t) &= \lim_{\epsilon \rightarrow 0^+} \frac{f(\psi(t) + \epsilon\psi(t)^{1-\alpha}) - f(\psi(t))}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{f(\psi(t)) + f'(\psi(t)) \cdot \epsilon\psi(t)^{1-\alpha} + o(\epsilon) - f(\psi(t))}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{f'(\psi(t)) \cdot \epsilon\psi(t)^{1-\alpha} + o(\epsilon)}{\epsilon} \\ &= f'(\psi(t)) \cdot \psi(t)^{1-\alpha}.\end{aligned}$$

To express this in terms of $\psi'(t)$, we observe that

$$\frac{d}{dt}(\psi(t)) = \psi'(t).$$

Thus, the conformable fractional derivative of f can be written as

$$\mathfrak{D}_{\alpha,\psi}(f)(t) = \psi'(t) \cdot \psi(t)^{1-\alpha} f'(\psi(t)),$$

which completes the proof. $\square$

Definition 6. Let f be an α -differentiable and absolutely continuous function on the interval $[p, q] \subseteq [0, \infty)$, and let $\psi(t) \in C^1[0, 1]$ be an increasing function defined on the interval $[p, q]$. The right-sided and left-sided generalized ψ -conformable fractional integral operators ${}^{\tau}_{\alpha}\mathcal{I}_{p^+}^{\beta,\psi}$ and ${}^{\tau}_{\alpha}\mathcal{I}_{q^-}^{\beta,\psi}$ of order $\beta > 0$ with $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$ are defined by

$${}^{\tau}_{\alpha}\mathcal{I}_{p^+}^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta-1} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s \text{ with } t > p, \quad (12)$$

and

$${}^{\tau}_{\alpha}\mathcal{I}_{q^-}^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_t^q \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta-1} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s \text{ with } q > t, \quad (13)$$

where Γ denotes the classical gamma function, and $d_{\alpha,\psi} s = \psi'(s) \psi(s)^{\alpha-1} ds$.

Definition 7. Let f be an α -differentiable and absolutely continuous function on the interval $[p, q] \subseteq [0, \infty)$, and let $\psi(t) \in C^1[0, 1]$ be an increasing function defined on the interval $[p, q]$. The right-sided and left-sided generalized ψ -conformable fractional derivative operators ${}^{\tau}_{\alpha}\mathfrak{D}_{p^+}^{\beta,\psi}$ and ${}^{\tau}_{\alpha}\mathfrak{D}_{q^-}^{\beta,\psi}$ of order $\beta \in (0, 1]$ with $\alpha \in (0, 1]$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$ are defined by

$${}^{\tau}_{\alpha}\mathfrak{D}_{p^+}^{\beta,\psi} f(t) = \frac{\psi(t)^{-\tau}}{\Gamma(1-\beta)} \mathfrak{D}_{\alpha,\psi} \left(\int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s \right) \text{ with } t > p, \quad (14)$$

and

$${}^{\tau}_{\alpha}\mathfrak{D}_{q^-}^{\beta,\psi} f(t) = \frac{\psi(t)^{-\tau}}{\Gamma(1-\beta)} \mathfrak{D}_{\alpha,\psi} \left(\int_t^q \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s \right) \text{ with } q > t, \quad (15)$$

where Γ denotes the classical gamma function, $\mathfrak{D}_{\alpha,\psi}$ is the conformable derivative of order α , and $d_{\alpha,\psi} s = \psi'(s) \psi(s)^{\alpha-1} ds$.

Remark 3. The generalized ψ -conformable fractional integral operators defined in this paper exhibit specific symmetry properties. The right-sided operator ${}^{\tau}_{\alpha}\mathcal{I}_{p^+}^{\beta,\psi}$ and the left-sided operator ${}^{\tau}_{\alpha}\mathcal{I}_{q^-}^{\beta,\psi}$ are

constructed to respect a balanced structure through their dependence on the auxiliary function $\psi(t)$ and its derivative. While these operators do not exhibit straightforward time-reversal symmetry due to the fixed integration limits and the nature of $\psi(t)$, they maintain a form of symmetry in their integral formulation. This is reflected in how they integrate the function $f(t)$ in a manner that preserves certain balanced characteristics of the fractional calculus framework. The inherent symmetry is thus more about maintaining structural consistency rather than classical symmetry transformations.

Theorem 3. Let f be an α -differentiable and absolutely continuous function on the interval $[p, q] \subseteq [0, \infty)$, and let $\psi(t) \in C^1[0, 1]$ be an increasing function defined on the interval $[p, q]$. Then,

$${}^{\tau}\mathfrak{D}_{p^+}^{\beta, \psi} f(t) = \psi'(t)^2 \left({}^{\tau}\mathcal{I}_{p^+}^{-\beta, \psi} (f(t) - f(p)) + \frac{f(p)}{\Gamma(1-\beta)} \left(\frac{\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} \right). \quad (16)$$

Proof. Let $\phi'(s) = \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s)$, and $\chi(s) = f(s) - f(p)$. Then,

$$\int_p^t \phi'(s) \chi(s) ds = \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) (f(s) - f(p)) ds, \quad (17)$$

and

$$\frac{d}{dt} \int_p^t \phi'(s) \chi(s) ds = \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) (f(s) - f(p)) ds. \quad (18)$$

Multiplying by $\frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}$, we obtain

$$\begin{aligned} & \frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \phi'(s) \chi(s) ds \\ &= \frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) (f(s) - f(p)) ds \\ &= \frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) f(s) ds \\ & \quad - \frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) f(p) ds \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{(\tau+\alpha)^{\beta} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} (\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha})^{-\beta} \psi'(s) f(s) ds \\ &= \frac{\psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) \psi(s)^{\tau} d_{\alpha, \psi} s \\ &= \frac{\psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{\psi(t)^{\alpha-1}}{\psi'(t)} \mathfrak{D}_{\alpha, \psi} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) \psi(s)^{\tau} d_{\alpha, \psi} s \\ &= \frac{1}{\psi'(t)} \frac{\psi(t)^{-\tau}}{\Gamma(1-\beta)} \mathfrak{D}_{\alpha, \psi} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} f(s) \psi(s)^{\tau} d_{\alpha, \psi} s \\ &= \frac{1}{\psi'(t)} {}^{\tau}\mathfrak{D}_{p^+}^{\beta, \psi} f(t), \end{aligned}$$

and

$$\begin{aligned}
 I_2 &= -\frac{(\tau + \alpha)^\beta \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \psi(s)^{\tau+\alpha-1} \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{-\beta} \psi'(s) f(p) ds \\
 &= -\frac{(\tau + \alpha)^{\beta-1} \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t (\tau + \alpha) \psi(s)^{\tau+\alpha-1} \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{-\beta} \psi'(s) f(p) ds \\
 &= -\frac{(\tau + \alpha)^{\beta-1} \psi(t)^{1-(\tau+\alpha)} f(p)}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{-\beta} d\psi(s)^{\tau+\alpha} \\
 &= -\frac{(\tau + \alpha)^{\beta-1} \psi(t)^{1-(\tau+\alpha)} f(p)}{\Gamma(1-\beta)} \frac{d}{dt} \left(\frac{\left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{-\beta+1}}{-\beta+1} \Bigg|_{s=t}^{s=p} \right) \\
 &= -\frac{(\tau + \alpha)^{\beta-1} \psi(t)^{1-(\tau+\alpha)} f(p)}{\Gamma(1-\beta)} \frac{d}{dt} \left(\frac{\left(\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha} \right)^{-\beta+1}}{-\beta+1} \right) \\
 &= -\frac{\psi'(t) f(p)}{\Gamma(1-\beta)} \left(\frac{\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta}.
 \end{aligned}$$

Hence,

$$\frac{(\tau + \alpha)^\beta \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \phi'(s) \chi(s) ds = \frac{1}{\psi'(t)} {}^\tau \mathfrak{D}_{p^+}^{\beta, \psi} f(t) - \frac{\psi'(t) f(p)}{\Gamma(1-\beta)} \left(\frac{\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta}. \quad (19)$$

Now, differentiating the integral in the right-hand side with respect to variable t , we obtain:

$$\begin{aligned}
 &\frac{(\tau + \alpha)^\beta \psi(t)^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dt} \int_p^t \phi'(s) \chi(s) ds \\
 &= \frac{-\beta \psi'(t)}{\Gamma(1-\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta-1} \psi(s)^\tau \psi'(s) \psi(s)^{\alpha-1} \chi(s) ds \\
 &= \frac{\psi'(t)}{\Gamma(-\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta-1} \psi(s)^\tau \chi(s) d_{\alpha, \psi} s \\
 &= \psi'(t) {}^\tau \mathcal{I}_{p^+}^{-\beta, \psi} (f(t) - f(p)).
 \end{aligned}$$

Therefore,

$${}^\tau \mathfrak{D}_{p^+}^{\beta, \psi} f(t) = \psi'(t)^2 \left({}^\tau \mathcal{I}_{p^+}^{-\beta, \psi} (f(t) - f(p)) + \frac{f(p)}{\Gamma(1-\beta)} \left(\frac{\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha}}{\tau + \alpha} \right)^{-\beta} \right). \quad (20)$$

□

Remark 4.

- (1) For $\psi(t) = t$ and $\tau \neq 0$ in Definitions 6 and 7, we obtain the generalized conformable fractional integral and derivative operators presented in [36].
- (2) For $\psi(t) = t$ and $\tau = 1 - \alpha$ in Definitions 6 and 7, we obtain the Riemann–Liouville fractional integral operators presented in [26].
- (3) For $\psi(t) = t$, then, by using L'Hospital, we obtain straightforwardly that, when $\alpha \rightarrow 0$ in Definitions 6 and 7, we obtain the Hadamard fractional integrals presented in [22].
- (4) For $\psi(t) = t, \tau = 0, \alpha = 1$ in Definition 6, we obtain the Riemann–Liouville fractional integrals presented in [26].

- (5) For $\psi(t) = t$, $\tau = 0$, $\alpha = 1$, and $\beta = 1$ in Definitions 6 and 7, we obtain the classical Riemann integral.

In the following, we prove some basic properties for the presented generalized ψ -conformable fractional operators.

Theorem 4. We consider the operators ${}^{\tau}\mathcal{D}_{p^{+}}^{\beta,\psi}$, ${}^{\tau}\mathcal{I}_{q^{-}}^{\beta,\psi}$, ${}^{\tau}\mathcal{D}_{p^{+}}^{\beta,\psi}$, and ${}^{\tau}\mathcal{D}_{q^{-}}^{\beta,\psi}$, and let f be an α -differentiable and absolutely continuous function on the interval $[p, q] \subseteq [0, \infty)$. If $\psi(t) \in C^1[0, 1]$ is an increasing function defined on the interval $[p, q]$, then

$${}^{\tau}\mathcal{D}_{p^{+}}^{\beta,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta,\psi} f(t) = f(t), \quad (21)$$

$${}^{\tau}\mathcal{D}_{q^{-}}^{\beta,\psi} {}^{\tau}\mathcal{I}_{q^{-}}^{\beta,\psi} f(t) = f(t), \quad (22)$$

Proof. We consider

$$\begin{aligned} & {}^{\tau}\mathcal{D}_{p^{+}}^{\beta,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta,\psi} f(t) \\ &= \frac{\psi(t)^{-\tau}}{\Gamma(1-\beta)} \mathcal{D}_{\alpha,\psi} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{-\beta} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta,\psi} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s, \\ &= \frac{\psi(t)^{-\tau}}{\Gamma(\beta)\Gamma(1-\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{-\beta} \psi(s)^{\tau} \\ &\quad \times \int_p^s \left(\frac{\psi(s)^{\tau+\alpha} - \psi(w)^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f(w) \psi(w)^{\tau} d_{\alpha,\psi} w d_{\alpha,\psi} s, \\ &= \frac{\psi(t)^{-\tau}(\tau+\alpha)}{\Gamma(\beta)\Gamma(1-\beta)} \int_p^t \int_p^s \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{-\beta} \\ &\quad \times \left(\psi(s)^{\tau+\alpha} - \psi(w)^{\tau+\alpha} \right)^{\beta-1} f(w) \psi(s)^{\tau} \psi(w)^{\tau} d_{\alpha,\psi} w d_{\alpha,\psi} s. \end{aligned}$$

By switching the order of integration and changing the variables to $\psi(u)$, let

$$\psi(s)^{\tau+\alpha} = \psi(w)^{\tau+\alpha} + \left(\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha} \right) \psi(u).$$

Then, we have

$$\begin{aligned} & {}^{\tau}\mathcal{D}_{p^{+}}^{\beta,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta,\psi} f(t) \\ &= \frac{\psi(t)^{-\tau}}{\Gamma(\beta)\Gamma(1-\beta)} \mathcal{D}_{\alpha,\psi} \int_p^t \int_0^1 (1-\psi(u))^{-\beta} \psi(u)^{\beta-1} f(w) \psi(w)^{\tau} du d_{\alpha,\psi} w, \\ &= \psi(t)^{-\tau} \mathcal{D}_{\alpha,\psi} \int_p^t f(w) \psi(w)^{\tau} d_{\alpha,\psi} w, \\ &= f(t). \end{aligned}$$

In a similar manner, we can prove the other case to be true. $\square$

Theorem 5. Let f be an α -differentiable and absolutely continuous function defined on the interval $[p, q] \subseteq [0, \infty)$, and let $\psi(t) \in C^1[0, 1]$ be an increasing function defined on the interval $[p, q]$. Then, for any $\alpha \in (0, 1]$ and $\beta > 0$, we have

$$\lim_{\beta \rightarrow 0} {}^{\tau}\mathcal{I}_{p^+}^{\beta, \psi} f(t) = {}^{\tau}\mathcal{I}_{p^+}^{0, \psi} f(t) = f(t), \quad (23)$$

$$\lim_{\beta \rightarrow 0} {}^{\tau}\mathcal{I}_{q^-}^{\beta, \psi} f(t) = {}^{\tau}\mathcal{I}_{q^-}^{0, \psi} f(t) = f(t), \quad (24)$$

$$\lim_{\beta \rightarrow 0} {}^{\tau}\mathfrak{D}_{p^+}^{\beta, \psi} f(t) = {}^{\tau}\mathfrak{D}_{p^+}^{0, \psi} f(t) = f(t), \quad (25)$$

$$\lim_{\beta \rightarrow 0} {}^{\tau}\mathfrak{D}_{q^-}^{\beta, \psi} f(t) = {}^{\tau}\mathfrak{D}_{q^-}^{0, \psi} f(t) = f(t). \quad (26)$$

Proof. By applying the relation in Remark 2 with respect to $\psi(t)$ and using integration by parts, we obtain

$${}^{\tau}\mathcal{I}_{p^+}^{\beta, \psi} f(t) \quad (27)$$

$$\begin{aligned} &= \frac{-1}{\Gamma(\beta + 1)} \int_p^t f(s) \frac{d}{ds} \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \\ &= \frac{-1}{\Gamma(\beta + 1)} \left(f(s) \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \Big|_{s=p}^{s=t} - \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \frac{d}{ds} f(s) \right) \\ &= \frac{-1}{\Gamma(\beta + 1)} \left(-f(p) \left(\frac{\psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} - \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \frac{d}{ds} f(s) \right). \end{aligned} \quad (28)$$

Taking the limit as $\beta \rightarrow 0$, we obtain ${}^{\tau}\mathcal{I}_{p^+}^{\beta, \psi} f(t)$, which tends to

$$\lim_{\beta \rightarrow 0} {}^{\tau}\mathcal{I}_{p^+}^{\beta, \psi} f(t) = \mathcal{I}_{p^+}^{0, \psi} f(t) = f(p) + \int_p^t \frac{d}{ds} f(s) = f(p) + f(s)|_{s=p}^{s=t} = f(t).$$

Similarly,

$$\begin{aligned} &{}^{\tau}\mathcal{I}_{q^-}^{\beta, \psi} f(t) \\ &= \frac{1}{\Gamma(\beta + 1)} \int_t^q f(s) \frac{d}{ds} \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \\ &= \frac{1}{\Gamma(\beta + 1)} \left(f(s) \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \Big|_{s=t}^{s=q} - \int_t^q \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \frac{d}{ds} f(s) \right) \\ &= \frac{1}{\Gamma(\beta + 1)} \left(f(q) \left(\frac{\psi(q)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} - \int_t^q \left(\frac{\psi(s)^{\tau+\alpha} - \psi(t)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta} \frac{d}{ds} f(s) \right). \end{aligned}$$

Taking the limit as $\beta \rightarrow 0$, we obtain

$${}^{\tau}\mathcal{I}_{q^-}^{\beta, \psi} f(t) \rightarrow {}^{\tau}\mathcal{I}_{q^-}^{0, \psi} f(t) = f(q) - \int_t^q \frac{d}{ds} f(s) = f(q) - f(s)|_{s=t}^{s=q} = f(t).$$

Form Theorem 4, we have

$${}^{\tau}\mathfrak{D}_{p^+}^{\beta, \psi} f(t) = {}^{\tau}\mathcal{I}_{p^+}^{-\beta, \psi} f(t) \text{ for } t > p,$$

and

$${}^{\tau}\mathfrak{D}_{q^{-}}^{\beta,\psi} f(t) = {}^{\tau}\mathcal{I}_{q^{-}}^{-\beta,\psi} f(t) \text{ for } t > p.$$

Hence, we deduce that

$${}^{\tau}\mathfrak{D}_{p^{+}}^{0,\psi} f(t) = {}^{\tau}\mathcal{I}_{p^{+}}^{0,\psi} f(t) = f(t), \text{ for all } t > p,$$

and

$${}^{\tau}\mathfrak{D}_{q^{-}}^{0,\psi} f(t) = {}^{\tau}\mathcal{I}_{q^{-}}^{0,\psi} f(t) = f(t), \text{ for all } q > t.$$

Thus, the proof is complete. $\square$

Theorem 6. Let $f : [p, q] \subseteq [0, \infty) \rightarrow \mathbb{R}$ be an α -differentiable function, and let $\psi(t) \in C^1[0, 1]$ be an increasing function defined on the interval $[p, q]$. Then, for $\beta_1, \beta_2 > 0$ and $\alpha \in (0, 1]$, we have

$${}^{\tau}\mathcal{I}_{p^{+}}^{\beta_1,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_2,\psi} f(t) = {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_1+\beta_2,\psi} f(t), \quad (29)$$

$${}^{\tau}\mathcal{I}_{q^{-}}^{\beta_1,\psi} {}^{\tau}\mathcal{I}_{q^{-}}^{\beta_2,\psi} f(t) = {}^{\tau}\mathcal{I}_{q^{-}}^{\beta_1+\beta_2,\psi} f(t), \quad (30)$$

$${}^{\tau}\mathfrak{D}_{p^{+}}^{\beta_1,\psi} {}^{\tau}\mathfrak{D}_{p^{+}}^{\beta_2,\psi} f(t) = {}^{\tau}\mathfrak{D}_{p^{+}}^{\beta_1+\beta_2,\psi} f(t), \quad (31)$$

$${}^{\tau}\mathfrak{D}_{q^{-}}^{\beta_1,\psi} {}^{\tau}\mathfrak{D}_{q^{-}}^{\beta_2,\psi} f(t) = {}^{\tau}\mathfrak{D}_{q^{-}}^{\beta_1+\beta_2,\psi} f(t). \quad (32)$$

Proof. Applying relation (12), we obtain

$$\begin{aligned} & {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_1,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_2,\psi} f(t) \\ &= \frac{1}{\Gamma(\beta_1)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta_1-1} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_2,\psi} f(w) \psi(w)^{\tau} d_{\alpha,\psi} w \\ &= \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta_1-1} \psi(w)^{\tau} \\ & \quad \int_p^w \left(\frac{\psi(w)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta_2-1} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s d_{\alpha,\psi} w. \end{aligned}$$

Now, using Fubini's theorem by switching the order of integration and applying the relation, we obtain

$$\begin{aligned} & {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_1,\psi} {}^{\tau}\mathcal{I}_{p^{+}}^{\beta_2,\psi} f(t) \\ &= \frac{(\tau + \alpha)^{2-(\beta_1+\beta_2)}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_p^t f(s) \psi(s)^{\tau} \\ & \quad \int_s^t \left(\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha} \right)^{\beta_1-1} \left(\psi(w)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{\beta_2-1} d_{\alpha,\psi} w d_{\alpha,\psi} s \\ &= \frac{(\tau + \alpha)^{2-(\beta_1+\beta_2)}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_p^t f(s) \psi(s)^{\tau} \\ & \quad \int_s^t \left(\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha} \right)^{\beta_1-1} \left(\psi(w)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{\beta_2-1} \psi'(w) \psi(w)^{\alpha-1+\tau} dw d_{\alpha,\psi} s. \end{aligned}$$

Changing the variables to u by using the substitution

$$\psi(s)^{\tau+\alpha} = \psi(w)^{\tau+\alpha} + \left(\psi(t)^{\tau+\alpha} - \psi(w)^{\tau+\alpha} \right) \psi(u),$$

and applying the relation $\int_0^1 \psi(u)^{\beta_2-1} (1-\psi(u))^{\beta_1-1} \psi'(u) du = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}$, we obtain

$$\begin{aligned} & {}^{\tau}\mathcal{I}_{p^+}^{\beta_1,\psi} {}^{\tau}\mathcal{I}_{p^+}^{\beta_2,\psi} f(t) \\ &= \frac{(\tau+\alpha)^{1-(\beta_1+\beta_2)}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_p^t \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{\beta_1+\beta_2-1} f(s) \psi(s)^{\tau} \\ & \quad \int_0^1 \psi(u)^{\beta_2-1} (1-\psi(u))^{\beta_1-1} \psi'(u) du d_{\alpha,\psi} s \\ &= \frac{1}{\Gamma(\beta_1+\beta_2)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta_1+\beta_2-1} f(s) \psi(s)^{\tau} d_{\alpha,\psi} s \\ &= {}^{\tau}\mathcal{I}_{p^+}^{\beta_1+\beta_2,\psi} f(t). \end{aligned}$$

Similarly, we can prove the other three cases to be true. $\square$

Notation 1. Throughout this paper, for $0 \leq p < q$, we define

$$L_{\alpha}[p, q] = \left\{ \Psi(s) : [p, q] \rightarrow \mathbb{R} : \int_p^q \Psi(s) d_{\alpha} s < \infty \right\}, \quad (33)$$

where $\alpha \in]0, 1]$, and $\int_p^q d_{\alpha} s$ is the conformable fractional integral. Furthermore, we denote by $\|\cdot\|$ the norm associated with $L_{\alpha}[p, q]$.

Theorem 7. The operators ${}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi}$ and ${}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi}$ are linear on $L_{\alpha}[p, q]$. That is, if we define ${}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi}, {}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} : L_{\alpha}[p, q] \rightarrow L_{\alpha}[p, q]$, then, for all $f_1, f_2 \in L_{\alpha}[p, q]$, $k_1, k_2 \in \mathbb{R}$, and, for the increasing function $\psi(t) \in C^1[0, 1]$ on the interval $[p, q]$, we have:

$${}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} (k_1 f_1 + k_2 f_2) = k_1 {}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} f_1 + k_2 {}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} f_2, \quad (34)$$

$${}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} (k_1 f_1 + k_2 f_2) = k_1 {}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} f_1 + k_2 {}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} f_2, \quad (35)$$

$${}^{\tau}\mathfrak{D}_{p^+}^{\beta,\psi} (k_1 f_1 + k_2 f_2) = k_1 {}^{\tau}\mathfrak{D}_{p^+}^{\beta,\psi} f_1 + k_2 {}^{\tau}\mathfrak{D}_{p^+}^{\beta,\psi} f_2, \quad (36)$$

$${}^{\tau}\mathfrak{D}_{q^-}^{\beta,\psi} (k_1 f_1 + k_2 f_2) = k_1 {}^{\tau}\mathfrak{D}_{q^-}^{\beta,\psi} f_1 + k_2 {}^{\tau}\mathfrak{D}_{q^-}^{\beta,\psi} f_2. \quad (37)$$

Proof. We consider

$$\begin{aligned} & {}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} (k_1 f_1 + k_2 f_2) \\ &= \frac{1}{\Gamma(\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \psi(s)^{\tau} (k_1 f_1 + k_2 f_2)(s) d_{\alpha,\psi} s \\ &= \frac{k_1}{\Gamma(\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \psi(s)^{\tau} f_1(s) d_{\alpha,\psi} s \\ & \quad + \frac{k_2}{\Gamma(\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \psi(s)^{\tau} f_2(s) d_{\alpha,\psi} s \\ &= k_1 {}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} f_1(t) + k_2 {}^{\tau}\mathcal{I}_{p^+}^{\beta,\psi} f_2(t). \end{aligned}$$

In a similar manner, we can show that ${}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} (k_1 f_1 + k_2 f_2)(t) = k_1 {}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} f_1(t) + k_2 {}^{\tau}\mathcal{I}_{q^-}^{\beta,\psi} f_2(t)$. $\square$

Theorem 8. The operators ${}^{\tau}\mathcal{I}_{p+}^{\beta,\psi}$ and ${}^{\tau}\mathcal{I}_{q-}^{\beta,\psi}$ are bounded on $L_{\alpha}[p, q]$. That is, if we define ${}^{\tau}\mathcal{I}_{p+}^{\beta,\psi}, {}^{\tau}\mathcal{I}_{q-}^{\beta,\psi} : L_{\alpha}[p, q] \rightarrow L_{\alpha}[p, q]$ for $\psi(t) \in C^1[0, 1]$ to be an increasing function defined on the interval $[p, q]$, then we have

$$\left\| {}^{\tau}\mathcal{I}_{p+}^{\beta,\psi} f \right\| \leq M \|f\|, \quad (38)$$

$$\left\| {}^{\tau}\mathcal{I}_{q-}^{\beta,\psi} f \right\| \leq M \|f\|, \quad (39)$$

$$\left\| {}^{\tau}\mathfrak{D}_{p+}^{\beta,\psi} f \right\| \leq M \|f\|, \quad (40)$$

$$\left\| {}^{\tau}\mathfrak{D}_{q-}^{\beta,\psi} f \right\| \leq M \|f\|. \quad (41)$$

with $\|f\| = \max_{t \in [p, q]} |f(t)|$ and $M = \left| \frac{(\tau+\alpha)^{1-\beta}}{\Gamma(\beta+1)} \left(\psi(q)^{\tau+\alpha} - \psi(p)^{\tau+\alpha} \right)^{\beta} \right|$.

Proof. We consider

$$\begin{aligned} & \left\| \frac{1}{\Gamma(\beta)} \int_p^t \left(\frac{\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha}}{\tau + \alpha} \right)^{\beta-1} \psi(s)^{\tau} f(s) d_{\alpha, \psi} s \right\| \\ & \leq \left| \frac{(\tau + \alpha)^{1-\beta}}{\Gamma(\beta)} \right| \|f\| \left\| \int_p^t \psi'(s) \left(\psi(t)^{\tau+\alpha} - \psi(s)^{\tau+\alpha} \right)^{\beta-1} \psi(s)^{\tau+\alpha-1} ds \right\| \\ & \leq \left| \frac{(\tau + \alpha)^{1-\beta}}{\Gamma(\beta + 1)} \left| \psi(t)^{\tau+\alpha} - \psi(p)^{\tau+\alpha} \right|^{\beta} \right|_{t=q}^{t=p} \|f\| \\ & \leq \left| \frac{(\tau + \alpha)^{1-\beta}}{\Gamma(\beta + 1)} \left(\psi(q)^{\tau+\alpha} - \psi(p)^{\tau+\alpha} \right)^{\beta} \right| \|f\| \\ & \leq M \|f\|, \end{aligned}$$

where $M = \left| \frac{(\tau+\alpha)^{1-\beta}}{\Gamma(\beta+1)} \left(\psi(q)^{\tau+\alpha} - \psi(p)^{\tau+\alpha} \right)^{\beta} \right|$.

In a similar manner, we can prove the other three cases. $\square$

Remark 5. Our work is considered as a comprehensive generalization of established fractional operators, including those by Khalil et al. [35], Katugampola [23], and Khan et al. [36]. The key innovation lies in the explicit incorporation of an increasing function $\psi(t)$, imparting the operators with a more versatile and robust framework capable of capturing a broader spectrum of real-world phenomena. In summary, our definitions extend beyond existing frameworks by offering a unified approach that includes and generalizes conformable fractional derivatives and integral operators, Riemann–Liouville, Hadamard, and Katugampola operators, thereby enriching the landscape of fractional calculus (see Remark 4).

3. Applications to Fractional Differential Boundary Value Problems

We consider the nonlinear implicit ψ -conformable fractional-order integro-differential equation ICFDE (1) given by

$$\frac{d^2}{dt^2} y(t) = f\left(t, y(t), {}^{\tau}\mathfrak{D}_{0+}^{\beta,\psi} y(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma,\psi} \left(\varphi(t, s) {}^{\tau}\mathfrak{D}_{0+}^{\theta,\psi} y(s) \right)(t)\right), t \in [0, 1], \quad (42)$$

and subjected to the following set of two boundary conditions:

$$\begin{cases} y(\eta) = {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(t, y(t)), \eta \in (0, 1], \\ y(\delta) = {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(t, y(t)), \delta \in (0, 1], \end{cases} \quad (43)$$

where $y(t) \in C^2[0, 1]$, the function $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is considered as α -differentiable with respect to $t \in [0, 1]$ and is absolutely continuous, $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, $\psi(t) \in C^1[0, 1]$ is strictly increasing on $[0, 1]$, and $G_i : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ for $i \in \{1, 2\}$ are functions with conformable integrability. Additionally, ${}^{\nu}\mathcal{I}_{0+}^{\sigma,\psi}$, ${}^{\tau}\mathcal{I}_{0+}^{\eta,\psi}$, and ${}^{\tau}\mathcal{I}_{0+}^{\delta,\psi}$ are generalized ψ -conformable fractional integral operators of respective orders $\alpha, \sigma, \delta, \eta \in (0, 1]$ with $\eta < \delta$, $\tau \in \mathbb{R}$, and $\alpha + \tau \neq 0$. Moreover, ${}^{\tau}\mathfrak{D}_{0+}^{\beta,\psi}$ and ${}^{\tau}\mathfrak{D}_{0+}^{\theta,\psi}$ are generalized ψ -conformable fractional derivative operators of respective orders $\beta, \theta \in (0, 1]$ with $\beta < \theta$.

Lemma 1. The implicit second-order generalized ψ -conformable fractional-order differential equation ICFDE (1) is equivalent to the following integral equation:

$$y(t) = A(t, y(t)) + \int_0^t G(t, s)u(s)ds, \quad (44)$$

with

$$u(t) = f\left(t, y(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta,\psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma,\psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta,\psi} u(s)\right)(t)\right), \quad (45)$$

$$A(t, y(t)) = \frac{(\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\gamma,\psi} G_1(\eta, y(\eta)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\gamma,\psi} G_2(\delta, y(\delta))}{(\delta - \eta)}, \quad (46)$$

and $G(t, s)$ is Green's function defined by

$$G(t, s) = \begin{cases} 0 & \text{if } 0 \leq s \leq t \leq \eta < \delta \leq 1, \\ (s - t) & \text{if } 0 \leq t \leq s \leq \eta < \delta \leq 1, \\ \frac{(\eta - t)(\delta - s)}{(\delta - \eta)} & \text{if } 0 \leq t \leq \eta \leq s < \delta \leq 1. \end{cases} \quad (47)$$

with $\mathfrak{G}_0 = \max\{|G(t, s)| \mid \forall (t, s) \in [0, 1] \times [0, 1]\}$.

Proof. It is clear that ${}^{\tau}\mathfrak{D}_{0+}^{\beta,\psi} y(t) = {}^{\tau}\mathcal{I}_{0+}^{2-\beta,\psi} \frac{d^2}{dt^2} y(t)$ and ${}^{\tau}\mathfrak{D}_{0+}^{\delta,\psi} y(t) = {}^{\tau}\mathcal{I}_{0+}^{2-\delta,\psi} \frac{d^2}{dt^2} y(t)$ for all $t \in (0, 1)$. Hence, if $y(t)$ is a solution of ICFDE (1) and $u(t) = \frac{d^2}{dt^2} y(t)$, then we obtain the following equality:

$$u(t) = f\left(t, y(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta,\psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma,\psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta,\psi} u(s)\right)(t)\right), \quad (48)$$

with

$$y(t) = c_0 + c_1 t + \int_0^t (t - s)u(s) ds. \quad (49)$$

Applying the boundary conditions in ICFDE (1) and performing straightforward calculations, we obtain that

$$\begin{cases} c_0 + c_1 \eta + \int_0^\eta (\eta - s)u(s)ds = {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)), \eta \in (0, 1], \\ c_0 + c_1 \delta + \int_0^\delta (\delta - s)u(s)ds = {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)), \delta \in (0, 1]. \end{cases} \quad (50)$$

Solving the above linear system for c_0 and c_1 , we obtain

$$c_0 = \frac{\delta {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - \eta {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) + \eta \int_0^\delta (\delta - s)u(s) ds - \delta \int_0^\eta (\eta - s)u(s) ds}{(\delta - \eta)}, \quad (51)$$

and

$$c_1 = \frac{- {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) + {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) - \int_0^{\delta} (\delta - s)u(s) ds + \int_0^{\eta} (\eta - s)u(s) ds}{(\delta - \eta)}. \quad (52)$$

Replacing c_0 and c_1 by their equivalents in (49), the solution of ICFDE (1) is provided by

$$\begin{aligned} y(t) &= \frac{\left((\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) \right. \\ &\quad \left. + (\delta - \eta) \int_0^t (t - s)u(s) ds + (\eta - t) \int_0^{\delta} (\delta - s)u(s) ds + (t - \delta) \int_0^{\eta} (\eta - s)u(s) ds \right)}{(\delta - \eta)} \\ &= \frac{\left((\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) \right. \\ &\quad \left. + (\delta - \eta) \int_0^t (t - s)u(s) ds + (\eta - t) \int_0^t (\delta - s)u(s) ds + (t - \delta) \int_0^t (\eta - s)u(s) ds \right. \\ &\quad \left. + (\eta - t) \int_t^{\eta} (\delta - s)u(s) ds + (\eta - t) \int_{\eta}^{\delta} (\delta - s)u(s) ds + (t - \delta) \int_t^{\eta} (\eta - s)u(s) ds \right)}{(\delta - \eta)} \\ &= \frac{\left((\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) \right. \\ &\quad \left. + \int_0^t [(\delta - \eta)(t - s) + (\eta - t)(\delta - s) + (t - \delta)(\eta - s)]u(s) ds \right. \\ &\quad \left. + \int_t^{\eta} (\eta - t)(\delta - s)u(s) ds + \int_t^{\eta} (t - \delta)(\eta - s)u(s) ds + \int_{\eta}^{\delta} (\eta - t)(\delta - s)u(s) ds \right)}{(\delta - \eta)} \\ &= \frac{\left((\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s)) \right. \\ &\quad \left. + \int_0^t 0u(s) ds + \int_t^{\eta} (s - t)(\delta - \eta)u(s) ds + \int_{\eta}^{\delta} (\eta - t)(\delta - s)u(s) ds \right)}{(\delta - \eta)}. \end{aligned}$$

This implies that

$$y(t) = A(t, y(t)) + \int_0^t G(t, s)u(s)ds,$$

where

$$A(t, y(t)) = \frac{(\delta - t) {}^{\tau}\mathcal{I}_{0+}^{\eta,\psi} G_1(s, y(s)) - (\eta - t) {}^{\tau}\mathcal{I}_{0+}^{\delta,\psi} G_2(s, y(s))}{(\delta - \eta)}, \quad (53)$$

and

$$G(t, s) = \begin{cases} 0 & \text{if } 0 \leq s \leq t \leq \eta < \delta \leq 1, \\ (s - t) & \text{if } 0 \leq t \leq s \leq \eta < \delta \leq 1, \\ \frac{(\eta - t)(\delta - s)}{(\delta - \eta)} & \text{if } 0 \leq t \leq \eta \leq s \leq \delta \leq 1. \end{cases} \quad (54)$$

The proof is complete. $\square$

Remark 6. The following appropriate assumptions will be used through this work in order to use Banach's and Krasnoselskii's fixed-point theorems to establish our main results:

(A₁) The functions $G_i : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuous and there exist positive constants $k_i \in (0, 1)$ such that

$$|G_i(t, x(t)) - G_i(t, y(t))| \leq k_i |x(t) - y(t)| \text{ for all } t \in [0, 1], \text{ and } i = 1, 2 \quad (55)$$

(A₂) The function $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and there exists $\omega(t) \in C([0, 1], \mathbb{R}^+)$, with norm $\|\omega\|$, such that:

$$|f(t, v_1, v_2, v_3) - f(t, u_1, u_2, u_3)| \leq \omega(t)(|v_1 - u_1| + |v_2 - u_2| + |v_3 - u_3|), \quad (56)$$

for every $t \in [0, 1]$, and $u_i, v_i \in \mathbb{R}$, ($i = 1, 2, 3$).

(A₃) $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous for all $(s, t) \in [0, 1] \times [0, 1]$ with $\Phi = \max_{t \in [0, 1]} |\varphi(t, s)|$.

Remark 7. From assumption (A₂), we can obtain the following observations:

- (1) $|f(t, u, v, w)| \leq \omega(t)(|u| + |v| + |w|) + \mathcal{F}$, where $\mathcal{F} = \sup_{t \in [0, 1]} f(t, 0, 0, 0)$.
- (2) $|G_i(t, x(t))| \leq \mathcal{G}_i + k_i|v_i|$, where $\mathcal{G}_i = \sup_{t \in [0, 1]} G_i(t, 0)$ for $i = 1, 2$.

Theorem 9. The function $A : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, is a Lipschitzian function with respect to the second variable with Lipschitz constant $c = \frac{(\tau+\alpha)^{-\gamma}}{\gamma+1} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right| \frac{|k_1(1-\delta) + k_2(1-\eta)|}{|\delta-\eta|}$, where $\gamma = \max\{\eta, \delta\}$.

Proof. We consider the variables $x(t), y(t) : [0, 1] \rightarrow \mathbb{R}$. Then,

$$\begin{aligned}
 & |A(t, x(t)) - A(t, y(t))| \\
 &= \left| \frac{(\delta - t) {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} G_1(t, x(t)) - (\eta - t) {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} G_2(s, x(s))}{(\delta - \eta)} \right. \\
 &\quad \left. - \frac{(\delta - t) {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} G_1(t, x(t)) - (\eta - t) {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} G_2(t, y(t))}{(\delta - \eta)} \right| \\
 &\leq \frac{|(\delta - t) {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} (G_1(t, x(t)) - G_1(t, y(t))) + (\eta - t) {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} (G_2(t, x(t)) - G_2(t, y(t)))|}{|\delta - \eta|} \\
 &\leq \left| \frac{1 - \delta}{\delta - \eta} \right| \left\| {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} (G_1(t, x(t)) - G_1(t, y(t))) \right\| + \left| \frac{1 - \eta}{\delta - \eta} \right| \left\| {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} (G_2(t, x(t)) - G_2(t, y(t))) \right\| \\
 &\leq \left| \frac{1 - \delta}{\delta - \eta} \right| M \|G_1(t, x(t)) - G_1(t, y(t))\| + \left| \frac{1 - \eta}{\delta - \eta} \right| M \|G_2(t, x(t)) - G_2(t, y(t))\| \\
 &\leq M \left| \frac{k_1(1 - \delta) + k_2(1 - \eta)}{\delta - \eta} \right| \|x - y\|,
 \end{aligned}$$

where $\|x - y\| = \max_{t \in [0, 1]} |x(t) - y(t)|$ and $M = \frac{(\tau+\alpha)^{-\gamma}}{\gamma+1} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right|$, and where $\gamma = \max\{\eta, \delta\}$. Hence, if the inequality

$$0 < \frac{(\tau + \alpha)^{-\gamma}}{\gamma + 1} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right| \frac{|k_1(1 - \delta) + k_2(1 - \eta)|}{|\delta - \eta|} < 1, \quad (57)$$

then $A(t, y(t))$ is Lipschitzian with a Lipschitz constant

$$c = \frac{(\tau + \alpha)^{-\gamma}}{\gamma + 1} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right| \frac{|k_1(1 - \delta) + k_2(1 - \eta)|}{|\delta - \eta|}. \quad (58)$$

□

3.1. Existence of Solutions

Theorem 10. If the assumptions $(A_1) - (A_3)$ hold, then the ICFDE (1) has at least one solution if

$$\frac{(\tau + \alpha)^{-\gamma}}{\gamma + 1} \left| \psi(1)^{\tau + \alpha} - \psi(0)^{\tau + \alpha} \right| \frac{|k_1(1 - \delta) + k_2(1 - \eta)|}{|\delta - \eta|} < 1. \quad (59)$$

Proof. The existence result for ICFDE (1) is based on the fixed-point theorem of Krasnoselskii [39]. We define the operator $T : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ by

$$Ty(t) = A(t, y(t)) + \int_0^t G(t, s)u(s)ds, \quad (60)$$

where $u \in C([0, 1], \mathbb{R})$ satisfies the implicit functional integral equation

$$u(t) = f\left(t, A(t, y(t)) + \int_0^t G(t, s)u(s)ds, {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi}\left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u(s)\right)(t)\right) \quad (61)$$

with $A(t, y(t))$ defined by (53).

Let ρ be chosen such that

$$\rho \geq \frac{N_1 + \frac{\mathfrak{G}_0 \mathcal{F}}{1 - \|\omega\|(M_1 + M_2)}}{1 - \left(N_2 + \frac{\mathfrak{G}_0 \|\omega\|}{1 - \|\omega\|(M_1 + M_2)}\right)}, \quad (62)$$

with

$$N_1 = \aleph(|1 - \delta|(\mathcal{G}_1) + |1 - \eta|(\mathcal{G}_2)), \quad (63)$$

$$N_2 = \aleph(k_1|1 - \delta| + k_2|1 - \eta|), \quad (64)$$

where

$$\aleph = \frac{M}{|\delta - \eta|} \frac{(\tau + \alpha)^{-\gamma}}{\gamma + 1} \left| \psi(1)^{\tau + \alpha} - \psi(0)^{\tau + \alpha} \right|. \quad (65)$$

Define the closed disk

$$B_\rho = \{y \in C([0, 1], \mathbb{R}) : \|y\| \leq \rho\}, \quad (66)$$

In addition, define for $t \in [0, 1]$ and $y(t) \in C([0, 1], \mathbb{R})$ the operators $T_1(y(t))$ and $T_2(y(t))$ on B_ρ as

$$T_1(y(t)) = A(t, y(t)), \quad (67)$$

and

$$T_2(y(t)) = \int_0^t G(t, s)u(s)ds. \quad (68)$$

Thus,

$$T(y(t)) = T_1(y(t)) + T_2(y(t)) \text{ for } t \in [0, 1]. \quad (69)$$

The proof will be decomposed into three steps:

- (1) Step 1: $T_1y_1 + T_2y_2 \in B_\rho$ for every $y_1, y_2 \in B_\rho$.
Let $y_1, y_2 \in B_\rho$, and $t \in [0, 1]$. Then, by using (A)

$$\begin{aligned} |T_1y_1(t) + T_2y_2(t)| &\leq |T_1y_1(t)| + |T_2y_2(t)| \\ &\leq |A(t, y_1(t))| + \int_0^t |G(t, s)||u(s)|ds, \end{aligned}$$

where

$$\begin{aligned}
 |A(t, y_1(t))| &= \frac{|(\delta - t) {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} G_1(t, y_1(t)) - (\eta - t) {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} G_2(t, y_1(t))|}{|\delta - \eta|} \\
 &\leq \frac{|\delta - t|}{|\delta - \eta|} \| {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} \| |G_1(t, y_1(t))| + \frac{|\eta - t|}{|\delta - \eta|} \| {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} \| |G_2(t, y_1(t))| \\
 &\leq \frac{|\delta - t|}{|\delta - \eta|} \| {}^{\tau} \mathcal{I}_{0+}^{\eta, \psi} \| (\mathcal{G}_1 + k_1 |y_1|) + \frac{|\eta - t|}{|\delta - \eta|} \| {}^{\tau} \mathcal{I}_{0+}^{\delta, \psi} \| (\mathcal{G}_2 + k_2 |y_1|) \\
 &\leq \aleph (|1 - \delta|(\mathcal{G}_1) + |1 - \eta|(\mathcal{G}_2)) + \aleph (k_1 |1 - \delta| + k_2 |1 - \eta|) \rho \\
 &\leq N_1 + N_2 \rho,
 \end{aligned}$$

and

$$\int_0^t |G(t, s)| |u(s)| ds \leq \mathfrak{G}_0 \|u\| \quad (70)$$

By applying (A₂), (A₃), and Remark 7, we have v

$$\begin{aligned}
 |u(t)| &= |f(t, y(t), {}^{\tau} \mathcal{I}_{0+}^{2-\beta, \psi} u(t), {}^{\nu} \mathcal{I}_{0+}^{\sigma, \psi} (\varphi(t, s) {}^{\tau} \mathcal{I}_{0+}^{2-\theta, \psi} u(s))(t))| \\
 &\leq \mathcal{F} + \omega(t)(|y(t)| + |{}^{\tau} \mathcal{I}_{0+}^{2-\beta, \psi} u(t)| + |{}^{\nu} \mathcal{I}_{0+}^{\sigma, \psi} (\varphi(t, s) {}^{\tau} \mathcal{I}_{0+}^{2-\theta, \psi} u(s))(t)|) \\
 &\leq \mathcal{F} + |\omega(t)|(|y(t)| + M_1 \|u\| + M_2 \|u\|),
 \end{aligned}$$

where

$$M_1 = \frac{(\tau + \alpha)^{-2+\beta}}{3 - \beta} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right|^{2-\beta}, \quad (71)$$

and

$$M_2 = \frac{(\tau + \alpha)^{-2-\sigma+\theta}}{(3 - \theta)(\sigma + 1)} \Phi \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right|^{2+\sigma-\theta}. \quad (72)$$

Taking the supremum for all $t \in [0, 1]$, we obtain

$$\|u\| \leq \frac{\mathcal{F} + \|\omega\| \rho}{1 - \|\omega\| (M_1 + M_2)}. \quad (73)$$

Hence, we obtain that

$$|T_1 y_1(t) + T_2 y_2(t)| \leq N_1 + N_2 \rho + \mathfrak{G}_0 \frac{\mathcal{F} + \|\omega\| \rho}{1 - \|\omega\| (M_1 + M_2)} \leq \rho. \quad (74)$$

Thus, $T_1 y_1 + T_2 y_2 \in B_\rho$ for every $y_1, y_2 \in B_\rho$ with

$$\rho \geq \frac{N_1 + \frac{\mathfrak{G}_0 \mathcal{F}}{1 - \|\omega\| (M_1 + M_2)}}{1 - \left(N_2 + \frac{\mathfrak{G}_0 \|\omega\|}{1 - \|\omega\| (M_1 + M_2)} \right)}. \quad (75)$$

- (2) Step 2: The operator T_1 is a contraction mapping on B_ρ .
Theorem 9 clearly indicates that the operator T_1 can be considered a contraction mapping for $c < 1$. This is evident when we observe that

$$c = \frac{(\tau + \alpha)^{-\gamma}}{\gamma + 1} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right| \frac{|k_1(1 - \delta) + k_2(1 - \eta)|}{|\delta - \eta|}. \quad (76)$$

- (3) Step 3: The operator T_2 is completely continuous on B_ρ .
To prove that T_2 is completely continuous on B_ρ , we need to establish its boundedness and its ability to map weakly convergent sequences to norm-convergent sequences.

It is clear that operator T_2 is continuous, since, for $\{y_n\}_{n \in \mathbb{N}}$, it is a sequence with $y_n \rightarrow y$ as $n \rightarrow \infty$ in $C([0, 1], \mathbb{R})$. Then, for $u_n, u \in C([0, 1], \mathbb{R})$ and for every $t \in [0, 1]$, we have

$$|T_2 y_n - T_2 y| = \int_0^t |G(t, s)| |u_n(s) - u(s)| ds, \quad (77)$$

where

$$u_n(t) = f\left(t, y_n(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u_n(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u_n(s)\right)(t)\right), \quad (78)$$

and

$$u(t) = f\left(t, y(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u(s)\right)(t)\right). \quad (79)$$

Hence,

$$\begin{aligned} |u_n(t) - u(t)| &= \left| f\left(t, y_n(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u_n(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u_n(s)\right)(t)\right) \right. \\ &\quad \left. - f\left(t, y(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u(s)\right)(t)\right) \right| \\ &\leq \omega(t) \left(|y_n(t) - y(t)| + \left| {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} (u_n(t) - u(t)) \right| \right. \\ &\quad \left. + \left| {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} (u_n(s) - u(s))\right)(t) \right| \right) \\ &\leq \|w\| (\|y_n - y\| + M_1 \|u_n - u\| + M_2 \|u_n - u\|), \end{aligned}$$

where M_1 and M_2 are given in (71) and (72).

Taking the supremum for all $t \in [0, 1]$, we obtain that, if $\|w\|(M_1 + M_2) < 1$, then

$$\|u_n - u\| \leq \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \|y_n - y\|. \quad (80)$$

Consequently, we can deduce that $\lim_{n \rightarrow \infty} u_n = u$, as a result of $\lim_{n \rightarrow \infty} y_n = y$ for every $t \in [0, 1]$, as $n \rightarrow \infty$.

Now, let $\varepsilon > 0$ be such that, for each $t \in [0, 1]$, we have $|u_n(t)| \leq \varepsilon/2$, $|u(t)| \leq \varepsilon/2$, and the function $s \rightarrow \varepsilon|G(t, s)|$ is integrable on $[0, 1]$. By applying Lebesgue's dominated convergence theorem, we obtain that

$$\begin{aligned} |T_2 y_n - T_2 y| &= \int_0^t |s - t| |u_n(s) - u(s)| ds \\ &\leq \int_0^t |G(t, s)| (|u_n(s)| + |u(s)|) ds \\ &\leq \varepsilon \int_0^t |G(t, s)| ds. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} \|T_2 y_n - T_2 y\| = 0$. This implies that T_2 is a continuous operator.

Moreover, since

$$\|T_2 y\| \leq \mathfrak{G}_0 \frac{\mathcal{F} + \|\omega\|\rho}{1 - \|\omega\|(M_1 + M_2)} \leq \rho, \quad (81)$$

then T_2 is uniformly bounded on B_ρ .

Finally, we demonstrate that bounded sets are transformed into equicontinuous sets in the space $C([0, 1], \mathbb{R})$ by the mapping T_2 . To establish this, we need to verify that the set B_ρ is equicontinuous.

We suppose that, for any given positive value ϵ , there exists a positive value δ such that, for every $y \in B_\rho$ and any pair of points $t_1, t_2 \in [0, 1]$, if $|t_2 - t_1| < \delta$, then

$$\begin{aligned} |T_2 y(t_2) - T_2 y(t_1)| &\leq \int_0^t |G(t_2, s) - G(t_1, s)| |u(s)| ds \\ &\leq \|u\| \int_0^t |G(t_2, s) - G(t_1, s)| ds \\ &\leq \frac{\mathcal{F} + \|\omega\| \rho}{1 - \|\omega\| (M_1 + M_2)} \int_0^t |G(t_2, s) - G(t_1, s)| ds \end{aligned}$$

It is clear that, as $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to zero. Thus,

$$\lim_{t_1 \rightarrow t_2} |T_2 y(t_2) - T_2 y(t_1)| = 0. \quad (82)$$

Consequently, the set $\{Ty\}$ demonstrates equicontinuity on B_ρ and, by virtue of the Arzela–Ascoli theorem [21], we establish that $T : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ is a compact operator.

Hence, all the conditions required by Krasnoselskii's fixed-point theorem are fulfilled, thereby demonstrating that $T = T_1 + T_2$ possesses a fixed point on B_ρ . Therefore, the ICFDE (1) has at least one solution. The proof is complete.

□

3.2. Uniqueness of Solutions

In the subsequent discussion, we will demonstrate the existence and uniqueness of solutions to the ICFDE (1) through the application of Banach's fixed-point theorem.

Theorem 11. *If the assumptions $(A_1) - (A_3)$ and the inequality in Theorem 10 hold, then the ICFDE (1) has a unique solution if*

$$c + \frac{\|w\|}{1 - \|w\| (M_1 + M_2)} < 1. \quad (83)$$

Proof. According to Theorem 10, the ICFDE (1) possesses at least one solution. This is accomplished by demonstrating that the operator $Ty(t) = A(t, y(t)) + \int_0^t (s - t)u(s)ds$ fulfills all the necessary conditions stated in Krasnoselskii's fixed-point theorem. Thus, the task of establishing the contraction property of the operator T is sufficient for the uniqueness of the solution.

Let $x(t), y(t) \in C([0, 1], \mathbb{R})$ such that

$$\begin{aligned} Tx(t) &= A(t, x(t)) + \int_0^t G(t, s)u(s)ds, \\ Ty(t) &= A(t, y(t)) + \int_0^t G(t, s)v(s)ds. \end{aligned}$$

Then, by using the results obtained in Theorem 9, we obtain that every $t \in [0, 1]$:

$$\begin{aligned} |Tx(t) - Ty(t)| &= |A(t, x(t)) - A(t, y(t)) + \int_0^t G(t, s)u(s)ds - \int_0^t G(t, s)v(s)ds| \\ &\leq |A(t, x(t)) - A(t, y(t))| + \int_0^t G(t, s)|u(s) - v(s)|ds. \end{aligned}$$

From Theorem 9, we have

$$|A(t, x(t)) - A(t, y(t))| \leq M \left| \frac{k_1(1 - \delta) + k_2(1 - \eta)}{\delta - \eta} \right| \|x - y\|, \quad (84)$$

and

$$\begin{aligned}
 |u(t) - v(t)| &= \left| f\left(t, x(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} u(s)\right)(t)\right) \right. \\
 &\quad \left. - f\left(t, y(t), {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} v(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} v(s)\right)(t)\right) \right| \\
 &\leq \omega(t) \left(|x(t) - y(t)| + \left| {}^{\tau}\mathcal{I}_{0+}^{2-\beta, \psi} (u(t) - v(t)) \right| \right. \\
 &\quad \left. + \left| {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{I}_{0+}^{2-\theta, \psi} (u(s) - v(s))\right)(t) \right| \right) \\
 &\leq \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \|x - y\|,
 \end{aligned}$$

where M, M_1 , and M_2 are defined in Theorems 8 and 10. Thus,

$$|Tx(t) - Ty(t)| \leq \mathfrak{S} \|x - y\|, \quad (85)$$

where

$$\mathfrak{S} = c + \frac{\|w\|}{1 - \|w\|(M_1 + M_2)}. \quad (86)$$

Based on the given assumption $\mathfrak{S} < 1$, it follows that the operator T is a contraction. Consequently, applying Banach's contraction principle, we deduce that T possesses a unique fixed point $y^*(t)$, which serves as the unique solution to ICFDE (1). Thus, the proof is finished. $\square$

3.3. Ulam–Hyers Stability of Solutions

In the following analysis, we examine the stability of ICFDE (1) according to the Ulam–Hyers criteria. Let $\varepsilon > 0$, $\Theta : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, and we consider the following inequalities:

$$\left| \frac{d^2}{dt^2} y(t) - f\left(t, y(t), {}^{\tau}\mathcal{D}_{0+}^{\beta, \psi} y(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{D}_{0+}^{\theta, \psi} y(s)\right)(t)\right) \right| \leq \varepsilon, \quad t \in [0, 1], \quad (87)$$

and

$$\left| \frac{d^2}{dt^2} y(t) - f\left(t, y(t), {}^{\tau}\mathcal{D}_{0+}^{\beta, \psi} y(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathcal{D}_{0+}^{\theta, \psi} y(s)\right)(t)\right) \right| \leq \Theta(t), \quad t \in [0, 1]. \quad (88)$$

Definition 8. [40] The ICFDE (1) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that there exists a solution $x \in C([0, 1], \mathbb{R})$ of (1) such that

$$|y(t) - x(t)| \leq \varepsilon c_f \quad \forall t \in [0, 1]. \quad (89)$$

for each solution $y \in C([0, 1], \mathbb{R})$ of the inequality (87).

Definition 9. [40] The ICFDE (1) is generalized to be Ulam–Hyers stable if there is $c_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $c_f(0) = 0$ so that there is a solution $x \in C([0, 1], \mathbb{R})$ with

$$|y(t) - x(t)| \leq c_f \varepsilon, \quad \forall t \in [0, 1]. \quad (90)$$

for each $\varepsilon > 0$ and for each solution $y \in C([0, 1], \mathbb{R})$ of the inequality (88).

Theorem 12. We suppose that the assumptions of Theorem (11) are satisfied. Then, ICFDE (1) is Ulam–Hyers stable.

Proof. Let $\varepsilon > 0$ and let $z \in C([0, 1], \mathbb{R})$ be a function which satisfies inequality (87), such that

$$\left| \frac{d^2}{dt^2} z(t) - f\left(t, z(t), {}^{\tau}\mathfrak{D}_{0+}^{\beta, \psi} z(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathfrak{D}_{0+}^{\theta, \psi} z(s) \right)(t) \right) \right| \leq \varepsilon, \quad \forall t \in [0, 1], \quad (91)$$

and let $y \in C([0, 1], \mathbb{R})$ be the unique solution of ICFDE (1), which, by Lemma 49, is equivalent to the fractional-order integral equation

$$y(t) = A(t, y(t)) + \int_0^t G(t, s) u(s) ds, \quad (92)$$

where u is the solution of the functional integral equation

$$u(t) = f\left(t, A(t, y(t)) + \int_0^t G(t, s) u(s) ds, {}^{\tau}\mathfrak{D}_{0+}^{\beta, \psi} u(t), {}^{\nu}\mathcal{I}_{0+}^{\sigma, \psi} \left(\varphi(t, s) {}^{\tau}\mathfrak{D}_{0+}^{\theta, \psi} u(s) \right)(t) \right). \quad (93)$$

Taking the left-sided ψ -conformable fractional integral operators ${}^{\tau}\mathcal{I}_{0+}^{2, \psi}$ on both sides of inequality (91), and then integrating, we obtain

$$|z(t) - A(t, y(t)) - \int_0^t G(t, s) u(s) ds| \leq \frac{\varepsilon}{2}. \quad (94)$$

For each $t \in [0, 1]$, we have

$$\begin{aligned} |z(t) - y(t)| &= |z(t) - A(t, y(t)) - \int_0^t G(t, s) u(s) ds| \\ &\leq |z(t) - A(t, z(t)) - \int_0^t G(t, s) v(s) ds \\ &\quad - A(t, y(t)) - \int_0^t G(t, s) u(s) ds + A(t, z(t)) + \int_0^t G(t, s) v(s) ds| \\ &\leq |z(t) - A(t, z(t)) - \int_0^t G(t, s) v(s) ds| + |A(t, z(t)) - A(t, y(t))| \\ &\quad + \int_0^t |G(t, s)| |v(s) - u(s)| ds \\ &\leq \frac{\varepsilon}{2} + c \|z - y\| + \mathfrak{G}_0 \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \|z - y\|. \end{aligned}$$

This implies that, for each $t \in [0, 1]$,

$$\|z - y\| \leq \frac{\varepsilon}{2} + \left(c + \mathfrak{G}_0 \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \right) \|z - y\|. \quad (95)$$

Thus,

$$\|z - y\| \leq \frac{\varepsilon}{2} \left[1 - \left(c + \mathfrak{G}_0 \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \right) \right]^{-1} = c_f \varepsilon. \quad (96)$$

If we take $c_f = \frac{1}{2} \left[1 - \left(c + \mathfrak{G}_0 \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \right) \right]^{-1}$, we obtain that the ICFDE (1) is Ulam–Hyers stable. $\square$

Remark 8. If we take $\Theta(\varepsilon) = \frac{\varepsilon}{2} \left[1 - \left(c + \mathfrak{G}_0 \frac{\|w\|}{1 - \|w\|(M_1 + M_2)} \right) \right]^{-1}$, then we obtain $\Theta(0) = 0$, which yields that the ICFDE (1) is generalized Ulam–Hyers stable.

3.4. Numerical Examples

Example 1. We consider the following non-linear implicit ψ -conformable fractional-order boundary value problem (ICFBVP) subject to dual nonlocal boundary conditions.

$$\left\{ \begin{array}{l} \frac{d^2}{dt^2} y(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}} \left[\frac{7+|y(t)| + \left| \frac{5}{2} \mathcal{D}_{0+}^{\frac{3}{5}, t^2+2t+1} y(t) \right| + \left| \frac{3}{4} \mathcal{I}_{0+}^{\frac{3}{5}, t^2+2t+1} \left(e^{3(t-s)} \frac{5}{2} \mathcal{D}_{0+}^{\frac{2}{5}, t^2+2t+1} y(s) \right) (t) \right|}{1+|y(t)| + \left| \frac{5}{2} \mathcal{D}_{0+}^{\frac{3}{5}, t^2+2t+1} y(t) \right| + \left| \frac{3}{4} \mathcal{I}_{0+}^{\frac{3}{5}, t^2+2t+1} \left(e^{3(t-s)} \frac{5}{2} \mathcal{D}_{0+}^{\frac{2}{5}, t^2+2t+1} y(s) \right) (t) \right|} \right] \quad \forall t \in (0, 1), \\ y\left(\frac{2}{7}\right) = \frac{5}{2} \mathcal{I}_{0+}^{\frac{2}{7}, t^2+2t+1} \left(\frac{e^{-3t}}{69+\sqrt{t}} + \frac{1}{23} |\cos \sqrt{y(t)}| \right), \\ y\left(\frac{4}{7}\right) = \frac{5}{2} \mathcal{I}_{0+}^{\frac{4}{7}, t^2+2t+1} \left(\frac{2}{\sqrt{69+t^2}} + \frac{2e^{-3t}}{23+t^2} |y(t)| \right), \end{array} \right. \quad (97)$$

In this problem, we are provided with the values $\tau = \frac{5}{7}$ and $\alpha = \frac{1}{2}$, where $\alpha + \tau$ is not equal to zero. Additionally, we have $\beta = \frac{3}{5} > 0$, $\theta = \frac{2}{5}$, $\nu = \frac{3}{4}$, $\varsigma = \frac{1}{3}$, $\sigma = \frac{3}{5}$, $\eta = \frac{2}{7}$, $\delta = \frac{4}{7}$, $\gamma = \max\{\eta, \delta\} = \frac{4}{7}$, $\varphi(t, s) = e^{3(t-s)}$ with $\Phi = \max_{t \in [0,1]} |\varphi(t, s)| \leq 21$. $G_1(t, y(t)) = \left(\frac{e^{-3t}}{699+\sqrt{t}} + \frac{1}{533} |\cos \sqrt{y(t)}| \right)$ with $k_1 = \frac{1}{533}$ and $\mathcal{G}_1 = \frac{1}{699}$, and $G_2(t, y(t)) = \left(\frac{1}{\sqrt{699+t^2}} + \frac{e^{-3t}}{533+t^2} |y(t)| \right)$ with $k_2 = \frac{1}{533}$ and $\mathcal{G}_2 = \frac{1}{\sqrt{699}}$. Furthermore, the function $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous. There exists $\omega(t) = \frac{\sqrt{2t+1}}{69e^{2t+1}}$ with a norm $\|\omega\| = \frac{1}{69e}$, and $\mathcal{F} = \frac{7}{69e}$ such that, for all $t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}$ ($i = 1, 2, 3$), the following inequalities hold:

$$|f(t, v_1, v_2, v_3) - f(t, u_1, u_2, u_3)| \leq \omega(t)(|v_1 - u_1| + |v_2 - u_2| + |v_3 - u_3|),$$

and

$$|f(t, u, v, w)| \leq \omega(t)(|u| + |v| + |w|) + \mathcal{F},$$

where $\mathcal{F} = \sup_{t \in [0,1]} f(t, 0, 0, 0)$. Thus, the conditions $(A_1) - (A_3)$ are met, and, as $c = 0.0187364 < 1$,

then, according to Theorem 10, we deduce that the initial and boundary value problem denoted as ICFBVP (97) has at least one solution. In addition, since $\Phi = \max_{t \in [0,1]} |\varphi(t, s)| \leq 21$,

$$M_1 = \frac{(\tau + \alpha)^{-2+\beta}}{3 - \beta} \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right|^{2-\beta} = \frac{35 \left(\frac{7}{17} \right)^{2/5} \left(4 \cdot 2^{3/7} - 1 \right)^{7/5}}{51 \cdot 2^{3/5}} = 2.51364,$$

and

$$M_2 = \frac{(\tau + \alpha)^{-2-\sigma+\theta}}{(3 - \theta)(\sigma + 1)} \Phi \left| \psi(1)^{\tau+\alpha} - \psi(0)^{\tau+\alpha} \right|^{2+\sigma-\theta} = \frac{1225 \sqrt[5]{\frac{7}{17}} \left(4 \cdot 2^{3/7} - 1 \right)^{11/5} e^3}{3468 \cdot 2^{4/5}} = 88.12.$$

Since $\mathfrak{G}_0 \approx 0.5714286$, this implies that

$$c + \mathfrak{G}_0 \frac{\|\omega\|}{1 - \|\omega\|(M_1 + M_2)} = 0.02463177 < 1.$$

Subsequently, as per Theorems 11 and 12, it can be deduced that the solution to the ICFBVP (97) is unique and exhibits Ulam–Hyers stability.

Example 2. We consider the following non-linear implicit ψ -conformable fractional-order boundary value problem (ICFBVP) subject to dual nonlocal boundary conditions:

$$\left\{ \begin{array}{l} \frac{d^2}{dt^2} y(t) = \frac{e^{-3t}}{e^{3t}+5} \left[\frac{|y(t)|}{2+|y(t)|} + \frac{\left| \frac{5}{2} \mathcal{D}_{0+}^{\frac{3}{5}, \log(1+2\sqrt{t})} y(t) \right|}{2+ \left| \frac{5}{2} \mathcal{D}_{0+}^{\frac{3}{5}, \log(1+2\sqrt{t})} y(t) \right|} + \frac{\left| \frac{3}{4} \mathcal{I}_{0+}^{\frac{3}{5}, \log(1+2\sqrt{t})} \left(\frac{\sqrt{t-s}}{2\sqrt{\pi e^s+1}} \frac{5}{2} \mathcal{D}_{0+}^{\frac{2}{5}, \log(1+2\sqrt{t})} y(s) \right) (t) \right|}{2+ \left| \frac{3}{4} \mathcal{I}_{0+}^{\frac{3}{5}, \log(1+2\sqrt{t})} \left(\frac{\sqrt{t-s}}{2\sqrt{\pi e^s+1}} \frac{5}{2} \mathcal{D}_{0+}^{\frac{2}{5}, \log(1+2\sqrt{t})} y(s) \right) (t) \right|} \right] \quad \forall t \in (0, 1), \\ y\left(\frac{1}{9}\right) = \frac{5}{2} \mathcal{I}_{0+}^{\frac{2}{9}, \log(1+2\sqrt{t})} \left(\frac{1}{2e^{t+1}} + \frac{2+|\sin y(t)|}{1+|\sin y(t)|} \right), \\ y\left(\frac{7}{9}\right) = \frac{5}{2} \mathcal{I}_{0+}^{\frac{4}{9}, \log(1+2\sqrt{t})} \left(\frac{2t+1}{69} + \frac{2+|\cos y(t)|}{1+|\cos y(t)|} \right). \end{array} \right. \quad (98)$$

In this problem, we are provided with the values $\tau = \frac{5}{7}$ and $\alpha = \frac{1}{2}$, where $\alpha + \tau$ is not equal to zero. Additionally, we have $\beta = \frac{3}{5} > 0$, $\theta = \frac{2}{5}$, $\nu = \frac{3}{4}$, $\varsigma = \frac{1}{3}$, $\sigma = \frac{3}{5}$, $\eta = \frac{1}{9}$, $\delta = \frac{7}{9}$, $\gamma = \max\{\eta, \delta\} = \frac{7}{9}$, and $\varphi(t, s) = \frac{\sqrt{t-s}}{2\sqrt{\pi e^s+1}}$ with $\Phi = \max_{t \in [0,1]} |\varphi(t, s)| \leq \frac{2}{2\sqrt{\pi e^s+1}}$. $G_1(t, y(t)) = \left(\frac{1}{2e^{t+1}} + \frac{2+|\sin y(t)|}{1+|\sin y(t)|}\right)$ with $k_1 = \frac{1}{2e^2}$ and $\mathcal{G}_1 = \frac{1}{1.64922}$, and $G_2(t, y(t)) = \left(\frac{2t+1}{69} + \frac{2+|\cos y(t)|}{1+|\cos y(t)|}\right)$ with $k_2 = \frac{1}{69}$ and $\mathcal{G}_2 = 2$.

In addition, the function $f(t, u, v, w) : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is mutually continuous. In particular, if $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$, and $t \in (0, 1)$, then the following inequalities hold:

$$f(t, u, v, w) = \frac{e^{-3t}}{e^{3t}+5} \left[2 + \frac{|u(t)|}{2+|u(t)|} + \frac{|v(t)|}{2+|v(t)|} + \frac{|w(t)|}{2+|w(t)|} \right],$$

$$|f(t, v_1, v_2, v_3) - f(t, u_1, u_2, u_3)| \leq \frac{e^{-3t}}{e^{3t}+5} (|v_1 - u_1| + |v_2 - u_2| + |v_3 - u_3|),$$

and

$$|f(t, u, v, w)| \leq \frac{e^{-3t}}{e^{3t}+5} (|u| + |v| + |w|) + \mathcal{F},$$

where $\mathcal{F} = \sup_{t \in [0,1]} f(t, 0, 0, 0)$. Thus, there exist $\omega(t) = \frac{e^{-3t}}{e^{3t}+5}$ with a norm $\|\omega\| = \frac{1}{6}$, and $\mathcal{F} = \frac{1}{3}$ such that, for all $t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}$ ($i = 1, 2, 3$) assumption (A_2) holds. In addition, assumption (A_3) holds with $\Phi = \frac{2}{2\sqrt{\pi e^s+1}}$. Therefore, assumptions (A_1) – (A_3) are met, and $c = 0.0227059 < 1$.

Therefore, according to Theorem 10, we deduce that the initial and boundary value problem denoted as ICFBVP (98) has at least one solution. In addition,

$$M_1 = \frac{35 \left(\frac{7}{17}\right)^{2/5} \log_{10}^{17}(3)}{51 \cdot 2^{3/5}} = 2.51364,$$

and

$$M_2 = \frac{1225 \sqrt[5]{\frac{7}{17}} \log_{70}^{187}(3)}{6936 \cdot 2^{4/5} e^2 \sqrt{\pi}} = 0.00833839.$$

Since $\mathfrak{G}_0 \approx 0.77778$, this implies that

$$c + \mathfrak{G}_0 \frac{\|\omega\|}{1 - \|\omega\|(M_1 + M_2)} = 0.113728 < 1.$$

Subsequently, as per Theorems 11 and 12, it can be deduced that the solution to the ICFBVP (98) is unique and exhibits Ulam–Hyers stability.

4. Conclusions

In conclusion, this research has introduced and rigorously analyzed innovative generalized conformable fractional integral and derivative operators with respect to the auxiliary function $\psi(t)$. Through a comprehensive investigation, we have established their key properties, including semigroup behavior, linearity, and boundedness, showcasing their significance within the realm of fractional calculus. These operators not only encompass and extend existing well-known fractional operators but also offer a versatile framework for addressing complex fractional boundary value problems.

The application of these operators to second-order nonlinear implicit ψ -conformable fractional differential equations under nonlocal fractional integral boundary conditions has highlighted their efficacy in tackling intricate mathematical challenges. The study has encompassed aspects such as solution existence, uniqueness, and Ulam–Hyers stability, underscoring the practical implications of these operators in the realm of fractional calculus methodologies.

Looking ahead, this research opens the door to several promising directions for future exploration. One avenue could involve further investigating the theoretical foundations and analytical properties of these generalized conformable operators, potentially leading to the discovery of new mathematical phenomena or relationships. Additionally, exploring numerical methods and computational techniques tailored to these operators could enhance their applicability to a broader range of practical problems. Lastly, the potential extension of this framework to systems of fractional differential equations and their connections to real-world phenomena presents an exciting avenue for interdisciplinary research, bridging the gap between mathematical theory and practical applications. As the field of fractional calculus continues to evolve, these operators hold the potential to play a pivotal role in advancing our understanding of complex phenomena across various domains.

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