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Soliton Molecules, Multi-Lumps and Hybrid Solutions in Generalized (2 + 1)-Dimensional Date–Jimbo–Kashiwara–Miwa Equation in Fluid Mechanics

Wei Zhu ¹ , Hai-Fang Song ², Wan-Li Wang ² and Bo Ren ^{2,*} 

¹ College of Teacher Education, Quzhou University, Quzhou 324000, China

² School of Mathematical Sciences, Zhejiang University of Technology, Hangzhou 310014, China

* Correspondence: renbo@zjut.edu.cn

Abstract: The generalized (2 + 1)-dimensional Date–Jimbo–Kashiwara–Miwa (gDJKM) equation, which can be used to describe some phenomena in fluid mechanics, is investigated based on the multi-soliton solution. Soliton molecules of the gDJKM equation are given by the velocity resonance mechanism. A soliton molecule containing three solitons is portrayed at different times. The invariance of the relative positions of three solitons confirms that they form a soliton molecule. Multi-order lumps are obtained by applying the long-wave limit method in the multi-soliton. By analyzing the dynamics of one-order and two-order lumps, the energy concentration and localization property for lump waves are displayed. In the meanwhile, a multi-soliton can transform into multi-order breathers by the complex conjugation relations of parameters. The interaction among lumps, breathers and soliton molecules can be constructed by combining the above comprehensive analysis. The interaction between a one-order lump and a soliton molecule is an elastic collision, which can be observed through investigating evolutionary processes. The results obtained in this paper are useful for explaining certain nonlinear phenomena in fluid dynamics.

Keywords: Date–Jimbo–Kashiwara–Miwa equation; Hirota bilinear method; long-wave limit method; soliton molecule; lump



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1. Introduction

In nonlinear science, the Kadomtsev–Petviashvili (KP) equation as a fundamental completely integrable model is used to describe the propagation of small amplitude surface waves in fluid mechanics. The KP equation has been widely investigated by several classical methods [1]. The corresponding integrable positive KP hierarchy of the KP equation reads as [2]

$$u_{t_1} = u_x, \quad (1a)$$

$$u_{t_2} = -2u_y, \quad (1b)$$

$$u_{t_3} = -6uu_x - u_{xxx} + 3\partial_x^{-1}u_{yy}, \quad (1c)$$

$$u_{t_4} = 12(2u_x\partial_x^{-1}u_y - \partial_x^{-2}u_{yyy} + u_{xxy} + 4uu_y). \quad (1d)$$

The third member of the above KP hierarchy will reduce to the usual KP equation with $t_3 = t$. Because the fourth member of the KP hierarchy becomes the usual (1 + 1)-dimensional Korteweg–de Vries (KdV) equation with $y = x$, the fourth member of the KP hierarchy can be called an extension of the KdV equation. By means of the linearized operator of the KP equation, i.e., combining the third member and fourth

member in the positive KP hierarchy, Lou proposed a novel $(2 + 1)$ -dimensional integrable Korteweg–de Vries (KdV) equation [3]

$$u_t - b(u_{xxy} + 2wu_x + 4uu_y - v_y) - a(u_{xxx} + 6uu_x - 3w_y) = 0, \quad (2a)$$

$$u_y = w_x, \quad u_{yy} = v_{xx}. \quad (2b)$$

The Lax pairs, Painlevé property and symmetry reductions have been systematically studied [3,4]. The results for bilinear equations including the bilinear KdV-type and modified KdV-type equations are systematically studied [5,6].

Recently, the generalized $(2 + 1)$ -dimensional Date–Jimbo–Kashiwara–Miwa (gDJKM) equation is given based on the $(2 + 1)$ -dimensional KdV Equation (2). The detail $(2 + 1)$ -dimensional gDJKM form reads as [7]

$$cu_t + b(u_{xxy} + 2\gamma u_x v_x + 4\gamma u u_y + \delta v_{yy}) + a(u_{xxx} + 6\gamma u u_x + 3\delta v_{xy}) = 0, \quad (3a)$$

$$v_{xx} = u_y, \quad (3b)$$

where a, b, c, γ and δ are arbitrary constants. Due to the special physical properties, the DJKM equation is an important mathematical model to depict the propagation of the two-dimensional nonlinear solitary wave in physical fields such as fluid mechanics, nonlinear optics, etc. For the usual DJKM equation, the explicit lump wave and the mixed solution consisting of the lump and multi-soliton are constructed by assuming the ansatz method [8]. Symmetry reductions and generalized group invariant solutions are investigated by the Lie group theoretic method [9]. Wronskian rational solutions of the gDJKM Equation (3) are presented by a direct and concise calculation [7]. Various analytical solutions are constructed by using the the Sardar subequation and the new Kudryashov methods [10].

Soliton molecules, namely compact aggregates of two or more individual solitons bound together, are currently attracting major attention for their potential applications in nonlinear optics, communications and plasma physics [11]. Soliton molecules are observed in mode-locked fiber laser [12,13] and optical microresonators [14]. Theoretically, the velocity resonance mechanism is proposed to extract the soliton molecule [15], and it has been employed in many integrable systems [16,17]. The dynamics of the hybrid of soliton molecules and other nonlinear waves are complex and can feature a wide range of intriguing phenomena [18,19]. To our knowledge, the soliton molecules, multi-order lumps and hybrid structures of the gDJKM Equation (3) have not yet been studied. In this paper, some theoretical results for the gDJKM equation would be stated by the viewpoints of the velocity resonance mechanism, the long-wave limit method and numerical simulations. The primary purpose of the work is to explore the soliton molecules and multi-order lumps, as well as the interaction among lumps, breathers and soliton molecules of the gDJKM equation.

This paper is organized as follows. In Section 2, soliton molecules are given by using a velocity resonance mechanism based on the multi-soliton of the gDJKM equation. The propagation of a soliton molecule composed of three solitons is displayed graphically. In Section 3, multi-order lumps are obtained by using the long-wave limit method in the multi-solitons. The dynamics of one lump, two lumps and the path of the lumps are analyzed by selecting explicit parameters. In Section 4, the interaction between a soliton molecule and a one-order lump is obtained in detail. It demonstrates this interaction is an elastic collision due to their shape and amplitude remaining unchanged after interactions. The interaction between a one-order lump and one-order breather is analyzed by the complex conjugation relations for parameters. In Section 5, we give conclusions and discussions based on the results developed in this work.

2. Soliton Molecules of gDJKM Equation

The (2 + 1)-dimensional gDJKM equation has the multi-linear form [20]

$$\begin{aligned} \frac{3a}{2} D_x [(D_x^4 + 3\delta D_y^2) f \cdot f] \cdot f^2 + D_x [(b D_x^3 D_y + \frac{3c}{2} D_x D_t) f \cdot f] \cdot f^2 \\ + \frac{b}{2} D_y [(D_x^4 + 3\delta D_y^2) f \cdot f] \cdot f^2 = 0, \end{aligned} \quad (4)$$

where D_t , D_x and D_y are Hirota's bilinear differential operators [21–24]. The above multi-linear form is equivalent to

$$\begin{aligned} a(f^2 f_{xxxx} + 2f f_{xx} f_{xxx} - 5f f_x f_{xxxx} - 6f_{xx}^2 f_x + 8f_{xxx} f_x^2 + 3\delta f^2 f_{xyy} - 3\delta f f_x f_{yy} - 6\delta f f_y f_{xy} \\ + 6\delta f_x f_y^2) + b(f^2 f_{xxy} - f f_{xxx} f_y + 2f f_{xx} f_{xy} - 4f f_x f_{xxy} + 4f_x f_{xxx} f_y - 2f_{xx}^2 f_y + 4f_x^2 f_{xxy} \\ - 4f_{xx} f_x f_{xy} + \delta f^2 f_{yyy} + 2\delta f_y^3 - 3\delta f f_y f_{yy}) + c f^2 f_{xt} - c f f_{xx} f_t - 2c f f_x f_{xt} + 2c f_x^2 f_t = 0. \end{aligned} \quad (5)$$

The relation between u , v and f writes as

$$u = \frac{2}{\gamma} (\ln f)_{xx}, \quad v = \frac{2}{\gamma} (\ln f)_y. \quad (6)$$

The form of the single soliton assumes as

$$f = f_1 = 1 + \exp(k_1(x + p_1 y + \omega_1 t) + \phi_1), \quad (7)$$

the dispersion relation could be obtained as the following form by substituting (7) into (5)

$$\omega_1 = -\frac{a}{c} k_1^2 - \frac{3a\delta}{c} p_1^2 - \frac{b}{c} p_1 k_1^2 - \frac{b\delta}{c} p_1^3. \quad (8)$$

A two-soliton solution is written as

$$\begin{aligned} f = f_2 = 1 + \exp(k_1(x + p_1 y + \omega_1 t) + \phi_1) + \exp(k_2(x + p_2 y + \omega_2 t) + \phi_2) \\ + \exp[(k_1 + k_2)x + (p_1 + p_2)y + (\omega_1 + \omega_2)t + \phi_1 + \phi_2 + A_{12}]. \end{aligned} \quad (9)$$

By substituting (9) into (5) and making them identically equal to zero, arbitrary constants meet the following conditions

$$\begin{aligned} \omega_m = -\frac{a}{c} k_m^2 - \frac{3a\delta}{c} p_m^2 - \frac{b}{c} p_m k_m^2 - \frac{b\delta}{c} p_m^3, \quad (m = 1, 2), \\ \exp(A_{12}) = \frac{\delta(p_1 - p_2)^2 - (k_1 - k_2)^2}{\delta(p_1 - p_2)^2 - (k_1 + k_2)^2}. \end{aligned} \quad (10)$$

A three-soliton solution is assumed as

$$\begin{aligned} f = f_3 = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \exp(\eta_1 + \eta_2 + A_{12}) + \exp(\eta_1 + \eta_3 + A_{13}) \\ + \exp(\eta_2 + \eta_3 + A_{23}) + \exp(\eta_1 + \eta_2 + \eta_3 + A_{12}A_{13}A_{23}), \end{aligned} \quad (11)$$

By substituting (11) into (5) and balancing the coefficients of exponential functions, one obtains the relations

$$\begin{aligned} \omega_n = -\frac{a}{c} k_n^2 - \frac{3a\delta}{c} p_n^2 - \frac{b}{c} p_n k_n^2 - \frac{b\delta}{c} p_n^3, \quad (n = 1, 2, 3), \\ \exp(A_{12}) = \frac{\delta(p_1 - p_2)^2 - (k_1 - k_2)^2}{\delta(p_1 - p_2)^2 - (k_1 + k_2)^2}, \quad \exp(A_{13}) = \frac{\delta(p_1 - p_3)^2 - (k_1 - k_3)^2}{\delta(p_1 - p_3)^2 - (k_1 + k_3)^2}, \\ \exp(A_{23}) = \frac{\delta(p_2 - p_3)^2 - (k_2 - k_3)^2}{\delta(p_2 - p_3)^2 - (k_2 + k_3)^2}. \end{aligned} \quad (12)$$

A multi-soliton solution can be derived while a system possesses three solitons [5,6]. Similar to the above procedure, the multi-soliton solution of the gDJKM Equation (4) is given as

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \eta_i + \sum_{1 \leq i \leq j}^N \mu_i \mu_j A_{ij}\right), \quad (13)$$

with $\mu_i, \mu_j = 0, 1$ and

$$\begin{aligned} \eta_i &= k_i(x + p_i y + \omega_i t) + \phi_i, \\ \exp(A_{ij}) &= \frac{\delta(p_i - p_j)^2 - (k_i - k_j)^2}{\delta(p_i - p_j)^2 - (k_i + k_j)^2}, \quad \omega_i = -\frac{a}{c} k_i^2 - \frac{3a\delta}{c} p_i^2 - \frac{b}{c} p_i k_i^2 - \frac{b\delta}{c} p_i^3. \end{aligned} \quad (14)$$

The velocity of soliton in the x -axis and y -axis reads

$$(v_x, v_y) = \left(\omega_i, \frac{\omega_i}{p_i}\right) = \left(-\frac{a}{c} k_i^2 - \frac{3a\delta}{c} p_i^2 - \frac{b}{c} p_i k_i^2 - \frac{b\delta}{c} p_i^3, -\frac{a}{c} \frac{k_i^2}{p_i} - \frac{3a\delta}{c} p_i - \frac{b}{c} k_i^2 - \frac{b\delta}{c} p_i^2\right). \quad (15)$$

By using the condition of velocity resonance [25], i.e.,

$$v_{x,i} = v_{x,j}, \quad v_{y,i} = v_{y,j}, \quad (16)$$

the parameters in the multi-soliton satisfy the following relation

$$p_i = p_j = -\frac{a}{b}. \quad (17)$$

To obtain the molecule consisting of three solitons, we select three solitons for (11). The three solitons for the gDJKM Equation (4) are derived by substituting (11) into (6). By adjusting the velocities of three solitons, the molecule consisting of three solitons can be constructed with the aid of the relation (17). The molecule composed by three solitons is plotted in Figure 1 with the parameters as

$$\begin{aligned} a &= 1, \quad b = 2, \quad c = \frac{1}{4}, \quad \gamma = \frac{1}{10}, \quad \delta = -1, \quad \phi_1 = 2, \\ \phi_2 &= 3, \quad \phi_3 = -1, \quad k_1 = \frac{1}{4}, \quad k_2 = -\frac{1}{6}, \quad k_3 = -\frac{1}{3}. \end{aligned} \quad (18)$$

The relative positions of three solitons keep the same distances as time evolutions. According to the definition of soliton molecules, these three solitons are named as the soliton molecule. We can also obtain the soliton molecule composed of any number of solitons by controlling the velocities of different solitons.

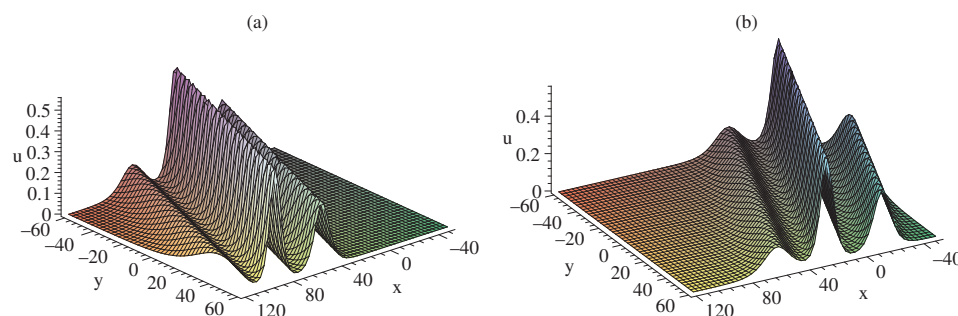


Figure 1. Three-dimensions of soliton molecule composed of three solitons with $t = -30$ and $t = 3$ in (a,b), respectively.

3. Multi-Order Lumps of gDJKM Equation

Based on the long-wave limit method, multi-solitons can be transformed to multi-lumps [26,27]. This method has been extensively investigated and widely applied to (2 + 1)-dimensional nonlinear systems [28,29]. Multi-order lumps of the (3 + 1)-dimensional nonlinear model are constructed by using the long-wave limit method [30]. The improved long-wave limit method is conveniently used to derive the higher-order rogue waves for the nonlinear Schrödinger equation [31]. Multi-lumps from the hydrodynamics have been observed in nonlinear optics [32]. In this section, the multi-order lumps of the gDJKM Equation (3) are constructed by using the long-wave limit approach. When the phase is selected as $\phi_i = i\pi$, the field f_N becomes

$$f_N = \sum_{\mu=0,1} \prod_{i=1}^N (-1)^{\mu_i} \exp(\mu_i \xi_i) \prod_{i < j}^N \exp(\mu_i \mu_j A_{ij}), \quad (19)$$

with $\xi_i = k_i[x + p_i y - (\frac{a}{c}k_i^2 + \frac{3a\delta}{c}p_i^2 + \frac{b}{c}p_i k_i^2 + \frac{b\delta}{c}p_i^3)t]$. By assuming a limit $k_i \rightarrow 0$ and omitting the constant factor $\prod_{i=1}^N k_i$, the above form of f_N will simplify as

$$f_N = \prod_{i=1}^N \theta_i + \frac{1}{2} \sum_{i,j}^N B_{ij} \prod_{r \neq i,j}^N \theta_r + \cdots + \frac{1}{M!2^M} \sum_{i,j,\dots,m,n}^N \overbrace{B_{ij}B_{kl} \cdots B_{mn}}^M \prod_{s \neq i,j,k,l,\dots,m,n}^N \theta_s + \cdots, \quad (20)$$

where $\theta_i = x + p_i y - \frac{\delta}{c}p_i^2(3a + bp_i)t$ and $B_{ij} = \frac{4}{\delta(p_i - p_j)^2}$. The symbolic of $\sum_{i,j,\dots,m,n}^N$ represents the summation over all possible combinations of $i, j, \dots, m, n$ which are selected with different values from $1, 2, \dots, N$. To establish the multi-order lumps, the parameter p_i meets the relation $p_{M+i} = p_i^*$, ($i = 1, 2, \dots, M$), $N = 2M$ and $B_{ij} > 0$, where the asterisk “*” denotes the complex conjugation.

By taking two solitons as $N = 2$, $M = 1$ and $\phi_1 = \phi_2 = i\pi$, (19) becomes a one-order lump solution

$$f_2 = 1 - \exp(\xi_1) - \exp(\xi_2) + \exp(\xi_1 + \xi_2 + A_{12}). \quad (21)$$

Selecting the long-wave limit $k_i \rightarrow 0$ and omitting the factor $k_1 k_2$ in (20), one derives

$$f_2 = \theta_1 \theta_2 + B_{12}, \quad (22)$$

where $\theta_i = x + p_i y - \frac{\delta}{c}p_i^2(3a + bp_i)t$, ($i = 1, 2$), $B_{12} = \frac{4}{\delta(p_1 - p_2)^2}$ and $p_1^* = p_2 = a_1 + b_1 i$. By substituting (22) into (3), a one-order lump wave writes as

$$u = \frac{2}{\gamma} \left[\frac{2}{\theta_1 \theta_2 + B_{12}} - \frac{\theta_1 \theta_2}{(\theta_1 \theta_2 + B_{12})^2} \right], \quad (23)$$

$$v = \frac{4}{\gamma} \left[\frac{a_1 x + (a_1^2 + b_1^2)y - \frac{\delta}{c}(a_1^2 + b_1^2)(a_1^2 b - b b_1^2 + 3a a_1)t}{\theta_1 \theta_2 + B_{12}} \right].$$

To track the moving path of a lump wave, the critical points of the lump wave are calculated by taking $u_x = u_y = 0$ or $v_x = v_y = 0$ [33]. The path of the lump wave moves along a straight line $y = \frac{bb_1^2 - 3a_1^2 b - 6aa_1}{2a_1^3 b + 2a_1 b b_1^2 + 3aa_1^2 + 3ab_1^2} x$. The profile of a one-order lump solution is exhibited in Figure 2 with $t = 0$. The parameters are selected as $a_1 = \frac{1}{5}$, $b_1 = \frac{1}{2}$, $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, $\delta = -1$, $\gamma = 2$ in Figure 2. The one-order lump wave travels along the line $y = -\frac{1135}{928}x$. We select different times in the contour plots of a one-order lump wave.

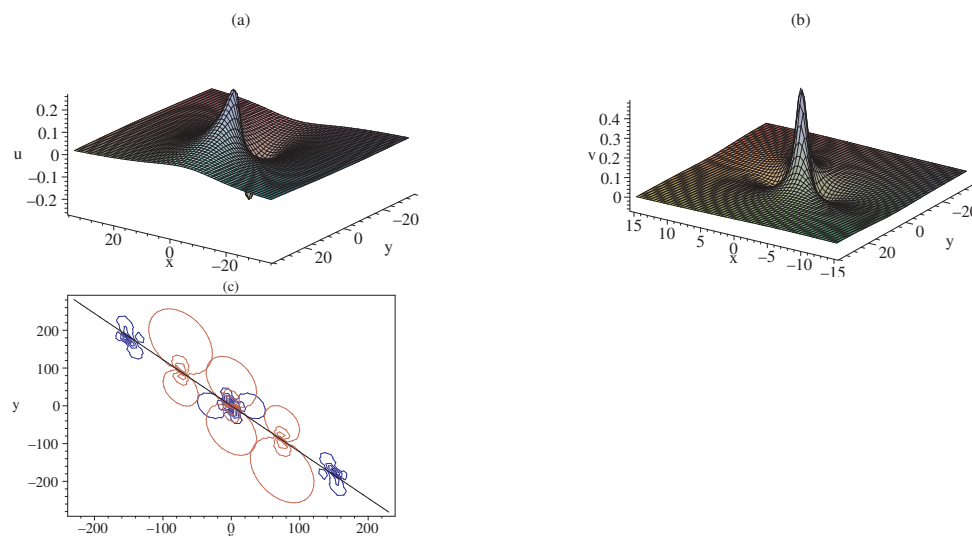


Figure 2. Three-dimensional visualization of a one-order lump for u and v with $t = 0$ are shown in (a,b). (c) The time of the contour plot of a one-order lump wave of field u is $t = -20, t = 0, t = 20$ (from bottom to top with orange color), while the time of the contour plot of v is $t = -40, t = 0, t = 40$ (from bottom to top with blue color), respectively. For the one-order lump wave, both u and v travel along the line $y = -\frac{1135}{928}x$ (black line).

To proceed with a similar procedure as $N = 4, M = 2$ in (20), a two-order lump writes as

$$f_4 = \theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 + B_{23}\theta_1\theta_4 + B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \quad (24)$$

where $\theta_i = x + p_i y - \frac{\delta}{c} p_i^2 (3a + b p_i) t$ and $B_{ij} = \frac{4}{\delta(p_i - p_j)^2}, (i, j = 1, 2, 3, 4)$. By taking $p_1 = p_2^* = a_1 + b_1 i, p_3 = p_4^* = a_2 + b_2 i$ and substituting (24) into (6), a two-order lump is given as

$$u = \frac{4}{\gamma} \frac{\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 + B_{12} + B_{13} + B_{14} + B_{23} + B_{24} + B_{34}}{f_4} - \frac{2}{\gamma} \frac{[\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 + B_{12}(\theta_3 + \theta_4) + B_{13}(\theta_2 + \theta_4) + B_{14}(\theta_2 + \theta_3) + B_{23}(\theta_1 + \theta_4) + B_{24}(\theta_1 + \theta_3) + B_{34}(\theta_1 + \theta_2)]^2}{f_4^2}, \quad (25)$$

$$v = \frac{4}{\gamma f_4} [p_4\theta_1\theta_2\theta_3 + p_3\theta_1\theta_2\theta_4 + p_2\theta_1\theta_3\theta_4 + p_1\theta_2\theta_3\theta_4 + B_{12}(p_4\theta_3 + p_3\theta_4) + B_{13}(p_4\theta_2 + p_2\theta_4) + B_{14}(p_3\theta_2 + p_2\theta_3) + B_{23}(p_4\theta_1 + p_1\theta_4) + B_{24}(p_3\theta_1 + p_1\theta_3) + B_{34}(p_2\theta_1 + p_1\theta_2)].$$

The parameters of the two-order lump in Figure 3 are $a_1 = -1, b_1 = 2, a_2 = -1, b_2 = -\frac{3}{2}, c = \frac{1}{2}, \delta = -3, \gamma = -1$. Three-dimensional visualizations of a two-order lump are given in Figure 3a and Figure 3b, respectively. The energy of a lump wave distributes in a local area from Figures 2 and 3. The lump wave is thus localized in all directions of space from Figures 2 and 3.

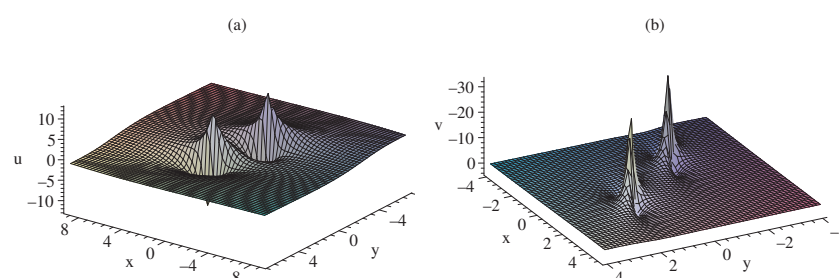


Figure 3. Three-dimensional visualization of a two-order lump for u and v with $t = 0$ is plotted in (a,b), respectively.

4. Interaction Among Lumps, Breathers and Soliton Molecules of gDJKM Equation

Complex physical phenomena are often described by interactions among different nonlinear excitations rather than a single localized wave [34–37]. While the partial long-wave limit method is applied to the multi-soliton, the interactions among different nonlinear excitations can be constructed [38,39]. Multi-order breathers, multi-order lumps and hybrid solutions of a (2 + 1)-dimensional nonlinear wave equation are given by the long-wave limit method in the multi-solitons [40]. For $N = 4$ and selecting a long-wave limit procedure to k_1, k_2 in (13), the interaction between a one-order lump and two solitons writes as

$$f_4 = \theta_1\theta_2 + B_{12} + (B_{13}B_{23} + B_{13}\theta_2 + B_{23}\theta_1 + B_{12} + \theta_1\theta_2)\exp(\eta_3) + (B_{14}B_{24} + B_{14}\theta_2 + B_{24}\theta_1 + B_{12} + \theta_1\theta_2)\exp(\eta_4) + A_{34}(B_{13}B_{23} + B_{14}B_{23} + B_{13}B_{24} + B_{14}B_{24})(B_{13} + B_{14})\theta_2 + (B_{23} + B_{24})\theta_1 + B_{12} + \theta_1\theta_2)\exp(\eta_3 + \eta_4), \quad (26)$$

with

$$\theta_i = x + p_i y - \frac{\delta}{c} p_i^2 (3a + b p_i) t, \quad \eta_j = k_j [x + p_j y - (\frac{a}{c} k_j^2 + \frac{3a\delta}{c} p_j^2 + \frac{b}{c} p_j k_j^2 + \frac{b\delta}{c} p_j^3) t] + \phi_j, \\ B_{12} = \frac{4}{\delta(p_1 - p_2)^2}, \quad B_{ij} = \frac{4}{k_j^2 - \delta(p_i - p_j)^2}, \quad (i = 1, 2; j = 3, 4), \quad p_1 = p_2^*, \quad A_{34} = \frac{\delta(p_3 - p_4)^2 - (k_3 - k_4)^2}{\delta(p_3 - p_4)^2 - (k_3 + k_4)^2}.$$

The interaction between a one-order lump wave and two solitons is given by substituting (26) into (6). By taking p_3 and p_4 as the condition (17), two solitons convert into a soliton molecule consisting of two solitons. For Figure 4, the parameters are selected as

$$p_1 = p_2^* = -\frac{1}{20} + \frac{1}{2}i, \quad k_3 = \frac{1}{2}, \quad \phi_3 = 15, \quad k_4 = \frac{1}{3}, \\ \phi_4 = 2, \quad a = 1, \quad b = 2, \quad c = 1, \quad \delta = -\frac{1}{2}, \quad \gamma = -\frac{1}{2}. \quad (27)$$

The shape and amplitude of a one-order lump and a soliton molecule consisting of two dark solitons remain unchanged after collision, which manifests as an elastic collision. Although soliton molecules have been observed in some experimental results [12–14], the type of the interaction between a one-order lump and a soliton molecule has not been found in experiments. Thus, this phenomenon deserves further exploration both experimentally and theoretically.

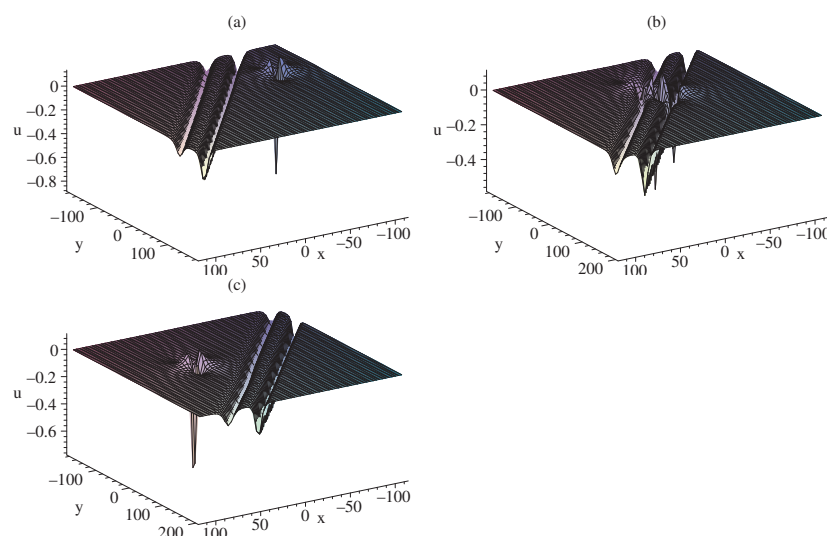


Figure 4. The interaction between a one-order lump and a soliton molecule is given at time $t = -200$, $t = -68$, $t = 80$ in (a–c), respectively.

A multi-soliton solution would become the multi-order breathers by selecting the complex conjugate relations of the parameters of a multi-soliton as

$$N = 2m, \quad k_{i+1} = k_i^* = a_i + ia_{ii}, \quad p_{i+1} = p_i^* = b_i + ib_{ii}, \\ \phi_{i+1} = \phi_i^* = n_i + in_{ii}, \quad i = 1, 3, 5, \dots, m. \quad (28)$$

The interaction between a one-order lump wave and one-order breather is obtained by choosing the parameters as (28). For Figure 5, the explicit parameters are selected as

$$p_1 = p_2^* = -\frac{1}{4} - \frac{1}{2}i, \quad k_3 = k_4^* = -\frac{1}{4} - \frac{1}{3}i, \quad p_3 = p_4^* = -\frac{1}{2} - \frac{1}{3}i, \\ \phi_3 = \phi_4^* = 8 + i, \quad a = 1, \quad b = 2, \quad c = 1, \quad \delta = -1, \quad \gamma = -1. \quad (29)$$

A one-order lump wave and one-order breather are dark forms for the field u , while the lump wave and the breather are symmetric about the plane $v = 0$ for the field v . Similar to the above approach, interactions among several multi-order lumps and soliton molecules can be constructed by combining the partial long-wave limit and the relation (28).

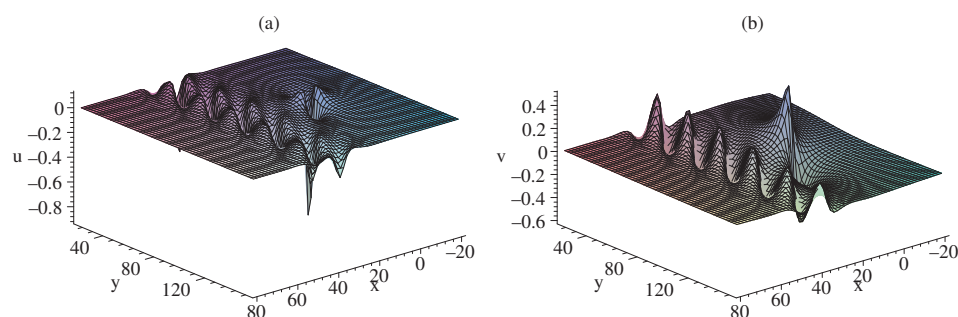


Figure 5. The interaction between a one-order lump and one-order breather for the field u and v is given at time $t = 0$ in (a,b), respectively.

5. Conclusions

In summary, the $(2 + 1)$ -dimensional gDJKM equation is studied by combining the long-wave limit method and velocity resonance mechanism. Soliton molecules are given by making the velocities of solitons equal. The evolution of a soliton molecule is exhibited in Figure 1. This type of the soliton molecule consists of three solitons. The relative positional invariance of three solitons verifies that they form a soliton molecule. A multi-soliton converts to the multi-order lumps through the long-wave limit method. A one-order lump, two-order lump and their paths of propagation are analyzed in detail. Multi-order breathers are derived by choosing the complex conjugate relations (28). Furthermore, hybrid solutions among multi-order lumps, multi-order breathers and soliton molecules can be constructed by the above analysis. The interaction between a one-order lump and a soliton molecule is an elastic collision which is shown in Figure 4. The study of these hybrid solutions possesses significant potential applications in physical and engineering fields [10]. On the other hand, the nonlinear transformed waves can be constructed by utilizing the characteristic lines of the breather solution [41,42]. The transformed waves possess unique spatial structure and dynamic properties, which may contribute to the understanding of the nonlinear wave behavior in higher-dimensional nonlinear systems. Further research will focus on the derivation and dynamics of the transformed waves.

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