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A Modified Analytical Data-Mapping Framework for Symmetric Multiscale Soliton and Chaotic Dynamics

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Abstract

The $(3 + 1)$ -dimensional KdV–Calogero–Bogoyavlenskii–Schiff equation, a model that describes long-wave interactions and has numerous applications in mathematics, engineering, and physics, is examined in this work. First, a wave transformation is used to reduce the equation to lower dimensions. The modified Khater method is then used to derive different types of solitary wave solutions, such as chirped, kink, periodic, and kink-bright types. By allocating suitable constant parameters, 3D, 2D, and contour plots are created to demonstrate the physical behavior of these solutions. Phase portraits are used to qualitatively analyze the undisturbed planar system using bifurcation theory. The system is then perturbed by an external force, resulting in chaotic dynamics. Chaos in the system is confirmed using multiple diagnostic tools, including time series plots, Poincaré sections, chaotic attractors, return maps, bifurcation diagrams, power spectra, and Lyapunov exponents. The stability of the model is further investigated with varying initial conditions. A bidirectional scatter plot technique, which efficiently reveals overlapping regions using data point distributions, is presented for comparing solution behaviors. Overall, this work offers useful tools for advancing applied mathematics research as well as a deeper understanding of nonlinear wave dynamics.

Keywords: dynamical analysis; phase portraits; power spectrum; iterative return maps; bifurcation diagram; wave solutions



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1. Introduction

An essential element of contemporary mathematical and physical modeling, nonlinear evolution equations (NLEEs) explain how nonlinear systems change over time. They can be found in a variety of fields, including data science [1], deep learning [2], fluid dynamics [3], neuroscience [4], and image processing [5]. The development of various analytical techniques, such as the bilinear [6], auxiliary equation [7], long-wave limit (LWL) [8], Nucci's reduction [9], generalized logistic equation [10], Darboux transform [11], trial function [12], Riemann–Hilbert [13], Bäcklund transformation [14], and modified Khater [15] methods, has been motivated by the need to find precise solutions to these equations.

In analytical research, one main emphasis is on obtaining soliton solutions to NLEEs. Solitary waves that are able to travel long distances without losing their form have long intrigued scientists, engineers, and mathematicians because of their interesting properties and vast number of possible applications. At the same time, the study of dynamical

systems has been of significant interest to researchers in many fields because of their complex behaviors, including bifurcation phenomena and chaotic dynamics [16,17].

The qualitative behavior of several nonlinear systems has been studied by many investigators, some significant work examples of which are: Shallow water wave [18], advection diffusion reaction model [19], Wazwaz-Benjamin-Bona-Mahony model [20], prey–predator population model [21], and Davey Stewartson Kadomtsev Petviashvili model [22]. A newly proposed (3 + 1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation was recently presented by Wazwaz [23], which models shallow wave propagation and fluid dynamics in diverse natural settings such as rivers and coastal regions. This equation has gained considerable attention for its broad applications in the fields of mathematics and engineering.

$$4\phi_{xt} + \alpha(4\phi_x\phi_{xy} + 2\phi_{xx}\phi_y + \phi_{xxx}) + \beta(4\phi_x\phi_{xz} + 2\phi_{xx}\phi_z + \phi_{xxx}) + \gamma(6\phi_x\phi_{xx} + \phi_{xxx}) + p\phi_{xx} + q\phi_{xy} + m\phi_{xz} = 0. \quad (1)$$

Here, $\phi = \phi(x, y, z, t)$, with $\alpha, \beta, \gamma, p, q$, and m being undetermined parameters. Equation (1) represents a generalized KdV–CBS-type nonlinear model that describes the interaction between nonlinear and dispersive effects in multidimensional media. The time-derivative term in Equation (1) defines the evolutionary nature of the model, governing how the wave amplitude changes over time under the combined effects of dispersion and nonlinearity [24]. These models are frequently used to explain nonlinear wave dynamics in fluid mechanics, shallow-water, and plasma physics [25–27]. The coefficients α, β , and γ quantify higher-order dispersion along different spatial directions, while p, q , and m represent nonlinear coupling strengths. Hence, Equation (1) serves as a flexible physical model for exploring nonlinear wave interactions and energy localization across different physical settings.

1.1. Literature Review

Several researchers have employed diverse methods to study the KdV-CBS in various dimensions. Wazwaz [23] formulated Equation (1) and assessed its integrability using the Painlevé test. Additionally, multi-wave solitons were constructed using the simplified Hirota technique. Lump and localized waves of Equation (1) are obtained using a symbolic computation approach. Furthermore, multiple solitons in complex form are derived by employing Hirota’s direct method along with the long-wave approximation technique [25]. Zhou et al. [26] employed the truncated Painlevé test to derive a series of auto-Bäcklund transformations, and also obtained multi-wave soliton solutions. In [27], a new general bilinear Bäcklund transformation, Lax pair, and conservation laws for Equation (1) are derived using the theory of binary Bell polynomials. It is also demonstrated that the equation is completely integrable in the Lax pair framework. Various types of mixed solutions are constructed through a combination of the homoclinic test approach. By modifying its coefficients, Equation (1) is transformed into various NLEEs, each exhibiting distinct physical characteristics.

- When $\alpha = \beta = p = q = m = 0$ and $\gamma = 4$, Equation (1) has been simplified to the well-known KdV equation [28].

$$\phi_t + \phi_{xxx} + 6\phi\phi_x = 0. \quad (2)$$

- When $\beta = \gamma = p = q = m = 0$ and $\alpha = 4$, Equation (1) has been simplified to the well-known CBS equation [29].

$$\phi_t + \phi_{xxy} + 4\phi\phi_y + 2\phi_x\partial_x^{-1}\phi_y = 0. \quad (3)$$

- When $\alpha = -d_1$, $\gamma = -d_2$ and $\beta = p = q = m = 0$, Equation (1) has been simplified to the (2 + 1) KdV equation [30].

$$4\phi_t - d_1(\phi_{xxy} + 4\phi\phi_y + 2\phi_x\partial_x^{-1}\phi_y) - d_2(\phi_{xxx} + 6\phi\phi_x) = 0. \quad (4)$$

1.2. Research Gaps and Contributions

Although substantial progress has been made in exploring soliton dynamics related to Equation (1), a significant gap persists in the application of thorough dynamical system analysis techniques. An overview of the work conducted previously on the KdV-CSS model shows that wave solutions have not been obtained by using the MKM. As indicated in Table 1, though some analytical methods were used in different systems by previous studies, they do not provide a comprehensive analysis involving all the necessary tools.

To bridge this gap in the existing studies, the main contributions of this paper are therefore detailed as follows: The KdV-CSS equation has been studied from different perspectives. In the first place, we use MKM to construct solitary waves as presented in [15]. The choice of this method is because it is simple, practical, and dependable for obtaining analytical solutions of NLEEs. Although there exist a number of extensions and alternative forms of the Khater method in literature, a modified form is adopted here which gives more freedom toward finding exact analytical solutions. This version of the method is best suited to derive an extensive set of nonlinear wave patterns through the possibility of introducing auxiliary constraints and nonlinear functional expansions within one unified analytical framework.

Next, the dynamics of the KdV-CSS model were examined by means of a qualitative analysis as discussed in [31]. In order to analyze the unpredictable dynamics of this system, numerical simulations were conducted with the help of a fourth-order Runge-Kutta integration scheme. A broad array of methods for identifying chaos were used that included:

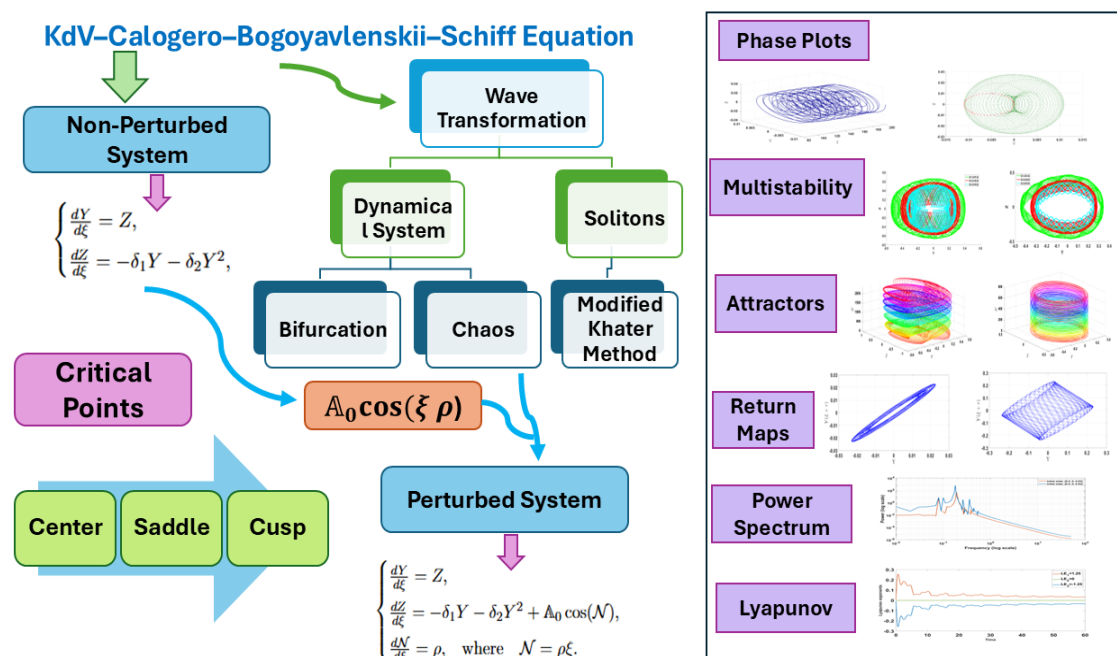
- Three-dimensional phase portraits (3D PP);
- Two-dimensional phase portraits (2D PP);
- Poincaré maps (PM);
- Time series graphs (TSG);
- Multistability analysis (MA);
- Chaotic attractor (CA);
- Iterative return mapping (IRM);
- Power spectrum (PS);
- Bifurcation diagram (BD);
- Lyapunov exponent (LE).

Table 1 provides a comparative view with previous research works highlighting the novelty aspects and unique contributions being carried out in the present work.

The paper is organized as follows, as displayed in Figure 1. The MKM is introduced in Section 2, and the soliton solutions are derived in Section 3. A thorough phase portrait analysis is presented in Section 5, while the physical interpretation of the results is covered in Section 4. In Section 6, the model's chaotic and irregular behavior is examined, and in Section 7, the bidirectional scatter plot method is used to highlight overlaps among various solutions. The paper concludes with final remarks in the Conclusion section. Layout is shown in the Figure 1.

Table 1. Evaluation of various analytical tools covered in the literature.

References	PP	PM	TSG	MA	CA	IRM	PS	BD	LE
[7]	✓	✓	✓	×	×	×	×	×	×
[10]	✓	✓	✓	✓	×	×	×	×	✓
[17]	✓	✓	✓	×	×	×	×	✓	✓
[18]	✓	✓	✓	×	×	×	×	×	×
[19]	✓	✓	✓	×	×	×	×	×	✓
[20]	✓	✓	✓	✓	×	×	×	×	✓
[21]	✓	×	×	×	×	×	×	✓	✓
[22]	✓	✓	✓	✓	×	×	×	×	×
[31]	✓	×	×	×	×	×	×	×	×
Our study	✓	✓	✓	✓	✓	✓	✓	✓	✓

**Figure 1.** Layout of the Paper.

2. Description of an Analytical Method

We use MKM [15], an effective analytical method that is ideal for dealing with NLEEs, to obtain wave solutions for the suggested model. A detailed explanation of the methodology is given below.

(Step 1): Consider a n th order general PDE.

$$H(\phi, \phi_x, \phi_y, \phi_z, \phi_t, \phi_{xx}, \phi_{tt}, \dots) = 0. \quad (5)$$

In this case, H represents a polynomial expression containing the unknown function $\phi(x, y, z, t)$.

(Step 2): The governing PDE in Equation (1) is reduced to the following ODE by using the wave transformation $\phi(x, y, z, t) = \Omega(\xi)$, where $\xi = a_1 x + a_2 y + a_3 z - \mu t$.

$$G(\Omega', \Omega'', \Omega''', \Omega'''' \dots) = 0. \quad (6)$$

(Step 3): The following functional form is assumed for the solution of Equation (6):

$$\Omega(\xi) = \sum_{j=0}^n \Lambda_j F^j(\xi), \quad (7)$$

where Λ_j are random constants, respectively. The function $F^j(\xi)$ satisfies the following equation.

$$F'(\xi) = c_1 \ln(v) + c_2 F(\xi) \ln(v) + c_3 F^2(\xi) \ln(v). \quad (8)$$

The condition $v \neq 0, 1$ is imposed, with c_1, c_2 , and c_3 denoting arbitrary constants. The discriminant of this equation is given by:

$$\Delta = c_2^2 - 4c_3c_1.$$

This definition of Δ is used consistently throughout the subsequent derivations and determines the nature of the resulting wave solutions. Specifically, $\Delta > 0$ corresponds to hyperbolic-type solitons, $\Delta = 0$ yields rational-type solutions, and $\Delta < 0$ leads to trigonometric-type wave structures.

(Step 4): Balancing techniques have been used to determine the value of n by comparing the highest-order derivative with the powers of nonlinear terms in the ODE. The value of j remains positive throughout.

(Step 5): Upon substituting Equations (7) and (8) into Equation (6) and setting the coefficients of F_j to zero, a solvable system of algebraic equations for Λ_j is obtained. The system is solved using Maple. By substituting the resulting Λ_j values together with Equation (8) and the wave transformation into Equation (1), the corresponding analytical solutions of Equation (1) are obtained.

3. Application of an Analytical Approach

The extended wave equation in (3 + 1) dimensions, known as the KdV–CBS equation, be formulated as:

$$4\phi_{xt} + \alpha(4\phi_x\phi_{xy} + 2\phi_{xx}\phi_y + \phi_{xxx}) + \beta(4\phi_x\phi_{xz} + 2\phi_{xx}\phi_z + \phi_{xxx}) + \gamma(6\phi_x\phi_{xx} + \phi_{xxx}) + p\phi_{xx} + q\phi_{xy} + m\phi_{xz} = 0. \quad (9)$$

By applying the traveling wave transformation:

$$\phi(x, y, z, t) = \Omega(\xi), \quad \xi = a_1x + a_2y + a_3z - \mu t. \quad (10)$$

In this regard, the traveling wave's characteristics are described by $\Omega(\xi)$ and μ indicates its speed. The parameters a_1, a_2, a_3 are treated as wave number in x, y and z directions. We get the resultant ODE by replacing the wave transformation in Equation (9).

$$K_1\Omega'' + K_2\Omega'\Omega'' + K_3\Omega^{iv} = 0. \quad (11)$$

By integrating Equation (11) with respect to ξ , we acquire

$$K_1\Omega' + K_2\frac{\Omega'^2}{2} + K_3\Omega''' = C_0. \quad (12)$$

Here, C_0 is the constant of integration, which is set to zero for simplicity under the assumption of localized wave solutions [32]. By choosing $\Omega' = Y$, Equation (12) becomes:

$$K_1Y + K_2\frac{Y^2}{2} + K_3Y'' = 0. \quad (13)$$

where, $K_1 = (-4a_1\mu + Pa_1^2 + qa_1a_2 + ma_1a_3)$, $K_2 = (6\alpha a_1^2a_2 + 6\beta a_1^2a_3 + 6\gamma a_1^3)$, and $K_3 = (\alpha a_2 + \beta a_3 + \gamma a_1)a_1^3$. Next, we will obtain the wave solutions of Equation (12).

Traveling Wave Solutions

This section addresses the application of the MKM for the extraction of analytical solutions. Balancing the nonlinear term Ω'^2 and the highest derivative term Ω''' in Equation (12) yields $n = 1$. The initial solution in this case is as follows.

$$\Omega(\xi) = \Lambda_0 + \Lambda_1 F. \quad (14)$$

Inserting Equations (8) and (14) in Equation (12) and solve the resulting system for Λ_0 , Λ_1 and c_1 :

$$\Lambda_0 = 2, \quad \Lambda_1 = \frac{-12c_3 \ln(v) K_3}{K_2}, \quad c_1 = \frac{\ln(v)^2 K_3 c_2^2 + K_1}{4 \ln(v)^2 K_3 c_3}. \quad (15)$$

Substituting the above values into Equation (14) and applying the transformation defined in Equation (10), the resulting solutions for Equation (1) are obtained as follows. The classification of these analytical solutions is governed by the discriminant $\Delta = c_2^2 - 4c_3c_1$ and by the sign combinations of c_1 , c_2 , and c_3 , which determine the qualitative behavior of the auxiliary function $F(\xi)$ in Equation (8).

Case 1: If $\Delta = c_2^2 - 4c_3c_1 < 0$ and $c_3 \neq 0$ then,

$$\begin{aligned} \phi_1(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} + \frac{\sqrt{-\Delta}}{2} \tan_v \left(\frac{\sqrt{-\Delta}}{2} \xi \right) \right) + 2, \\ \phi_2(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{-\Delta}}{2} \cot_v \left(\frac{\sqrt{-\Delta}}{2} \xi \right) \right) + 2, \\ \phi_3(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} + \frac{\sqrt{-\Delta}}{2} (\tan_v(\sqrt{-\Delta} \xi) \pm \sqrt{rs} \sec_v(\sqrt{-\Delta} \xi)) \right) + 2, \\ \phi_4(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{-\Delta}}{2} (\cot_v(\sqrt{-\Delta} \xi) \pm \sqrt{rs} \csc_v(\sqrt{-\Delta} \xi)) \right) + 2, \\ \phi_5(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} + \frac{\sqrt{-\Delta}}{4} \left(\tan_v \left(\frac{\sqrt{-\Delta} \xi}{4} \right) - \cot_v \left(\frac{\sqrt{-\Delta} \xi}{4} \right) \right) \right) + 2. \end{aligned} \quad (16)$$

Case 2: If $\Delta = c_2^2 - 4c_3c_1 > 0$ and $c_3 \neq 0$ then,

$$\begin{aligned} \phi_6(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} \tanh_v \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) + 2, \\ \phi_7(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} \coth_v \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) + 2, \\ \phi_8(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} (\tanh_v(\sqrt{\Delta} \xi) \pm i \sqrt{rs} \sec h_v(\sqrt{\Delta} \xi)) \right) + 2, \\ \phi_9(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} (\coth_v(\sqrt{\Delta} \xi) \pm \sqrt{rs} \csc h_v(\sqrt{\Delta} \xi)) \right) + 2, \\ \phi_{10}(x, y, z, t) &= \frac{-12 \ln(v) K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{-\Delta}}{4} \left(\tan h_v \left(\frac{\sqrt{\Delta} \xi}{4} \right) + \cot h_v \left(\frac{\sqrt{\Delta} \xi}{4} \right) \right) \right) + 2. \end{aligned} \quad (17)$$

Case 3: If $c_3c_1 > 0$ and $c_2 = 0$ then,

$$\begin{aligned}
\phi_{11}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\sqrt{\frac{c_1}{c_3}} \tan_v(\sqrt{c_3 c_1} \xi) \right) + 2, \\
\phi_{12}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\sqrt{\frac{c_1}{c_3}} \cot_v(\sqrt{c_3 c_1} \xi) \right) + 2, \\
\phi_{13}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\sqrt{\frac{c_1}{c_3}} (\tan_v(2\sqrt{c_3 c_1} \xi) \pm \sqrt{rs} \sec_v(2\sqrt{c_3 c_1} \xi)) \right) + 2, \\
\phi_{14}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\sqrt{\frac{c_1}{c_3}} (-\cot_v(2\sqrt{c_3 c_1} \xi) \pm \sqrt{rs} \csc_v(2\sqrt{c_3 c_1} \xi)) \right) + 2, \\
\phi_{15}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{1}{2} \sqrt{\frac{c_1}{c_3}} \left(\tan_v\left(\frac{\sqrt{c_3 c_1}(\xi)}{2}\right) - \cot_v\left(\frac{\sqrt{c_3 c_1}(\xi)}{2}\right) \right) \right) + 2.
\end{aligned} \tag{18}$$

Case 4: If $c_3 c_1 < 0$ and $c_2 = 0$ then,

$$\begin{aligned}
\phi_{16}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\sqrt{\frac{-c_1}{c_3}} \tanh_v(\sqrt{-c_3 c_1} \xi) \right) + 2, \\
\phi_{17}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\sqrt{\frac{-c_1}{-c_3}} \coth_v(\sqrt{-c_3 c_1} \xi) \right) + 2, \\
\phi_{18}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\sqrt{\frac{-c_1}{c_3}} (\tanh_v(2\sqrt{-c_3 c_1} \xi) \pm \iota \sqrt{rs} \sec h_v(2\sqrt{-c_3 c_1} \xi)) \right) + 2, \\
\phi_{19}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\sqrt{\frac{c_1}{c_3}} (\coth_v(2\sqrt{-c_3 c_1} \xi) \pm \sqrt{rs} \csc h_v(2\sqrt{-c_3 c_1} \xi)) \right) + 2, \\
\phi_{20}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\frac{1}{2} \sqrt{\frac{-c_1}{c_3}} \left(\tanh_v\left(\frac{\sqrt{-c_3 c_1}(\xi)}{2}\right) + \coth_v\left(\frac{\sqrt{-c_3 c_1}(\xi)}{2}\right) \right) \right) + 2.
\end{aligned} \tag{19}$$

Case 5: If $c_2 = 0$ and $c_3 = c_1$ then,

$$\begin{aligned}
\phi_{21}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (\tan_v(c_1 \xi)) + 2, \\
\phi_{22}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\cot_v(c_1 \xi)) + 2, \\
\phi_{23}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (\tan_v(2c_1 \xi) \pm \sqrt{rs} \sec_v(2c_1 \xi)) + 2, \\
\phi_{24}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\cot_v(2c_1 \xi) \pm \sqrt{rs} \csc_v(2c_1 \xi)) + 2, \\
\phi_{25}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{1}{2} (\tan_v(\frac{c_1 \xi}{2}) - \cot_v(\frac{c_1 \xi}{2})) \right) + 2.
\end{aligned} \tag{20}$$

Case 6: If $c_2 = 0$ and $c_3 = -c_1$ then,

$$\begin{aligned}
\phi_{26}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\tanh_v(c_1 \xi)) + 2, \\
\phi_{27}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\coth_v(c_1 \xi)) + 2, \\
\phi_{28}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\tanh_v(2c_1 \xi) \pm \iota \sqrt{rs} \sec_v(2c_1 \xi)) + 2, \\
\phi_{29}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} (-\coth_v(2c_1 \xi) \pm \sqrt{rs} \csc h_v(2c_1 \xi)) + 2, \\
\phi_{30}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{-1}{2} (\tanh_v(\frac{c_1 \xi}{2}) + \coth_v(\frac{c_1 \xi}{2})) \right) + 2.
\end{aligned} \tag{21}$$

Case 7: If $c_2^2 = 4c_3c_1$ then,

$$\phi_{31}(x, y, z, t) = \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{-2c_1(c_2 \xi \ln(v) + 2)}{c_2^2 \xi \ln(v)} \right) + 2. \quad (22)$$

Case 8: If $c_2 = \theta, c_1 = p\theta$ ($p \neq 0$) and $c_3 = 0$, then,

$$\phi_{32}(x, y, z, t) = \frac{-12c_3 \ln(v) K_3}{K_2} (v^{\theta \xi} - p) + 2. \quad (23)$$

Case 9: If $c_2 = c_3 = 0$ then,

$$\phi_{33}(x, y, z, t) = \frac{-12c_3 \ln(v) K_3}{K_2} (c_1 \xi \ln(v)) + 2. \quad (24)$$

Case 10: If $c_2 = c_1 = 0$ then,

$$\phi_{34}(x, y, z, t) = \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{-1}{c_3 \xi \ln(v)} \right) + 2. \quad (25)$$

Case 11: If $c_1 = 0$ and $c_2 \neq 0$ then,

$$\begin{aligned} \phi_{35}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\frac{rc_2}{c_3 (\cosh_v(c_2 \xi) - \sinh_v(c_2 \xi) + r)} \right) + 2, \\ \phi_{36}(x, y, z, t) &= \frac{-12c_3 \ln(v) K_3}{K_2} \left(-\frac{c_2 (\sinh_v(c_2 \xi) + \cosh_v(c_2 \xi))}{c_3 (\sinh_v(c_2 \xi) + \cosh_v(c_2 \xi) + s)} \right) + 2. \end{aligned} \quad (26)$$

Case 12: If $c_2 = \theta, c_3 = p\theta$ and $c_1 = 0$ then,

$$\phi_{37}(x, y, z, t) = \frac{-12c_3 \ln(v) K_3}{K_2} \left(\frac{rv^{\theta \xi}}{s - prv^{\theta \xi}} \right) + 2. \quad (27)$$

Here, the trigonometric and hyperbolic functions are discussed below:

$$\begin{aligned} \sinh_v(\xi) &= \frac{rv^{\xi} - sv^{-\xi}}{2}, & \cosh_v(\xi) &= \frac{rv^{\xi} + sv^{-\xi}}{2}, \\ \tanh_v(\xi) &= \frac{rv^{\xi} - sv^{-\xi}}{rv^{\xi} + sv^{-\xi}}, & \coth_v(\xi) &= \frac{rv^{\xi} + sv^{-\xi}}{rv^{\xi} - sv^{-\xi}}, \\ \operatorname{csch}_v(\xi) &= \frac{2}{rv^{\xi} - sv^{-\xi}}, & \operatorname{sech}_v(\xi) &= \frac{2}{rv^{\xi} + sv^{-\xi}}, \\ \sin_v(\xi) &= \frac{rv^{\iota \xi} - sv^{-\iota \xi}}{2\iota}, & \cos_v(\iota \xi) &= \frac{rv^{\iota \xi} + sv^{-\iota \xi}}{2}, \\ \tan_v(\xi) &= -\iota \frac{rv^{\xi} - sv^{-\xi}}{rv^{\xi} + sv^{-\xi}}, & \cot_v(\xi) &= \iota \frac{rv^{\xi} + sv^{-\xi}}{rv^{\xi} - sv^{-\xi}}, \\ \operatorname{csc}_v(\xi) &= \frac{2\iota}{rv^{\iota \xi} - sv^{-\iota \xi}}, & \sec_v(\xi) &= \frac{2}{rv^{\iota \xi} + sv^{-\iota \xi}}. \end{aligned}$$

4. Physical Interpretation and Discussion of the Results

Four primary nonlinear wave forms—kink, periodic, kink–bright, and chirped solitons—are revealed by the analytical and graphical results produced by the modified Khater method. Because nonlinearity and dispersion are balanced, each pattern exhibits unique nonlinear dynamics.

A kink soliton is a front-like wave created by a smooth transition between two constant states. A periodic soliton mimics a recurring pattern of wave crests and troughs and displays regular, continuous oscillations. A kink–bright soliton, which combines a kink background with a sharp localized peak, reflects the interaction between large- and small-scale wave features. A localized pulse with a relationship between wave amplitude and phase, as well as a frequency variation along its profile as a result of phase modulation, is called a chirped soliton.

When combined, these structures demonstrate how the system can sustain periodic and localized waves under various dispersive and nonlinear effects. Physically, periodic solitons correlate to coherent oscillations, chirped solitons occur in optical communication and energy-transport systems, and kink and kink–bright solitons describe front propagation and density changes in fluids and plasmas. The dynamics of Equation (1) are visualized by the contour, 2D, and 3D plots. Four regimes are depicted in Figures 2–5:

- (i) Sharp kink or kink-bright forms are produced by strong nonlinearity;
- (ii) Extended periodic waves are produced by dominant dispersion;
- (iii) Modulated or chirped pulses are produced by combined effects;
- (iv) Symmetrical structures are created by symmetric coefficients.

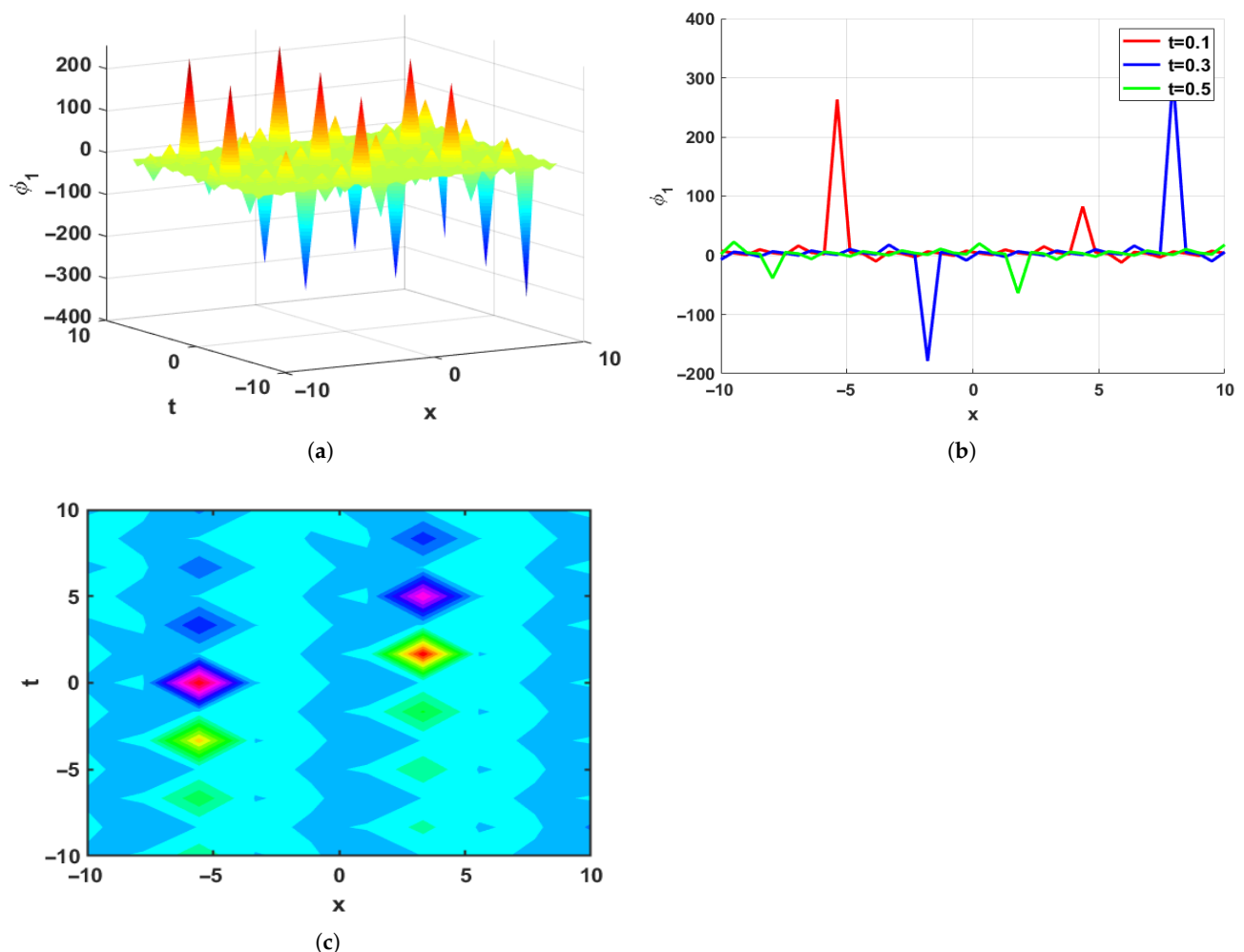


Figure 2. Visualization of the solution $\phi_1(x, t)$, displaying periodic soliton under the parameter value $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3$ and $\gamma = 2$. (a) 3D Plot. (b) 2D Plot. (c) Contour Plot.

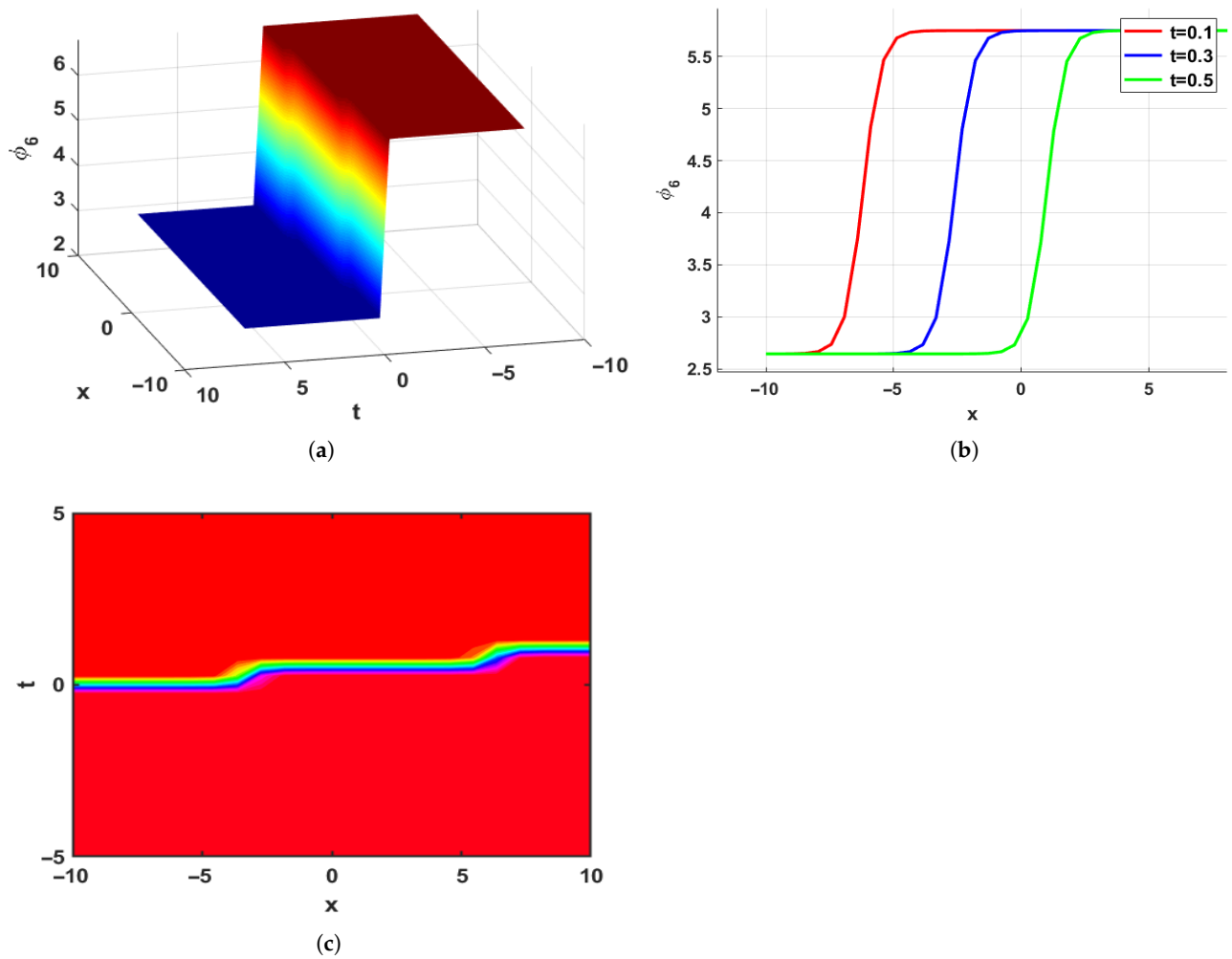


Figure 3. Visualization of the solution $\phi_6(x, t)$, displaying kink type solitons under the parameter value $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3$ and $\gamma = 2$. (a) 3D Plot. (b) 2D Plot. (c) Contour Plot.

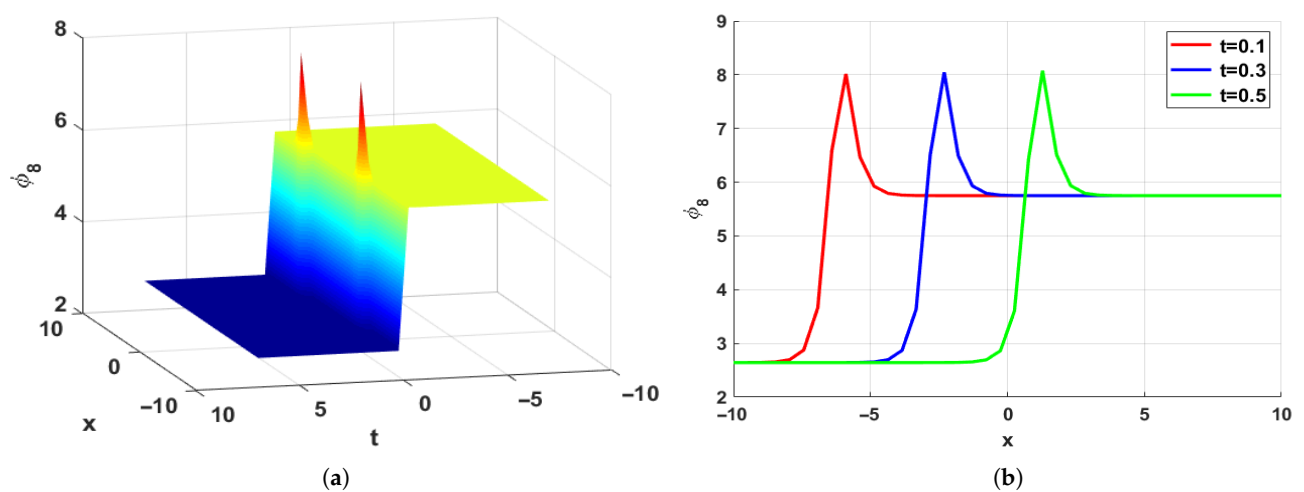


Figure 4. Cont.

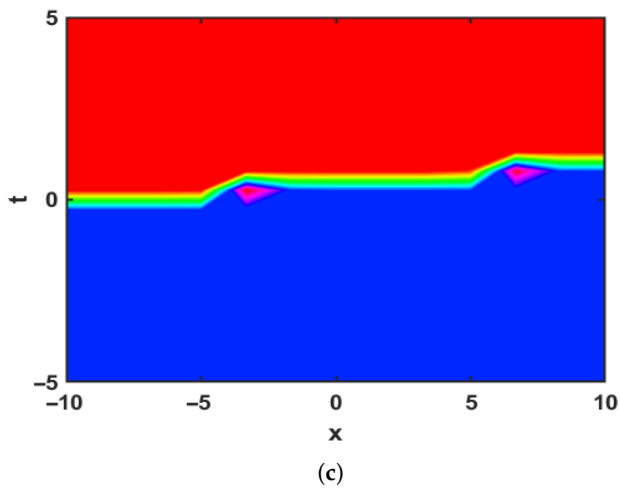


Figure 4. Visualization of the solution $\phi_8(x, t)$, displaying bright-kink soliton under the parameter value $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3$ and $\gamma = 2$. (a) 3D Plot. (b) 2D Plot. (c) Contour Plot.

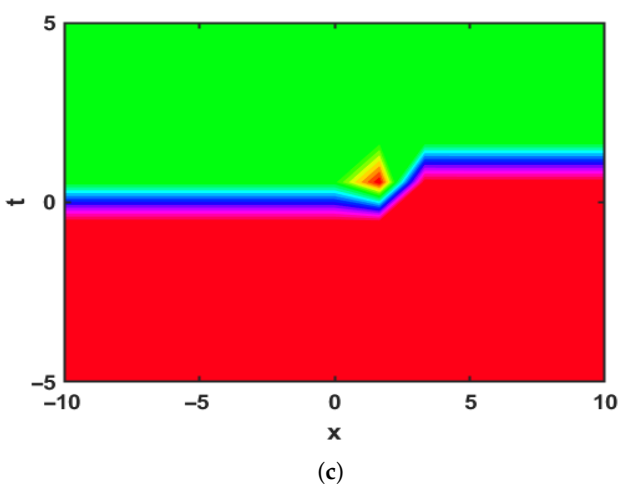
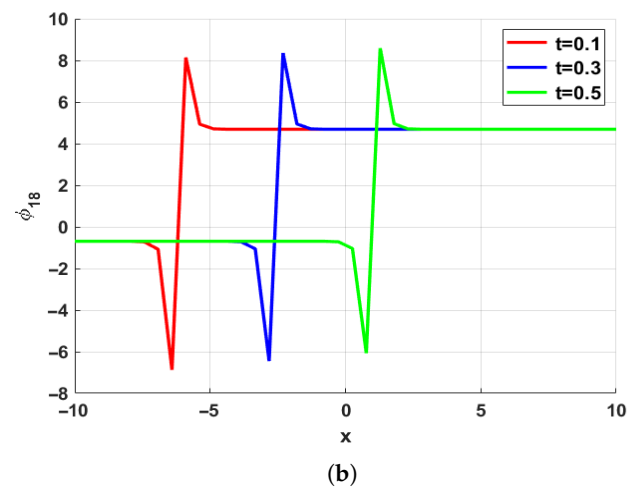
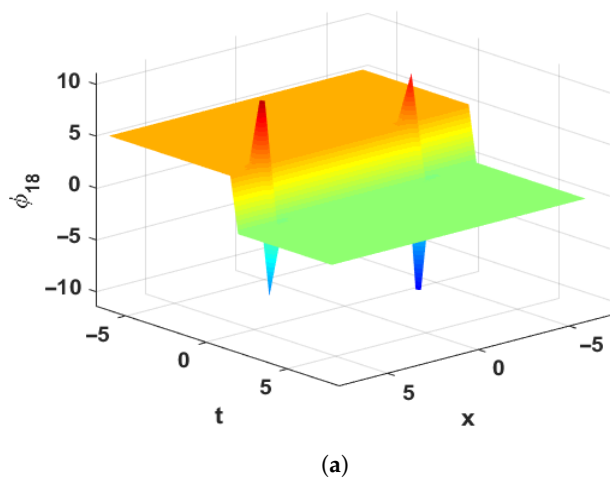


Figure 5. Visualization of the solution $\phi_{18}(x, t)$, displaying chirped soliton solitons under the parameter value $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3$ and $\gamma = 2$. (a) 3D Plot. (b) 2D Plot. (c) Contour Plot.

These patterns show how adaptable the system is and how it can display a variety of nonlinear wave behaviors.

5. Phase Portraits and System Dynamics

A bifurcation is a qualitative shift in the behavior of the trajectories that happens when the system parameters are modified in a differential equation system. By letting $Y' = Z$ in Equation (13), the associated planar system can be formulated as follows:

$$\begin{cases} \frac{dY}{d\xi} = Z, \\ \frac{dZ}{d\xi} = -\delta_1 Y - \delta_2 Y^2, \end{cases} \quad (28)$$

where $\delta_1 = \frac{K_1}{K_3} = \frac{(-4a_1\mu + pa_1^2 + qa_1a_2 + ma_1a_3)}{(aa_2 + \beta a_3 + \gamma a_1)a_1^3}$, and $\delta_2 = \frac{K_2}{2K_3} = \frac{3}{a_1}$. The Hamiltonian corresponding to Equation (28) can be written as:

$$\mathcal{H}(Y, Z) = \frac{Z^2}{2} + \delta_1 \frac{Y^2}{2} + \delta_2 \frac{Y^3}{3}, \quad (29)$$

which satisfies

$$\frac{\partial \mathcal{H}}{\partial Y} = -Z', \quad \frac{\partial \mathcal{H}}{\partial Z} = Y'.$$

As a starting point, the analysis involves locating the equilibrium points $\mathfrak{E}_i(Y^*, Z^*)$ of system Equation (28) by setting $\frac{dY}{d\xi} = 0$ and $\frac{dZ}{d\xi} = 0$, which gives two points based on the chosen parameters.

$$\mathfrak{E}_1(Y^*, Z^*) = \mathfrak{E}_1(0, 0), \quad \mathfrak{E}_2(Y^*, Z^*) = \mathfrak{E}_2\left(\frac{-\delta_1}{\delta_2}, 0\right).$$

Moreover, the Jacobian associated with system Equation (28) is given by:

$$\mathfrak{J}(Y^*, Z^*) = \begin{pmatrix} 0 & 1 \\ -\delta_1 - 2\delta_2 Y & 0 \end{pmatrix} \quad (30)$$

The corresponding eigenvalues of the Jacobian are obtained from its characteristic equation and are given by:

$$\kappa_{\pm}(Y^*, Z^*) = \pm i\sqrt{\delta_1 + 2\delta_2 Y_e}.$$

The classification of the equilibrium points is presented in Table 2.

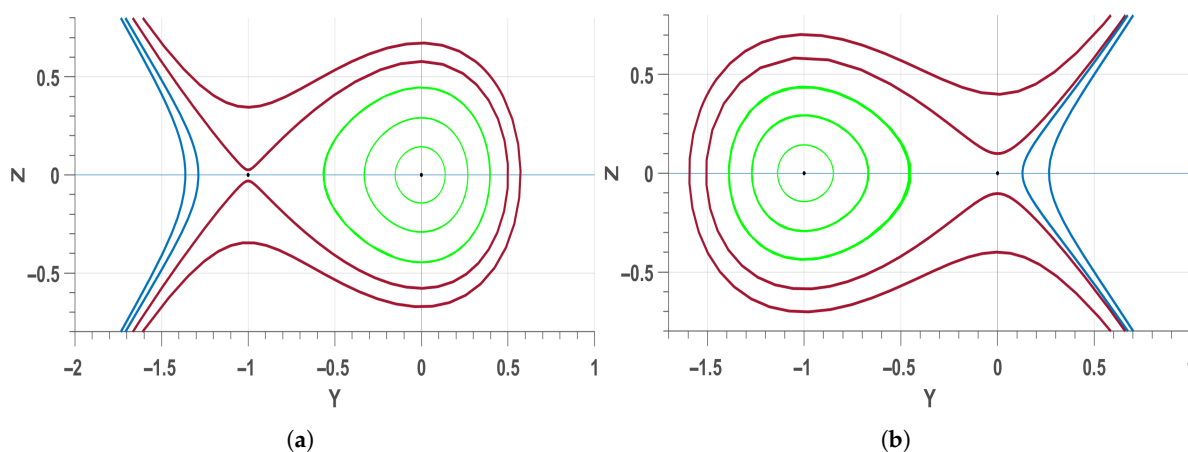
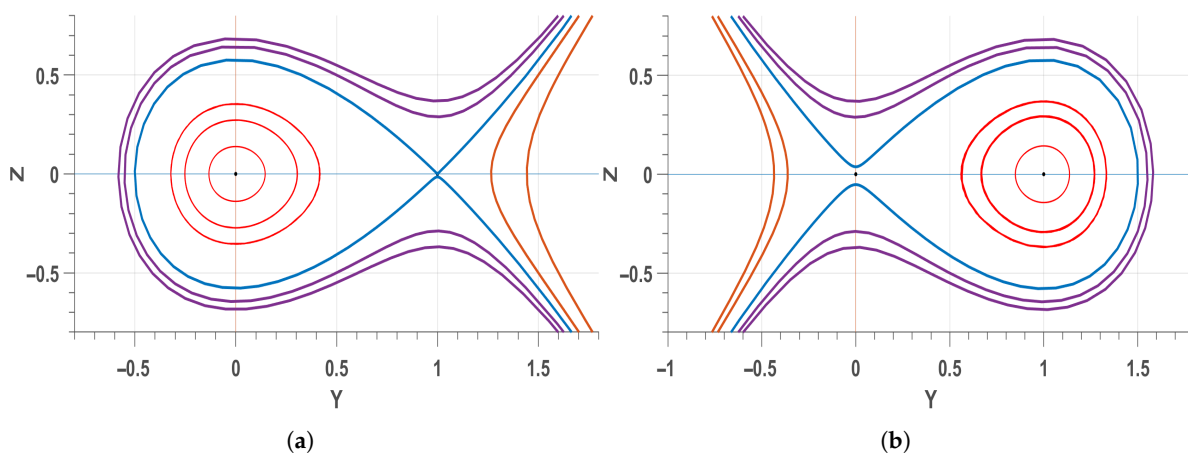
Table 2. Classification of Equilibrium Points.

Condition	Jacobian Determinant	Eigenvalues $\rho_{1,2}$	Type of Equilibrium
$\delta_1 + 2\delta_2 Y_e > 0$	$ \mathfrak{J}(Y, Z) > 0$	$\kappa_{\pm} = \pm i\sqrt{\delta_1 + 2\delta_2 Y_e}$	Center
$\delta_1 + 2\delta_2 Y_e < 0$	$ \mathfrak{J}(Y, Z) < 0$	$\kappa_{\pm} = \pm \sqrt{-(\delta_1 + 2\delta_2 Y_e)}$	Saddle
$\delta_1 + 2\delta_2 Y_e = 0$	$ \mathfrak{J}(Y, Z) = 0$	$\kappa_{\pm} = 0$	Degenerate (Neutral)

Adjusting the parameters as $a_2 = a_3 = p = q = m = 1, \beta = -1$, the corresponding results are derived and displayed below in Table 3 and Figures 6 and 7.

Table 3. Overview of equilibrium points, and categorizations for various scenarios.

Case	Conditions	Parameters	Critical Points
1	$\delta_1 > 0, \delta_2 > 0$	$a_1 = 3, \mu = 1, \gamma = \frac{1}{27}$	$(0, 0)$: Center, $(-1, 0)$: Saddle
2	$\delta_1 < 0, \delta_2 < 0$	$a_1 = -3, \mu = \frac{-1}{3}, \gamma = \frac{1}{81}$	$(0, 0)$: Saddle, $(-1, 0)$: Center
3	$\delta_1 > 0, \delta_2 < 0$	$a_1 = -3, \mu = \frac{-1}{3}, \gamma = -\frac{1}{81}$	$(0, 0)$: Center, $(1, 0)$: Saddle
4	$\delta_1 < 0, \delta_2 > 0$	$a_1 = 3, \mu = 1, \gamma = -\frac{1}{27}$	$(0, 0)$: Saddle, $(1, 0)$: Center

**Figure 6.** Phase plots for the system Equation (28), when (a) $\delta_1 > 0$, and $\delta_2 > 0$, (b) $\delta_1 < 0$, and $\delta_2 < 0$.**Figure 7.** Phase plots for the system Equation (28), when (a) $\delta_1 > 0$, and $\delta_2 < 0$, (b) $\delta_1 < 0$, and $\delta_2 > 0$.

6. Chaotic Phenomena

Chaotic dynamics arise when deterministic systems exhibit irregular behavior. To study this in Equation (1), the system Equation (28) is modified by adding a periodic forcing term, $A_0 \cos(\rho \zeta)$, as follows:

$$\begin{cases} \frac{dY}{d\zeta} = Z, \\ \frac{dZ}{d\zeta} = -\delta_1 Y - \delta_2 Y^2 + A_0 \cos(\mathcal{N}), \\ \frac{d\mathcal{N}}{d\zeta} = \rho, \quad \text{where } \mathcal{N} = \rho \zeta. \end{cases} \quad (31)$$

In this study, chaotic dynamics were detected using several well-established methods from previous research [31]. A summary of the main techniques used in the analysis is provided below.

6.1. Phase Space Analysis and Poincaré Section Mapping

A phase plot offers a 3D view of the system's phase space, showing how state variables evolve over time. It helps identify various motion types, including chaotic behavior, as summarized in Table 4. Figures 8 and 9 present the phase plots and time series of system Equation (31) for different values of A_0 , ρ , and other parameters.

To further examine irregular dynamics, Poincaré sections are used. These provide discrete snapshots of the phase space, where periodic motion appears as orderly points, while chaotic motion forms scattered, irregular patterns. In Figure 8b, the red closed curve signifies quasi-periodic behavior, while the red scattered points in Figure 9b confirm the emergence of chaos.

Table 4. Classification of Motion Types Based on Phase Portrait Features.

Type of Motion	Phase Portrait Characteristic
Periodic	Single Point
Quasi-Periodic	Closed Curve
Chaotic	Scattered Points

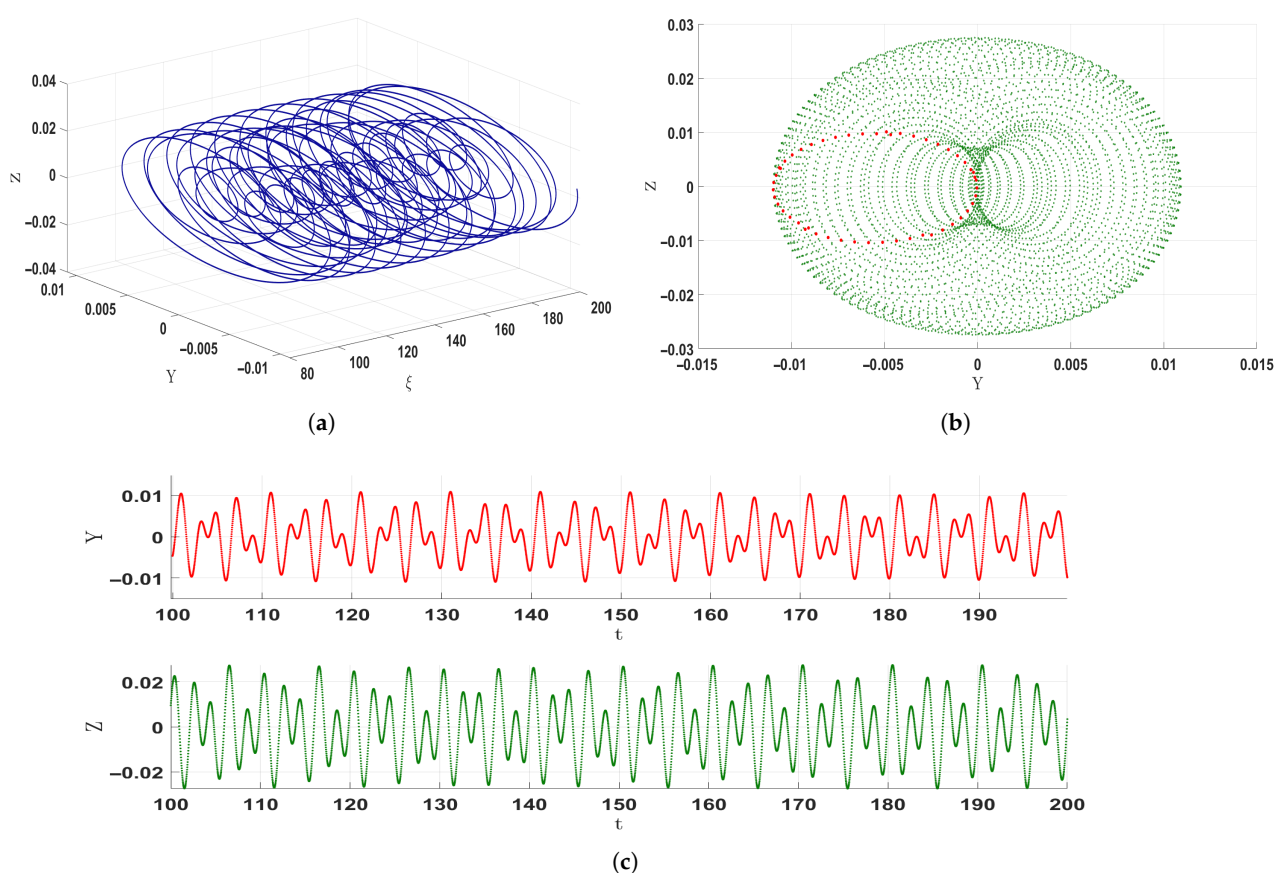


Figure 8. Visualization of chaotic dynamics in system Equation (31): (a) three-dimensional phase portrait, (b) two-dimensional projection with the Poincaré section presented in red, and (c) time series graph. The analysis is based on the parameter set: $\delta_1 = 3.5$, $\delta_2 = 1.15$, $A_0 = 0.035$, and $\rho = \pi$.

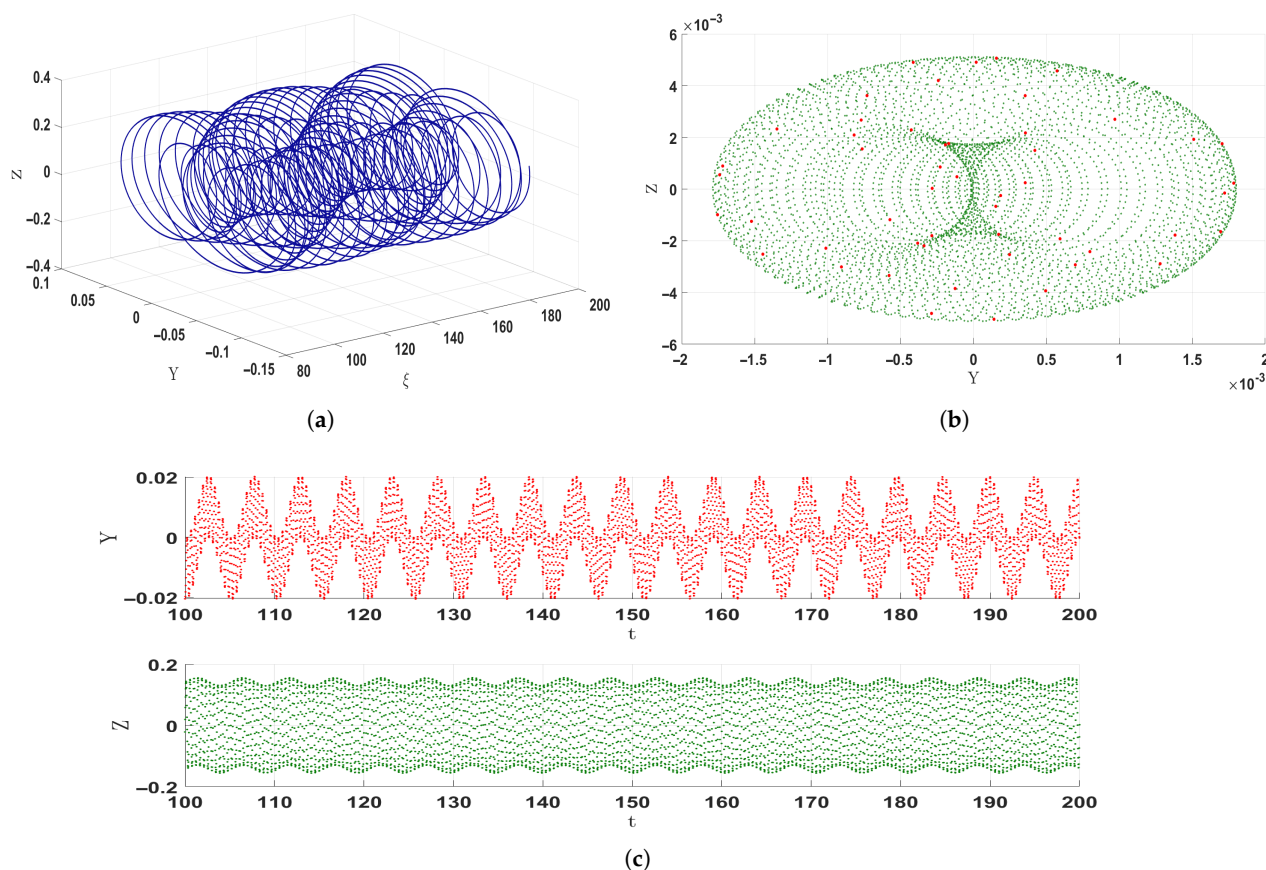


Figure 9. Visualization of chaotic dynamics in system Equation (31): (a) three-dimensional phase portrait, (b) two-dimensional projection with the Poincaré section presented in red, and (c) time series graph. The analysis is based on the parameter set: $\delta_1 = 3.5$, $\delta_2 = 1.15$, $A_0 = 0.3$, and $\rho = 2\pi$.

6.2. Multistability

Multistability refers to the presence of various stable states that can exist simultaneously in a nonlinear dynamical system, depending on the initial conditions, even when the system parameters remain unchanged. In the current analysis, four return maps are used to investigate this phenomenon by tracking system trajectories from three different initial states: $(Y, Z, \mathcal{N}) = (0.1, 0.01, 0)$ (green), $(Y, Z, \mathcal{N}) = (0.2, 0.02, 0)$ (red), and $(Y, Z, \mathcal{N}) = (0.3, 0.03, 0)$ (cyan). The parameters A_0 and ρ are varied for each subplot, as specified below the corresponding diagrams, to highlight how initial conditions influence the system's dynamic response.

The green, red, and cyan trajectories in Figure 10a exhibit periodic motion, creating smooth closed loops that signify steady oscillations. The motion becomes quasi-periodic in Figure 10b, indicating increased sensitivity to initial conditions and parameters. Figure 10c,d show chaotic behavior, where the unpredictable nature of chaos is revealed by overlapping and densely packed paths. These findings demonstrate the rich multistability of nonlinear dynamics by demonstrating how slight modifications to initial values or parameters can change the system's evolution.

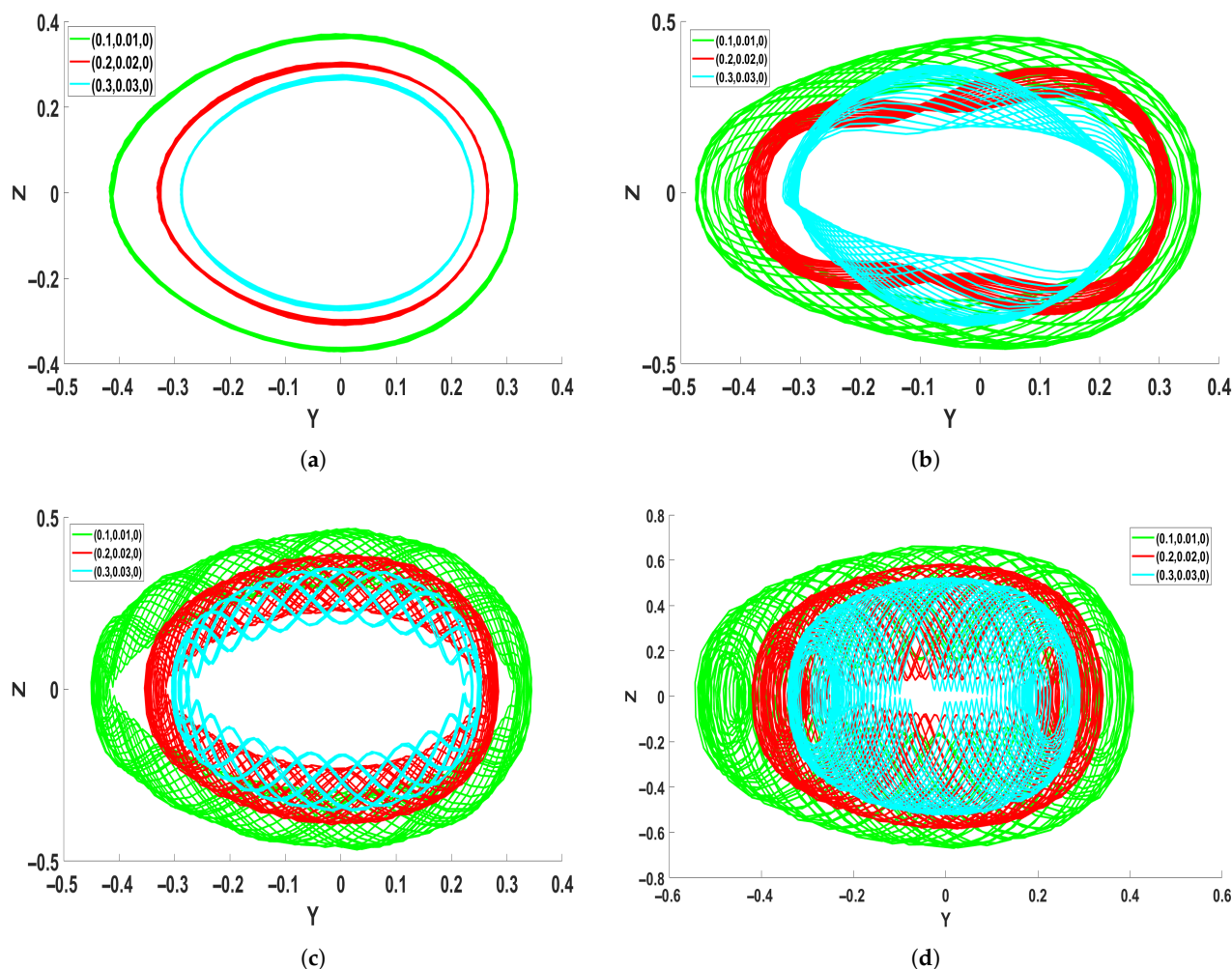


Figure 10. Examination of chaotic behavior in system Equation (31) via multistability diagrams, employing the parameters $\delta_1 = 1.1$ and $\delta_2 = 1.5$. (a) $A_0 = 0.01$ and $\rho = 0.1$. (b) $A_0 = 0.1$ and $\rho = 1.5$. (c) $A_0 = 0.5$ and $\rho = \pi$. (d) $A_0 = 1.5$ and $\rho = 2\pi$.

6.3. Iterative Return Map

A basic analytical method for examining how time delays affect system dynamics is the return map. One can learn a lot about the system's temporal evolution by graphing the values of $Y(\xi)$ against $Y(\xi + \tau)$ for different delay values τ . Figure 11 demonstrates this relationship for four selected delay values: $\tau = 0.1, 1.4, 1.8$, and 2.4 , with system parameters held constant at $(A_0, \rho) = (0.03, \pi)$ and initial conditions specified as $(Y, Z, \mathcal{N}) = (0.01, 0.02, 0.03)$.

As the delay parameter τ is raised, the system makes an evident transition into chaotic behavior from periodic behavior. When $\tau = 1.4$, very minor irregularities start appearing on the return map, showing the start of nonlinear effects and small deviations from periodic motion. When $\tau = 1.8$, the trajectories become more entangled, showing stronger nonlinearities and higher sensitivity to initial conditions, which lead to the formation of chaotic tendencies. At $\tau = 2.4$, the return map is very densely filled with overlapping and irregular trajectories, typical of a completely chaotic regime in which the system is highly sensitive to initial conditions and has fractal-like properties. In general, smaller values of τ preserve stability, while larger delays increase instability and complexity, propelling the system toward complete chaos.

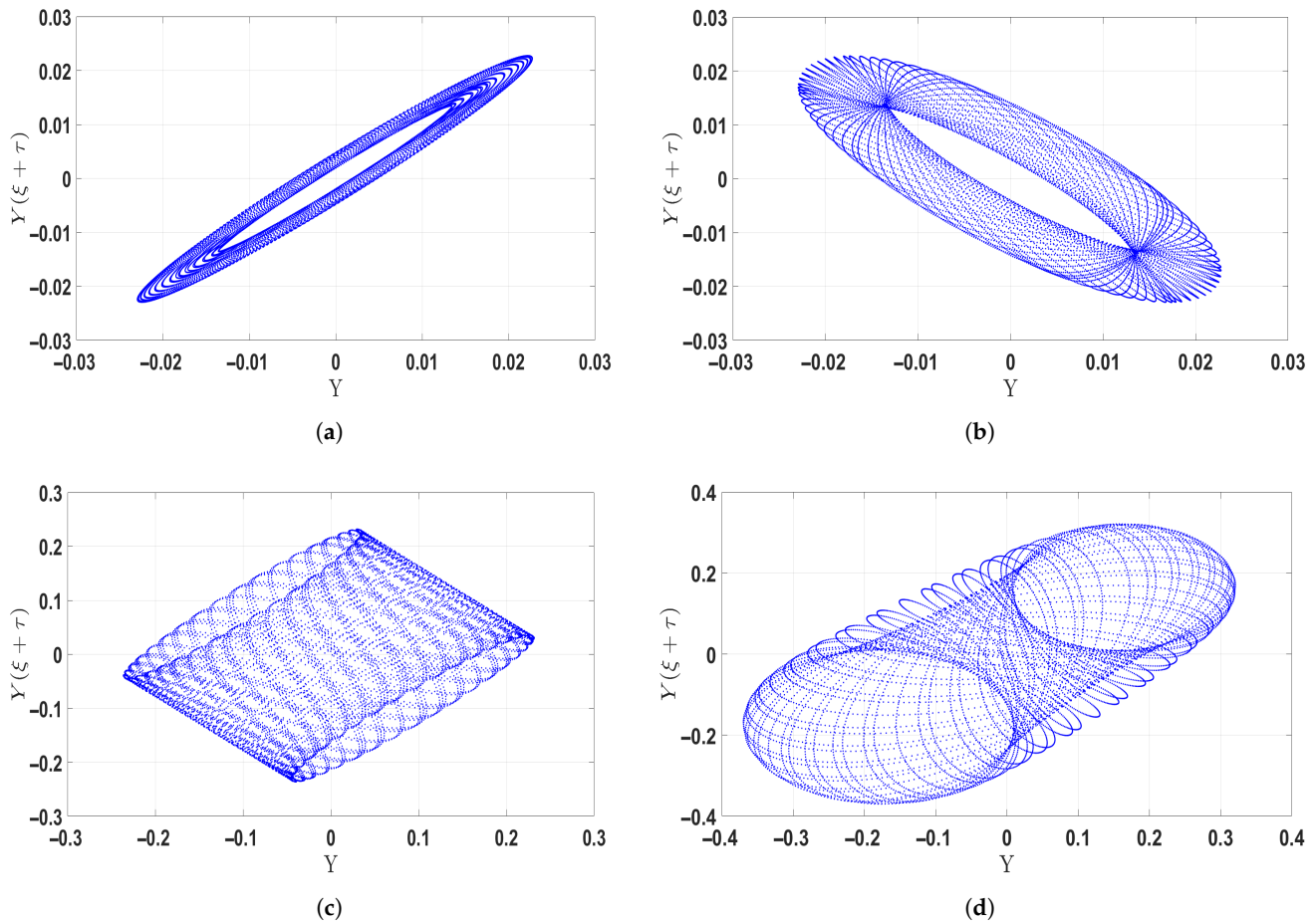


Figure 11. Return map analysis illustrating chaotic dynamics in system Equation (31) for parameter settings $\delta_1 = 3.5$ and $\delta_2 = 1.15$. (a) For $\tau = 0.1$ (b) For $\tau = 1.4$. (c) For $\tau = 1.8$. (d) For $\tau = 2.4$.

6.4. Lyapunov Exponent

Lyapunov exponents (LEs) measure how quickly nearby trajectories in a system diverge, revealing its chaotic behavior. In this study, LEs were calculated using the Wolf algorithm with Gram-Schmidt orthogonalization for precision. They indicate system stability and sensitivity to initial conditions—where a positive LE confirms chaos. The following formulation defines the dynamical system used for calculating the LEs:

$$\frac{dY}{d\xi} = Z = \mathcal{P}, \quad \frac{dZ}{d\xi} = -\delta_1 Y - \delta_2 Y^2 + \mathbb{A}_0 \cos(\mathcal{N}) = \mathcal{Q}, \quad \frac{d\mathcal{N}}{d\xi} = \rho = \mathcal{R}. \quad (32)$$

The Jacobian matrix takes the form:

$$\mathfrak{J} = \begin{bmatrix} \frac{\partial \mathcal{P}}{\partial Y} & \frac{\partial \mathcal{P}}{\partial Z} & \frac{\partial \mathcal{P}}{\partial \mathcal{N}} \\ \frac{\partial \mathcal{Q}}{\partial Y} & \frac{\partial \mathcal{Q}}{\partial Z} & \frac{\partial \mathcal{Q}}{\partial \mathcal{N}} \\ \frac{\partial \mathcal{R}}{\partial Y} & \frac{\partial \mathcal{R}}{\partial Z} & \frac{\partial \mathcal{R}}{\partial \mathcal{N}} \end{bmatrix}$$

The Jacobian matrix associated with the system is given by:

$$\mathfrak{J} = \begin{bmatrix} 0 & 1 & 0 \\ -\delta_1 - 2\delta_2 Y & 0 & -\mathbb{A}_0 \sin \mathcal{N} \\ 0 & 0 & 0 \end{bmatrix}$$

Subsequently, κI is subtracted from Jacobian, where I is the identity matrix and κ stands for the eigenvalue.

$$\mathfrak{J} - \kappa I = \begin{bmatrix} -\kappa & 1 & 0 \\ -\delta_1 - 2\delta_2 Y & -\kappa & -\mathbb{A}_0 \sin \mathcal{N} \\ 0 & 0 & -\kappa \end{bmatrix}$$

By evaluating the determinant $|\mathfrak{J} - \kappa I|$, the result is as follows:

$$\kappa_1 = \sqrt{-\delta_1 - 2\delta_2 Y}, \quad \kappa_2 = 0, \quad \kappa_3 = -\sqrt{-\delta_1 - 2\delta_2 Y}. \quad (33)$$

Since two eigenvalues of the system have equal magnitudes but opposite signs, their corresponding LEs show mirrored behavior. As κ_2 remains zero, the analysis focuses on κ_1 and κ_3 . Figure 12 shows the time evolution of the LEs for two parameter sets under identical initial conditions.

For $(\delta_1, \delta_2) = (1, -2)$ and the initial state $(1, 1.2, 1.5)$, Figure 12a illustrates that when $(\mathbb{A}_0, \rho) = (0.03, 2)$, κ_1 and κ_3 fluctuate near zero before diverging— κ_1 stabilizes at a positive value while κ_3 settles at an equal negative one. In the second case, with $(\mathbb{A}_0, \rho) = (0.5, \pi)$, Figure 12b shows that both exponents initially oscillate but later converge in opposite directions: one positive, one negative. The coexistence of one positive, one negative, and one zero LE confirms the presence of chaos in the system.

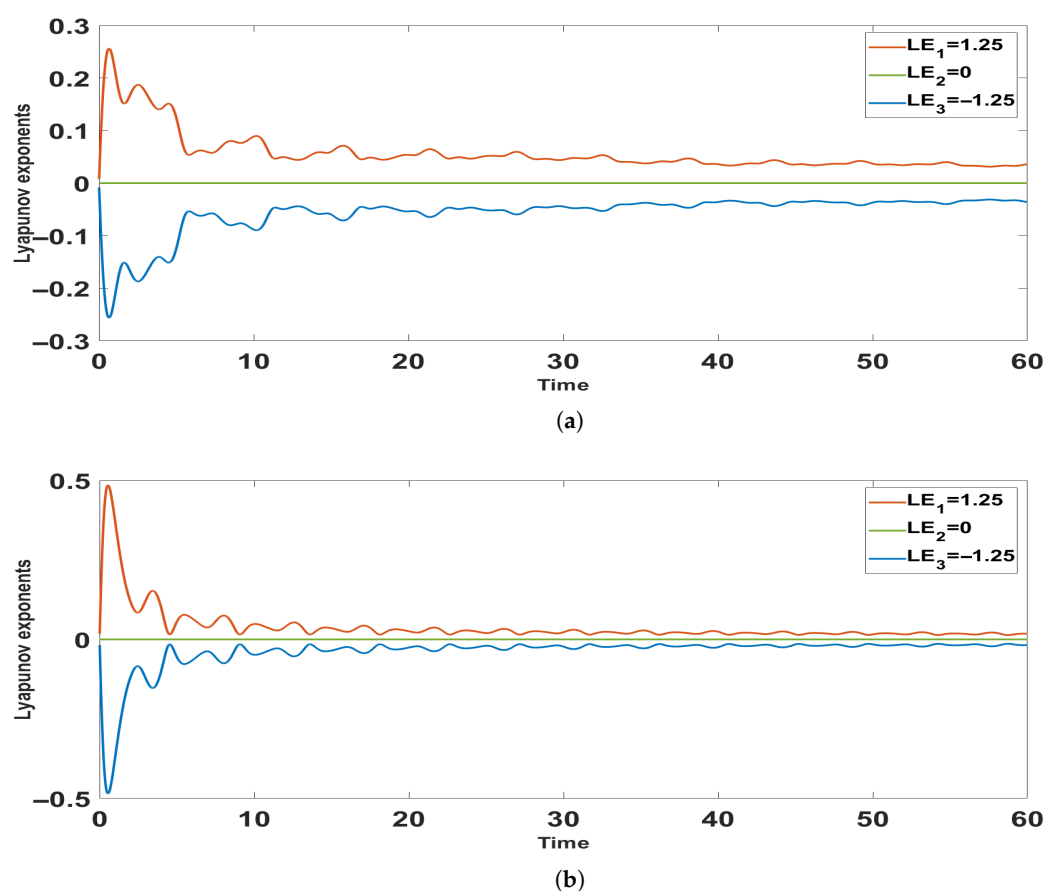


Figure 12. Investigation of chaotic behavior in system Equation (31) using Lyapunov exponents under two parameter configurations: (a) $(\mathbb{A}_0, \rho) = (0.03, 2)$ and (b) $(\mathbb{A}_0, \rho) = (0.5, \pi)$.

6.5. Strange Attractors

The chaotic attractors generated by the planar system for various initial values $(Y, Z, \mathcal{N})$ and parameters $(\mathbb{A}_0, \rho)$ are shown in Figure 13. Each subplot highlights the

system's response to initial conditions, showcasing a variety of intricate, constrained, and chaotic paths.

Figure 13a depicts strange attractors for $(\mathbb{A}_0, \rho) = (0.05, 0.5)$ with initial condition $(Y, Z, \mathcal{N}) = (0.1, 0, 0.03)$, exhibiting a multi-loop structure indicative of quasi-periodic behavior. However, Figure 13b, with $(Y, Z, \mathcal{N}) = (0.2, -0.2, -1.3)$ and adjusted parameters $(\mathbb{A}_0, \rho) = (1.5, 2\pi)$, displays attractors with closely packed loops and overlapping trajectories, emphasizing the high degree of unpredictability and chaotic dynamics. The complex attractor geometry highlights how sensitive the system is to initial values, leading to nonlinear behavior.

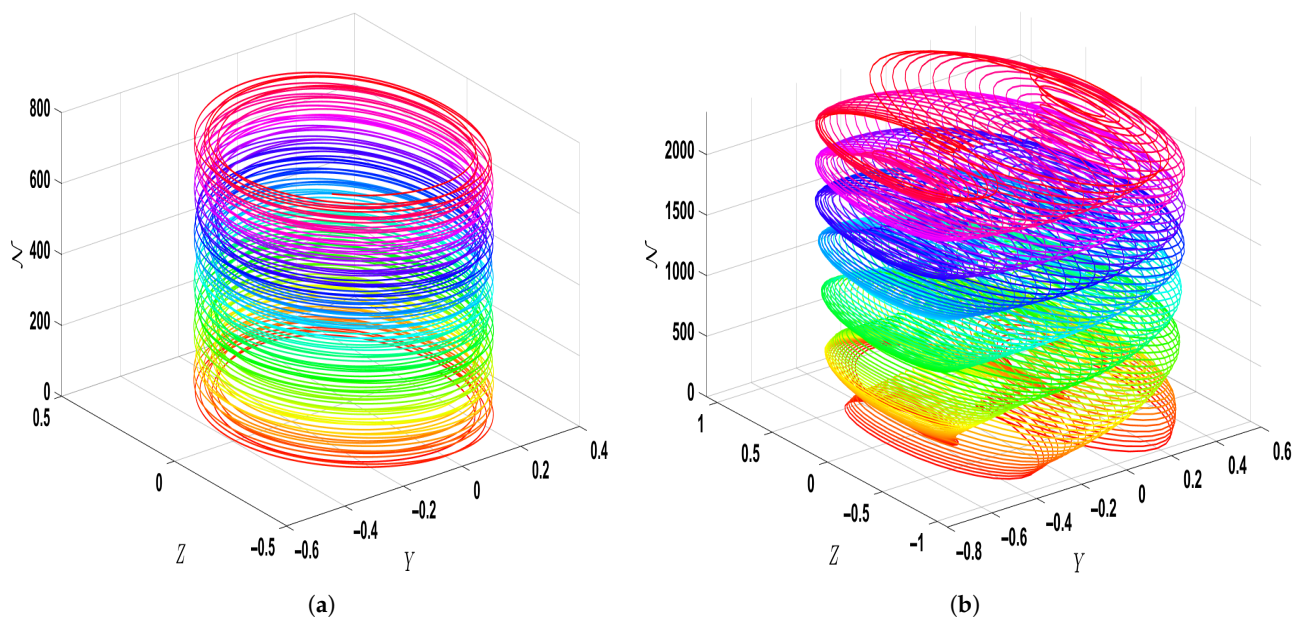


Figure 13. Strange attractors illustrating chaotic dynamics in system Equation (31) for parameter settings $\delta_1 = 1.15$ and $\delta_2 = 1.5$. (a) $(\mathbb{A}_0, \rho) = (0.05, 0.5)$ and (b) $(\mathbb{A}_0, \rho) = (1.5, 2\pi)$.

6.6. Power Spectrum

Power spectrum analysis gives important information about how energy is distributed in terms of frequency within the system and aids in recognizing chaotic dynamics. An irregular, wide spectrum shows an aperiodic and sensitive motion characteristic of chaos, whereas narrow peaks represent periodic motion. Placing the axes logarithmically, the analysis shows how initial conditions $(Y, Z, \mathcal{N})$ under the parameter set $(\mathbb{A}_0, \rho, \delta_1, \delta_2) = (0.05, 0.5, 1.15, 1.5)$ influence the resulting energy distribution. Figure 14a plots two initial conditions, $(Y, Z, \mathcal{N}) = (0.1, 0, 0.03)$ (blue) and $(Y, Z, \mathcal{N}) = (0.1, 0, 0.3)$ (orange), the only difference being in $\mathcal{N}$. At low frequencies (10^{-2} – 10^{-1}), the two curves both reflect weak power, with the blue one slightly higher. In the mid-frequency range (10^{-1} – 10^0), both oscillate, with the blue curve reflecting stronger peaks. At higher frequencies (10^0 – 10^2), power decreases gradually for both cases.

Also, Figure 14b shows the spectra for $(Y, Z, \mathcal{N}) = (0.2, 0, 0.03)$ (orange) and $(Y, Z, \mathcal{N}) = (0.5, 0, 0.03)$ (blue), with only Y changed. Both have low power at low frequency, with the blue one varying slightly more. Both have mid-frequency fluctuations and then a smooth decline in power at higher frequencies.

In general, the continuous reduction of power with increasing frequency and the broadband, aperiodic character of the spectra verify the occurrence of chaos. Energy is spread out in a broad range of frequency, and this is an indicator of strong sensitivity to initial conditions and the simultaneous existence of several dynamic states.

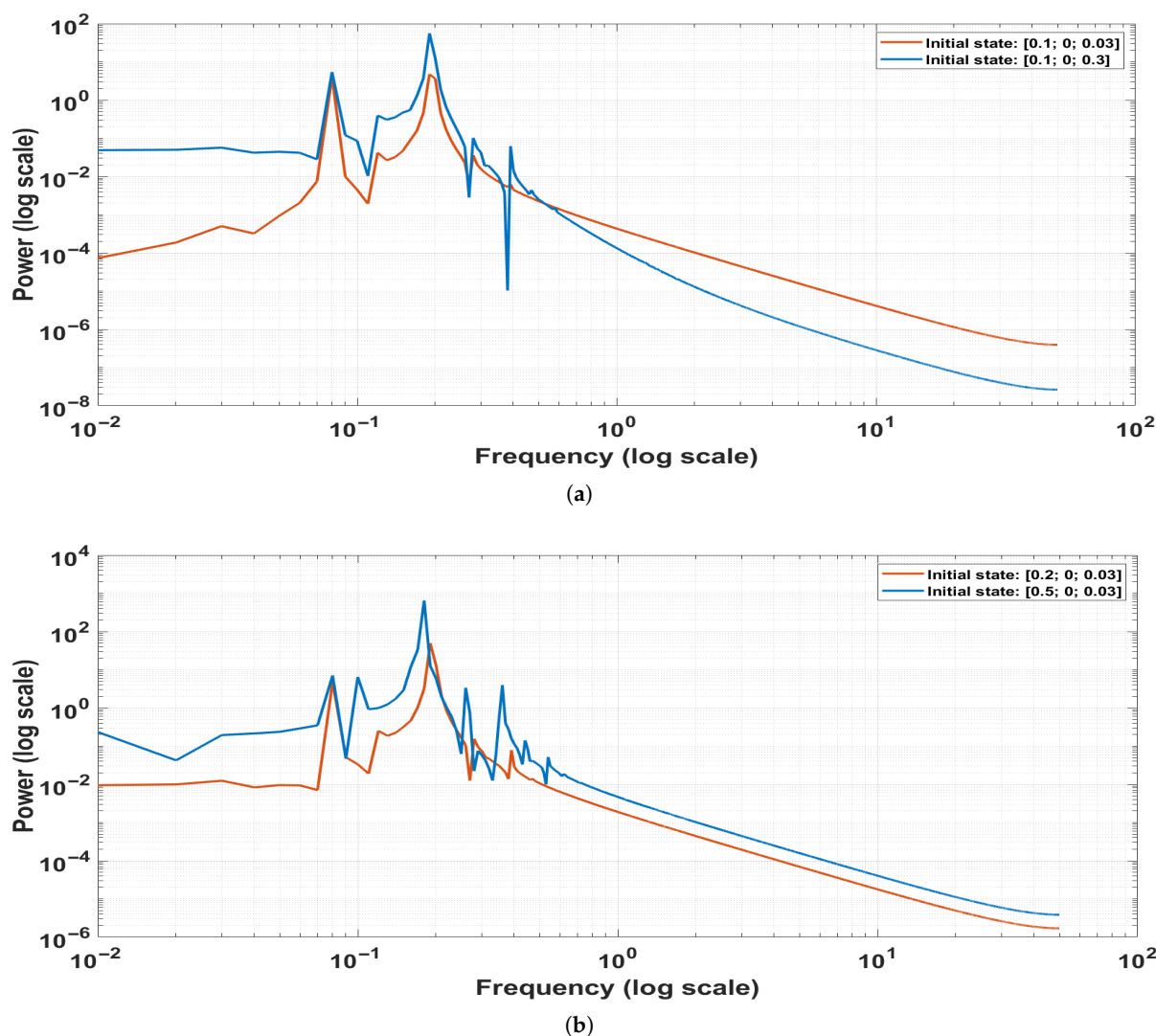


Figure 14. Power spectrum analysis illustrating chaotic dynamics in system Equation (31) for parameter settings $\delta_1 = 1.15$, $\delta_2 = 1.5$, $A_0 = 0.05$, and $\rho = 0.5$. (a) $(Y, Z, \mathcal{N}) = (0.1, 0, 0.03)$ (blue) and $(Y, Z, \mathcal{N}) = (0.1, 0, 0.3)$ (orange) and (b) $(Y, Z, \mathcal{N}) = (0.2, 0, 0.03)$ (orange) and $(Y, Z, \mathcal{N}) = (0.5, 0, 0.03)$ (blue).

6.7. Bifurcation Diagram

A bifurcation diagram shows how the dynamics of a dynamical system evolve as a control parameter is changed. In a perturbed plane system, the parameter usually denotes the magnitude of the perturbation. By looking at the bifurcation diagram, various dynamical regimes like steady states, periodic oscillations (limit cycles), and chaotic motion can be determined. Figure 15 presents the bifurcation diagram of the perturbed system Equation (31), plotted as Y versus Z with the parameters $\delta_1 = 1.15$, $\delta_2 = 1.5$, $A_0 = 1.5$, and $\rho = 2\pi$. The diagram exhibits irregular and complex patterns, indicating the presence of chaotic dynamics.

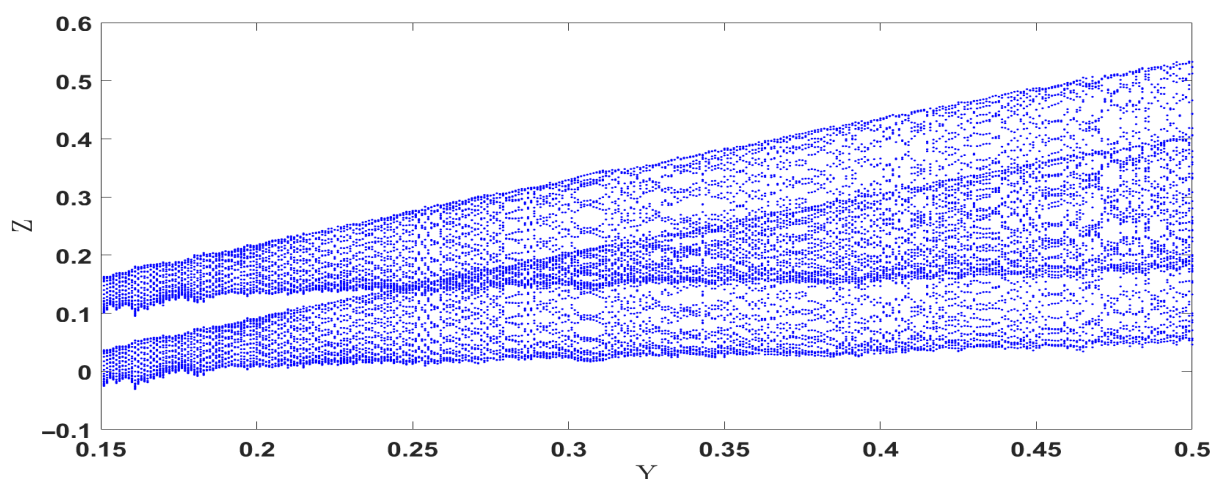


Figure 15. Investigation of chaotic behavior in system Equation (31) using bifurcation diagram.

7. Analysis of Solution Overlaps

Using data points, this section plots the behaviors of different wave solutions against one another to show how they interact and share traits. The information tells us where the solutions start to diverge and where they possess similar features. Using specific parameter sets and selected temporal and spatial intervals, four soliton solutions— ϕ_6 , ϕ_7 , ϕ_8 , and ϕ_{18} —are examined. Different data points are produced for each scenario by carefully adjusting these parameters in addition to the coordinates (x, y, z, t) . The main goal is to investigate whether solutions with different spatial distributions behave similarly when given particular parameter values.

- Solution 1:

$$\phi_6(x, y) = \frac{-12\ln(v)K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} \tanh_v \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) + 2, \quad (34)$$

with parameters $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3, \gamma = 2, t = 0.5, z = 2$.

- Solution 2:

$$\phi_7(x, y) = \frac{-12\ln(v)K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} \coth_v \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) + 2, \quad (35)$$

with parameters $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3, \gamma = 2, t = 0.5, z = 2$.

- Solution 3:

$$\phi_8(x, z) = \frac{-12\ln(v)K_3}{K_2} \left(\frac{-c_2}{2} - \frac{\sqrt{\Delta}}{2} (\tanh_v(\sqrt{\Delta}\xi) \pm i\sqrt{rs} \operatorname{sech}_v(\sqrt{\Delta}\xi)) \right) + 2, \quad (36)$$

with parameters $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3, \gamma = 2, y = 1, t = 1$.

- Solution 4:

$$\phi_{18}(x, z) = \frac{-12c_3\ln(v)K_3}{K_2} \left(-\sqrt{\frac{-c_1}{c_3}} (\tanh_v(2\sqrt{-c_3c_1}\xi) \pm i\sqrt{rs} \operatorname{sech}_v(2\sqrt{-c_3c_1}\xi)) \right) + 2, \quad (37)$$

with parameters $a_1 = 1, a_2 = 2, a_3 = 4, \alpha = 2, \beta = 3, \gamma = 2, y = 1, t = 1$.

The spatial variables are defined within the ranges $x \in [2, 4]$, $y \in [1, 4]$, and $z \in [-1.5, 1.5]$, with step sizes $x = 0.6$, $y = 0.4$, and $z = 0.3$. These pairs produce a set of data points for each solution, enabling comparison of their overlapping and non-

overlapping behaviors in different spatial regions. To display such comparisons, bidirectional scatter plots are given in Figure 16a,b, illustrating how the soliton solutions interact and differ in different spatial–temporal domains.

- Figure 16a: compares $\phi_6(x, y)$ and $\phi_7(x, y)$, both with the same parameters. Magenta points indicate ϕ_6 , and green points indicate ϕ_7 . There seems to be strong overlap in the range $x \in [0, 2]$ and $y \in [0.5, 2.5]$, where they have the same amplitude patterns. Outside this range, their behavior differs because of the different hyperbolic components in their representations— $\tanh$ for ϕ_6 and $\coth$ for ϕ_7 . Therefore, ϕ_7 penetrates deeper into areas where ϕ_6 rapidly falls.
- Figure 16b: indicates the comparison of $\phi_8(x, z)$ (red) and $\phi_{18}(x, z)$ (blue). The two solutions overlap near $x \in [-0.5, 0.5]$ and $z \in [-0.3, 0.3]$, exhibiting similar distributions. Outside this region, because of their different functional structures— ϕ_8 consists of $\tanh$ and sech terms scaled by $\sqrt{\Delta}$, whereas ϕ_{18} consists of a more involved root form with c_1 and c_3 —the two solutions deviate. Consequently, ϕ_{18} extends further along the negative x -axis, while ϕ_8 is clustered in the positive realm.

In total, these results demonstrate how even structurally dissimilar solutions might share transitional commonalities in particular regions. Outside of these overlap regions, however, each solution has a distinctive dynamic behavior, determined by its mathematical nature and parameters.

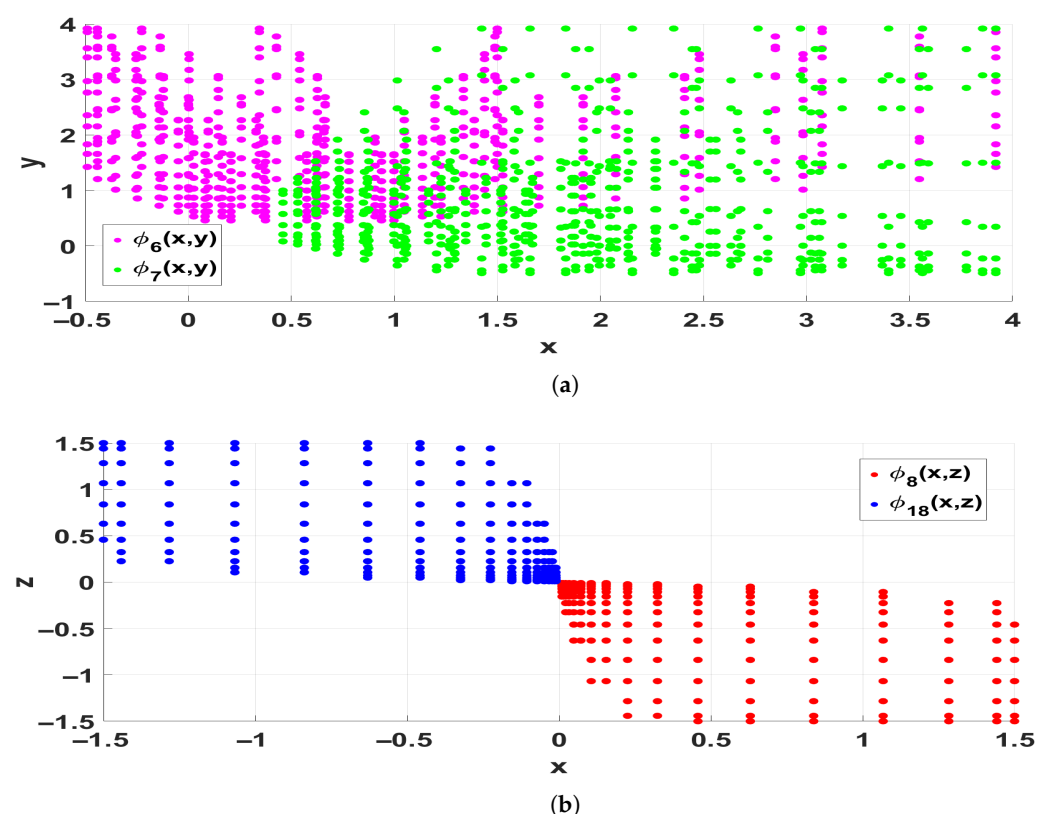


Figure 16. Comparison of soliton solution behaviors based on data point distribution: (a) Scatter plot comparing $\phi_6(x, y, z, t)$ and $\phi_7(x, y, z, t)$ in xy -plane. (b) Scatter plot comparing $\phi_8(x, y, z, t)$ and $\phi_{18}(x, y, z, t)$ in xz -plane.

8. Conclusions

The $(3 + 1)$ -dimensional KdV–CBS equation, a nonlinear model that describes long-wave interactions with applications in engineering and physics, is examined in this work. The modified Khater method (MKM) was used to derive analytical soliton solutions af-

ter the equation was reduced to a lower-dimensional form using an appropriate wave transformation. Plots in three dimensions, two dimensions, and contours (Figures 2–5) show different nonlinear structures, such as chirped, periodic, kink, and kink–bright solitons, with symmetric amplitude patterns that are in line with the inherent symmetry of the system.

Transitions between stable and unstable regimes were confirmed by bifurcation analysis around equilibrium points (Figures 6–7). The RK4 scheme and several nonlinear diagnostic tools, such as phase portraits, Poincaré sections, time series, chaotic attractors, multistability plots, iterative return maps, bifurcation diagrams, power spectra, and Lyapunov exponents, were used to analyze the system’s irregular and chaotic dynamics under external perturbations (Figures 8–15). A bidirectional scatter plot approach (Figure 16) was used to quantitatively highlight the overlaps and differences between soliton profiles and to further compare them.

Providing a unified analytical–numerical framework for investigating both stable and chaotic behaviors in nonlinear multidimensional systems was the primary driving force behind the work. The MKM offers greater flexibility than traditional methods and produces compact, accurate, and physically meaningful solutions that directly satisfy the original equation. Overall, this work advances our understanding of complex dynamics, energy localization, and nonlinear wave propagation by making methodological and conceptual contributions to soliton theory and nonlinear dynamics.

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