

Article

On Bivariate-Balancing and Lucas-Balancing Hybrinomials

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Abstract: In this paper, we introduce and study new hybrid generalizations of bivariate-balancing and Lucas-balancing polynomials. We give some properties of bivariate-balancing and Lucas-balancing hybrinomials, among other Binet-type formulas, Catalan, Cassini, d'Ocagne, Vajda, Ruggles, and Honsberger identities. Moreover, we present matrix generators and generating functions of these hybrinomials.

Keywords: balancing numbers; balancing polynomials; bivariate-balancing polynomials; hybrid numbers; hybrinomials

MSC: 11B37; 11B39; 11D04

1. Introduction and Preliminary Results

A positive integer, n , is called a balancing number with a positive balancer, r , if it is the solution of the Diophantine equation $1 + 2 + \dots + (n - 1) = (n + 1) + (n + 2) + \dots + (n + r)$. The balancing sequence, denoted by $\{B_n\}$, was introduced in [1]. The balancing numbers satisfy the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$ with initial terms $B_0 = 0$ and $B_1 = 1$.

Moreover, the sequence of balancing numbers possesses the property of symmetry, more precisely, $B_{-n} = -B_n$ for every integer n ; see [2].

In [3], the author introduced the n th Lucas-balancing number C_n as follows: $C_n = \sqrt{8B_n^2 + 1}$. The Lucas-balancing numbers can be defined recursively for $n \geq 2$, using the same recurrence relation as balancing numbers, namely, $C_n = 6C_{n-1} - C_{n-2}$, with $C_0 = 1$, $C_1 = 3$.

Balancing numbers have diverse applications across various mathematical and computational domains. According to the authors in [4], a fourth-order linear recursive sequence connected to the concept of subbalancing numbers was presented. This sequence is constructed by utilizing the third balancing number in the Diophantine equation of subbalancing numbers. Balancing numbers also have applications in cryptography, as discussed in [5]. Furthermore, the literature includes numerous studies on hypercomplex numbers, where the coefficients are balancing numbers or their generalizations; for example, harmonic complex balancing numbers [6] and dual bicomplex balancing-type numbers [7].

Natural extensions of balancing and Lucas-balancing numbers are given by balancing and Lucas-balancing polynomials of one or two variables. The n th-balancing polynomial $B_n(x)$ of one variable x was introduced in [8] by $B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x)$ for $n \geq 2$ with $B_0(x) = 0$, $B_1(x) = 1$. In [9], Lucas-balancing polynomials of one variable were considered. They were defined in the following way: $C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x)$ for $n \geq 2$ with initial terms $C_0(x) = 1$, $C_1(x) = 3x$.



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The sequence of bivariate-balancing polynomials was defined recursively in [10] as follows:

$$B_n(x, y) = 6xB_{n-1}(x, y) - yB_{n-2}(x, y) \quad \text{for } n \geq 2 \quad (1)$$

with $B_0(x, y) = 0$, $B_1(x, y) = 1$. In a similar way, the bivariate Lucas-balancing polynomials were defined in [11] as follows:

$$C_n(x, y) = 6xC_{n-1}(x, y) - yC_{n-2}(x, y) \quad \text{for } n \geq 2 \quad (2)$$

where $C_0(x, y) = 1$, $C_1(x, y) = 3x$.

Using the above equalities, we obtain $B_0(x, y) = 0$, $B_1(x, y) = 1$, $B_2(x, y) = 6x$, $B_3(x, y) = 36x^2 - y$, $B_4(x, y) = 216x^3 - 12xy$, $B_5(x, y) = 1296x^4 - 108x^2y + y^2$ and $C_0(x, y) = 1$, $C_1(x, y) = 3x$, $C_2(x, y) = 18x^2 - y$, $C_3(x, y) = 108x^3 - 9xy$, $C_4(x, y) = 648x^4 - 72x^2y + y^2$, $C_5(x, y) = 3888x^5 - 540x^3y + 15xy^2$.

It is easy to see that $B_n(x, 1) = B_n(x)$, $B_n(1, 1) = B_n$, $C_n(x, 1) = C_n(x)$ and $C_n(1, 1) = C_n$ for each n .

Binet-type formulas for the bivariate-balancing polynomials and bivariate Lucas-balancing polynomials have the following forms:

$$B_n(x, y) = \frac{\lambda^n(x, y) - \gamma^n(x, y)}{\lambda(x, y) - \gamma(x, y)}, \quad (3)$$

$$C_n(x, y) = \frac{1}{2}(\lambda^n(x, y) + \gamma^n(x, y)), \quad (4)$$

where

$$\lambda(x, y) = 3x + \sqrt{9x^2 - y}, \quad \gamma(x, y) = 3x - \sqrt{9x^2 - y} \quad (5)$$

with $9x^2 - y > 0$.

In this study, we explore the idea of bivariate-balancing polynomials and bivariate Lucas-balancing polynomials within the framework of the hybrid number theory.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$, where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \boldsymbol{\varepsilon} + \mathbf{i}. \quad (6)$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$, are any two hybrid numbers, then equality, addition, subtraction, and multiplication by scalar are defined as follows:

$$\begin{aligned} \mathbf{Z}_1 &= \mathbf{Z}_2 \text{ if and only if } a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2, \\ \mathbf{Z}_1 \pm \mathbf{Z}_2 &= (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\boldsymbol{\varepsilon} + (d_1 \pm d_2)\mathbf{h}, \\ s\mathbf{Z}_1 &= sa_1 + sb_1\mathbf{i} + sc_1\boldsymbol{\varepsilon} + sd_1\mathbf{h}, \text{ where } s \in \mathbb{R}. \end{aligned}$$

To multiply two hybrid numbers, we need to use Formula (6). Table 1 presents the products of operators $\mathbf{i}$, $\boldsymbol{\varepsilon}$, and $\mathbf{h}$.

Table 1. The hybrid number multiplication.

.	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\boldsymbol{\varepsilon} + \mathbf{i}$
$\boldsymbol{\varepsilon}$	$\mathbf{h} + 1$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-\boldsymbol{\varepsilon} - \mathbf{i}$	$\boldsymbol{\varepsilon}$	1

Using the rules given in Table 1, the multiplication of hybrid numbers can be carried out analogously to the multiplication of algebraic expressions. For details on hybrid numbers, see [12].

One of the best-known generalizations of balancing and Lucas-balancing numbers are Horadam numbers, as introduced in [13]. Importantly, the Horadam numbers also generalize other known numbers, including Fibonacci and Lucas numbers, and all numbers defined by second-order recurrence relations with constant coefficients and arbitrary initial conditions. Some of the properties obtained for the balancing numbers can be written as special cases of those for the Horadam numbers, but many relationships hold only between balancing or Lucas-balancing numbers. Similarly, Horadam polynomials (of one or two variables) serve as generalizations of both balancing and Lucas-balancing polynomials. Horadam polynomials of one variable were studied in [14] and Horadam polynomials of two variables were defined and investigated in [15]. The definitions of these polynomials will be recalled later in this paper.

The balancing and Lucas-balancing hybrid numbers were defined in [16]. In [2], balancing and Lucas-balancing hybrinomials of one variable were studied. Properties of Horadam hybrinomials of one and two variables can be found in [17] and [15], respectively. In this paper, we define and study the balancing and Lucas-balancing hybrinomials of two variables. Some properties of bivariate-balancing and Lucas-balancing hybrinomials are special cases of the properties of bivariate Horadam hybrinomials. However, many properties and interconnections specific to bivariate-balancing and Lucas-balancing hybrinomials are entirely new.

For an integer $n \geq 0$, the n th bivariate-balancing hybrinomial $BH_n(x, y)$ and the n th bivariate Lucas-balancing hybrinomial $CH_n(x, y)$ are defined by the following:

$$BH_n(x, y) = B_n(x, y) + B_{n+1}(x, y)\mathbf{i} + B_{n+2}(x, y)\boldsymbol{\varepsilon} + B_{n+3}(x, y)\mathbf{h} \quad (7)$$

and

$$CH_n(x, y) = C_n(x, y) + C_{n+1}(x, y)\mathbf{i} + C_{n+2}(x, y)\boldsymbol{\varepsilon} + C_{n+3}(x, y)\mathbf{h}, \quad (8)$$

where $B_n(x, y)$ is the n th bivariate-balancing polynomial, $C_n(x, y)$ is the n th bivariate Lucas-balancing polynomial, and $\mathbf{i}$, $\boldsymbol{\varepsilon}$, $\mathbf{h}$ are hybrid units that satisfy (6).

It can be easily observed that $BH_n(x, 1)$ is the n th-balancing hybrinomial, $CH_n(x, 1)$ is the n th Lucas-balancing hybrinomial, $BH_n(1, 1)$ is the n th-balancing hybrid number, and $CH_n(1, 1)$ is the n th Lucas-balancing hybrid number, for each n .

Theorem 1. For any variables x, y , we have the following:

$$BH_n(x, y) = 6xBH_{n-1}(x, y) - yBH_{n-2}(x, y) \text{ for } n \geq 2 \quad (9)$$

with $BH_0(x, y) = \mathbf{i} + 6x\boldsymbol{\varepsilon} + (36x^2 - y)\mathbf{h}$

and $BH_1(x, y) = 1 + 6x\mathbf{i} + (36x^2 - y)\boldsymbol{\varepsilon} + (216x^3 - 12xy)\mathbf{h}$.

Proof. For $n = 2$, we have the following:

$$\begin{aligned} BH_2(x, y) &= 6xBH_1(x, y) - yBH_0(x, y) \\ &= 6x(1 + 6x\mathbf{i} + (36x^2 - y)\boldsymbol{\varepsilon} + (216x^3 - 12xy)\mathbf{h}) \\ &\quad - y(\mathbf{i} + 6x\boldsymbol{\varepsilon} + (36x^2 - y)\mathbf{h}) \\ &= 6x + (36x^2 - y)\mathbf{i} + (216x^3 - 12xy)\boldsymbol{\varepsilon} + (1296x^4 - 108x^2y + y^2)\mathbf{h} \\ &= B_2(x, y) + B_3(x, y)\mathbf{i} + B_4(x, y)\boldsymbol{\varepsilon} + B_5(x, y)\mathbf{h}. \end{aligned}$$

Let $n \geq 3$. Then, by the definition of balancing polynomials, we have the following:

$$\begin{aligned}
 BH_n(x, y) &= B_n(x, y) + B_{n+1}(x, y)\mathbf{i} + B_{n+2}(x, y)\epsilon + B_{n+3}(x, y)\mathbf{h} \\
 &= (6xB_{n-1}(x, y) - yB_{n-2}(x, y)) + (6xB_n(x, y) - yB_{n-1}(x, y))\mathbf{i} \\
 &\quad + (6xB_{n+1}(x, y) - yB_n(x, y))\epsilon + (6xB_{n+2}(x, y) - yB_{n+1}(x, y))\mathbf{h} \\
 &= 6x(B_{n-1}(x, y) + B_n(x, y)\mathbf{i} + B_{n+1}(x, y)\epsilon + B_{n+2}(x, y)\mathbf{h}) \\
 &\quad - y(B_{n-2}(x, y) + B_{n-1}(x, y)\mathbf{i} + B_n(x, y)\epsilon + B_{n+1}(x, y)\mathbf{h}) \\
 &= 6xBH_{n-1}(x, y) - yBH_{n-2}(x, y),
 \end{aligned}$$

which completes the proof. $\square$

Similarly, the next theorem can be proven.

Theorem 2. For any variables, x, y , we have the following:

$$CH_n(x, y) = 6xCH_{n-1}(x, y) - yCH_{n-2}(x, y) \text{ for } n \geq 2$$

with $CH_0(x, y) = 1 + 3x\mathbf{i} + (18x^2 - 1)\epsilon + (108x^3 - 9x)\mathbf{h}$
 and $CH_1(x, y) = 3x + (18x^2 - 1)\mathbf{i} + (108x^3 - 9x)\epsilon + (648x^4 - 72x^2 + y)\mathbf{h}$.

As pointed out earlier, the bivariate Horadam polynomials generalize the bivariate-balancing polynomials and bivariate Lucas-balancing polynomials. For a positive integer, n , the n th bivariate Horadam polynomial $w_n(x, y)$ was defined in [15] as $w_n(x, y) = xpw_{n-1}(x, y) + yqw_{n-2}(x, y)$ for $n \geq 3$ with initial values $w_1(x, y) = a$ and $w_2(x, y) = bx$. Note that bivariate Horadam polynomials were defined for $n \geq 1$. Putting $w_1(x, y) = 1$, $w_2(x, y) = 6x$, $p = 6$ and $q = -1$, we obtain bivariate-balancing polynomials, and by taking $w_1(x, y) = 3x$, $w_2(x, y) = 18x^2 - y$, $p = 6$ and $q = -1$, we derive bivariate Lucas-balancing polynomials.

2. Binet-Type Formulas and Some Identities

Now, we will present Binet-type formulas for bivariate-balancing hybrinomials and bivariate Lucas-balancing hybrinomials. Moreover, we will give general bilinear index-reduction formulas for these hybrinomials.

Theorem 3. Let $n \geq 0$ be an integer. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned}
 BH_n(x, y) &= \frac{\lambda^n(x, y)}{\lambda(x, y) - \gamma(x, y)} \left(1 + \lambda(x, y)\mathbf{i} + \lambda^2(x, y)\epsilon + \lambda^3(x, y)\mathbf{h} \right) \\
 &\quad - \frac{\gamma^n(x, y)}{\lambda(x, y) - \gamma(x, y)} \left(1 + \gamma(x, y)\mathbf{i} + \gamma^2(x, y)\epsilon + \gamma^3(x, y)\mathbf{h} \right),
 \end{aligned} \tag{10}$$

where $\lambda(x, y), \gamma(x, y)$ are given by (5).

Proof. Using (7) and (3), we have the following:

$$\begin{aligned}
 BH_n(x, y) &= B_n(x, y) + B_{n+1}(x, y)\mathbf{i} + B_{n+2}(x, y)\boldsymbol{\varepsilon} + B_{n+3}(x, y)\mathbf{h} \\
 &= \frac{\lambda^n(x, y) - \gamma^n(x, y)}{\lambda(x, y) - \gamma(x, y)} + \frac{\lambda^{n+1}(x, y) - \gamma^{n+1}(x, y)}{\lambda(x, y) - \gamma(x, y)}\mathbf{i} \\
 &\quad + \frac{\lambda^{n+2}(x, y) - \gamma^{n+2}(x, y)}{\lambda(x, y) - \gamma(x, y)}\boldsymbol{\varepsilon} + \frac{\lambda^{n+3}(x, y) - \gamma^{n+3}(x, y)}{\lambda(x, y) - \gamma(x, y)}\mathbf{h} \\
 &= \frac{\lambda^n(x, y)}{\lambda(x, y) - \gamma(x, y)}(1 + \lambda(x, y)\mathbf{i} + \lambda^2(x, y)\boldsymbol{\varepsilon} + \lambda^3(x, y)\mathbf{h}) \\
 &\quad - \frac{\gamma^n(x, y)}{\lambda(x, y) - \gamma(x, y)}(1 + \gamma(x, y)\mathbf{i} + \gamma^2(x, y)\boldsymbol{\varepsilon} + \gamma^3(x, y)\mathbf{h}),
 \end{aligned}$$

and, thus, the proof is complete. $\square$

A similar reasoning applies to prove the next theorem.

Theorem 4. Let $n \geq 0$ be an integer. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned}
 CH_n(x, y) &= \frac{1}{2}(\lambda^n(x, y)(1 + \lambda(x, y)\mathbf{i} + \lambda^2(x, y)\boldsymbol{\varepsilon} + \lambda^3(x, y)\mathbf{h}) \\
 &\quad + \gamma^n(x, y)(1 + \gamma(x, y)\mathbf{i} + \gamma^2(x, y)\boldsymbol{\varepsilon} + \gamma^3(x, y)\mathbf{h})),
 \end{aligned} \tag{11}$$

where $\lambda(x, y)$, $\gamma(x, y)$ are given by (5).

To simplify the notation, let

$$\hat{\lambda}(x, y) = 1 + \lambda(x, y)\mathbf{i} + \lambda^2(x, y)\boldsymbol{\varepsilon} + \lambda^3(x, y)\mathbf{h}, \tag{12}$$

$$\hat{\gamma}(x, y) = 1 + \gamma(x, y)\mathbf{i} + \gamma^2(x, y)\boldsymbol{\varepsilon} + \gamma^3(x, y)\mathbf{h}. \tag{13}$$

Moreover,

$$\lambda(x, y) - \gamma(x, y) = 2\sqrt{9x^2 - y},$$

$$\lambda(x, y) \cdot \gamma(x, y) = y.$$

Thus, we can write (10) and (11) as

$$BH_n(x, y) = \frac{\lambda^n(x, y)\hat{\lambda}(x, y) - \gamma^n(x, y)\hat{\gamma}(x, y)}{2\sqrt{9x^2 - y}}, \tag{14}$$

$$CH_n(x, y) = \frac{\lambda^n(x, y)\hat{\lambda}(x, y) + \gamma^n(x, y)\hat{\gamma}(x, y)}{2}, \tag{15}$$

respectively.

We will now give general bilinear index-reduction formulas for bivariate-balancing and Lucas-balancing hybrinomials, also called Johnson identities:

Theorem 5. Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned}
 &BH_a(x, y) \cdot BH_b(x, y) - BH_c(x, y) \cdot BH_d(x, y) \\
 &= \frac{1}{36x^2 - 4y} \left[(\lambda^c(x, y)\gamma^d(x, y) - \lambda^a(x, y)\gamma^b(x, y))\hat{\lambda}(x, y)\hat{\gamma}(x, y) \right. \\
 &\quad \left. + (\gamma^c(x, y)\lambda^d(x, y) - \gamma^a(x, y)\lambda^b(x, y))\hat{\gamma}(x, y)\hat{\lambda}(x, y) \right],
 \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Proof. By Formula (14), we obtain the following:

$$\begin{aligned} & BH_a(x, y) \cdot BH_b(x, y) - BH_c(x, y) \cdot BH_d(x, y) \\ &= \frac{1}{36x^2 - 4y} \left[\lambda^{a+b}(x, y)(\hat{\lambda}(x, y))^2 - \lambda^a(x, y)\gamma^b(x, y)\hat{\lambda}(x, y)\hat{\gamma}(x, y) \right. \\ &\quad - \gamma^a(x, y)\lambda^b(x, y)\hat{\gamma}(x, y)\hat{\lambda}(x, y) + \gamma^{a+b}(x, y)(\hat{\gamma}(x, y))^2 \\ &\quad - \lambda^{c+d}(x, y)(\hat{\lambda}(x, y))^2 + \lambda^c(x, y)\gamma^d(x, y)\hat{\lambda}(x, y)\hat{\gamma}(x, y) \\ &\quad \left. + \gamma^c(x, y)\lambda^d(x, y)\hat{\gamma}(x, y)\hat{\lambda}(x, y) - \gamma^{c+d}(x, y)(\hat{\gamma}(x, y))^2 \right]. \end{aligned}$$

Using the fact that $a + b = c + d$, we obtain the desired formula. $\square$

Using the same approach, one can prove the next theorem.

Theorem 6. Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_a(x, y) \cdot CH_b(x, y) - CH_c(x, y) \cdot CH_d(x, y) \\ &= \frac{1}{4} \left[\left(\lambda^a(x, y)\gamma^b(x, y) - \lambda^c(x, y)\gamma^d(x, y) \right) \hat{\lambda}(x, y)\hat{\gamma}(x, y) \right. \\ &\quad \left. + \left(\gamma^a(x, y)\lambda^b(x, y) - \gamma^c(x, y)\lambda^d(x, y) \right) \hat{\gamma}(x, y)\hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

It is easily seen that for special values of a, b, c, d , by Theorems 5 and 6, we obtain some identities for bivariate-balancing and Lucas-balancing hybrinomials, i.e., Catalan (Corollaries 1 and 7), Cassini (Corollaries 2 and 8), d'Ocagne (Corollaries 3 and 9), Vajda (Corollaries 4 and 10) first-Halton (Corollaries 5 and 11), and second-Halton (Corollaries 6 and 12) identities.

Corollary 1. Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{n-m}(x, y) \cdot BH_{n+m}(x, y) - (BH_n(x, y))^2 \\ &= \frac{y^n}{36x^2 - 4y} \left[\left(1 - \left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^m \right) \hat{\lambda}(x, y)\hat{\gamma}(x, y) \right. \\ &\quad \left. + \left(1 - \left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^m \right) \hat{\gamma}(x, y)\hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 2. Let $n \geq 1$ be an integer. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{n-1}(x, y) \cdot BH_{n+1}(x, y) - (BH_n(x, y))^2 \\ &= \frac{y^{n-1}}{36x^2 - 4y} \left[\left(2y - 18x^2 + 6x\sqrt{9x^2 - y} \right) \hat{\lambda}(x, y)\hat{\gamma}(x, y) \right. \\ &\quad \left. + \left(2y - 18x^2 - 6x\sqrt{9x^2 - y} \right) \hat{\gamma}(x, y)\hat{\lambda}(x, y) \right], \end{aligned}$$

where $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (12), (13), respectively.

Corollary 3. Let $n \geq 0, m \geq 0$ be integers such that $m \geq n$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_n(x, y) \cdot BH_{m+1}(x, y) - BH_{n+1}(x, y) \cdot BH_m(x, y) \\ &= \frac{y^n}{2\sqrt{9x^2 - y}} [\gamma^{m-n}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) - \lambda^{m-n}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y)], \end{aligned}$$

where $\lambda(x, y), \gamma(x, y), \hat{\lambda}(x, y), \hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 4. Let $n \geq 0, m \geq 0, k \geq 0$ be integers such that $n \geq m$ and $n \geq k$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{m+k}(x, y) \cdot BH_{n-k}(x, y) - BH_m(x, y) \cdot BH_n(x, y) \\ &= \frac{y^m}{36x^2 - 4y} \left[\gamma^{n-m}(x, y) \left(1 - \left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^k \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ & \quad \left. + \lambda^{n-m}(x, y) \left(1 - \left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^k \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y), \gamma(x, y), \hat{\lambda}(x, y), \hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 5. Let $n \geq 0, m \geq 0, r \geq 0$ be integers. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{m+r}(x, y) \cdot BH_n(x, y) - BH_r(x, y) \cdot BH_{m+n}(x, y) \\ &= \frac{\gamma^m(x, y) - \lambda^m(x, y)}{36x^2 - 4y} [\lambda^r(x, y) \gamma^n(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ & \quad - \lambda^n(x, y) \gamma^r(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y)], \end{aligned}$$

where $\lambda(x, y), \gamma(x, y), \hat{\lambda}(x, y), \hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 6. Let $n \geq 0, k \geq 0, s \geq 0$ be integers such that $n \geq k$ and $n \geq s$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{n+k}(x, y) \cdot BH_{n-k}(x, y) - BH_{n+s}(x, y) \cdot BH_{n-s}(x, y) \\ &= \frac{y^n}{36x^2 - 4y} \left[\left(\left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^s - \left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^k \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ & \quad \left. + \left(\left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^s - \left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^k \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y), \gamma(x, y), \hat{\lambda}(x, y), \hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 7. Let $n \geq 0, m \geq 0$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{n-m}(x, y) \cdot CH_{n+m}(x, y) - (CH_n(x, y))^2 \\ &= \frac{y^n}{4} \left[\left(\left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^m - 1 \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ & \quad \left. + \left(\left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^m - 1 \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 8. Let $n \geq 1$ be an integer. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{n-1}(x, y) \cdot CH_{n+1}(x, y) - (CH_n(x, y))^2 \\ &= \frac{y^{n-1}}{4} \left[\left(18x^2 - 2y - 6x\sqrt{9x^2 - y} \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ &\quad \left. + \left(18x^2 - 2y + 6x\sqrt{9x^2 - y} \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (12) and (13), respectively.

Corollary 9. Let $n \geq 0$ and $m \geq 0$ be integers such that $m \geq n$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_n(x, y) \cdot CH_{m+1}(x, y) - CH_{n+1}(x, y) \cdot CH_m(x, y) \\ &= -\frac{y^n \sqrt{9x^2 - y}}{2} (\gamma^{m-n}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) - \lambda^{m-n}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y)), \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 10. Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$ and $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{m+k}(x, y) \cdot CH_{n-k}(x, y) - CH_m(x, y) \cdot CH_n(x, y) \\ &= \frac{y^m}{4} \left[\gamma^{n-m}(x, y) \left(\left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^k - 1 \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ &\quad \left. + \lambda^{n-m}(x, y) \left(\left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^k - 1 \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 11. Let $n \geq 0$, $m \geq 0$, and $r \geq 0$ be integers. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{m+r}(x, y) \cdot CH_n(x, y) - CH_r(x, y) \cdot CH_{m+n}(x, y) \\ &= \frac{\lambda^m(x, y) - \gamma^m(x, y)}{4} [\lambda^r(x, y) \gamma^n(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ &\quad - \lambda^n(x, y) \gamma^r(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y)], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 12. Let $n \geq 0$, $k \geq 0$, $s \geq 0$ be integers such that $n \geq k$ and $n \geq s$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{n+k}(x, y) \cdot CH_{n-k}(x, y) - CH_{n+s}(x, y) \cdot CH_{n-s}(x, y) \\ &= \frac{y^n}{4} \left[\left(\left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^k - \left(\frac{\lambda(x, y)}{\gamma(x, y)} \right)^s \right) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \right. \\ &\quad \left. + \left(\left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^k - \left(\frac{\gamma(x, y)}{\lambda(x, y)} \right)^s \right) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

In a similar manner, using Binet's formulas, we can prove two additional identities, namely, the Ruggles identity (Corollaries 13 and 15) and the Honsberger identity (Corollaries 14 and 16).

Corollary 13. Let $n \geq 0$ and $m \geq 1$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{m-1}(x, y) \cdot BH_{n-m}(x, y) + BH_m(x, y) \cdot BH_{n-m+1}(x, y) \\ &= \frac{1}{36x^2 - 4y} \left[\left(6x + \frac{1-y}{y} \gamma(x, y) \right) \lambda^n(x, y) (\hat{\lambda}(x, y))^2 \right. \\ &\quad + \left(6x + \frac{1-y}{y} \lambda(x, y) \right) \gamma^n(x, y) (\hat{\gamma}(x, y))^2 \\ &\quad - \frac{1+y}{y} \lambda^m(x, y) \gamma^{n-m+1}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ &\quad \left. - \frac{1+y}{y} \gamma^m(x, y) \lambda^{n-m+1}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 14. Let $n \geq 0$, $m \geq 1$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & BH_{m-1}(x, y) \cdot BH_n(x, y) + BH_m(x, y) \cdot BH_{n+1}(x, y) \\ &= \frac{1}{36x^2 - 4y} \left[\left(6x + \frac{1-y}{y} \gamma(x, y) \right) \lambda^{m+n}(x, y) (\hat{\lambda}(x, y))^2 \right. \\ &\quad + \left(6x + \frac{1-y}{y} \lambda(x, y) \right) \gamma^{m+n}(x, y) (\hat{\gamma}(x, y))^2 \\ &\quad - \frac{1+y}{y} \lambda^m(x, y) \gamma^{n+1}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ &\quad \left. - \frac{1+y}{y} \gamma^m(x, y) \lambda^{n+1}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 15. Let $n \geq 0$, $m \geq 1$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{m-1}(x, y) \cdot CH_{n-m}(x, y) + CH_m(x, y) \cdot CH_{n-m+1}(x, y) \\ &= \frac{1}{4} \left[\left(6x + \frac{1-y}{y} \gamma(x, y) \right) \lambda^n(x, y) (\hat{\lambda}(x, y))^2 \right. \\ &\quad + \left(6x + \frac{1-y}{y} \lambda(x, y) \right) \gamma^n(x, y) (\hat{\gamma}(x, y))^2 \\ &\quad + \frac{1+y}{y} \lambda^m(x, y) \gamma^{n-m+1}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ &\quad \left. + \frac{1+y}{y} \gamma^m(x, y) \lambda^{n-m+1}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollary 16. Let $n \geq 0, m \geq 1$ be integers such that $n \geq m$. Then, for $9x^2 - y > 0$, we have the following:

$$\begin{aligned} & CH_{m-1}(x, y) \cdot CH_n(x, y) + CH_m(x, y) \cdot CH_{n+1}(x, y) \\ &= \frac{1}{4} \left[\left(6x + \frac{1-y}{y} \gamma(x, y) \right) \lambda^{m+n}(x, y) (\hat{\lambda}(x, y))^2 \right. \\ &\quad + \left(6x + \frac{1-y}{y} \lambda(x, y) \right) \gamma^{m+n}(x, y) (\hat{\gamma}(x, y))^2 \\ &\quad + \frac{1+y}{y} \lambda^m(x, y) \gamma^{n+1}(x, y) \hat{\lambda}(x, y) \hat{\gamma}(x, y) \\ &\quad \left. + \frac{1+y}{y} \gamma^m(x, y) \lambda^{n+1}(x, y) \hat{\gamma}(x, y) \hat{\lambda}(x, y) \right], \end{aligned}$$

where $\lambda(x, y)$, $\gamma(x, y)$, $\hat{\lambda}(x, y)$, $\hat{\gamma}(x, y)$ are given by (5), (12), and (13), respectively.

Corollaries 1–3 and 7–9 can also be obtained as special cases of the analogous properties for bivariate Horadam hybrinomials presented in [15].

3. Generating Functions, Matrix Representations, and Summation Formulas

Now, we will give ordinary generating functions for bivariate-balancing and Lucas-balancing hybrinomials. Generating functions are important tools in solving problems, including counting and combinatorial problems. There are several types of generating functions, such as ordinary generating functions, exponential generating functions, the Bell series, the Dirichlet series, and the Lambert series; see [18–24]. An ordinary generating function, $g(t)$, of the sequence $\{a_n\}$ is a power series, $g(t) = \sum_{n=0}^{\infty} a_n t^n$. It is also worth noting that when we consider the generating function of a sequence, we often ignore the issue of convergence of the series.

Theorem 7. The generating function for the bivariate-balancing hybrinomial sequence $\{BH_n(x, y)\}$ is as follows:

$$G(t) = \frac{BH_0(x, y) + (BH_1(x, y) - 6x BH_0(x, y))t}{1 - 6xt + yt^2},$$

where $9x^2 - y > 0$.

Proof. Assume that the generating function of the bivariate-balancing hybrinomial sequence $\{BH_n(x, y)\}$ has the form $G(t) = \sum_{n=0}^{\infty} BH_n(x, y) t^n$. Then, we have the following:

$$G(t) = BH_0(x, y) + BH_1(x, y)t + BH_2(x, y)t^2 + \dots$$

Hence, we obtain the following:

$$\begin{aligned} -6xt G(t) &= -6x BH_0(x, y)t - 6x BH_1(x, y)t^2 - 6x BH_2(x, y)t^3 - \dots \\ yt^2 G(t) &= y BH_0(x, y)t^2 + y BH_1(x, y)t^3 + y BH_2(x, y)t^4 + \dots \end{aligned}$$

By adding these three equalities above, we obtain the following:

$$G(t)(1 - 6xt + yt^2) = BH_0(x, y) + (BH_1(x, y) - 6x BH_0(x, y))t$$

since $BH_n(x, y) = 6xBH_{n-1}(x, y) - yBH_{n-2}(x, y)$ (see (9)), and the coefficients of t^n for $n \geq 2$ are equal to zero. Hence, we obtain the result. $\square$

Following the same steps, we can prove the next theorem.

Theorem 8. *The generating function for the bivariate Lucas-balancing hybrinomial sequence $\{CH_n(x, y)\}$ is as follows:*

$$g(t) = \frac{CH_0(x, y) + (CH_1(x, y) - 6xCH_0(x, y))t}{1 - 6xt + yt^2},$$

where $9x^2 - y > 0$.

Note that we can also obtain Theorems 7 and 8 as special cases of Theorem 3 from [15].

Matrix generators are another useful tool in combinatorics, apart from generating functions. Applications of matrix calculus for finding relationships between known numbers can be found in sources such as [23,25–27]. Now, we give matrix representations of bivariate-balancing and Lucas-balancing hybrinomials.

Theorem 9. *Let $n \geq 1$ be an integer. Then, we have the following:*

$$\begin{aligned} & \begin{bmatrix} BH_{n+1}(x, y) & -BH_n(x, y) \\ BH_n(x, y) & -BH_{n-1}(x, y) \end{bmatrix} \\ &= \begin{bmatrix} BH_2(x, y) & -BH_1(x, y) \\ BH_1(x, y) & -BH_0(x, y) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ y & 0 \end{bmatrix}^{n-1}. \end{aligned} \quad (16)$$

Proof. (By induction on n) If $n = 1$, then the result is obvious. Assuming that Formula (16) holds for $n \geq 1$, we shall prove it for $n + 1$. Using the induction hypothesis and Formula (9), we have the following:

$$\begin{aligned} & \begin{bmatrix} BH_2(x, y) & -BH_1(x, y) \\ BH_1(x, y) & -BH_0(x, y) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ y & 0 \end{bmatrix}^n \\ &= \begin{bmatrix} BH_{n+1}(x, y) & -BH_n(x, y) \\ BH_n(x, y) & -BH_{n-1}(x, y) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ y & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6xBH_{n+1}(x, y) - yBH_n(x, y) & -BH_{n+1}(x, y) \\ 6xBH_n(x, y) - yBH_{n-1}(x, y) & -BH_n(x, y) \end{bmatrix} \\ &= \begin{bmatrix} BH_{n+2}(x, y) & -BH_{n+1}(x, y) \\ BH_{n+1}(x, y) & -BH_n(x, y) \end{bmatrix}, \end{aligned}$$

this finalizes the proof. $\square$

Following the same steps, we can prove the next theorem:

Theorem 10. *Let $n \geq 1$ be an integer. Then, we have the following:*

$$\begin{aligned} & \begin{bmatrix} CH_{n+1}(x, y) & -CH_n(x, y) \\ CH_n(x, y) & -CH_{n-1}(x, y) \end{bmatrix} \\ &= \begin{bmatrix} CH_2(x, y) & -CH_1(x, y) \\ CH_1(x, y) & -CH_0(x, y) \end{bmatrix} \cdot \begin{bmatrix} 6x & -1 \\ y & 0 \end{bmatrix}^{n-1}. \end{aligned}$$

It is worth noting that, due to the non-commutativity of the multiplication of hybrid numbers, determinant properties cannot be used. However, we can use algebraic operations on matrices to derive the properties of hybrinomials.

In [15], summation formulas for bivariate Horadam polynomials and bivariate Horadam hybrinomials were provided. However, both formulas included $w_0(x, y)$, which is not defined in the paper. Here, we will present the correct formulas for bivariate-balancing and Lucas-balancing polynomials and hybrinomials. These formulas will be proven using a different approach.

Remark 1. For any integer $n \geq 0$, the bivariate-balancing polynomials provide the summation formula:

$$\sum_{l=0}^n B_l(x, y) = \frac{B_{n+1}(x, y) - yB_n(x, y) - 1}{6x - y - 1}.$$

Proof. Using (1), we have the following:

$$\begin{aligned} \sum_{l=0}^n B_l(x, y) &= B_0(x, y) + B_1(x, y) + B_2(x, y) + \cdots + B_n(x, y) \\ &= B_0(x, y) + B_1(x, y) + (6xB_1(x, y) - yB_0(x, y)) + \cdots \\ &\quad + (6xB_{n-1}(x, y) - yB_{n-2}(x, y)) \\ &= B_0(x, y) + B_1(x, y) + 6x(B_0(x, y) + B_1(x, y) + \cdots + B_n(x, y)) \\ &\quad - y(B_0(x, y) + B_1(x, y) + \cdots + B_n(x, y)) \\ &\quad - 6xB_0(x, y) - 6xB_n(x, y) + yB_{n-1}(x, y) + yB_n(x, y) \end{aligned}$$

and then we obtain the following:

$$(6x - y - 1) \sum_{l=0}^n B_l(x, y) = B_{n+1}(x, y) - yB_n(x, y) - B_1(x, y) + (6x - 1)B_0(x, y),$$

and, thus, the proof is complete. $\square$

The following sum formulas are proven through the same method.

Remark 2. For any integer $n \geq 0$, the bivariate Lucas-balancing polynomials provide the summation formula:

$$\sum_{l=0}^n C_l(x, y) = \frac{C_{n+1}(x, y) - yC_n(x, y) + 3x - 1}{6x - y - 1}.$$

Remark 3. For any integer $n \geq 0$, the bivariate-balancing hybrinomials provide the summation formula:

$$\sum_{l=0}^n BH_l(x, y) = \frac{BH_{n+1}(x, y) - yBH_n(x, y) - BH_1(x, y) + (6x - 1)BH_0(x, y)}{6x - y - 1}.$$

Remark 4. For any integer $n \geq 0$, the bivariate Lucas-balancing hybrinomials provide the summation formula:

$$\sum_{l=0}^n CH_l(x, y) = \frac{CH_{n+1}(x, y) - yCH_n(x, y) - CH_1(x, y) + (6x - 1)CH_0(x, y)}{6x - y - 1}.$$

4. Concluding Remarks

As with any sequence defined by recurrence relations, we can also define bivariate-balancing polynomials with negative indices. In [10], it was shown that for any positive integer, n , we have $B_n(x, y) = -y^2 B_{-n}(x, y)$. If n is negative, we obtain rational functions instead of polynomials. Consequently, we can define and study the corresponding bivariate-balancing ‘hybrationals’.

Balancing and Lucas-balancing polynomials of one variable are ‘rescaled’ Chebyshev polynomials of one variable, see [28]. Chebyshev polynomials of several variables were presented in [29]. In particular, properties and applications of Chebyshev polynomials of two variables can be found in [30,31]. It will be interesting to study the relationship between the balancing and Lucas-balancing polynomials of two variables and the Chebyshev polynomials of two variables, defining bivariate Chebyshev hybrinomials and examining the relationships between the mentioned hybrinomials. The balancing and Lucas-balancing polynomials also have many other properties, as presented in [10]. Building upon these concepts, the investigation of novel properties of balancing-type hybrinomials presents itself as a promising direction for future research.

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