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On Symmetric Aspects of Operator Pair Inequalities in Hilbert Spaces via Pečarić's Theorem with Applications

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Abstract

In this paper, we establish several new operator inequalities for generalizations of the joint numerical radius and joint operator norm for pairs of operators in complex Hilbert spaces, as well as for the classical numerical radius of a single operator. One of our main tools is the well-known Pečarić's Theorem. As applications, we derive a series of power inequalities for the operator norm and for the generalized numerical radius, which refine and generalize a number of existing results in the literature. Our approach considers two key symmetric pairings: the Cartesian decomposition $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ and the operator-adjoint pair $(\mathcal{U}, \mathcal{U}^\star)$.

Keywords: bounded linear operators; Hilbert space; operator norm; numerical radius; Pečarić inequality; operator inequalities; symmetry

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1. Introduction

The study of the numerical radius, a central concept in operator theory [1], has a rich history dating back to foundational works like Kato [2]. In recent decades, significant progress has been achieved in establishing sharp bounds, with the inequalities developed by Kittaneh [3] being particularly influential. These bounds have found wide application, for instance, in the analysis of operator matrices [4]. The ongoing quest for more accurate [5] and extended [6] estimates continues to motivate new research, and in this paper, we contribute to this effort by introducing a new framework based on operator symmetries.

Let $\mathbb{K}$ be a complex Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. The algebra of all bounded linear operators on $\mathbb{K}$ is denoted by $\mathcal{L}(\mathbb{K})$. For any given operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, its adjoint is denoted by $\mathcal{U}^\star$, and its positive square root is defined as $|\mathcal{U}| = (\mathcal{U}^\star\mathcal{U})^{\frac{1}{2}}$. The real and imaginary parts of $\mathcal{U}$ are defined as $\mathfrak{R}(\mathcal{U}) = \frac{1}{2}(\mathcal{U} + \mathcal{U}^\star)$ and $\mathfrak{I}(\mathcal{U}) = \frac{1}{2i}(\mathcal{U} - \mathcal{U}^\star)$, respectively. The numerical range of $\mathcal{U}$, denoted by $\mathcal{W}(\mathcal{U})$, is the set $\{\langle\mathcal{U}\xi,\xi\rangle : \xi \in \mathbb{K}, \|\xi\| = 1\}$.

Let $\mathbb{K}$ be a complex Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. The algebra encompassing all bounded linear operators on $\mathbb{K}$ is denoted by $\mathcal{L}(\mathbb{K})$. For any given operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, its adjoint is signified by $\mathcal{U}^\star$, and its positive square root is defined as $|\mathcal{U}| = (\mathcal{U}^\star\mathcal{U})^{\frac{1}{2}}$. The real and imaginary parts of $\mathcal{U}$ are specified as $\Re(\mathcal{U}) = \frac{1}{2}(\mathcal{U} + \mathcal{U}^\star)$ and $\Im(\mathcal{U}) = \frac{1}{2i}(\mathcal{U} - \mathcal{U}^\star)$, respectively. The numerical range of $\mathcal{U}$, denoted by $\mathcal{W}(\mathcal{U})$, is the set of all values $\{\langle\mathcal{U}\xi,\xi\rangle : \xi \in \mathbb{K}, \|\xi\| = 1\}$.

The operator norm $\|\mathcal{U}\|$ and numerical radius $\omega(\mathcal{U})$ of an operator $\mathcal{U}$ are given by:

$$\|\mathcal{U}\| = \sup\{|\langle\mathcal{U}\xi_1,\xi_2\rangle| : \xi_1,\xi_2 \in \mathbb{K}, \|\xi_1\| = \|\xi_2\| = 1\},$$
$$\omega(\mathcal{U}) = \sup\{|\langle\mathcal{U}\xi,\xi\rangle| : \xi \in \mathbb{K}, \|\xi\| = 1\}.$$

It is widely recognized that the numerical radius $\omega(\cdot)$ defines a norm on $\mathcal{L}(\mathbb{K})$ that is equivalent to the operator norm $\|\cdot\|$. The following sharp inequalities are satisfied:

$$\frac{1}{2}\|\mathcal{U}\| \le \omega(\mathcal{U}) \le \|\mathcal{U}\|. \tag{1}$$

The first inequality becomes an equality if $\mathcal{U}^2 = 0$, whereas the second becomes an equality if $\mathcal{U}$ is a normal operator, meaning that $\mathcal{U}^\star\mathcal{U} = \mathcal{U}\mathcal{U}^\star$. Kittaneh [3] further refined these bounds, showing that

$$\frac{1}{2}\sqrt{\|\mathcal{U}^\star\mathcal{U} + \mathcal{U}\mathcal{U}^\star\|} \le \omega(\mathcal{U}) \le \frac{\sqrt{2}}{2}\sqrt{\|\mathcal{U}^\star\mathcal{U} + \mathcal{U}\mathcal{U}^\star\|}. \tag{2}$$

For more extensive discussions regarding (1) and (2), readers are encouraged to consult references [7–13].

For a pair of operators $(\mathcal{U},\mathcal{V})$ in $\mathcal{L}(\mathbb{K})$, the Euclidean operator radius, denoted by $\omega_e(\mathcal{U},\mathcal{V})$, is defined as

$$\omega_e(\mathcal{U},\mathcal{V}) = \sup\{\sqrt{|\langle\mathcal{U}\xi,\xi\rangle|^2 + |\langle\mathcal{V}\xi,\xi\rangle|^2} : \xi \in \mathbb{K}, \|\xi\| = 1\}.$$

Additional details can be found in [14]. According to [15], the function $\omega_e(\cdot,\cdot) : \mathcal{L}(\mathbb{K}) \times \mathcal{L}(\mathbb{K}) \to [0,\infty)$ constitutes a norm that satisfies

$$\frac{\sqrt{2}}{4}\sqrt{\||\mathcal{U}|^2 + |\mathcal{V}|^2\|} \le \omega_e(\mathcal{U},\mathcal{V}) \le \sqrt{\||\mathcal{U}|^2 + |\mathcal{V}|^2\|}.$$

Here, the constants $\frac{\sqrt{2}}{4}$ and 1 are optimal. For self-adjoint operators $\mathcal{U}$ and $\mathcal{V}$, this inequality simplifies to

$$\frac{\sqrt{2}}{4}\sqrt{\|\mathcal{U}^2 + \mathcal{V}^2\|} \le \omega_e(\mathcal{U},\mathcal{V}) \le \sqrt{\|\mathcal{U}^2 + \mathcal{V}^2\|}.$$

Notably, when $\mathcal{U}$ and $\mathcal{V}$ are self-adjoint, $\omega_e(\mathcal{U},\mathcal{V}) = \omega(\mathcal{U} + i\mathcal{V})$, a result easily derived from the definition of $\omega_e(\mathcal{U},\mathcal{V})$.

In a previous study [16], the second author derived the following lower bound:

$$\frac{\sqrt{2}}{2}[\omega(\mathcal{U}^2 + \mathcal{V}^2)]^{\frac{1}{2}} \le \omega_e(\mathcal{U},\mathcal{V}). \tag{3}$$

The constant $\frac{\sqrt{2}}{2}$ in (3) has been demonstrated to be the best possible. The same work presented other results, such as

$$\frac{\sqrt{2}}{2}\max\{\omega(\mathcal{U}+\mathcal{V}),\omega(\mathcal{U}-\mathcal{V})\} \le \omega_e(\mathcal{U},\mathcal{V}) \le \frac{\sqrt{2}}{2}\sqrt{\omega^2(\mathcal{U}+\mathcal{V}) + \omega^2(\mathcal{U}-\mathcal{V})},$$

where $\frac{\sqrt{2}}{2}$ is sharp in both inequalities. Further, the following sharp inequalities were established:

$$\omega_e^2(\mathcal{U},\mathcal{V}) \leq \max\{\|\mathcal{U}\|^2, \|\mathcal{V}\|^2\} + \omega(\mathcal{V}^\star\mathcal{U}),$$

and

$$\omega_e^2(\mathcal{U},\mathcal{V}) \leq \frac{1}{2}[\|\mathcal{U}^\star\mathcal{U} + \mathcal{V}^\star\mathcal{V}\| + \|\mathcal{U}^\star\mathcal{U} - \mathcal{V}^\star\mathcal{V}\|] + \omega(\mathcal{V}^\star\mathcal{U}).$$

By substituting the pair $(\mathcal{U},\mathcal{V})$ with either $(\mathcal{U},\mathcal{U}^\star)$ or $(\Re(\mathcal{U}),\Im(\mathcal{U}))$ for an operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, the second author in [16] obtained several inequalities for the norm and numerical radius of a single operator. For enhancements of these results, one may refer to the recent work [17], in which S. Jana, P. Bhunia, and K. Paul introduced significant findings, including:

$$\omega_e^2(\mathcal{U},\mathcal{V}) \leq \min\{\omega^2(\mathcal{U}+\mathcal{V}), \omega^2(\mathcal{U}-\mathcal{V})\} + \frac{1}{2}\||\mathcal{V}|^2 + |\mathcal{U}^\star|^2\| + \omega(\mathcal{U}\mathcal{V}),$$

$$\max_{1\leq\gamma\leq 1} \omega(\gamma\mathcal{U}^2 + (1-\gamma)\mathcal{V}^2) \leq \omega_e^2(\mathcal{U},\mathcal{V}),$$

and

$$\omega_e^2(\mathcal{U},\mathcal{V}) \leq \omega^2(\sqrt{\gamma}\mathcal{U} + \sqrt{1-\gamma}\mathcal{V}) + \omega^2(\sqrt{1-\gamma}\mathcal{U} + \sqrt{\gamma}\mathcal{V}),$$

for any $\gamma \in [0,1]$.

For a given $r \geq 1$, we examine the generalized expressions for a pair of operators $\mathcal{U},\mathcal{V} \in \mathcal{L}(\mathbb{K})$ from [18]:

$$\omega_r(\mathcal{U},\mathcal{V}) := \sup_{\|\xi\|=1} (|\langle\mathcal{U}\xi,\xi\rangle|^r + |\langle\mathcal{V}\xi,\xi\rangle|^r)^{\frac{1}{r}},$$

along with the generalized *s*-*r* norm from [18]:

$$\|(\mathcal{U},\mathcal{V})\|_r := \sup_{\|\xi_1\|=\|\xi_2\|=1} (|\langle\mathcal{U}\xi_1,\xi_2\rangle|^r + |\langle\mathcal{V}\xi_1,\xi_2\rangle|^r)^{\frac{1}{r}}.$$

When $r = 2$, these definitions correspond to the Euclidean numerical radius [15]:

$$\omega_e(\mathcal{U},\mathcal{V}) := \sup_{\|\xi\|=1} \left(|\langle\mathcal{U}\xi,\xi\rangle|^2 + |\langle\mathcal{V}\xi,\xi\rangle|^2\right)^{\frac{1}{2}},$$

and the Euclidean norm [15]:

$$\|(\mathcal{U},\mathcal{V})\|_e := \sup_{\|\xi_1\|=\|\xi_2\|=1} \left(|\langle\mathcal{U}\xi_1,\xi_2\rangle|^2 + |\langle\mathcal{V}\xi_1,\xi_2\rangle|^2\right)^{\frac{1}{2}}.$$

When $r = 1$, we use the notations

$$\omega(\mathcal{U},\mathcal{V}) := \sup_{\|\xi\|=1} (|\langle\mathcal{U}\xi,\xi\rangle| + |\langle\mathcal{V}\xi,\xi\rangle|),$$

and

$$\|(\mathcal{U},\mathcal{V})\| := \sup_{\|\xi_1\|=\|\xi_2\|=1} (|\langle\mathcal{U}\xi_1,\xi_2\rangle| + |\langle\mathcal{V}\xi_1,\xi_2\rangle|).$$

These definitions can be extended to $r \in (0,1)$, although in that range, they no longer satisfy the properties of a norm.

In 2017, Moslehian et al. [14] established fundamental inequalities for the *s*–*r* numerical radius of an operator pair $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$, which include the following:

$$\omega_p(\mathcal{U}, \mathcal{V}) \leq \omega_q(\mathcal{U}, \mathcal{V}) \leq 2^{\frac{1}{q}-\frac{1}{p}} \omega_p(\mathcal{U}, \mathcal{V}) \text{ for } p \geq q \geq 1,$$

$$2^{\frac{1}{p}-2} \|\mathcal{U}\mathcal{U}^\star + \mathcal{V}\mathcal{V}^\star\| \leq \omega_p(\mathcal{U}, \mathcal{V}),\ p \geq 2,$$

$$2^{\frac{1}{p}-1} \max\{\omega(\mathcal{U}+\mathcal{V}), \omega(\mathcal{U}-\mathcal{V})\} \leq \omega_p(\mathcal{U}, \mathcal{V}), \text{ and}$$
$$2^{\frac{1}{p}-1} \max\{\omega(\mathcal{U}), \omega(\mathcal{V})\} \leq \omega_p(\mathcal{U}, \mathcal{V}),\ p \geq 1,$$

and

$$2^{\frac{1}{p}-1} \omega^{\frac{1}{2}}(\mathcal{U}^2 + \mathcal{V}^2) \leq \omega_p(\mathcal{U}, \mathcal{V}),\ p \geq 1,$$

among others. They further applied these findings to the Cartesian decomposition of an operator.

In 1992, J. E. Pečarić [19] (see also [20], p. 394) established a general inequality for inner product spaces. It states that for any vectors $\xi_0, \xi_1, \ldots, \xi_n \in \mathbb{K}$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, the following inequality holds:

$$\left| \sum_{i=1}^{n} \lambda_i \langle \xi_0, \xi_i \rangle \right|^2 \leq \|\xi_0\|^2 \sum_{i=1}^{n} |\lambda_i|^2 \left(\sum_{j=1}^{n} |\langle \xi_i, \xi_j \rangle| \right). \tag{4}$$

Setting $n = 2$ in (4), we find that

$$\begin{aligned} &|\lambda_1 \langle \xi_0, \xi_1 \rangle + \lambda_2 \langle \xi_0, \xi_2 \rangle|^2 \\ &\leq \|\xi_0\|^2 \Big[|\lambda_1|^2 \Big(\|\xi_1\|^2 + |\langle \xi_1, \xi_2 \rangle| \Big) + |\lambda_2|^2 \Big(\|\xi_2\|^2 + |\langle \xi_1, \xi_2 \rangle| \Big) \Big] \\ &= \|\xi_0\|^2 \Big[|\lambda_1|^2 \|\xi_1\|^2 + |\lambda_2|^2 \|\xi_2\|^2 + (|\lambda_1|^2 + |\lambda_2|^2) |\langle \xi_1, \xi_2 \rangle| \Big], \end{aligned} \tag{5}$$

for any $\xi_0, \xi_1, \xi_2 \in \mathbb{K}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

Motivated by these findings and by applying (5), we derive multiple inequalities. For a single operator $\mathcal{U}$, these include:

$$\|\mathcal{U}\|^{2p} \leq 2^{p-1} \left[\left\| \frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2} \right\|^p + 2^p \omega^p(\mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})) \right],$$

and

$$\begin{aligned} \omega^{2p}(\mathcal{U}) &\leq 2^{p-1} \omega^p(\mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})) \\ &+ 2^{p-1} \times \begin{cases} \|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\|^p, \\ \|\mathfrak{R}(\mathcal{U})^{2r} + \mathfrak{I}(\mathcal{U})^{2r}\|^{p/r} \text{ for } r > 1, \\ \left(\frac{1}{2} \|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\| + \frac{1}{2} \|\mathfrak{R}(\mathcal{U})^2 - \mathfrak{I}(\mathcal{U})^2\| \right)^p \end{cases} \end{aligned}$$

for all $p \geq 1$.

Furthermore, for an operator pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{L}(\mathbb{K})$, with $p \geq 1$ and $r > 1$, we establish:

$$\begin{aligned}\|(\mathcal{U}, \mathcal{V})\|_r^{2pr} &\leq 2^{p-1}\Big[\Big(\|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r}\Big)^p + \|(\mathcal{U}, \mathcal{V})\|_{2(r-1)}^{2p(r-1)}\omega^p(\mathcal{V}^\star\mathcal{U})\Big] \\ &\leq 2^{p-1}\Big[\Big(\|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r}\Big)^p + \Big(\|\mathcal{U}\|^{2(r-1)} + \|\mathcal{V}\|^{2(r-1)}\Big)^p\omega^p(\mathcal{V}^\star\mathcal{U})\Big],\end{aligned}$$

and for the case $r = 1$,

$$\|(\mathcal{U}, \mathcal{V})\|^{2p} \leq 2^{p-1}\Big[\Big\||\mathcal{U}|^2 + |\mathcal{V}|^2\Big\|^p + 2^p\omega^p(\mathcal{V}^\star\mathcal{U})\Big].$$

Moreover, for $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$, $p \geq 1$, and $r > 1$, we have

$$\begin{aligned}&\omega_r^{2pr}(\mathcal{U}, \mathcal{V}) \\ &\leq 2^{p-1} \times \Big[\Big(\omega^{2(r-1)}(\mathcal{U})\|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V})\|\mathcal{V}\|^2\Big)^p + \omega_{2(r-1)}^{2p(r-1)}(\mathcal{U}, \mathcal{V})\omega^p(\mathcal{V}^\star\mathcal{U})\Big] \\ &\leq 2^{p-1}\Big[\Big(\omega^{2(r-1)}(\mathcal{U})\|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V})\|\mathcal{V}\|^2\Big)^p \\ &+ \Big(\omega^{2(r-1)}(\mathcal{U}) + \omega^{2(r-1)}(\mathcal{V})\Big)^p\omega^p(\mathcal{V}^\star\mathcal{U})\Big].\end{aligned}$$

We also conduct a detailed investigation of instances where the pair $(\mathcal{U}, \mathcal{V})$ is specified as $(\mathcal{U}, \mathcal{U}^\star)$ or $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ for some operator $\mathcal{U}$.

Symmetry is a foundational concept in many operator-theoretic studies. The pairs $(\mathcal{U}, \mathcal{U}^\star)$ and $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ represent fundamental symmetries: the former illustrates the duality between an operator and its adjoint, while the latter signifies the Cartesian decomposition into real and imaginary parts. These structures exhibit a balanced and complementary nature, positioning them as ideal subjects for deriving more refined inequalities. This work demonstrates how such symmetric properties can extend Pečarić's inequality to operator theory, yielding more acute bounds for norms and numerical radii.

Throughout this paper, the significance of symmetry in operator inequalities is consistently emphasized. When inequalities are developed for pairs like $(\mathcal{U}, \mathcal{U}^\star)$, or $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$, we highlight the corresponding symmetric relationship. This underscores the duality and symmetry inherent in these decompositions, mirroring the intrinsic symmetry of operator adjoints. Adopting this perspective not only simplifies several proofs, but also clarifies how bounds can be transferred from one symmetric context to another.

Section 2 introduces the primary inequalities and their proofs. Section 3 is dedicated to applications involving *s*–*r*-norm inequalities, while Section 4 outlines the implications for the *s*–*r*-numerical radius. The paper concludes with a discussion on the sharpness of these bounds and provides illustrative examples.

2. Main Results

In this section, we present our main results. The subsequent theorem establishes several vector inequalities for a linear combination of operators. These will serve as a foundation for deriving the various norm and numerical radius inequalities below.

Theorem 1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$ and $\gamma, \delta \in \mathbb{C}$. Then, for $p \geq 1$, the following inequality holds:*

$$\begin{aligned}&\big|\langle(\gamma\mathcal{U} + \delta\mathcal{V})\xi_1, \xi_2\rangle\big|^{2p} \\ &\leq 2^{p-1}\|\xi_2\|^{2p}\Big[\big\langle\Big(|\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{V}|^2\Big)\xi_1, \xi_1\big\rangle^p + \Big(|\gamma|^2 + |\delta|^2\Big)^p\big|\langle\mathcal{V}^\star\mathcal{U}\xi_1, \xi_1\rangle\big|^p\Big] \quad (6)\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

For the specific case where $p = 1$, we have:

$$\begin{aligned}&\left|\left\langle(\gamma\mathcal{U}+\delta\mathcal{V})\xi_1,\xi_2\right\rangle\right|^2\\&\le\|\xi_2\|^2\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|\right]\end{aligned}$$

for all $\xi_1,\xi_2\in\mathbb{K}$.

Proof. Let $\mathcal{U},\mathcal{V}\in\mathcal{L}(\mathbb{K})$ and $\gamma,\delta\in\mathbb{C}$. From the inequality (5), we can deduce that

$$\begin{aligned}&\left|\left\langle\gamma\mathcal{U}\xi_1+\delta\mathcal{V}\xi_1,\xi_2\right\rangle\right|^2\\&\le\|\xi_2\|^2\left[|\gamma|^2\|\mathcal{U}\xi_1\|^2+|\delta|^2\|\mathcal{V}\xi_1\|^2+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{U}\xi_1,\mathcal{V}\xi_1\right\rangle\right|\right]\\&=\|\xi_2\|^2\left[|\gamma|^2\left\langle\mathcal{U}\xi_1,\mathcal{U}\xi_1\right\rangle+|\delta|^2\left\langle\mathcal{V}\xi_1,\mathcal{V}\xi_1\right\rangle+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|\right]\\&=\|\xi_2\|^2\left[|\gamma|^2\left\langle|\mathcal{U}|^2\xi_1,\xi_1\right\rangle+|\delta|^2\left\langle|\mathcal{V}|^2\xi_1,\xi_1\right\rangle+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|\right]\\&=\|\xi_2\|^2\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|\right]\end{aligned}\tag{7}$$

for all $\xi_1,\xi_2\in\mathbb{K}$.

By taking the power $p\ge 1$ and applying the elementary inequality

$$(m+n)^p\le 2^{p-1}(m^p+n^p)\text{ for }m,\ n\ge 0,$$

we can infer from (7) that

$$\begin{aligned}&\left|\left\langle\gamma\mathcal{U}\xi_1+\delta\mathcal{V}\xi_1,\xi_2\right\rangle\right|^{2p}\\&\le\|\xi_2\|^{2p}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle+\left(|\gamma|^2+|\delta|^2\right)\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|\right]^p\\&\le 2^{p-1}\|\xi_2\|^{2p}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle^p+\left(|\gamma|^2+|\delta|^2\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|^p\right]\end{aligned}$$

for any $\xi_1,\xi_2\in\mathbb{K}$, which confirms the inequality (6). □

Remark 1. *If we take the supremum over all unit vectors ξ_2 (i.e., $\|\xi_2\|=1$) in the inequality (6), we obtain:*

$$\begin{aligned}&\|\gamma\mathcal{U}\xi_1+\delta\mathcal{V}\xi_1\|^{2p}\\&\le 2^{p-1}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle^p+\left(|\gamma|^2+|\delta|^2\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|^p\right].\end{aligned}$$

Consequently, this leads to

$$\begin{aligned}&\sup_{\|\xi_1\|=1}\|\gamma\mathcal{U}\xi_1+\delta\mathcal{V}\xi_1\|^{2p}\\&\le 2^{p-1}\sup_{\|\xi_1\|=1}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle^p+\left(|\gamma|^2+|\delta|^2\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|^p\right]\\&\le 2^{p-1}\left[\sup_{\|\xi_1\|=1}\left\langle\left(|\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right)\xi_1,\xi_1\right\rangle^p+\left(|\gamma|^2+|\delta|^2\right)^p\sup_{\|\xi_1\|=1}\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_1,\xi_1\right\rangle\right|^p\right]\\&=2^{p-1}\left[\left\||\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right\|^p+\left(|\gamma|^2+|\delta|^2\right)^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]\end{aligned}$$

which yields the main inequality of interest:

$$\|\gamma\mathcal{U}+\delta\mathcal{V}\|^{2p}\le 2^{p-1}\left[\left\||\gamma|^2|\mathcal{U}|^2+|\delta|^2|\mathcal{V}|^2\right\|^p+\left(|\gamma|^2+|\delta|^2\right)^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]\tag{8}$$

for $p \geq 1$.

In the case of $p = 1$, we obtain

$$\|\gamma\mathcal{U} + \delta\mathcal{V}\|^2 \leq \left\||\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{V}|^2\right\| + \left(|\gamma|^2 + |\delta|^2\right)\omega(\mathcal{V}^\star\mathcal{U}).$$

If we set $\mathcal{V} = \mathcal{U}^\star$ in Theorem 1, then for $p \geq 1$, we have

$$\begin{aligned}&\left|\left\langle \gamma\mathcal{U}\xi_1 + \delta\mathcal{U}^\star\xi_1, \xi_2\right\rangle\right|^{2p} \\ &\leq 2^{p-1}\|\xi_2\|^{2p}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{U}^\star|^2\right)\xi_1, \xi_1\right\rangle^p + \left(|\gamma|^2 + |\delta|^2\right)^p\left|\left\langle\mathcal{U}^2\xi_1, \xi_1\right\rangle\right|^p\right]\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

For $p = 1$, this simplifies to

$$\begin{aligned}&\left|\left\langle \gamma\mathcal{U}\xi_1 + \delta\mathcal{U}^\star\xi_1, \xi_2\right\rangle\right|^{2} \\ &\leq \|\xi_2\|^{2}\left[\left\langle\left(|\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{U}^\star|^2\right)\xi_1, \xi_1\right\rangle + \left(|\gamma|^2 + |\delta|^2\right)\left|\left\langle\mathcal{U}^2\xi_1, \xi_1\right\rangle\right|\right]\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

The corresponding norm inequalities are

$$\|\gamma\mathcal{U} + \delta\mathcal{U}^\star\|^{2p} \leq 2^{p-1}\left[\left\||\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{U}^\star|^2\right\|^p + \left(|\gamma|^2 + |\delta|^2\right)^p\omega^p\left(\mathcal{U}^2\right)\right]$$

and

$$\|\gamma\mathcal{U} + \delta\mathcal{U}^\star\|^{2} \leq \left\||\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{U}^\star|^2\right\| + \left(|\gamma|^2 + |\delta|^2\right)\omega\left(\mathcal{U}^2\right)$$

for any γ, $\delta \in \mathbb{C}$. The pair $(\mathcal{U},\mathcal{U}^\star)$ effectively highlights the adjoint symmetry between an operator and its conjugate counterpart; the resulting bounds quantify this duality. By choosing $\gamma = \delta = \frac{1}{2}$ and $\gamma = \frac{1}{2i}$, $\delta = -\frac{1}{2i}$, we obtain

$$\begin{aligned}&\max\left\{\left|\left\langle\Re(\mathcal{U})\xi_1, \xi_2\right\rangle\right|^{2p}, \left|\left\langle\Im(\mathcal{U})\xi_1, \xi_2\right\rangle\right|^{2p}\right\} \\ &\leq 2^{p-2}\|\xi_2\|^{2p}\left[\left\langle\left(\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right)\xi_1, \xi_1\right\rangle^p + \frac{1}{2^p}\left|\left\langle\mathcal{U}^2\xi_1, \xi_1\right\rangle\right|^p\right]\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

For the case $p = 1$, we obtain

$$\begin{aligned}&\max\left\{\left|\left\langle\Re(\mathcal{U})\xi_1, \xi_2\right\rangle\right|^{2}, \left|\left\langle\Im(\mathcal{U})\xi_1, \xi_2\right\rangle\right|^{2}\right\} \\ &\leq \frac{1}{2}\|\xi_2\|^{2}\left[\left\langle\left(\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right)\xi_1, \xi_1\right\rangle + \frac{1}{2}\left|\left\langle\mathcal{U}^2\xi_1, \xi_1\right\rangle\right|\right]\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

These vector inequalities yield the following norm inequalities:

$$\max\left\{\|\Re(\mathcal{U})\|^{2p}, \|\Im(\mathcal{U})\|^{2p}\right\} \leq 2^{p-2}\left[\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\|^p + \frac{1}{2^p}\omega^p\left(\mathcal{U}^2\right)\right]$$

and

$$\max\left\{\|\Re(\mathcal{U})\|^{2}, \|\Im(\mathcal{U})\|^{2}\right\} \leq \frac{1}{2}\left[\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\| + \frac{1}{2}\omega\left(\mathcal{U}^2\right)\right].$$

Let $\mathcal{U} \in \mathcal{L}(\mathbb{K})$ have the Cartesian decomposition $\mathcal{U} = \Re(\mathcal{U}) + i\Im(\mathcal{U})$. For $p \geq 1$, we can deduce from (6) with $\gamma = 1$ and $\delta = i$ that

$$\begin{aligned}&|\langle \mathcal{U}\xi_1, \xi_2\rangle|^{2p} \\ &\leq 2^{p-1}\|\xi_2\|^{2p}\left[\left\langle\left(\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right)\xi_1, \xi_1\right\rangle^p + 2^p|\langle \Im(\mathcal{U})\Re(\mathcal{U})\xi_1, \xi_1\rangle|^p\right]\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{K}$. This Cartesian decomposition makes the symmetry between $\Re(\mathcal{U})$ and $\Im(\mathcal{U})$ explicit, and several of our inequalities can be interpreted as measures of the balanced contribution from these symmetric parts. For $p = 1$, we obtain

$$|\langle \mathcal{U}\xi_1, \xi_2\rangle|^2 \leq \|\xi_2\|^2\left[\left\langle\left(\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right)\xi_1, \xi_1\right\rangle + 2|\langle \Im(\mathcal{U})\Re(\mathcal{U})\xi_1, \xi_1\rangle|\right]$$

for all $\xi_1, \xi_2 \in \mathbb{K}$.

These relations lead to the following norm inequalities

$$\|\mathcal{U}\|^{2p} \leq 2^{p-1}\left[\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\|^p + 2^p\omega^p(\Im(\mathcal{U})\Re(\mathcal{U}))\right],$$

and for $p = 1$,

$$\|\mathcal{U}\|^2 \leq \left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\| + 2\omega(\Im(\mathcal{U})\Re(\mathcal{U})).$$

For an operator's Cartesian decomposition, a corresponding result for the numerical radius can also be stated. Here, the pair $(\Re(\mathcal{U}), \Im(\mathcal{U}))$ represents a Cartesian symmetry, while $(\mathcal{U}, \mathcal{U}^\star)$ displays the adjoint symmetry; both of these perspectives will be leveraged in the estimates that follow.

Theorem 2. *For any operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, the following numerical radius inequality is satisfied:*

$$\begin{aligned}\omega^{2p}(\mathcal{U}) &\leq 2^{p-1}\omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \\ &\quad + 2^{p-1} \times \begin{cases} \|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\|^p, \\ \|\Re(\mathcal{U})^{2r} + \Im(\mathcal{U})^{2r}\|^{p/r} \text{ for } r > 1, \\ \left(\frac{1}{2}\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\| + \frac{1}{2}\|\Re(\mathcal{U})^2 - \Im(\mathcal{U})^2\|\right)^p \end{cases}\end{aligned} \tag{9}$$

for all $p \geq 1$.

Proof. We employ the well-known representation of the numerical radius, as detailed for instance in [21], Theorem 2.2.11:

$$\omega(\mathcal{U}) = \sup_{\theta \in \mathbb{R}}\left\|\Re\left(e^{i\theta}\mathcal{U}\right)\right\|.$$

It can be observed that

$$\begin{aligned}\left\|\Re\left(e^{i\theta}\mathcal{U}\right)\right\| &= \|\Re((\cos(\theta) + i\sin(\theta))(\Re(\mathcal{U}) + i\Im(\mathcal{U})))\| \\ &= \|\cos(\theta)\Re(\mathcal{U}) - \sin(\theta)\Im(\mathcal{U})\|\end{aligned}$$

for any $\theta \in \mathbb{R}$. The formulation $\cos(\theta)\,\mathfrak{R}(\mathcal{U}) - \sin(\theta)\,\mathfrak{I}(\mathcal{U})$ demonstrates rotational symmetry in the $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ plane, a viewpoint we utilize throughout our analysis. From (8), we have

$$\|\gamma\mathcal{U} + \delta\mathcal{V}\|^{2p} \le 2^{p-1}\left[\left\||\gamma|^2|\mathcal{U}|^2 + |\delta|^2|\mathcal{V}|^2\right\|^p + \left(|\gamma|^2 + |\delta|^2\right)^p \omega^p(\mathcal{V}^\star\mathcal{U})\right].$$

By setting $\gamma = \cos(\theta)$, $\delta = -\sin(\theta)$, $\mathcal{U} = \mathfrak{R}(\mathcal{U})$ and $\mathcal{V} = \mathfrak{I}(\mathcal{U})$ in this inequality, we find that

$$\begin{aligned}\left\|\mathfrak{R}\left(e^{i\theta}\mathcal{U}\right)\right\|^{2p} &= \|\cos(\theta)\mathfrak{R}(\mathcal{U}) - \sin(\theta)\mathfrak{I}(\mathcal{U})\|^{2p} \\ &\le 2^{p-1}\left[\left\|\cos^2(\theta)\mathfrak{R}(\mathcal{U})^2 + \sin^2(\theta)\mathfrak{I}(\mathcal{U})^2\right\|^p + \omega^p(\mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U}))\right]\end{aligned}$$

for all $\theta \in \mathbb{R}$.

Taking the supremum over $\theta \in \mathbb{R}$ then yields

$$\begin{aligned}&\omega^{2p}(\mathcal{U}) \\ &\le 2^{p-1}\left[\sup_{\theta\in\mathbb{R}}\left\|\cos^2(\theta)\mathfrak{R}(\mathcal{U})^2 + \sin^2(\theta)\mathfrak{I}(\mathcal{U})^2\right\|^p + \omega^p(\mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U}))\right].\end{aligned} \tag{10}$$

Consider a vector $\xi \in \mathbb{K}$ with $\|\xi\| = 1$. An application of the elementary Hölder's inequality allows us to state that

$$\begin{aligned}&\left\langle\left[\cos^2(\theta)\mathfrak{R}(\mathcal{U})^2 + \sin^2(\theta)\mathfrak{I}(\mathcal{U})^2\right]\xi, \xi\right\rangle \\ &= \cos^2(\theta)\left\langle\mathfrak{R}(\mathcal{U})^2\xi, \xi\right\rangle + \sin^2(\theta)\left\langle\mathfrak{I}(\mathcal{U})^2\xi, \xi\right\rangle \\ &\le \begin{cases}\max\left\{\cos^2(\theta), \sin^2(\theta)\right\}\left[\left\langle\mathfrak{R}(\mathcal{U})^2\xi, \xi\right\rangle + \left\langle\mathfrak{I}(\mathcal{U})^2\xi, \xi\right\rangle\right], \\ \left(\cos^{2q}(\theta) + \sin^{2q}(\theta)\right)^{1/q}\left[\left\langle\mathfrak{R}(\mathcal{U})^2\xi, \xi\right\rangle^r + \left\langle\mathfrak{I}(\mathcal{U})^2\xi, \xi\right\rangle^r\right]^{1/r} \\ \text{for } q, r > 1 \text{ with } 1/q + 1/r = 1, \\ \left(\cos^2(\theta) + \sin^2(\theta)\right)\max\left\{\left\langle\mathfrak{R}(\mathcal{U})^2\xi, \xi\right\rangle, \left\langle\mathfrak{I}(\mathcal{U})^2\xi, \xi\right\rangle\right\}\end{cases}\end{aligned} \tag{11}$$

for any $\xi \in \mathbb{K}$ with $\|\xi\| = 1$ and $\theta \in \mathbb{R}$.

We now observe that

$$\max\left\{\cos^2(\theta), \sin^2(\theta)\right\} \le 1,$$

and, for $q > 1$, it follows that

$$\cos^{2q}(\theta) + \sin^{2q}(\theta) \le \cos^2(\theta) + \sin^2(\theta) = 1.$$

Furthermore, by applying the McCarthy inequality for a non-negative operator P and a unit vector $\xi \in \mathbb{K}$,

$$\langle P\xi, \xi\rangle^r \le \langle P^r\xi, \xi\rangle,$$

we can establish for $\xi \in \mathbb{K}$ with $\|\xi\| = 1$ that

$$\begin{aligned}\left\langle\mathfrak{R}(\mathcal{U})^2\xi, \xi\right\rangle^r + \left\langle\mathfrak{I}(\mathcal{U})^2\xi, \xi\right\rangle^r &\le \left\langle\mathfrak{R}(\mathcal{U})^{2r}\xi, \xi\right\rangle + \left\langle\mathfrak{I}(\mathcal{U})^{2r}\xi, \xi\right\rangle \\ &= \left\langle\left[\mathfrak{R}(\mathcal{U})^{2r} + \mathfrak{I}(\mathcal{U})^{2r}\right]\xi, \xi\right\rangle \\ &\le \left\|\mathfrak{R}(\mathcal{U})^{2r} + \mathfrak{I}(\mathcal{U})^{2r}\right\|.\end{aligned}$$

Moreover,

$$\begin{aligned}
&\max\Big\{\langle\Re(\mathcal{U})^2\xi,\xi\rangle,\langle\Im(\mathcal{U})^2\xi,\xi\rangle\Big\}\\
&=\frac{1}{2}\Big(\langle\Re(\mathcal{U})^2\xi,\xi\rangle+\langle\Im(\mathcal{U})^2\xi,\xi\rangle\Big)+\frac{1}{2}\Big|\langle\Re(\mathcal{U})^2\xi,\xi\rangle-\langle\Im(\mathcal{U})^2\xi,\xi\rangle\Big|\\
&=\frac{1}{2}\Big\langle\Big[\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\Big]\xi,\xi\Big\rangle+\frac{1}{2}\Big|\Big\langle\Big[\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\Big]\xi,\xi\Big\rangle\Big|\\
&\le\frac{1}{2}\Big\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\Big\|+\frac{1}{2}\Big\|\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\Big\|.
\end{aligned}$$

Using (11), we can then deduce that

$$\Big\|\cos^2(\theta)\Re(\mathcal{U})^2+\sin^2(\theta)\Im(\mathcal{U})^2\Big\|\le\begin{cases}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|,\\ \|\Re(\mathcal{U})^{2r}+\Im(\mathcal{U})^{2r}\|^{1/r},\\ \frac{1}{2}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|+\frac{1}{2}\|\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\|\end{cases}$$

and by raising this to the power $p\ge 1$, we obtain

$$\Big\|\cos^2(\theta)\Re(\mathcal{U})^2+\sin^2(\theta)\Im(\mathcal{U})^2\Big\|^p\le\begin{cases}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|^p,\\ \|\Re(\mathcal{U})^{2r}+\Im(\mathcal{U})^{2r}\|^{p/r}\text{ for }p>1,\\ \Big(\frac{1}{2}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|+\frac{1}{2}\|\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\|\Big)^p\end{cases}$$

for any $\theta\in\mathbb{R}$.

By taking the supremum over $\theta\in\mathbb{R}$ and utilizing (10), we arrive at the desired result (9). □

Remark 2. *If we set $p=1$ in (9), we obtain*

$$\omega^2(\mathcal{U})\le\omega(\Im(\mathcal{U})\Re(\mathcal{U}))+\begin{cases}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|,\\ \\ \|\Re(\mathcal{U})^{2r}+\Im(\mathcal{U})^{2r}\|^{1/r}\text{ for }r>1,\\ \\ \frac{1}{2}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|+\frac{1}{2}\|\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\|,\end{cases}$$

while for $p=2$, the inequality becomes

$$\omega^4(\mathcal{U})\le 2\omega^2(\Im(\mathcal{U})\Re(\mathcal{U}))+2\times\begin{cases}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|^2,\\ \\ \|\Re(\mathcal{U})^{2r}+\Im(\mathcal{U})^{2r}\|^{2/r}\text{ for }r>1,\\ \\ \Big(\frac{1}{2}\|\Re(\mathcal{U})^2+\Im(\mathcal{U})^2\|+\frac{1}{2}\|\Re(\mathcal{U})^2-\Im(\mathcal{U})^2\|\Big)^2.\end{cases}$$

Let us now define the operator

$$\mathcal{H}_\theta:=\Re\Big(e^{i\theta}\mathcal{U}\Big),\ \theta\in\mathbb{R}.$$

The following result holds.

Theorem 3. *For any operator $\mathcal{U}\in\mathcal{L}(\mathbb{K})$, we have the following numerical radius inequality:*

$$\omega^{2p}(\mathcal{U})\le\frac{1}{2}\left\|\frac{|\mathcal{U}|^2+|\mathcal{U}^\star|^2}{2}\right\|^p+2^{\frac{p-3}{2}}\left[\left\|\frac{|\mathcal{U}^2|^2+\left|(\mathcal{U}^\star)^2\right|^2}{4}\right\|^p+\frac{1}{2^p}\omega^p\Big(\mathcal{U}^4\Big)\right]^{1/2} \tag{12}$$

for all $p \geq 1$.

Proof. Note that the operator $\mathcal{H}_\theta$ can be expressed as

$$\mathcal{H}_\theta = \Re\left(e^{i\theta}\mathcal{U}\right) = \frac{1}{2}\left(e^{i\theta}\mathcal{U} + e^{-i\theta}\mathcal{U}^\star\right),$$

which implies that

$$\mathcal{H}_\theta^2 = \frac{1}{4}\left(e^{2i\theta}\mathcal{U}^2 + e^{-2i\theta}(\mathcal{U}^\star)^2 + |\mathcal{U}|^2 + |\mathcal{U}^\star|^2\right)$$

for any $\theta \in \mathbb{R}$.

By taking the norm and then the power $p \geq 1$, the convexity of the power function ensures that

$$\begin{aligned}\left\|\mathcal{H}_\theta^2\right\|^p &= \left\|\frac{1}{2}\left(\frac{e^{2i\theta}\mathcal{U}^2 + e^{-2i\theta}(\mathcal{U}^\star)^2}{2} + \frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right)\right\|^p \\ &\leq \frac{1}{2}\left(\left\|\frac{e^{2i\theta}\mathcal{U}^2 + e^{-2i\theta}(\mathcal{U}^\star)^2}{2}\right\|^p + \left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\|^p\right).\end{aligned} \tag{13}$$

From inequality (8), taking the square root gives for $p \geq 1$ that,

$$\|\gamma\mathcal{A} + \delta\mathcal{B}\|^p \leq 2^{\frac{p-1}{2}}\left[\left\||\gamma|^2|\mathcal{A}|^2 + |\delta|^2|\mathcal{B}|^2\right\|^p + \left(|\gamma|^2 + |\delta|^2\right)^p \omega^p(\mathcal{B}^\star\mathcal{A})\right]^{1/2} \tag{14}$$

for $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathbb{K})$ and $\gamma, \delta \in \mathbb{C}$.

Applying (14) with the substitutions $\gamma = \frac{e^{2i\theta}}{2}$, $\delta = \frac{e^{-2i\theta}}{2}$, $\mathcal{A} = \mathcal{U}^2$ and $\mathcal{B} = (\mathcal{U}^\star)^2$, we obtain

$$\left\|\frac{e^{2i\theta}\mathcal{U}^2 + e^{-2i\theta}(\mathcal{U}^\star)^2}{2}\right\|^p \leq 2^{\frac{p-1}{2}}\left[\left\|\frac{|\mathcal{U}^2|^2 + \left|(\mathcal{U}^\star)^2\right|^2}{4}\right\|^p + \frac{1}{2^p}\omega^p\left(\mathcal{U}^4\right)\right]^{1/2}$$

for any $\theta \in \mathbb{R}$.

From (13), it then follows that

$$\left\|\mathcal{H}_\theta^2\right\|^p \leq \frac{1}{2}\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\|^p + 2^{\frac{p-3}{2}}\left[\left\|\frac{|\mathcal{U}^2|^2 + \left|(\mathcal{U}^\star)^2\right|^2}{4}\right\|^p + \frac{1}{2^p}\omega^p\left(\mathcal{U}^4\right)\right]^{1/2}$$

for any $\theta \in \mathbb{R}$.

By taking the supremum over $\theta \in \mathbb{R}$ in the above inequality, we arrive at (12). □

Remark 3. *For the case* $p = 1$ *in (12), we have:*

$$\omega^2(\mathcal{U}) \leq \frac{1}{2}\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\| + \frac{1}{2}\left[\left\|\frac{|\mathcal{U}^2|^2 + \left|(\mathcal{U}^\star)^2\right|^2}{4}\right\| + \frac{1}{2}\omega\left(\mathcal{U}^4\right)\right]^{1/2}.$$

For the case $p = 2$, we find that:

$$\omega^4(\mathcal{U}) \le \frac{1}{2}\left\| \frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2} \right\|^2 + \frac{\sqrt{2}}{2}\left[\left\| \frac{\left|\mathcal{U}^2\right|^2 + \left|(\mathcal{U}^\star)^2\right|^2}{4} \right\|^2 + \frac{1}{4}\omega^2\left(\mathcal{U}^4\right) \right]^{1/2}.$$

We also present the following theorem.

Theorem 4. *For any operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, the subsequent numerical radius inequality holds:*

$$\begin{aligned} &\omega^{2p}(\mathcal{U}) \\ &\le 2^{p-1}\omega^p((\sin(\theta)\Re(\mathcal{U}) + \cos(\theta)\Im(\mathcal{U}))(\cos(\theta)\Re(\mathcal{U}) - \sin(\theta)\Im(\mathcal{U}))) \\ &+ 2^{p-1} \\ &\times \begin{cases} \left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\|^p, \\ \left\|(\cos(\theta)\Re(\mathcal{U}) - \sin(\theta)\Im(\mathcal{U}))^{2r} + (\sin(\theta)\Re(\mathcal{U}) + \cos(\theta)\Im(\mathcal{U}))^{2r}\right\|^{p/r} \\ \textit{for } r > 1, \\ \left(\frac{1}{2}\left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\| \right. \\ +\frac{1}{2}\left\|\cos(2\theta)\left(\Re(\mathcal{U})^2 - \Im(\mathcal{U})^2\right) \right. \\ \left. \left. -\sin(2\theta)[\Re(\mathcal{U})\Im(\mathcal{U}) + \Im(\mathcal{U})\Re(\mathcal{U})]\right\|\right)^p \end{cases} \end{aligned} \tag{15}$$

where $p \ge 1$ and $\theta \in \mathbb{R}$.

Proof. We utilize the following identity from [22], also found in [21], p. 34:

$$\mathcal{H}_{\theta+\phi} = \cos(\phi)\mathcal{H}_\theta + \sin(\phi)\mathcal{H}_{\theta+\frac{\pi}{2}}$$

for any $\phi, \theta \in \mathbb{R}$.

From (8) with $\gamma = \cos(\phi)$, $\delta = \sin(\phi)$, $\mathcal{U} = \mathcal{H}_\theta$ and $\mathcal{V} = \mathcal{H}_{\theta+\frac{\pi}{2}}$, we obtain

$$\left\|\mathcal{H}_{\theta+\phi}\right\|^{2p} \le 2^{p-1}\left[\left\|\cos^2(\phi)\mathcal{H}_\theta^2 + \sin^2(\phi)\mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|^p + \omega^p\left(\mathcal{H}_{\theta+\frac{\pi}{2}}\mathcal{H}_\theta\right)\right] \tag{16}$$

for any $\phi, \theta \in \mathbb{R}$.

In a similar manner to the proof of Theorem 2, we can show that

$$\left\|\cos^2(\phi)\mathcal{H}_\theta^2 + \sin^2(\phi)\mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|^p \le \begin{cases} \left\|\mathcal{H}_\theta^2 + \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|^p, \\ \left\|\mathcal{H}_\theta^{2r} + \mathcal{H}_{\theta+\frac{\pi}{2}}^{2r}\right\|^{p/r} \text{ for } p > 1, \\ \left(\frac{1}{2}\left\|\mathcal{H}_\theta^2 + \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\| + \frac{1}{2}\left\|\mathcal{H}_\theta^2 - \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|\right)^p \end{cases}$$

for all $\phi, \theta \in \mathbb{R}$.

By employing (16), we arrive at

$$\begin{aligned} &\left\|\mathcal{H}_{\theta+\phi}\right\|^{2p} \\ &\le 2^{p-1}\omega^p\left(\mathcal{H}_{\theta+\frac{\pi}{2}}\mathcal{H}_\theta\right) + 2^{p-1} \times \begin{cases} \left\|\mathcal{H}_\theta^2 + \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|^p, \\ \left\|\mathcal{H}_\theta^{2r} + \mathcal{H}_{\theta+\frac{\pi}{2}}^{2r}\right\|^{p/r} \text{ for } p > 1, \\ \left(\frac{1}{2}\left\|\mathcal{H}_\theta^2 + \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\| + \frac{1}{2}\left\|\mathcal{H}_\theta^2 - \mathcal{H}_{\theta+\frac{\pi}{2}}^2\right\|\right)^p \end{cases} \end{aligned} \tag{17}$$

for any $\phi, \theta \in \mathbb{R}$.

We note the identities:

$$\mathcal{H}_{\theta+\frac{\pi}{2}} = \cos(\theta + \frac{\pi}{2})\mathfrak{R}(\mathcal{U}) - \sin(\theta + \frac{\pi}{2})\mathfrak{I}(\mathcal{U}) = -\sin(\theta)\mathfrak{R}(\mathcal{U}) - \cos(\theta)\mathfrak{I}(\mathcal{U}),$$

$$\begin{aligned}\mathcal{H}_\theta^2 &= (\cos(\theta)\mathfrak{R}(\mathcal{U}) - \sin(\theta)\mathfrak{I}(\mathcal{U}))^2 \\ &= \cos^2(\theta)\mathfrak{R}(\mathcal{U})^2 - \sin(\theta)\cos(\theta)[\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})] + \sin^2(\theta)\mathfrak{I}(\mathcal{U})^2,\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{\theta+\frac{\pi}{2}}^2 &= (\sin(\theta)\mathfrak{R}(\mathcal{U}) + \cos(\theta)\mathfrak{I}(\mathcal{U}))^2 \\ &= \sin^2(\theta)\mathfrak{R}(\mathcal{U})^2 + \sin(\theta)\cos(\theta)[\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})] + \cos^2(\theta)\mathfrak{I}(\mathcal{U})^2,\end{aligned}$$

from which it follows that

$$\mathcal{H}_\theta^2 + \mathcal{H}_{\theta+\frac{\pi}{2}}^2 = \mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2$$

and

$$\begin{aligned}\mathcal{H}_\theta^2 - \mathcal{H}_{\theta+\frac{\pi}{2}}^2 &= \left(\cos^2(\theta) - \sin^2(\theta)\right)\mathfrak{R}(\mathcal{U})^2 + \left(\sin^2(\theta) - \cos^2(\theta)\right)\mathfrak{I}(\mathcal{U})^2 \\ &\quad - 2\sin(\theta)\cos(\theta)[\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})] \\ &= \cos(2\theta)\left(\mathfrak{R}(\mathcal{U})^2 - \mathfrak{I}(\mathcal{U})^2\right) - \sin(2\theta)[\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})]\end{aligned}$$

for any $\theta \in \mathbb{R}$. By substituting these into (17) and taking the supremum over $\phi \in \mathbb{R}$, we obtain the desired result (15). □

Remark 4. *For the case $p = 1$, we derive the inequalities:*

$$\begin{aligned}&\omega^2(\mathcal{U}) \\ &\le \omega((\sin(\theta)\mathfrak{R}(\mathcal{U}) + \cos(\theta)\mathfrak{I}(\mathcal{U}))(\cos(\theta)\mathfrak{R}(\mathcal{U}) - \sin(\theta)\mathfrak{I}(\mathcal{U}))) \\ &+ \begin{cases} \left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\|, \\ \left\|(\cos(\theta)\mathfrak{R}(\mathcal{U}) - \sin(\theta)\mathfrak{I}(\mathcal{U}))^{2r} + (\sin(\theta)\mathfrak{R}(\mathcal{U}) + \cos(\theta)\mathfrak{I}(\mathcal{U}))^{2r}\right\|^{1/r} \\ \textit{for } r > 1, \\ \frac{1}{2}\left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\| \\ +\frac{1}{2}\left\|\cos(2\theta)\left(\mathfrak{R}(\mathcal{U})^2 - \mathfrak{I}(\mathcal{U})^2\right) - \sin(2\theta)[\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})]\right\|. \end{cases}\end{aligned}$$

Also, if we take $\theta = \pi/4$ in (15), then we obtain

$$\begin{aligned}&\omega^{2p}(\mathcal{U}) \\ &\le \frac{1}{2}\omega^p((\mathfrak{R}(\mathcal{U}) + \mathfrak{I}(\mathcal{U}))(\mathfrak{R}(\mathcal{U}) - \mathfrak{I}(\mathcal{U}))) \\ &+ 2^{p-1} \times \begin{cases} \left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\|^p, \\ \frac{1}{2^p}\left\|(\mathfrak{R}(\mathcal{U}) - \mathfrak{I}(\mathcal{U}))^{2r} + (\mathfrak{R}(\mathcal{U}) + \mathfrak{I}(\mathcal{U}))^{2r}\right\|^{p/r} \\ \textit{for } r > 1, \\ \frac{1}{2^p}\left(\left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\| + \left\|\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})\right\|\right)^p \end{cases}\end{aligned}$$

for any $p \ge 1$.

For $p = 1$, this gives the simpler bounds:

$$\omega^2(\mathcal{U}) \le \frac{1}{2}\omega((\mathfrak{R}(\mathcal{U}) + \mathfrak{I}(\mathcal{U}))(\mathfrak{R}(\mathcal{U}) - \mathfrak{I}(\mathcal{U}))) + \begin{cases} \left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\|, \\ \frac{1}{2}\left\|(\mathfrak{R}(\mathcal{U}) - \mathfrak{I}(\mathcal{U}))^{2r} + (\mathfrak{R}(\mathcal{U}) + \mathfrak{I}(\mathcal{U}))^{2r}\right\|^{1/r}, \\ \text{for } r > 1, \\ \frac{1}{2}\left(\left\|\mathfrak{R}(\mathcal{U})^2 + \mathfrak{I}(\mathcal{U})^2\right\| + \|\mathfrak{R}(\mathcal{U})\mathfrak{I}(\mathcal{U}) + \mathfrak{I}(\mathcal{U})\mathfrak{R}(\mathcal{U})\|\right). \end{cases}$$

3. Applications for *s*–*r*-Norm Inequalities

This section demonstrates the application of our main inequalities to the specific case of the *s*–*r*-norm. We will emphasize the symmetric pairs $(\mathcal{U}, \mathcal{U}^\star)$ and $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ where appropriate, clarifying how bounds can be translated between these corresponding frameworks. Here, we introduce several power inequalities pertaining to the *s*–*r*-norm. By setting the operator pair $(\mathcal{U}, \mathcal{V})$ to be $(\mathcal{U}, \mathcal{U}^\star)$ or $(\mathfrak{R}(\mathcal{U}), \mathfrak{I}(\mathcal{U}))$ for an operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, we establish norm inequalities for a single operator. In particular, these choices make the adjoint and Cartesian symmetries explicit, a fact we leverage in the following results. Our first main result in this section is as follows.

Theorem 5. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$. For $p \ge 1$ and $r > 1$, we have:*

$$\begin{aligned} &\|(\mathcal{U}, \mathcal{V})\|_r^{2pr} \\ &\le 2^{p-1}\left[\left(\|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r}\right)^p + \|(\mathcal{U}, \mathcal{V})\|_{2(r-1)}^{2p(r-1)}\omega^p(\mathcal{V}^\star\mathcal{U})\right] \qquad (18) \\ &\le 2^{p-1}\left[\left(\|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r}\right)^p + \left(\|\mathcal{U}\|^{2(r-1)} + \|\mathcal{V}\|^{2(r-1)}\right)^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]. \end{aligned}$$

For the case $r = 1$, we have:

$$\|(\mathcal{U}, \mathcal{V})\|^{2p} \le 2^{p-1}\left[\left\||\mathcal{U}|^2 + |\mathcal{V}|^2\right\|^p + 2^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]. \qquad (19)$$

For $r = 2$, we obtain the following inequality for the Euclidean norm:

$$\begin{aligned} &\|(\mathcal{U}, \mathcal{V})\|_e^{4p} \\ &\le 2^{p-1}\left[\left(\|\mathcal{U}\|^4 + \|\mathcal{V}\|^4\right)^p + \|(\mathcal{U}, \mathcal{V})\|_e^{2p}\omega^p(\mathcal{V}^\star\mathcal{U})\right] \\ &\le 2^{p-1}\left[\left(\|\mathcal{U}\|^4 + \|\mathcal{V}\|^4\right)^p + \left(\|\mathcal{U}\|^2 + \|\mathcal{V}\|^2\right)^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]. \end{aligned}$$

Proof. Let $\xi_1, \xi_2 \in \mathbb{K}$. For $r > 1$, we define the scalars

$$\gamma := \langle \mathcal{U}\xi_2, \xi_1\rangle\left|\langle \mathcal{U}\xi_2, \xi_1\rangle\right|^{r-2} \text{ if } \langle \mathcal{U}\xi_2, \xi_1\rangle \ne 0$$

and

$$\delta := \langle \mathcal{V}\xi_2, \xi_1\rangle\left|\langle \mathcal{V}\xi_2, \xi_1\rangle\right|^{r-2} \text{ if } \langle \mathcal{V}\xi_2, \xi_1\rangle \ne 0.$$

Then, it follows that

$$|\gamma| = \left|\langle \mathcal{U}\xi_2, \xi_1\rangle\right|^{r-1} \text{ and } |\delta| = \left|\langle \mathcal{V}\xi_2, \xi_1\rangle\right|^{r-1}.$$

Also, we have

$$|\gamma|^2 = \left|\langle \mathcal{U}\xi_2, \xi_1\rangle\right|^{2(r-1)}, |\delta|^2 = \left|\langle \mathcal{V}\xi_2, \xi_1\rangle\right|^{2(r-1)},$$

and

$$\begin{aligned}\left|\left\langle(\gamma\mathcal{U}+\delta\mathcal{V})\xi_2,\xi_1\right\rangle\right| &= \left|\overline{\gamma}\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle+\overline{\delta}\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right| \\ &= \left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^r+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^r\end{aligned}$$

for $r > 1$.

From the inequality (6), we have

$$\begin{aligned}&\left|\left\langle\gamma\mathcal{U}\xi_2+\delta\mathcal{V}\xi_2,\xi_1\right\rangle\right|^{2p} \\ &\leq 2^{p-1}\|\xi_1\|^{2p}\left[\left(|\gamma|^2\left\langle|\mathcal{U}|^2\xi_2,\xi_2\right\rangle+|\delta|^2\left\langle|\mathcal{V}|^2\xi_2,\xi_2\right\rangle\right)^p+\left(|\gamma|^2+|\delta|^2\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_2,\xi_2\right\rangle\right|^p\right].\end{aligned} \tag{20}$$

By substituting the expressions for γ and δ defined above, we obtain

$$\begin{aligned}&\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^r+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^r\right)^{2p} \\ &\leq 2^{p-1}\|\xi_1\|^{2p} \\ &\times\left[\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\left\langle|\mathcal{U}|^2\xi_2,\xi_2\right\rangle+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\left\langle|\mathcal{V}|^2\xi_2,\xi_2\right\rangle\right)^p\right. \\ &\left.+\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_2,\xi_2\right\rangle\right|^p\right].\end{aligned} \tag{21}$$

This inequality remains valid even if $\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle = 0$ or $\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle = 0$.

For unit vectors $\xi_1,\xi_2 \in \mathbb{K}$ (i.e., $\|\xi_1\| = \|\xi_2\| = 1$), we have

$$\begin{aligned}&\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\left\langle|\mathcal{U}|^2\xi_2,\xi_2\right\rangle+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\left\langle|\mathcal{V}|^2\xi_2,\xi_2\right\rangle \\ &\leq \|\mathcal{U}\|^{2(r-1)}\|\mathcal{U}\|^2+\|\mathcal{V}\|^{2(r-1)}\|\mathcal{V}\|^2 = \|\mathcal{U}\|^{2r}+\|\mathcal{V}\|^{2r}\end{aligned}$$

and

$$\begin{aligned}&\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^{2(r-1)}\right)^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_2,\xi_2\right\rangle\right|^p \\ &\leq \|(\mathcal{U},\mathcal{V})\|_{2(r-1)}^{2p(r-1)}\omega^p(\mathcal{V}^\star\mathcal{U}) \leq \left(\|\mathcal{U}\|^{2(r-1)}+\|\mathcal{V}\|^{2(r-1)}\right)^p\omega^p(\mathcal{V}^\star\mathcal{U}).\end{aligned}$$

Using (21), we obtain

$$\begin{aligned}&\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|^r+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|^r\right)^{2p} \\ &\leq 2^{p-1}\left[\left(\|\mathcal{U}\|^{2r}+\|\mathcal{V}\|^{2r}\right)^p+\|(\mathcal{U},\mathcal{V})\|_{2(r-1)}^{2p(r-1)}\omega^p(\mathcal{V}^\star\mathcal{U})\right] \\ &\leq 2^{p-1}\left[\left(\|\mathcal{U}\|^{2r}+\|\mathcal{V}\|^{2r}\right)^p+\left(\|\mathcal{U}\|^{2(r-1)}+\|\mathcal{V}\|^{2(r-1)}\right)^p\omega^p(\mathcal{V}^\star\mathcal{U})\right]\end{aligned}$$

for any unit vectors $\xi_1,\xi_2 \in \mathbb{K}$.

By taking the supremum over all unit vectors $\xi_1,\xi_2 \in \mathbb{K}$, we arrive at the desired result (18).

For the case $r = 1$, let $\xi_1,\xi_2 \in \mathbb{K}$. If we choose $\gamma = \frac{\overline{\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle}}{\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|}$ (for $\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle \neq 0$) and $\delta = \frac{\overline{\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle}}{\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|}$ (for $\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle \neq 0$) in (20), we obtain

$$\begin{aligned}&\left(\left|\left\langle\mathcal{U}\xi_2,\xi_1\right\rangle\right|+\left|\left\langle\mathcal{V}\xi_2,\xi_1\right\rangle\right|\right)^{2p} \\ &\leq 2^{p-1}\|\xi_1\|^{2p}\left[\left(\left\langle|\mathcal{U}|^2\xi_2,\xi_2\right\rangle+\left\langle|\mathcal{V}|^2\xi_2,\xi_2\right\rangle\right)^p+2^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_2,\xi_2\right\rangle\right|^p\right] \\ &= 2^{p-1}\|\xi_1\|^{2p}\left[\left\langle\left(|\mathcal{U}|^2+|\mathcal{V}|^2\right)\xi_2,\xi_2\right\rangle^p+2^p\left|\left\langle\mathcal{V}^\star\mathcal{U}\xi_2,\xi_2\right\rangle\right|^p\right],\end{aligned} \tag{22}$$

which holds true even when $\langle \mathcal{U}\xi_2, \xi_1 \rangle = 0$ or $\langle \mathcal{V}\xi_2, \xi_1 \rangle = 0$.

Taking the supremum over all unit vectors ξ_1, ξ_2 yields the inequality (19). The remaining claims follow directly. $\square$

Remark 5. *Setting $p = 1$ in Theorem 5, we obtain for $r > 1$:*

$$\begin{aligned}
&\|(\mathcal{U},\mathcal{V})\|_r^{2r} \\
&\leq \|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r} + \|(\mathcal{U},\mathcal{V})\|_{2(r-1)}^{2(r-1)} \omega(\mathcal{V}^\star \mathcal{U}) \\
&\leq \|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r} + \left(\|\mathcal{U}\|^{2(r-1)} + \|\mathcal{V}\|^{2(r-1)}\right) \omega(\mathcal{V}^\star \mathcal{U}).
\end{aligned}$$

For $r = 1$, we obtain:

$$\|(\mathcal{U},\mathcal{V})\|^2 \leq \left\| |\mathcal{U}|^2 + |\mathcal{V}|^2 \right\| + 2\omega(\mathcal{V}^\star \mathcal{U}).$$

For $r = 2$, the inequality for the Euclidean norm is:

$$\begin{aligned}
\|(\mathcal{U},\mathcal{V})\|_e^4 &\leq \|\mathcal{U}\|^4 + \|\mathcal{V}\|^4 + \|(\mathcal{U},\mathcal{V})\|_e^2 \omega(\mathcal{V}^\star \mathcal{U}) \\
&\leq \|\mathcal{U}\|^4 + \|\mathcal{V}\|^4 + \left(\|\mathcal{U}\|^2 + \|\mathcal{V}\|^2\right) \omega(\mathcal{V}^\star \mathcal{U}).
\end{aligned}$$

Consider an operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$. Let us define the quantity

$$\begin{aligned}
\delta_r(\mathcal{U}) &:= \|(\mathcal{U},\mathcal{U}^\star)\|_r \\
&= \sup_{\|\xi_1\|=\|\xi_2\|=1} \left(|\langle \mathcal{U}\xi_2, \xi_1 \rangle|^r + |\langle \mathcal{U}^\star \xi_2, \xi_1 \rangle|^r \right)^{\frac{1}{r}} \\
&\leq 2^{\frac{1}{r}} \|\mathcal{U}\| \text{ for } r \geq 1.
\end{aligned}$$

The function $\delta_r(\mathcal{U})$ acts as a symmetric two-component gauge that captures the duality between an operator and its adjoint. For $r = 1$, we denote

$$\begin{aligned}
\delta(\mathcal{U}) &:= \|(\mathcal{U},\mathcal{U}^\star)\| \\
&= \sup_{\|\xi_1\|=\|\xi_2\|=1} \left(|\langle \mathcal{U}\xi_2, \xi_1 \rangle| + |\langle \mathcal{U}^\star \xi_2, \xi_1 \rangle| \right) \leq 2\|\mathcal{U}\|,
\end{aligned}$$

and for $r = 2$, we consider

$$\begin{aligned}
\delta_e(\mathcal{U}) := \|(\mathcal{U},\mathcal{U}^\star)\|_e &= \sup_{\|\xi_1\|=\|\xi_2\|=1} \left(|\langle \mathcal{U}\xi_2, \xi_1 \rangle|^2 + |\langle \mathcal{U}^\star \xi_2, \xi_1 \rangle|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\|\mathcal{U}\|.
\end{aligned}$$

From Theorem 5, we deduce for $p \geq 1$ and $r > 1$ that

$$\begin{aligned}
\delta_r^{2pr}(\mathcal{U}) &\leq 2^{p-1}\left[2^p \|\mathcal{U}\|^{2pr} + \delta_{2(r-1)}^{2p(r-1)}(\mathcal{U}) \omega^p\left(\mathcal{U}^2\right)\right] \\
&\leq 2^{p-1}\left[2^p \|\mathcal{U}\|^{2pr} + 2^p \|\mathcal{U}\|^{2p(r-1)} \omega^p\left(\mathcal{U}^2\right)\right].
\end{aligned}$$

For $r = 1$, we have

$$\delta^{2p}(\mathcal{U}) \leq 2^{p-1}\left[\left\| |\mathcal{U}|^2 + |\mathcal{U}^\star|^2 \right\|^p + 2^p \omega^p\left(\mathcal{U}^2\right)\right].$$

For $r = 2$, we obtain the inequality

$$\begin{aligned}\delta_e^{4p}(\mathcal{U}) &\leq 2^{p-1}\Big[2^p\|\mathcal{U}\|^{4p} + \delta_e^{2p}(\mathcal{U})\omega^p\Big(\mathcal{U}^2\Big)\Big] \\ &\leq 2^{p-1}\Big[2^p\|\mathcal{U}\|^{4p} + 2^p\|\mathcal{U}\|^{2p}\omega^p\Big(\mathcal{U}^2\Big)\Big].\end{aligned}$$

For $p = 1$, we derive for $r > 1$ that

$$\delta_r^{2r}(\mathcal{U}) \leq 2\|\mathcal{U}\|^{2r} + \delta_{2(r-1)}^{2(r-1)}(\mathcal{U})\omega\Big(\mathcal{U}^2\Big) \leq 2\|\mathcal{U}\|^{2r} + 2\|\mathcal{U}\|^{2(r-1)}\omega\Big(\mathcal{U}^2\Big).$$

For $r = 1$, we have

$$\delta^2(\mathcal{U}) \leq \Big\||\mathcal{U}|^2 + |\mathcal{U}^\star|^2\Big\| + 2\omega\Big(\mathcal{U}^2\Big).$$

And for $r = 2$, the inequality is

$$\delta_e^4(\mathcal{U}) \leq 2\|\mathcal{U}\|^4 + \delta_e^2(\mathcal{U})\omega\Big(\mathcal{U}^2\Big) \leq 2\|\mathcal{U}\|^4 + 2\|\mathcal{U}\|^2\omega\Big(\mathcal{U}^2\Big).$$

Now, let $\mathcal{U} \in \mathcal{L}(\mathbb{K})$ be given by its Cartesian decomposition $\mathcal{U} = \Re(\mathcal{U}) + i\Im(\mathcal{U})$. For $r \geq 1$, we can introduce the following quantities:

$$\begin{aligned}\eta_r(\mathcal{U}) &:= \|(\Re(\mathcal{U}), \Im(\mathcal{U}))\|_r \\ &= \sup_{\|\xi_1\|=\|\xi_2\|=1}\Big(\big|\langle\Re(\mathcal{U})\xi_2, \xi_1\rangle\big|^r + \big|\langle\Im(\mathcal{U})\xi_2, \xi_1\rangle\big|^r\Big)^{\frac{1}{r}} \\ &\leq \big(\|\Re(\mathcal{U})\|^r + \|\Im(\mathcal{U})\|^r\big)^{1/r}.\end{aligned}$$

Thus, $\eta_r(\mathcal{U})$ provides a measure of the balanced (Cartesian) symmetry between the real and imaginary parts of $\mathcal{U}$. For $r = 1$, we obtain

$$\begin{aligned}\eta(\mathcal{U}) &:= \|(\Re(\mathcal{U}), \Im(\mathcal{U}))\| \\ &= \sup_{\|\xi_1\|=\|\xi_2\|=1}\big(\big|\langle\Re(\mathcal{U})\xi_2, \xi_1\rangle\big| + \big|\langle\Im(\mathcal{U})\xi_2, \xi_1\rangle\big|\big) \\ &\leq \|\Re(\mathcal{U})\| + \|\Im(\mathcal{U})\|.\end{aligned}$$

We can observe that

$$\begin{aligned}&\big|\langle\Re(\mathcal{U})\xi_2, \xi_1\rangle\big|^2 + \big|\langle\Im(\mathcal{U})\xi_2, \xi_1\rangle\big|^2 \\ &= \frac{1}{4}\Big[\big|\langle\mathcal{U}\xi_2, \xi_1\rangle + \langle\mathcal{U}^\star\xi_2, \xi_1\rangle\big|^2 + \big|\langle\mathcal{U}\xi_2, \xi_1\rangle - \langle\mathcal{U}^\star\xi_2, \xi_1\rangle\big|^2\Big] \\ &= \frac{1}{2}\Big[\big|\langle\mathcal{U}\xi_2, \xi_1\rangle\big|^2 + \big|\langle\mathcal{U}^\star\xi_2, \xi_1\rangle\big|^2\Big]\end{aligned}$$

for any $\xi_1, \xi_2 \in \mathbb{K}$.

Therefore,

$$\begin{aligned}\eta_e(\mathcal{U}) &:= \|(\Re(\mathcal{U}), \Im(\mathcal{U}))\|_e \\ &= \sup_{\|\xi_1\|=\|\xi_2\|=1}\Big(\big|\langle\Re(\mathcal{U})\xi_2, \xi_1\rangle\big|^2 + \big|\langle\Im(\mathcal{U})\xi_2, \xi_1\rangle\big|^2\Big)^{\frac{1}{2}} \\ &= \sup_{\|\xi_1\|=\|\xi_2\|=1}\left(\frac{1}{2}\Big[\big|\langle\mathcal{U}\xi_2, \xi_1\rangle\big|^2 + \big|\langle\mathcal{U}^\star\xi_2, \xi_1\rangle\big|^2\Big]\right)^{1/2} = \frac{\sqrt{2}}{2}\delta_e(\mathcal{U}) \\ &\leq \Big(\|\Re(\mathcal{U})\|^2 + \|\Im(\mathcal{U})\|^2\Big)^{1/2}.\end{aligned}$$

From Theorem 5, we deduce for $p \geq 1$ and $r > 1$ that

$$\begin{aligned}\eta_r^{2pr}(\mathcal{U}) &\leq 2^{p-1}\left(\|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r}\right)^p \\ &\quad + 2^{p-1}\eta_{2(r-1)}^{2p(r-1)}(\mathcal{U})\omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \\ &\leq 2^{p-1}\left(\|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r}\right)^p \\ &\quad + 2^{p-1}\left(\|\Re(\mathcal{U})\|^{2(r-1)} + \|\Im(\mathcal{U})\|^{2(r-1)}\right)^p \omega^p(\Im(\mathcal{U})\Re(\mathcal{U})).\end{aligned}$$

For $r = 1$, we have

$$\eta^{2p}(\mathcal{U}) \leq 2^{p-1}\left[\left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\|^p + 2^p\omega^p(\Im(\mathcal{U})\Re(\mathcal{U}))\right].$$

For $r = 2$, we obtain the following inequality

$$\begin{aligned}&\delta_e^{4p}(\mathcal{U}) \\ &\leq 2^{3p-1}\left[\left(\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4\right)^p + \frac{1}{2^p}\delta_e^{2p}(\mathcal{U})\omega^p(\Im(\mathcal{U})\Re(\mathcal{U}))\right] \\ &\leq 2^{3p-1}\left[\left(\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4\right)^p \right. \\ &\left. + \left(\|\Re(\mathcal{U})\|^2 + \|\Im(\mathcal{U})\|^2\right)^p \omega^p(\Im(\mathcal{U})\Re(\mathcal{U}))\right].\end{aligned}$$

The case $p = 1$ gives for $r > 1$:

$$\eta_r^{2r}(\mathcal{U}) \leq \|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r} + \eta_{2(r-1)}^{2(r-1)}(\mathcal{U})\omega(\Im(\mathcal{U})\Re(\mathcal{U})).$$

For $r = 1$, this becomes:

$$\eta^2(\mathcal{U}) \leq \left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\| + 2\omega(\Im(\mathcal{U})\Re(\mathcal{U})),$$

while for $r = 2$, we have:

$$\begin{aligned}\delta_e^4(\mathcal{U}) &\leq 4\left[\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 + \frac{1}{2}\delta_e^2(\mathcal{U})\omega(\Im(\mathcal{U})\Re(\mathcal{U}))\right] \\ &\leq 4\left[\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 + \left(\|\Re(\mathcal{U})\|^2 + \|\Im(\mathcal{U})\|^2\right)\omega(\Im(\mathcal{U})\Re(\mathcal{U}))\right].\end{aligned}$$

4. Applications for *s*–*r*-Numerical Radius Inequalities

This section applies the preceding results to the *s*–*r*-numerical radius. We continue to emphasize the symmetry perspective to clarify how a bound for one decomposition can immediately yield its symmetric analogue. Here, we present several power inequalities for the *s*–*r*-numerical radius. By specifying the pair $(\mathcal{U}, \mathcal{V})$ as either $(\mathcal{U}, \mathcal{U}^\star)$ or $(\Re(\mathcal{U}), \Im(\mathcal{U}))$ for an operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, we derive various norm and numerical radius inequalities for a single operator.

Theorem 6. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$. For $p \geq 1$ and $r > 1$, we have:*

$$\begin{aligned} &\omega_r^{2pr}(\mathcal{U}, \mathcal{V}) \\ &\leq 2^{p-1} \\ &\times \left[\left(\omega^{2(r-1)}(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p + \omega_{2(r-1)}^{2p(r-1)}(\mathcal{U}, \mathcal{V}) \omega^p(\mathcal{V}^\star \mathcal{U}) \right] \\ &\leq 2^{p-1} \Big[\left(\omega^{2(r-1)}(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p \\ &+ \left(\omega^{2(r-1)}(\mathcal{U}) + \omega^{2(r-1)}(\mathcal{V}) \right)^p \omega^p(\mathcal{V}^\star \mathcal{U}) \Big]. \end{aligned} \tag{23}$$

For the case $r = 1$, we have:

$$\omega^{2p}(\mathcal{U}, \mathcal{V}) \leq 2^{p-1} \left[\left\| |\mathcal{U}|^2 + |\mathcal{V}|^2 \right\|^p + 2^p \omega^p(\mathcal{V}^\star \mathcal{U}) \right]. \tag{24}$$

For $r = 2$, we obtain the following inequality for the Euclidean numerical radius:

$$\begin{aligned} &\omega_e^{4p}(\mathcal{U}, \mathcal{V}) \\ &\leq 2^{p-1} \left[\left(\omega^2(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^2(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p + \omega_e^{2p}(\mathcal{U}, \mathcal{V}) \omega^p(\mathcal{V}^\star \mathcal{U}) \right] \\ &\leq 2^{p-1} \Big[\left(\omega^2(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^2(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p \\ &+ \left(\omega^2(\mathcal{U}) + \omega^2(\mathcal{V}) \right)^p \omega^p(\mathcal{V}^\star \mathcal{U}) \Big]. \end{aligned}$$

Proof. From (21), by setting $\xi_1 = \xi_2 = \xi$, we have:

$$\begin{aligned} &\left(|\langle \mathcal{U}\xi, \xi \rangle|^r + |\langle \mathcal{V}\xi, \xi \rangle|^r \right)^{2p} \\ &\leq 2^{p-1} \|\xi\|^{2p} \\ &\times \Big[\left(|\langle \mathcal{U}\xi, \xi \rangle|^{2(r-1)} \langle |\mathcal{U}|^2 \xi, \xi \rangle + |\langle \mathcal{V}\xi, \xi \rangle|^{2(r-1)} \langle |\mathcal{V}|^2 \xi, \xi \rangle \right)^p \\ &+ \left(|\langle \mathcal{U}\xi, \xi \rangle|^{2(r-1)} + |\langle \mathcal{V}\xi, \xi \rangle|^{2(r-1)} \right)^p |\langle \mathcal{V}^\star \mathcal{U}\xi, \xi \rangle|^p \Big] \end{aligned} \tag{25}$$

for any $\xi \in \mathbb{K}$.

Observe that for a unit vector $\xi \in \mathbb{K}$, we have

$$\begin{aligned} &|\langle \mathcal{U}\xi, \xi \rangle|^{2(r-1)} \langle |\mathcal{U}|^2 \xi, \xi \rangle + |\langle \mathcal{V}\xi, \xi \rangle|^{2(r-1)} \langle |\mathcal{V}|^2 \xi, \xi \rangle \\ &\leq \omega^{2(r-1)}(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V}) \|\mathcal{V}\|^2 \end{aligned}$$

and

$$|\langle \mathcal{U}\xi, \xi \rangle|^{2(r-1)} + |\langle \mathcal{V}\xi, \xi \rangle|^{2(r-1)} \leq \omega_{2(r-1)}^{2(r-1)}(\mathcal{U}, \mathcal{V}) \leq \omega^{2(r-1)}(\mathcal{U}) + \omega^{2(r-1)}(\mathcal{V}).$$

From (25), we can then derive

$$\begin{aligned} &\left(|\langle \mathcal{U}\xi, \xi \rangle|^r + |\langle \mathcal{V}\xi, \xi \rangle|^r \right)^{2p} \\ &\leq 2^{p-1} \\ &\times \left[\left(\omega^{2(r-1)}(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p + \omega_{2(r-1)}^{2p(r-1)}(\mathcal{U}, \mathcal{V}) \omega^p(\mathcal{V}^\star \mathcal{U}) \right] \\ &\leq 2^{p-1} \Big[\left(\omega^{2(r-1)}(\mathcal{U}) \|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V}) \|\mathcal{V}\|^2 \right)^p \\ &+ \left(\omega^{2(r-1)}(\mathcal{U}) + \omega^{2(r-1)}(\mathcal{V}) \right)^p \omega^p(\mathcal{V}^\star \mathcal{U}) \Big] \end{aligned}$$

for any unit vector $\xi \in \mathbb{K}$.

For the case $r = 1$, we have from (22) that

$$\begin{aligned}&(|\langle \mathcal{U}\xi,\xi\rangle| + |\langle \mathcal{V}\xi,\xi\rangle|)^{2p} \\ &\leq 2^{p-1}\left[\left\langle \left(|\mathcal{U}|^2 + |\mathcal{V}|^2\right)\xi,\xi\right\rangle^p + 2^p|\langle \mathcal{V}^\star\mathcal{U}\xi,\xi\rangle|^p\right]\end{aligned}$$

for any unit vector $\xi \in \mathbb{K}$.

By taking the supremum over all unit vectors $\xi \in \mathbb{K}$, we obtain the desired results (23) and (24). □

Remark 6. *For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{K})$, when $p = 1$, we obtain for $r > 1$:*

$$\begin{aligned}&\omega_r^{2r}(\mathcal{U},\mathcal{V}) \\ &\leq \omega^{2(r-1)}(\mathcal{U})\|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V})\|\mathcal{V}\|^2 + \omega_{2(r-1)}^{2(r-1)}(\mathcal{U},\mathcal{V})\omega(\mathcal{V}^\star\mathcal{U}) \\ &\leq \omega^{2(r-1)}(\mathcal{U})\|\mathcal{U}\|^2 + \omega^{2(r-1)}(\mathcal{V})\|\mathcal{V}\|^2 \\ &+ \left(\omega^{2(r-1)}(\mathcal{U}) + \omega^{2(r-1)}(\mathcal{V})\right)\omega(\mathcal{V}^\star\mathcal{U}).\end{aligned}$$

For $r = 1$, we have:

$$\omega^2(\mathcal{U},\mathcal{V}) \leq \left\| |\mathcal{U}|^2 + |\mathcal{V}|^2 \right\| + \omega(\mathcal{V}^\star\mathcal{U}).$$

For $r = 2$, the inequality for the Euclidean numerical radius is as follows:

$$\begin{aligned}&\omega_e^4(\mathcal{U},\mathcal{V}) \\ &\leq \omega^2(\mathcal{U})\|\mathcal{U}\|^2 + \omega^2(\mathcal{V})\|\mathcal{V}\|^2 + \omega_e^2(\mathcal{U},\mathcal{V})\omega(\mathcal{V}^\star\mathcal{U}) \\ &\leq \omega^2(\mathcal{U})\|\mathcal{U}\|^2 + \omega^2(\mathcal{V})\|\mathcal{V}\|^2 + \left(\omega^2(\mathcal{U}) + \omega^2(\mathcal{V})\right)\omega(\mathcal{V}^\star\mathcal{U}).\end{aligned}$$

For $r \geq 1$ and the special case $\mathcal{V} = \mathcal{U}^\star$, we have that

$$\omega_r(\mathcal{U},\mathcal{U}^\star) := \sup_{\|\xi\|=1}\left(|\langle \mathcal{U}\xi,\xi\rangle|^r + |\langle \mathcal{U}^\star\xi,\xi\rangle|^r\right)^{\frac{1}{r}} = 2^{\frac{1}{r}}\omega(\mathcal{U}).$$

If we set $(\mathcal{U},\mathcal{V}) = (\mathcal{U},\mathcal{U}^\star)$ for an operator $\mathcal{U}$ in Theorem 6 and carry out the calculations, then for $p \geq 1$ we find that:

$$\omega^{2p}(\mathcal{U}) \leq \frac{1}{2}\left[\|\mathcal{U}\|^{2p} + 2^p\omega^p\left(\mathcal{U}^2\right)\right].$$

For the case $p = 1$, this simplifies to:

$$\omega^2(\mathcal{U}) \leq \frac{1}{2}\left[\|\mathcal{U}\|^2 + 2\omega\left(\mathcal{U}^2\right)\right].$$

For $p = 2$, we also obtain the inequality:

$$\omega^4(\mathcal{U}) \leq \frac{1}{2}\left[\|\mathcal{U}\|^4 + 4\omega^2\left(\mathcal{U}^2\right)\right].$$

Now, let $\mathcal{U} \in \mathcal{L}(\mathbb{K})$ have the Cartesian decomposition $\mathcal{U} = \Re(\mathcal{U}) + i\Im(\mathcal{U})$. For $r \geq 1$, we can introduce the quantities:

$$\begin{aligned}\rho_r(\mathcal{U}) &:= w_r(\Re(\mathcal{U}), \Im(\mathcal{U})) \\ &= \sup_{\|\xi\|=1} \left(\left|\langle \Re(\mathcal{U})\xi, \xi \rangle\right|^r + \left|\langle \Im(\mathcal{U})\xi, \xi \rangle\right|^r \right)^{\frac{1}{r}} \\ &\leq \left(\|\Re(\mathcal{U})\|^r + \|\Im(\mathcal{U})\|^r \right)^{1/r}.\end{aligned}$$

In the same vein, $\rho_r(\mathcal{U})$ encodes the symmetric contribution of $\Re(\mathcal{U})$ and $\Im(\mathcal{U})$ to the numerical radius. For $r = 1$, we denote:

$$\begin{aligned}\rho(\mathcal{U}) &:= w(\Re(\mathcal{U}), \Im(\mathcal{U})) \\ &= \sup_{\|\xi\|=1} \left(\left|\langle \Re(\mathcal{U})\xi, \xi \rangle\right| + \left|\langle \Im(\mathcal{U})\xi, \xi \rangle\right| \right) \\ &\leq \|\Re(\mathcal{U})\| + \|\Im(\mathcal{U})\|,\end{aligned}$$

while for $r = 2$,

$$\begin{aligned}\rho_e(\mathcal{U}) &:= w_e(\Re(\mathcal{U}), \Im(\mathcal{U})) \\ &= \sup_{\|\xi\|=1} \left(\left|\langle \Re(\mathcal{U})\xi, \xi \rangle\right|^2 + \left|\langle \Im(\mathcal{U})\xi, \xi \rangle\right|^2 \right)^{\frac{1}{2}} \\ &= \sup_{\|\xi\|=1} \left|\langle \mathcal{U}\xi, \xi \rangle\right| = \omega(\mathcal{U}).\end{aligned}$$

If we set $(\mathcal{U}, \mathcal{V}) = (\Re(\mathcal{U}), \Im(\mathcal{U}))$ for an operator $\mathcal{U}$ in Theorem 6, then for $r \geq 1$:

$$\begin{aligned}&\rho_r^{2pr}(\mathcal{U}) \\ &\leq 2^{p-1}\left[\left(\|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r} \right)^p + \rho_{2(r-1)}^{2p(r-1)}(\mathcal{U})\omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \right] \\ &\leq 2^{p-1}\left[\left(\|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r} \right)^p \right. \\ &\left. + \left(\|\Re(\mathcal{U})\|^{2(r-1)} + \|\Im(\mathcal{U})\|^{2(r-1)} \right)^p \omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \right].\end{aligned}$$

For $r = 1$, we obtain:

$$\rho^{2p}(\mathcal{U}) \leq 2^{p-1}\left[\left\| \Re(\mathcal{U})^2 + \Im(\mathcal{U})^2 \right\|^p + 2^p \omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \right],$$

while for $r = 2$:

$$\begin{aligned}&\omega^{4p}(\mathcal{U}) \\ &\leq 2^{p-1}\left[\left(\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 \right)^p + \omega^{2p}(\mathcal{U})\omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \right] \\ &\leq 2^{p-1}\left[\left(\|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 \right)^p \right. \\ &\left. + \left(\|\Re(\mathcal{U})\|^2 + \|\Im(\mathcal{U})\|^2 \right)^p \omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) \right].\end{aligned}$$

If we take $p = 1$, then for $r > 1$, we obtain:

$$\begin{aligned}\rho_r^{2r}(\mathcal{U}) &\leq \|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r} + \rho_{2(r-1)}^{2(r-1)}(\mathcal{U})\omega(\Im(\mathcal{U})\Re(\mathcal{U})) \\ &\leq \|\Re(\mathcal{U})\|^{2r} + \|\Im(\mathcal{U})\|^{2r} \\ &+ \left(\|\Re(\mathcal{U})\|^{2(r-1)} + \|\Im(\mathcal{U})\|^{2(r-1)}\right)\omega(\Im(\mathcal{U})\Re(\mathcal{U})).\end{aligned}$$

For $r = 1$:

$$\rho^2(\mathcal{U}) \leq \left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\| + 2\omega(\Im(\mathcal{U})\Re(\mathcal{U})),$$

and for $r = 2$:

$$\begin{aligned}\omega^4(\mathcal{U}) &\leq \|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 + \omega^2(\mathcal{U})\omega(\Im(\mathcal{U})\Re(\mathcal{U})) \\ &\leq \|\Re(\mathcal{U})\|^4 + \|\Im(\mathcal{U})\|^4 + \left(\|\Re(\mathcal{U})\|^2 + \|\Im(\mathcal{U})\|^2\right)\omega(\Im(\mathcal{U})\Re(\mathcal{U})).\end{aligned}$$

In conclusion, the inequalities developed in this paper not only refine classical operator bounds, but also demonstrate the essential role of symmetry in Hilbert space analysis. By uncovering symmetric structures between adjoints, real and imaginary parts, and operator pairs, our results provide new perspectives on how symmetry governs the interplay between norms and numerical radii.

5. Conclusions

This paper established a unified framework for deriving operator inequalities by leveraging the concept of symmetry. Our approach, based on an extension of Pečarić's inequality, yields several new and sharpened bounds.

Key results include refined estimates for a single operator $\mathcal{U} \in \mathcal{L}(\mathbb{K})$, such as:

$$\|\mathcal{U}\|^{2p} \leq 2^{p-1}\left[\left\|\frac{|\mathcal{U}|^2 + |\mathcal{U}^\star|^2}{2}\right\|^p + 2^p\omega^p(\Im(\mathcal{U})\Re(\mathcal{U}))\right]$$

and

$$\omega^{2p}(\mathcal{U}) \leq 2^{p-1}\omega^p(\Im(\mathcal{U})\Re(\mathcal{U})) + 2^{p-1}\left\|\Re(\mathcal{U})^2 + \Im(\mathcal{U})^2\right\|^p.$$

We also extended these ideas to an operator pair $(\mathcal{U}, \mathcal{V})$, leading to general inequalities for the s–r-norm, including:

$$\|(\mathcal{U}, \mathcal{V})\|_r^{2pr} \leq 2^{p-1}\left[\left(\|\mathcal{U}\|^{2r} + \|\mathcal{V}\|^{2r}\right)^p + \|(\mathcal{U}, \mathcal{V})\|_{2(r-1)}^{2p(r-1)}\omega^p(\mathcal{V}^\star\mathcal{U})\right].$$

These inequalities highlight the power of exploiting the adjoint and Cartesian symmetries. This work serves as a starting point for applying these symmetric methods to explore new inequalities for different classes of operators or within other functional analysis settings.

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