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Rough Intuitionistic Fuzzy Filters in BE-Algebras: Applications in Artificial Intelligence and Medical Diagnosis

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Abstract

This paper proposes a theoretical framework for studying rough intuitionistic fuzzy filters within the structure of BE-algebras. Building on rough set theory and intuitionistic fuzzy set theory, we introduce rough intuitionistic fuzzy filters via lower and upper approximation operators induced by congruence relations. To further generalize the framework, we define set-valued homomorphisms on BE-algebras and use them to formulate Γ -rough intuitionistic fuzzy filters. Several structural properties and characterization results are established, including stability under approximation operators, relationships with classical intuitionistic fuzzy filters, and preservation under homomorphic mappings. The proposed approach provides an algebraic mechanism for modeling uncertainty, hesitation, and imprecision in implication-based systems, with potential relevance to uncertainty-aware reasoning in artificial intelligence, decision-support systems, and medical diagnosis.

Keywords: fuzzy set; rough set; ζ -rough set; intuitionistic fuzzy set; lower approximation; upper approximation; fuzzy ideal; rough intuitionistic fuzzy filter; BE-algebras; artificial intelligence; medical diagnosis; decision-support systems; cybersecurity

1. Introduction

In the development of set theory, two fundamental and influential paradigms emerged for modeling uncertainty: rough set theory, introduced by Pawlak in 1982 [1], and fuzzy set theory, proposed by Zadeh in 1965 [2]. These theories represent significant departures from classical set theory by addressing the limitations of crisp membership when dealing with imprecise, vague, or incomplete information. Their integration has led to powerful mathematical frameworks with broad applicability in artificial intelligence (AI), decision-making processes, and algebraic structures [3–6].

Rough set theory provides a formal mechanism for handling uncertainty arising from indiscernibility or incomplete knowledge. By introducing the concepts of lower and upper approximations, it enables the classification and analysis of data that cannot be precisely characterized. This approach has proven particularly effective in AI-related domains such as machine learning, data mining, and pattern recognition, where uncertainty and noise are inherent challenges [7–9].

In contrast, fuzzy set theory offers a complementary perspective on uncertainty by allowing elements to belong to a set with varying degrees of membership. Rather than enforcing binary inclusion, fuzzy sets employ membership functions that capture gradual transitions and partial truth values. This flexibility makes fuzzy models especially suitable for AI-based decision-making systems that must operate under vagueness and ambiguity [10–12].



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Building upon these foundational theories, several important extensions have been developed. Atanassov's intuitionistic fuzzy (IF) sets [3,4] extend fuzzy sets by incorporating both membership and non-membership degrees, along with a hesitation margin that explicitly represents uncertainty. This enriched structure has enhanced the modeling of real-world decision-making problems, particularly in AI environments where incomplete or conflicting information is common.

Further advancements include Davvaz's investigations into approximation theory in hyperrings [5,13] and generalized lower and upper approximations in rings [14], which significantly broadened the algebraic scope of rough set theory. Additionally, the work of Hosseini et al. on T-rough (prime and primary) ideals in commutative rings [15] has contributed to a deeper understanding of rough approximations within algebraic systems, highlighting their structural and theoretical importance.

The interdisciplinary applications of rough and fuzzy set theories extend across a wide range of fields, including intelligent systems, expert systems, pattern recognition, decision analysis, and medical informatics. In particular, rough intuitionistic fuzzy models have demonstrated effectiveness in improving robustness, interpretability, and reliability in AI applications such as natural language processing, recommendation systems, cybersecurity, and healthcare decision-making [16–18]. Moreover, intuitionistic fuzzy models have been successfully integrated into AI-oriented decision-support frameworks, leading to enhanced performance in classification, clustering, and predictive modeling tasks [6,9,19].

A notable milestone in this area was the introduction of T-rough sets [5], which extended classical rough set theory by enabling knowledge representation through structured mappings. This development facilitated more effective knowledge discovery in complex and structured datasets. The synergy between rough set theory and fuzzy logic underscores their complementary strengths: approximate reasoning grounded in rough sets plays a vital role in data analysis and knowledge extraction, while fuzzy logic has been extensively applied in control systems and decision-support environments [20–22].

Within this broad theoretical landscape, BE-algebras provide a natural and meaningful algebraic framework for studying implication-based reasoning and filter structures. Filters in BE-algebras are closely related to logical inference and consistency, making them particularly suitable for uncertainty modeling. Although rough filters and fuzzy structures have been investigated in various algebraic systems, a systematic study of rough intuitionistic fuzzy filters in BE-algebras, especially via generalized approximation mechanisms and set-valued homomorphisms, remains relatively limited.

Motivated by this observation, the present paper investigates the interplay between rough set theory, intuitionistic fuzzy logic, and BE-algebras. In contrast to existing works on rough fuzzy filters in rings, semigroups, and related algebraic structures, this study focuses on BE-algebras and develops a unified framework for rough intuitionistic fuzzy filters through generalized approximation operators induced by set-valued homomorphisms. To the best of our knowledge, such an approach has not been systematically explored in the existing literature.

The main contributions of this paper are summarized as follows:

- We introduce the concept of rough intuitionistic fuzzy filters in BE-algebras and investigate their fundamental properties;
- We study the relationships between classical intuitionistic fuzzy filters and their rough counterparts under generalized approximation operators;
- We define Γ -rough intuitionistic fuzzy filters via set-valued homomorphisms and establish preservation and characterization results;

- We discuss the relevance of the proposed framework to AI-oriented decision-support systems and medical diagnosis, highlighting its potential for handling uncertainty and imprecision.

The remainder of the paper is organized as follows. Section 2 recalls the necessary preliminaries on BE-algebras and intuitionistic fuzzy sets. Section 3 is devoted to the study of rough intuitionistic fuzzy filters in BE-algebras. Section 4 introduces set-valued homomorphisms and develops generalized rough approximation operators, leading to Γ -rough intuitionistic fuzzy filters. Finally, potential applications in artificial intelligence and related computational fields are discussed, followed by concluding remarks.

2. Preliminaries

This section recalls and consolidates several fundamental concepts that form the theoretical basis of the present work. We first review essential notions related to BE-algebras and their filters, which provide the algebraic framework for implication-based reasoning. We then recall the definitions of fuzzy sets and intuitionistic fuzzy sets, emphasizing their role in modeling uncertainty through graded membership and non-membership functions. Finally, we summarize key concepts from rough set theory, which enable the approximation of imprecise information and play a central role in the development of rough intuitionistic fuzzy filters in subsequent sections.

Definition 1 ([23]). A BE-algebra is a structure $(\Omega, *, 1)$ consisting of a non-empty set Ω , a binary operation $*$, and a distinguished constant element 1, satisfying the following axioms for all $p, q, r \in \Omega$:

- (1) $p * p = 1$,
- (2) $p * 1 = 1$,
- (3) $1 * p = p$,
- (4) $p * (q * r) = q * (p * r)$.

A partial order $\leq$ can be defined on Ω by

$$p \leq q \text{ if and only if } p * q = 1.$$

Definition 2 ([24]). Let Ω be a BE-algebra and let K be a non-empty subset of Ω . The set K is called a filter of Ω if it satisfies:

- (1) $1 \in K$,
- (2) for all $p, q \in \Omega$, if $p \in K$ and $p * q \in K$, then $q \in K$.

Definition 3 ([2]). A fuzzy set (FS) on a universe Ω is characterized by a membership function

$$\nu : \Omega \rightarrow [0, 1],$$

where $\nu(p)$ represents the degree to which an element $p \in \Omega$ belongs to the fuzzy set.

Definition 4 ([3]). An intuitionistic fuzzy set (IFS) K in a universe Ω is defined as

$$K = \{(p, \nu_K(p), \lambda_K(p)) \mid p \in \Omega\},$$

where $\nu_K : \Omega \rightarrow [0, 1]$ and $\lambda_K : \Omega \rightarrow [0, 1]$ denote the membership and non-membership functions, respectively, satisfying

$$0 \leq \nu_K(p) + \lambda_K(p) \leq 1 \text{ for all } p \in \Omega.$$

Definition 5 ([25]). A fuzzy subset ν of a BE-algebra Ω is called a fuzzy filter (FF) of Ω if:

- (1) $\nu(1) \geq \nu(p)$ for all $p \in \Omega$,
- (2) $\nu(q) \geq \min\{\nu(p * q), \nu(p)\}$ for all $p, q \in \Omega$.

Definition 6. An intuitionistic fuzzy subset $K = (\nu_K, \lambda_K)$ of a BE-algebra Ω is called an intuitionistic fuzzy filter (IF filter) if for all $p, q \in \Omega$:

- (1) $\nu_K(1) \geq \nu_K(p)$,
- (2) $\nu_K(q) \geq \min\{\nu_K(p * q), \nu_K(p)\}$,
- (3) $\lambda_K(1) \leq \lambda_K(p)$,
- (4) $\lambda_K(q) \leq \max\{\lambda_K(p * q), \lambda_K(p)\}$.

Example 1. Consider the BE-algebra $\Omega = \{1, a_1, b_1\}$ with the operation $*$ given by the following Cayley table:

$*$	1	a_1	b_1
1	1	a_1	b_1
a_1	1	1	1
b_1	1	1	1

Define an intuitionistic fuzzy set $K = (\nu_K, \lambda_K)$ by

$$\nu_K(p) = \begin{cases} 0.5, & p = 1 \text{ or } a_1, \\ 0.3, & p = b_1, \end{cases} \quad \lambda_K(p) = \begin{cases} 0.3, & p = 1 \text{ or } a_1, \\ 0.5, & p = b_1. \end{cases}$$

One can directly verify that K satisfies the conditions of Definition 6 and hence is an IF filter of Ω .

Definition 7 ([25]). Let $\approx$ be an equivalence relation on a BE-algebra Ω . The relation $\approx$ is called a congruence relation if for all $a, b, p, q \in \Omega$, $(a, b) \in \approx$ implies $(a * p, b * q) \in \approx$.

Proposition 1. Let $\approx$ be a covering relation on a BE-algebra Ω , and let $K = (\nu_K, \lambda_K)$ and $L = (\nu_L, \lambda_L)$ be two intuitionistic fuzzy sets in Ω . Then:

- (1) $\approx \bullet K \subseteq K \subseteq \approx \bullet K$,
- (2) if $K \subseteq L$, then $\approx \bullet K \subseteq \approx \bullet L$,
- (3) if $K \subseteq L$, then $\approx \bullet K \subseteq \approx \bullet L$,
- (4) $\approx \bullet (K \cap L) = \approx \bullet K \cap \approx \bullet L$,
- (5) $\approx \bullet (K \cup L) = \approx \bullet K \cup \approx \bullet L$.

3. Rough Intuitionistic Fuzzy Filters in BE-Algebras

Throughout this section, let $(\Omega, *, 1)$ be a BE-algebra. In particular, the binary operation $*$ is closed on Ω , 1 denotes the identity element, and the defining axioms of BE-algebras are assumed to hold for all $p, q, r \in \Omega$. Unless otherwise stated, all results are valid for arbitrary elements $p, q \in \Omega$.

In this section, we explore the concept of rough intuitionistic fuzzy (IF) filters within the framework of BE-algebras, a branch of algebraic structures that has garnered significant attention due to its wide-ranging applications in logic, computer science, and information theory. The integration of rough set theory and IF sets into BE-algebras offers a rich avenue for analyzing and handling uncertainty, imprecision, and vagueness in mathematical modeling. By extending the notion of fuzzy filters (FFs) through rough set approximations and intuitionistic frameworks, we aim to develop a more nuanced understanding of how these filters operate under the influence of both fuzziness and roughness. This exploration not only broadens the theoretical landscape of BE-algebras but also provides new tools for

addressing complex problems in various applied domains. In the following subsections, we define rough IF filters, discuss their properties, and investigate their relationships with other algebraic structures.

Definition 8. Let $\mathbb{N}$ be a congruence relation (CR) on a BE-algebra Ω and $K = (v_K, \lambda_K)$ be an IF set of Ω . The IF sets

$$\mathbb{N}_\bullet(K) = (\mathbb{N}_\bullet(v_K), \mathbb{N}_\bullet(\lambda_K))$$

and

$$\mathbb{N}^\bullet(K) = (\mathbb{N}^\bullet(v_K), \mathbb{N}^\bullet(\lambda_K))$$

are called the $\mathbb{N}$ -bottom and $\mathbb{N}$ -exceeding approximations of the IF set K , where:

$$\mathbb{N}^\bullet(v_K)(p) = \bigvee_{a \in [p]_{\mathbb{N}}} v_K(a),$$

$$\mathbb{N}^\bullet(\lambda_K)(p) = \bigwedge_{a \in [p]_{\mathbb{N}}} \lambda_K(a),$$

$$\mathbb{N}_\bullet(v_K)(p) = \bigwedge_{a \in [p]_{\mathbb{N}}} v_K(a),$$

$$\mathbb{N}_\bullet(\lambda_K)(p) = \bigvee_{a \in [p]_{\mathbb{N}}} \lambda_K(a)$$

for all $p \in \Omega$. The IF set $K = (v_K, \lambda_K)$ is called a rough intuitionistic fuzzy filter if both $\mathbb{N}_\bullet(K)$ and $\mathbb{N}^\bullet(K)$ are intuitionistic fuzzy filters.

In the following theorem, we examine the behavior of intuitionistic fuzzy filters under the action of a complete covering relation (CR) on a BE-algebra. Specifically, we consider a BE-algebra Ω and an IF filter K within this structure. The theorem asserts that if $\mathbb{N}$ is a complete CR on Ω , then applying $\mathbb{N}^\bullet$ to K results in another IF filter.

Theorem 1. Let $\mathbb{N}$ be a complete CR on a BE-algebra Ω . If K is an IF filter of Ω , then $\mathbb{N}^\bullet(K)$ is also an IF filter.

The key idea is that the defining conditions of an intuitionistic fuzzy filter are stable under the $\mathcal{N}$ -bottom and $\mathcal{N}$ -exceeding approximations. Since these approximations are constructed using infimum and supremum over $\mathcal{N}$ -equivalence classes, the filter inequalities for v_K and λ_K are preserved when passing from K to $\mathbb{N}_\bullet(K)$ and $\mathbb{N}^\bullet(K)$. The proof formalizes this observation by exploiting the closure of the operation $*$ and the order-preserving behavior of the approximation operators.

Proof. Let $K = (v_K, \lambda_K)$ and $\mathbb{N}^\bullet(K) = (\mathbb{N}^\bullet(v_K), \mathbb{N}^\bullet(\lambda_K))$. First, we examine the function $\mathbb{N}^\bullet(v_K)$:

$$\mathbb{N}^\bullet(v_K)(1) = \bigvee_{r \in [1]_{\mathbb{N}}} v_K(r) \geq \bigvee_{p \in [p]_{\mathbb{N}}} v_K(p) = \mathbb{N}^\bullet(v_K)(p).$$

For any $p, q \in \Omega$, we have:

$$\mathbb{N}^\bullet(v_K)(q) = \bigvee_{q \in [q]_{\mathbb{N}}} v_K(q) \geq \bigvee_{p * q \in [p]_{\mathbb{N}} * [q]_{\mathbb{N}}, p \in [p]_{\mathbb{N}}} \min(v_K(p * q), v_K(p)).$$

This simplifies to:

$$\mathbb{N}^\bullet(v_K)(q) = \min(\mathbb{N}^\bullet(v_K)(p * q), \mathbb{N}^\bullet(v_K)(p)).$$

Next, for $\aleph^\bullet(\lambda_K)$:

$$\aleph^\bullet(\lambda_K)(1) = \bigwedge_{r \in [1]_\aleph} \lambda_K(r) \leq \bigwedge_{p \in [p]_\aleph} \lambda_K(p) = \aleph^\bullet(\lambda_K)(p).$$

For any $p, q \in \Omega$, we have:

$$\aleph^\bullet(\lambda_K)(q) = \max(\aleph^\bullet(\lambda_K)(p * q), \aleph^\bullet(\lambda_K)(p)).$$

Thus, $\aleph^\bullet(K)$ is an IF filter. $\square$

The next theorem investigates the effect of a complete covering relation (CR) on intuitionistic fuzzy filters within a BE-algebra.

Theorem 2. Let $\aleph$ be a complete CR on a BE-algebra Ω . If K is an IF filter of Ω , then $\aleph_\bullet(K)$ is also an IF filter.

This result relies on the monotonic behavior of the membership and non-membership functions ν_K and λ_K on $\mathcal{N}$ -equivalence classes. Taking infimum and supremum over these classes ensures that the relative order between values of ν_K and λ_K is maintained. Consequently, the approximation operators $\aleph_\bullet$ and $\aleph^\bullet$ preserve the inequalities required for the intuitionistic fuzzy filter conditions.

Proof. Let $K = (\nu_K, \lambda_K)$ and $\aleph_\bullet(K) = (\aleph_\bullet(\nu_K), \aleph_\bullet(\lambda_K))$. We proceed by showing the filter properties for $\aleph_\bullet(\nu_K)$ and $\aleph_\bullet(\lambda_K)$. First, for $\aleph_\bullet(\nu_K)$:

$$\aleph_\bullet(\nu_K)(1) = \bigwedge_{r \in [1]_\aleph} \nu_K(r) \geq \bigwedge_{p \in [p]_\aleph} \nu_K(p).$$

Similarly, for any $p, q \in \Omega$:

$$\aleph_\bullet(\nu_K)(q) = \min(\aleph_\bullet(\nu_K)(p * q), \aleph_\bullet(\nu_K)(p)).$$

For $\aleph_\bullet(\lambda_K)$:

$$\aleph_\bullet(\lambda_K)(1) = \bigvee_{r \in [1]_\aleph} \lambda_K(r) \leq \bigvee_{p \in [p]_\aleph} \lambda_K(p).$$

Thus, $\aleph_\bullet(K)$ is an IF filter. $\square$

Example 2. Consider the IF filter K defined in Example 1. Define a CR $\aleph$ on Ω as $\aleph = \{(1, 1), (a, a), (b, b), (1, a), (a, 1), (a, b), (b, a)\}$. The functions $\aleph^\bullet(\nu_K)$, $\aleph^\bullet(\lambda_K)$, $\aleph_\bullet(\nu_K)$, and $\aleph_\bullet(\lambda_K)$ are given by:

$$\aleph^\bullet(\nu_K)(p) = 0.5 \quad \text{for } p = 1, a, b,$$

$$\aleph^\bullet(\lambda_K)(p) = 0.3 \quad \text{for } p = 1, a, b,$$

$$\aleph_\bullet(\nu_K)(p) = \begin{cases} 0.5 & \text{if } p = 1, \\ 0.3 & \text{if } p = a, b, \end{cases}$$

$$\aleph_\bullet(\lambda_K)(p) = \begin{cases} 0.3 & \text{if } p = 1, \\ 0.5 & \text{if } p = a, b. \end{cases}$$

Clearly, $\aleph^\bullet(K)$ is an IF filter, and $\aleph_\bullet(K)$ is also an IF filter.

In this, we present a detailed finite example showing how the lower and upper approximations of an intuitionistic fuzzy (IF) set with respect to a congruence relation produce rough intuitionistic fuzzy filters.

Example 3. Let $\Omega = \{1, a, b\}$ be the BE-algebra defined in Example 2.1, and let $K = (\nu_K, \lambda_K)$ be an intuitionistic fuzzy set on Ω . Consider the congruence relation $\mathcal{N}$ with equivalence classes $[1]_{\mathcal{N}} = \{1, a\}$ and $[b]_{\mathcal{N}} = \{b\}$.

The lower and upper approximations $\mathcal{N}_{\bullet}(K)$ and $\mathcal{N}^{\bullet}(K)$ are computed explicitly and are shown to satisfy the axioms of intuitionistic fuzzy filters. Hence, K is a rough intuitionistic fuzzy filter with respect to $\mathcal{N}$.

The detailed step-by-step verification is provided in Appendix A.

Definition 9 ([4]). Let $K = (\nu_K, \lambda_K)$ be an intuitionistic fuzzy set in a BE-algebra Ω . For $\alpha, \beta \in [0, 1]$, the (α, β) -cut of K is defined by

$$K_{\alpha, \beta} = \{p \in \Omega \mid \nu_K(p) \geq \alpha, \lambda_K(p) \leq \beta\}.$$

The strong (α, β) -cut is denoted by $K_s^{\alpha, \beta}$ and is defined as

$$K_s^{\alpha, \beta} = \{p \in \Omega \mid \nu_K(p) > \alpha \text{ and } \lambda_K(p) < \beta\}.$$

Theorem 3. Let $K = (\nu_K, \lambda_K)$ be an IF subset. Then K is an IF filter of a BE-algebra Ω if and only if $K^{\alpha, \beta}$ is non-empty and is a filter of Ω for some α and β such that $\nu_K(1) \geq \alpha$ and $\lambda_K(1) \leq \beta$.

The essential idea is to transfer the filter axioms from an intuitionistic fuzzy filter K to its rough approximations. Because the operation $*$ is closed in the BE-algebra and the approximation operators act uniformly on $\mathcal{N}$ -equivalence classes, the filter inequalities involving $p * q$ are inherited by both $\mathcal{N}_{\bullet}(K)$ and $\mathcal{N}^{\bullet}(K)$. The proof verifies this transfer explicitly for each axiom.

Proof. Suppose $K = (\nu_K, \lambda_K)$ is an IF filter of Ω and that $\nu_K(1) \geq \alpha$ and $\lambda_K(1) \leq \beta$. Clearly, $1 \in K^{\alpha, \beta}$, and for any $p, q \in K^{\alpha, \beta}$, it follows that $p * q \in K^{\alpha, \beta}$. Therefore, $K^{\alpha, \beta}$ is a filter. Conversely, suppose $K^{\alpha, \beta}$ is non-empty and is a filter. It follows that K is an IF filter. $\square$

Theorem 4. Let $\mathfrak{N}$ be a CR on a BE-algebra Ω . If K is an IF set of Ω , and $\alpha, \beta \in [0, 1]$, then:

- (1) $(\mathfrak{N}_{\bullet}(K))^{\alpha, \beta} = \mathfrak{N}_{\bullet}(K^{\alpha, \beta})$,
- (2) $(\mathfrak{N}^{\bullet}(K))^{\alpha, \beta} = \mathfrak{N}^{\bullet}(K^{\alpha, \beta})$.

This theorem establishes an equivalence between algebraic filter conditions and their rough intuitionistic fuzzy counterparts. The forward direction shows that the rough approximations of an intuitionistic fuzzy filter retain the filter structure, while the converse direction reconstructs the filter properties of K from those of $\mathcal{N}_{\bullet}(K)$ and $\mathcal{N}^{\bullet}(K)$. Hence, the characterization follows from the precise interplay between BE-algebra operations and $\mathcal{N}$ -based rough approximations.

Proof. 1. For any point p , we have:

$$p \in (\mathfrak{N}_{\bullet}(K))^{\alpha, \beta} \iff \mathfrak{N}_{\bullet}(\nu_K)(p) \geq \alpha \text{ and } \mathfrak{N}_{\bullet}(\lambda_K)(p) \leq \beta,$$

which is equivalent to:

$$\bigwedge_{a \in [p]_{\mathfrak{N}}} \nu_K(a) \geq \alpha \text{ and } \bigvee_{a \in [p]_{\mathfrak{N}}} \lambda_K(a) \leq \beta.$$

This further simplifies to:

$$[p]_{\aleph} \subseteq K^{\alpha, \beta} \iff p \in \aleph_{\bullet}(K^{\alpha, \beta}).$$

2. For the upper approximation, we similarly have:

$$p \in (\aleph^{\bullet}(K))^{\alpha, \beta} \iff \aleph^{\bullet}(\nu_K)(p) \geq \alpha \text{ and } \aleph^{\bullet}(\lambda_K)(p) \leq \beta,$$

which simplifies to:

$$[p]_{\aleph} \cap K^{\alpha, \beta} \neq \emptyset \iff p \in \aleph^{\bullet}(K_s^{\alpha, \beta}).$$

Thus, the theorem is proved. $\square$

Corollary 1. Let $\mathcal{N}$ be a congruence relation on a BE-algebra Ω , and let $K = (\nu_K, \lambda_K)$ be an intuitionistic fuzzy set in Ω . For any $\alpha, \beta \in [0, 1]$, the following equalities hold:

1. $(\mathcal{N}_{\bullet}(K))_{\alpha, \beta}^s = \mathcal{N}_{\bullet}(K_{\alpha, \beta}^s)$,
2. $(\mathcal{N}^{\bullet}(K))_{\alpha, \beta}^s = \mathcal{N}^{\bullet}(K_{\alpha, \beta}^s)$.

Proof. The proof follows the same steps as Theorem 3.4 by replacing the non-strict inequalities $\nu_K(\cdot) \geq \alpha$ and $\lambda_K(\cdot) \leq \beta$ with the strict inequalities $\nu_K(\cdot) > \alpha$ and $\lambda_K(\cdot) < \beta$. $\square$

Theorem 5. Let φ be an epimorphism of BE-algebra Ω_1 to Ω_2 , and let $\aleph_2$ be a CR on Ω_2 . Define $\aleph_1$ on Ω_1 as:

$$\aleph_1 = \{(a, b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \aleph_2\}.$$

Then:

- (1) $\aleph_1$ is a CR on Ω_1 ,
- (2) if $\aleph_2$ is complete and φ is one-to-one, then $\aleph_1$ is complete,
- (3) $\varphi(\aleph_1^{\bullet}(K)) = \aleph_2^{\bullet}(\varphi(K))$.

Proof. (1) It is clear that $\aleph_1$ is a CR on Ω_1 .

(2) We show that $[p_1 * p_2]_{\aleph_1} = [p_1]_{\aleph_1} * [p_2]_{\aleph_1}$. Suppose that r is an element of $[p_1 * p_2]_{\aleph_1}$. Since $\aleph_2$ is complete, we have $\varphi(r) \in [\varphi(p_1 * p_2)]_{\aleph_2} = [\varphi(p_1)]_{\aleph_2} * [\varphi(p_2)]_{\aleph_2}$. Since φ is surjective, there exist $a, b \in \Omega_1$ such that $\varphi(a) \in [\varphi(p_1)]_{\aleph_2}$, $\varphi(b) \in [\varphi(p_2)]_{\aleph_2}$, and $\varphi(r) = \varphi(a * b)$. Since φ is one-to-one, we have $a \in [p_1]_{\aleph_1}$ and $b \in [p_2]_{\aleph_1}$, and $r = a * b$. Thus, $r \in [p_1]_{\aleph_1} * [p_2]_{\aleph_1}$. Therefore, $\aleph_1$ is complete.

(3) Let $q \in \varphi(\aleph_1^{\bullet}(K))$, then there exists $p \in \aleph_1^{\bullet}(K)$ such that $\varphi(p) = q$. Hence, $[p]_{\aleph_1} \cap K \neq \emptyset$. There exists $a \in [p]_{\aleph_1} \cap K$. Then, $\varphi(a) \in \varphi(K)$ and $\varphi(a) \in [\varphi(p)]_{\aleph_2}$. Thus, $[\varphi(p)]_{\aleph_2} \cap \varphi(K) \neq \emptyset$, which implies $q = \varphi(p) \in \aleph_2^{\bullet}(\varphi(K))$. Hence, $\varphi(\aleph_1^{\bullet}(K)) \subseteq \aleph_2^{\bullet}(\varphi(K))$.

Conversely, let $q \in \aleph_2^{\bullet}(\varphi(K))$, then there exists $p \in \Omega_1$ such that $\varphi(p) = q$. Since $[\varphi(p)]_{\aleph_2} \cap \varphi(K) \neq \emptyset$, there exists $a \in K$ such that $\varphi(a) \in \varphi(K)$, and $\varphi(a) \in [\varphi(p)]_{\aleph_2}$. By the definition of $\aleph_1$, we have $a \in [p]_{\aleph_1}$. Thus, $[p]_{\aleph_1} \cap K \neq \emptyset$, which implies that $p \in \aleph_1^{\bullet}(K)$. Therefore, $q = \varphi(p) \in \varphi(\aleph_1^{\bullet}(K))$, and hence, $\aleph_2^{\bullet}(\varphi(K)) \subseteq \varphi(\aleph_1^{\bullet}(K))$. Therefore, $\varphi(\aleph_1^{\bullet}(K)) = \aleph_2^{\bullet}(\varphi(K))$. $\square$

In the following theorem, we analyze the interaction between epimorphisms and filters in BE-algebras. Let φ be an epimorphism from BE-algebra Ω_1 to Ω_2 , and let $\aleph_2$ be a covering relation on Ω_2 . Define $\aleph_1$ on Ω_1 by $\aleph_1 = \{(a, b) \in \Omega_1 \times \Omega_2 \mid (\varphi(a), \varphi(b)) \in \aleph_2\}$. The theorem states that the filter $\aleph_1^{\bullet}(K)$ in Ω_1 is valid if and only if the filter $\aleph_2^{\bullet}(\varphi(K))$ in Ω_2 is valid. This result shows that the filter property is preserved under the image of an epimorphism, providing a useful link between the structure of filters in different BE-algebras.

Theorem 6. Let φ be an epimorphism of BE-algebra Ω_1 to Ω_2 , and let $\aleph_2$ be a CR on Ω_2 . Let K be a subset of Ω_1 . If

$$\aleph_1 = \{(a, b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \aleph_2\},$$

then $\aleph_1^\bullet(K)$ is a filter of Ω_1 if and only if $\aleph_2^\bullet(\varphi(K))$ is a filter of Ω_2 .

Proof. Forward Direction:

Assume that $\aleph_1^\bullet(K)$ is a filter in Ω_1 . Suppose $p * q \in \aleph_2^\bullet(\varphi(K))$ and $x \in \aleph_2^\bullet(\varphi(K))$. Since φ is an epimorphism, by Theorem 5, we have:

$$x * q \in \varphi(\aleph_1^\bullet(K)).$$

Since φ is surjective, there exist elements $a, b \in \aleph_1^\bullet(K)$ such that:

$$p = \varphi(a), \quad q = \varphi(b), \quad \text{and} \quad p * q = \varphi(a * b).$$

Given $q = \varphi(b)$ and $b \in \aleph_1^\bullet(K)$, and since $\aleph_1^\bullet(K)$ is a filter, $a * b \in \aleph_1^\bullet(K)$ and $b \in \aleph_1^\bullet(K)$. Thus, we have:

$$q = \varphi(b) \in \varphi(\aleph_1^\bullet(K)).$$

Since $\varphi(\aleph_1^\bullet(K)) \subseteq \aleph_2^\bullet(\varphi(K))$, it follows that:

$$q \in \aleph_2^\bullet(\varphi(K)).$$

Hence, $\aleph_2^\bullet(\varphi(K))$ is a filter.

Conversely:

Assume $\aleph_2^\bullet(\varphi(K))$ is a filter in Ω_2 . Suppose $p * q \in \aleph_1^\bullet(K)$ and $p \in \aleph_1^\bullet(K)$. Since φ is an epimorphism, we have:

$$\varphi(p * q) \in \varphi(\aleph_1^\bullet(K)) \quad \text{and} \quad \varphi(p) \in \varphi(\aleph_1^\bullet(K)).$$

By Theorem 5, it follows that:

$$\varphi(p * q) \in \aleph_2^\bullet(\varphi(K)) \quad \text{and} \quad \varphi(p) \in \aleph_2^\bullet(\varphi(K)).$$

Since $\aleph_2^\bullet(\varphi(K))$ is a filter, we have:

$$\varphi(q) \in \aleph_2^\bullet(\varphi(K)).$$

Since φ is surjective, there exists $b \in \aleph_1^\bullet(K)$ such that:

$$\varphi(q) = \varphi(b).$$

Therefore:

$$[b]_{\aleph_1} \cap K \neq \emptyset \quad \text{so} \quad q \in [b]_{\aleph_1}.$$

Thus:

$$[q]_{\aleph_1} \cap K \neq \emptyset \quad \text{so} \quad q \in \aleph_1^\bullet(K).$$

Therefore, $\aleph_1^\bullet(K)$ is a filter. $\square$

This theorem explores the preservation of the intuitionistic FF property under epimorphisms between BE-algebras. Let φ be an epimorphism from BE-algebra Ω_1 to Ω_2 , and let $\aleph_2$ be a complete covering relation on Ω_2 . Define $\aleph_1$ on Ω_1 as $\aleph_1 = \{(a, b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \aleph_2\}$. The theorem asserts that the intuitionistic FF $\aleph_1^\bullet(K)$ in Ω_1 is valid if

and only if $\aleph_2^\bullet(\varphi(K))$ is a valid intuitionistic FF in Ω_2 . This result highlights that the filter property is preserved through the epimorphism, ensuring the coherence of filter structures across different BE-algebras.

Theorem 7. Let φ be an epimorphism from Ω_1 to Ω_2 , and let $\aleph_2$ be a complete CR on Ω_2 . Let K be a fuzzy subset of Ω_1 . If $\aleph_1 = \{(a, b) \in \Omega_1 \times \Omega_1 \mid (\varphi(a), \varphi(b)) \in \aleph_2\}$, then $(\aleph_1^\bullet(K))$ is an IF filter of Ω_1 if and only if $(\aleph_2^\bullet(\varphi(K)))$ is an IF filter of Ω_2 .

Proof. By Theorem 6, we have that $(\aleph_1^\bullet(K))$ is an IF filter of Ω_1 if and only if $(\aleph_1^\bullet(K))_s^{\alpha, \beta}$ is, if non-empty, a filter of Ω_1 . By Corollary 1, it follows that

$$(\aleph_1^\bullet(K))_s^{\alpha, \beta} = \aleph_1^\bullet(K_s^{\alpha, \beta})$$

By Theorem 6, $\aleph_1^\bullet(K_s^{\alpha, \beta})$ is a filter of Ω_1 if and only if:

$$\aleph_2^\bullet(\varphi(K_s^{\alpha, \beta}))$$

is a filter of Ω_2 . Since $\varphi(K_s^{\alpha, \beta}) = (\varphi(K))_s^{\alpha, \beta}$, we can substitute to obtain:

$$\aleph_2^\bullet(\varphi(K_s^{\alpha, \beta})) = (\aleph_2^\bullet(\varphi(K)))_s^{\alpha, \beta}$$

By Theorem 7, we conclude that $(\aleph_2^\bullet(\varphi(K)))_s^{\alpha, \beta}$ is a filter of Ω_2 if and only if $(\aleph_2^\bullet(\varphi(K)))$ is an IF filter of Ω_2 for all $\alpha, \beta \in [0, 1]$. $\square$

Table 1 summarizes the main structural relationships among intuitionistic fuzzy filters, rough intuitionistic fuzzy filters, and T -rough intuitionistic fuzzy filters, highlighting their closure properties and stability under homomorphisms.

Table 1. Comparison of IF filters, rough IF filters, and T -rough IF filters.

Property	IF Filters	Rough IF Filters	T -rough IF Filters
Underlying algebra	BE-algebra $(\Omega, *, 1)$	BE-algebra $(\Omega, *, 1)$	BE-algebra $(\Omega, *, 1)$
Associated relation	None	Congruence $\mathcal{N}$	Tolerance T
Lower approximation	Not defined	$\mathcal{N}_\bullet(K)$	$T_\bullet(K)$
Upper approximation	Not defined	$\mathcal{N}^\bullet(K)$	$T^\bullet(K)$
Closure under intersection $\cap$	Preserved	Preserved	Preserved
Closure under union $\cup$	Not preserved (in general)	Not preserved (in general)	Not preserved (in general)
Filter axioms stability	Directly satisfied	Preserved via approximations	Preserved via tolerance-based approximations
Homomorphic image	IF filter (under surjective homomorphism)	Rough IF filter with induced congruence	T -rough IF filter with induced tolerance
Homomorphic preimage	IF filter	Rough IF filter	T -rough IF filter
Generalization level	Base structure	Extension of IF filters	Extension of rough IF filters

4. Γ -Rough Intuitionistic Fuzzy Filters in BE-Algebras

In this section, we introduce the concept of set-valued homomorphism in BE-algebras and explore the fundamental properties of generalized bottom and exceeding approximation operators. A set-valued homomorphism extends the traditional notion of a homomorphism by mapping each element in the algebra to a set of elements, rather than

a single element. This extension is particularly useful for handling uncertainty within algebraic structures.

We commence by defining set-valued homomorphisms and their role in the framework of BE-algebras. We then examine the interaction of these homomorphisms with generalized bottom and exceeding approximation operators, which serve as tools to approximate elements in algebraic settings.

Prior to discussing these concepts, we will introduce essential definitions to aid in understanding the material covered in this section.

Definition 10. Let U and W be non-empty sets and $\Gamma : U \rightarrow \mathcal{P}(W)$, where $\mathcal{P}(W)$ denotes the power set of W . The triplet (U, W, Γ) forms a generalized approximation space or a generalized rough set. A set-valued function Γ from U to $\mathcal{P}(W)$ induces a binary relation between U and W , defined by $R_\Gamma = \{(p, q) \in U \times W \mid q \in \Gamma(p)\}$. For any subset $K \subseteq W$, the bottom and exceeding approximations, denoted by $\Gamma^+(K)$ and $\Gamma^{-1}(K)$, are defined as follows:

$$\Gamma^+(K) = \{p \in U \mid \Gamma(p) \subseteq K\},$$

$$\Gamma^{-1}(K) = \{p \in U \mid \Gamma(p) \cap K \neq \emptyset\}.$$

These operators form a generalized rough set, where $\Gamma^+(K)$ and $\Gamma^{-1}(K)$ are called the bottom and exceeding generalized approximation operators, respectively.

Definition 11. Let Ω_1 and Ω_2 be BE-algebras and $\Gamma : \Omega_1 \rightarrow \mathcal{P}^*(\Omega_2)$, where $\mathcal{P}^*(\Omega_2)$ represents the collection of all non-empty subsets of Ω_2 .

The bottom and exceeding inverses of a subset $B \subseteq \Omega_2$ under Γ are defined as follows:

- The bottom inverse of B under Γ is:

$$\Gamma^+(B) = \{p \in \Omega_1 \mid \Gamma(p) \subseteq B\}.$$

- The exceeding inverse of B under Γ is:

$$\Gamma^{-1}(B) = \{p \in \Omega_1 \mid \Gamma(p) \cap B \neq \emptyset\}.$$

Before introducing set-valued homomorphisms, we clarify that whenever a negation operation appears in a BE-algebra, it is understood in the sense commonly adopted in implication-based algebras; namely, for $a \in \Omega$, the element $-a$ denotes a pseudo-complement of a , whenever such an element exists.

Definition 12. Let Ω_1 and Ω_2 be two BE-algebras, and let $\Gamma : \Omega_1 \rightarrow \mathcal{P}^*(\Omega_2)$ be a set-valued mapping. The mapping Γ is called a set-valued homomorphism if the following conditions hold:

1. $\Gamma(a) * \Gamma(b) \subseteq \Gamma(a * b)$,
2. $\Gamma(-a) \supseteq -\Gamma(a)$,

for all $a, b \in \Omega_1$, whenever the negation is defined.

Moreover, Γ is called a strong set-valued homomorphism if

1. $\Gamma(a) * \Gamma(b) = \Gamma(a * b)$,
2. $\Gamma(-a) = -\Gamma(a)$,

for all $a, b \in \Omega_1$.

Theorem 8. Let Ω_1 and Ω_2 be two BE-algebras and $\Gamma : \Omega_1 \rightarrow \mathcal{P}^*(\Omega_2)$ be a strong set-valued homomorphism. If A is a filter in Ω_2 , then:

- $\Gamma^+(A)$, if nonempty, is a filter of Ω_1 .

- $\Gamma^{-1}(K)$, if nonempty, is a filter of Ω_1 .

Proof.

- (1) Suppose that K is a filter, but $\Gamma^+(K)$ is not. Let $p, q \in \Omega_1$ be such that $p * q \in \Gamma^+(K)$ and $p \in \Gamma^+(K)$, but $q \notin \Gamma^+(K)$. This implies there exist $a \in \Gamma(p)$ and $b \in \Gamma(q)$, with $b \notin K$. Since $a * b \in \Gamma(p) * \Gamma(q) = \Gamma(p * q) \subseteq K$ and K is a filter, $a * b \in K$ and $a \in K$ imply $b \in K$. This leads to a contradiction, as b was initially assumed not to be in K . Therefore, if $\Gamma^+(K)$ is nonempty, it must be a filter in Ω_1 .
- (2) Assume that $p * q \in \Gamma^{-1}(K)$ and $p \in \Gamma^{-1}(K)$. This means $\Gamma(p * q) \cap K \neq \emptyset$ and $\Gamma(p) \cap K \neq \emptyset$. Since Γ is a strong set-valued homomorphism, we have $\Gamma(p) * \Gamma(q) = \Gamma(p * q)$. Consequently, there exist $a \in \Gamma(p)$ and $b \in \Gamma(q)$ such that $a * b \in K$ and $a \in K$. Given that K is a filter, $a \in K$ implies $b \in K$. Therefore, $b \in \Gamma(p) \cap K$ and $q \in \Gamma^{-1}(K)$. Hence, if $\Gamma^{-1}(K)$ is nonempty, it must be a filter in Ω_1 .

□

Definition 13. Let Ω_1 and Ω_2 be two BE-algebras and $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be set-valued homomorphism. Let $K = (\nu_K, \lambda_K)$ be IF subset of Ω_2 . For every $p \in \Omega_1$, we define

$$\Gamma^+(\nu_K)(p) = \bigwedge_{a \in \Gamma(p)} (\nu_K)(a)$$

$$\Gamma^+(\lambda_K)(p) = \bigvee_{a \in \Gamma(p)} (\lambda_K)(a)$$

$$\Gamma^{-1}(\nu_K)(p) = \bigvee_{a \in \Gamma(p)} (\nu_K)(a)$$

$$\Gamma^{-1}(\lambda_K)(p) = \bigwedge_{a \in \Gamma(p)} (\lambda_K)(a)$$

$\Gamma^+(K)$ and $\Gamma^{-1}(K)$ are called, respectively, the T-rough bottom and the T-rough exceeding IF subset of Ω_1 . If $\Gamma^+(K)$ and $\Gamma^{-1}(K)$ are IF filter, $(\Gamma^+(K), \Gamma^{-1}(K))$ is said to be T-rough IF filter of Ω_1 .

This theorem investigates the preservation of intuitionistic fuzzy (IF) filters under set-valued homomorphisms between BE-algebras. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If K is an IF filter in Ω_2 , the theorem asserts that $\Gamma^{-1}(K)$ will be an IF filter in Ω_1 . This result demonstrates that the IF filter structure is maintained when transferring through the homomorphism, ensuring that the properties of IF filters are preserved across different BE-algebras.

Theorem 9. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If K is an intuitionistic FF in Ω_2 , then the set $\Gamma^{-1}(K)$ forms an intuitionistic FF in Ω_1 .

Proof. Let $K = (\nu_K, \lambda_K)$, and define $\Gamma^{-1}(K) = (\Gamma^{-1}(\nu_K), \Gamma^{-1}(\lambda_K))$.

$$\Gamma^{-1}(\nu_K)(1) = \bigvee_{r \in \Gamma(1)} \nu_K(r) \geq \bigvee_{p \in \Gamma(p)} \nu_K(p) = \Gamma^{-1}(\nu_K)(p)$$

For any $p, q \in \Omega_1$, the following holds:

$$\Gamma^{-1}(\nu_K)(q) = \bigvee_{q \in \Gamma(q)} \nu_K(q) \geq \bigvee_{p * q \in \Gamma(p) * \Gamma(q), p \in \Gamma(p)} \min(\nu_K(p * q), \nu_K(p))$$

$$\begin{aligned}
&= \bigvee_{p*q \in \Gamma(p*q), p \in \Gamma(p)} \min(v_K(p*q), v_K(p)) \\
&= \min \left(\bigvee_{p*q \in \Gamma(p*q)} v_K(p*q), \bigvee_{p \in \Gamma(p)} v_K(p) \right) \\
&= \min(\Gamma^{-1}(v_K)(p*q), \Gamma^{-1}(v_K)(p))
\end{aligned}$$

Similarly,

$$\Gamma^{-1}(\lambda_K)(1) = \bigwedge_{r \in \Gamma(1)} \lambda_K(r) \leq \bigwedge_{p \in \Gamma(p)} \lambda_K(p) = \Gamma^{-1}(\lambda_K)(p)$$

For any $p, q \in \Omega_1$, the following holds:

$$\begin{aligned}
\Gamma^{-1}(\lambda_K)(q) &= \bigwedge_{q \in \Gamma(q)} \lambda_K(q) \leq \bigwedge_{p*q \in \Gamma(p)*\Gamma(q), p \in \Gamma(p)} \max(\lambda_K(p*q), \lambda_K(p)) \\
&= \bigwedge_{p*q \in \Gamma(p*q), p \in \Gamma(p)} \max(\lambda_K(p*q), \lambda_K(p)) \\
&= \max \left(\bigwedge_{p*q \in \Gamma(p*q)} \lambda_K(p*q), \bigwedge_{p \in \Gamma(p)} \lambda_K(p) \right) \\
&= \max(\Gamma^{-1}(\lambda_K)(p*q), \Gamma^{-1}(\lambda_K)(p))
\end{aligned}$$

Thus, $\Gamma^{-1}(K)$ is an IF filter. $\square$

This theorem explores the preservation of intuitionistic fuzzy (IF) filters under set-valued homomorphisms between BE-algebras. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If K is an IF filter in Ω_2 , then $\Gamma^+(K)$ will be an IF filter in Ω_1 . This result indicates that the IF filter structure is preserved when applying the homomorphism, ensuring that the characteristics of IF filters are maintained across different BE-algebras.

Theorem 10. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If K is an intuitionistic FF in Ω_2 , then $\Gamma^+(K)$ is an intuitionistic FF in Ω_1 .

Proof. Let $K = (v_K, \lambda_K)$, and define $\Gamma^+(K) = (\Gamma^+(v_K), \Gamma^+(\lambda_K))$.

$$\Gamma^+(v_K)(1) = \bigwedge_{r \in \Gamma(1)} v_K(r) \geq \bigwedge_{p \in \Gamma(p)} v_K(p) = \Gamma^+(v_K)(p)$$

For any $p, q \in \Omega_1$, the following holds:

$$\begin{aligned}
\Gamma^+(v_K)(q) &= \bigwedge_{q \in \Gamma(q)} v_K(q) \geq \bigwedge_{p*q \in \Gamma(p)*\Gamma(q), p \in \Gamma(p)} \min(v_K(p*q), v_K(p)) \\
&= \bigwedge_{p*q \in \Gamma(p*q), p \in \Gamma(p)} \min(v_K(p*q), v_K(p)) \\
&= \min \left(\bigwedge_{p*q \in \Gamma(p*q)} v_K(p*q), \bigwedge_{p \in \Gamma(p)} v_K(p) \right)
\end{aligned}$$

$$= \min(\Gamma^+(v_K)(p * q), \Gamma^+(v_K)(p))$$

Similarly,

$$\Gamma^+(\lambda_K)(1) = \bigvee_{r \in \Gamma(1)} \lambda_K(r) \leq \bigvee_{p \in \Gamma(p)} \lambda_K(p) = \Gamma^+(\lambda_K)(p)$$

For any $p, q \in \Omega_1$, the following holds:

$$\begin{aligned} \Gamma^+(\lambda_K)(q) &= \bigvee_{q \in \Gamma(q)} \lambda_K(q) \leq \bigvee_{p * q \in \Gamma(p) * \Gamma(q), p \in \Gamma(p)} \max(\lambda_K(p * q), \lambda_K(p)) \\ &= \bigvee_{p * q \in \Gamma(p * q), p \in \Gamma(p)} \max(\lambda_K(p * q), \lambda_K(p)) \\ &= \max \left(\bigvee_{p * q \in \Gamma(p * q)} \lambda_K(p * q), \bigvee_{p \in \Gamma(p)} \lambda_K(p) \right) \\ &= \max(\Gamma^+(\lambda_K)(p * q), \Gamma^+(\lambda_K)(p)) \end{aligned}$$

Therefore, $\Gamma^+(K)$ is an IF filter. $\square$

Definition 14. Let $K = (v_K, \lambda_K)$ and $B = (v_B, \lambda_B)$ are any two IFS of a BE-algebra Ω . Then the composition $K \circ B$ is defined by

$$K \circ B = (v_K \circ v_B, \lambda_K \circ \lambda_B)$$

where for all $p \in \Omega$,

$$\begin{aligned} (v_K \circ v_B)(p) &= \bigvee_{p=qr} [v_K(q) \wedge v_B(r)] \\ (\lambda_K \circ \lambda_B)(p) &= \bigwedge_{p=qr} [\lambda_K(q) \vee \lambda_B(r)] \end{aligned}$$

This theorem addresses the relationship between intuitionistic fuzzy sets (IFS) under set-valued homomorphisms between BE-algebras. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If A and B are two IFSs in Ω_2 , then the homomorphic preimage of their composition is contained within the composition of their individual preimages. Formally, this is expressed as $\Gamma^{-1}(K) \circ \Gamma^{-1}(B) \subseteq \Gamma^{-1}(K \circ B)$. This result highlights how the composition of FSs is preserved under the homomorphism, maintaining the structure of their relationships within different BE-algebras.

Theorem 11. Let Ω_1 and Ω_2 be two BE-algebras and $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be set-valued homomorphism. If A and B are two IFS of Ω_2 , then

$$\Gamma^{-1}(K) \circ \Gamma^{-1}(B) \subseteq \Gamma^{-1}(K \circ B)$$

Proof. Let $K = (v_K, \lambda_K)$ and $B = (v_B, \lambda_B)$ are any two IFS of a BE-algebra Ω_2 . Then,

$$\Gamma^{-1}(K) \circ \Gamma^{-1}(B) = (\Gamma^{-1}(v_K) \circ \Gamma^{-1}(v_B), \Gamma^{-1}(\lambda_K) \circ \Gamma^{-1}(\lambda_B))$$

and

$$\Gamma^{-1}(A \circ B) = (\Gamma^{-1}(v_K \circ v_B), \Gamma^{-1}(\lambda_K \circ \lambda_B))$$

To show $\Gamma^{-1}(K) \circ \Gamma^{-1}(B) \subseteq \Gamma^{-1}(K \circ B)$ we have to prove that for all $p \in \Omega_1$

$$(\Gamma^{-1}(\nu_K) \circ \Gamma^{-1}(\nu_B))(p) \leq \Gamma^{-1}(\nu_K \circ \nu_B)(p)$$

and

$$(\Gamma^{-1}(\lambda_K) \circ \Gamma^{-1}(\lambda_B))(p) \geq \Gamma^{-1}(\lambda_K \circ \lambda_B)(p)$$

Now for all $p \in \Omega_1$,

$$\begin{aligned} (\Gamma^{-1}(\nu_K) \circ \Gamma^{-1}(\nu_B))(p) &= \bigvee_{p=qr} [\Gamma^{-1}(\nu_K)(q) \wedge \Gamma^{-1}(\nu_B)(r)] = \bigvee_{p=qr} [(\bigvee_{a \in \Gamma(q)} \nu_K(a)) \wedge (\bigvee_{b \in \Gamma(r)} \nu_B(b))] \\ &= \bigvee_{p=qr} [\bigvee_{a \in \Gamma(q), b \in \Gamma(r)} (\nu_K(a) \wedge \nu_B(b))] \leq \bigvee_{p=qr} [\bigvee_{ab \in \Gamma(qr)} (\nu_K(a) \wedge \nu_B(b))], \\ &\text{since } ab \in \Gamma(q)\Gamma(r) \subseteq \Gamma(qr) \\ &= \bigvee_{ab \in \Gamma(p)} (\nu_K(a) \wedge \nu_B(b)), \text{ since } qr = p \\ &= \bigvee_{\alpha \in \Gamma(p), \alpha=ab} (\nu_K(a) \wedge \nu_B(b)) \\ &= \bigvee_{\alpha \in \Gamma(p)} \bigvee_{\alpha=ab} (\nu_K(a) \wedge \nu_B(b)) \\ &= \bigvee_{\alpha \in \Gamma(p)} (\nu_K(a) \circ \nu_B(b))(\alpha) = \Gamma^{-1}(\nu_K \circ \nu_B)(p) \end{aligned}$$

Again,

$$\begin{aligned} (\Gamma^{-1}(\lambda_K) \circ \Gamma^{-1}(\lambda_B))(p) &= \bigwedge_{p=qr} [\Gamma^{-1}(\lambda_K)(q) \vee \Gamma^{-1}(\lambda_B)(r)] = \bigwedge_{p=qr} [(\bigwedge_{a \in \Gamma(q)} \lambda_K(a)) \vee (\bigwedge_{b \in \Gamma(r)} \lambda_B(b))] \\ &= \bigwedge_{p=qr} [\bigwedge_{a \in \Gamma(q), b \in \Gamma(r)} (\lambda_K(a) \vee \lambda_B(b))] \geq \bigwedge_{p=qr} [\bigwedge_{ab \in \Gamma(qr)} (\lambda_K(a) \vee \lambda_B(b))], \\ &\text{since } ab \in \Gamma(q)\Gamma(r) \subseteq \Gamma(qr) \\ &= \bigwedge_{ab \in \Gamma(p)} (\lambda_K(a) \vee \lambda_B(b)), \text{ since } qr = p \\ &= \bigwedge_{\alpha \in \Gamma(p), \alpha=ab} (\lambda_K(a) \vee \lambda_B(b)) \\ &= \bigwedge_{\alpha \in \Gamma(p)} \bigwedge_{\alpha=ab} (\lambda_K(a) \vee \lambda_B(b)) \\ &= \bigwedge_{\alpha \in \Gamma(p)} (\lambda_K(a) \circ \lambda_B(b))(\alpha) = \Gamma^{-1}(\lambda_K \circ \lambda_B)(p) \end{aligned}$$

Thus, we have $\Gamma^{-1}(K) \circ \Gamma^{-1}(B) \subseteq \Gamma^{-1}(K \circ B)$ $\square$

This theorem explores the behavior of intuitionistic fuzzy sets (IFS) under strong set-valued homomorphisms between BE-algebras. Let Ω_1 and Ω_2 be BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a strong set-valued homomorphism. If K and B are IFSs in Ω_2 , then the composition of the images of these IFSs under Γ^+ is contained within the image of their composition under Γ^+ . In other words, $\Gamma^+(K) \circ \Gamma^+(B) \subseteq \Gamma^+(K \circ B)$. This result demonstrates how the structure of FS compositions is preserved and reflected under strong homomorphisms between BE-algebras.

Theorem 12. Let Ω_1 and Ω_2 be two BE-algebras and $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be strong set-valued homomorphism. If K and B are two IFS of Ω_2 , then

$$\Gamma^+(K) \circ \Gamma^+(B) \subseteq \Gamma^+(K \circ B)$$

Proof. Let $K = (\nu_K, \lambda_K)$ and $B = (\nu_B, \lambda_B)$ are any two IFS of a BE-algebra Ω_2 . Then,

$$\Gamma^+(K) \circ \Gamma^+(B) = (\Gamma^+(\nu_K) \circ \Gamma^+(\nu_B), \Gamma^+(\lambda_K) \circ \Gamma^+(\lambda_B))$$

and

$$\Gamma^+(K \circ B) = (\Gamma^+(\nu_K \circ \nu_B), \Gamma^+(\lambda_K \circ \lambda_B))$$

To show $\Gamma^+(K) \circ \Gamma^+(B) \subseteq \Gamma^+(K \circ B)$ we have to prove that for all $p \in \Omega_1$

$$(\Gamma^+(\nu_K) \circ \Gamma^+(\nu_B))(p) \leq \Gamma^+(\nu_K \circ \nu_B)(p)$$

and

$$\Gamma^+(\lambda_K) \circ \Gamma^+(\lambda_B)(p) \geq \Gamma^+(\lambda_K \circ \lambda_B)(p)$$

Now, for all $p \in \Omega_1$,

$$(\Gamma^+(\nu_K) \circ \Gamma^+(\nu_B))(p) = \bigvee_{p=qr} [\Gamma^+(\nu_K)(q) \wedge \Gamma^+(\nu_B)(r)] = \bigvee_{p=qr} [(\bigwedge_{a \in \Gamma(q)} \nu_K(a)) \wedge (\bigwedge_{b \in \Gamma(r)} \nu_B(b))]$$

$$= \bigvee_{p=qr} [\bigwedge_{a \in \Gamma(q), b \in \Gamma(r)} (\nu_K(a) \wedge \nu_B(b))] \leq \bigvee_{p=qr} [\bigwedge_{a \in \Gamma(q), b \in \Gamma(r)} \bigvee_{ab=\alpha\beta} (\nu_K(\alpha) \wedge \nu_B(\beta))],$$

where $\alpha, \beta \in \Omega_1$

$$= \bigvee_{p=qr} [\bigwedge_{a \in \Gamma(q), b \in \Gamma(r)} (\nu_K \circ \nu_B)(ab)] = \bigvee_{p=qr} [\bigwedge_{ab \in \Gamma(qr)} (\nu_K \circ \nu_B)(ab)],$$

since $ab \in \Gamma(q)\Gamma(r) = \Gamma(qr)$

$$= \bigvee_{p=qr} \Gamma^+(\nu_K \circ \nu_B)(qr) = \Gamma^+(\nu_K \circ \nu_B)(p)$$

Again,

$$(\Gamma^+(\lambda_K) \circ \Gamma^+(\lambda_B))(p) = \bigwedge_{p=qr} [\Gamma^+(\lambda_K)(q) \vee \Gamma^+(\lambda_B)(r)] = \bigwedge_{p=qr} [(\bigvee_{a \in \Gamma(q)} \lambda_K(a)) \vee (\bigvee_{b \in \Gamma(r)} \lambda_B(b))]$$

$$= \bigwedge_{p=qr} [\bigvee_{a \in \Gamma(q), b \in \Gamma(r)} (\lambda_K(a) \vee \lambda_B(b))] \geq \bigwedge_{p=qr} [\bigvee_{a \in \Gamma(q), b \in \Gamma(r)} \bigwedge_{ab=\alpha\beta} (\lambda_K(\alpha) \vee \lambda_B(\beta))],$$

where $\alpha, \beta \in \Omega_1$

$$= \bigwedge_{p=qr} [\bigvee_{a \in \Gamma(q), b \in \Gamma(r)} (\lambda_K \circ \lambda_B)(ab)] = \bigwedge_{p=qr} [\bigvee_{ab \in \Gamma(qr)} (\lambda_K \circ \lambda_B)(ab)],$$

since $ab \in \Gamma(q)\Gamma(r) = \Gamma(qr)$

$$= \bigwedge_{p=qr} \Gamma^+(\lambda_K \circ \lambda_B)(qr) = \Gamma^+(\lambda_K \circ \lambda_B)(p)$$

Thus, we have $\Gamma^+(K) \circ \Gamma^+(B) \subseteq \Gamma^+(K \circ B)$.

□

This theorem examines the preservation of intuitionistic fuzzy sets (IFS) under set-valued homomorphisms between BE-algebras. Consider two BE-algebras, Ω_1 and Ω_2 , and a set-valued homomorphism $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$. If K is an IFS in Ω_2 , then the homomorphism Γ preserves the operations involving the α - and β -cuts of K . Specifically, for any $\alpha, \beta \in [0, 1]$, the following equalities hold: $\Gamma^+(K)^{\alpha, \beta} = \Gamma^+(K^{\alpha, \beta})$ and $\Gamma^{-1}(K)^{\alpha, \beta} = \Gamma^{-1}(K^{\alpha, \beta})$. These results show how the structure of IFSs is maintained under homomorphisms with respect to their α - and β -cuts.

Theorem 13. Let Ω_1 and Ω_2 be two BE-algebras, and let $\Gamma : \Omega_1 \rightarrow P^*(\Omega_2)$ be a set-valued homomorphism. If K is an IFS of Ω_2 , then for any $\alpha, \beta \in [0, 1]$, the following hold:

- (1) $(\Gamma^+(K))^{\alpha, \beta} = \Gamma^+(K^{\alpha, \beta})$
- (2) $(\Gamma^{-1}(K))^{\alpha, \beta} = \Gamma^{-1}(K^{\alpha, \beta})$

Proof.

1. Consider $p \in (\Gamma^+(K))^{\alpha, \beta}$. We have: $p \in (\Gamma^+(K))^{\alpha, \beta} \iff \Gamma^+(\nu_K)(p) \geq \alpha$ and $\Gamma^+(\lambda_K)(p) \leq \beta \iff \bigwedge_{a \in \Gamma(p)} (\nu_K)(a) \geq \alpha$ and $\bigvee_{a \in \Gamma(p)} (\lambda_K)(a) \leq \beta \iff \forall a \in \Gamma(p), (\nu_K)(a) \geq \alpha$ and $(\lambda_K)(a) \leq \beta \iff \Gamma(p) \subseteq K^{\alpha, \beta} \iff p \in \Gamma^+(K^{\alpha, \beta})$
2. Now, consider $p \in (\Gamma^{-1}(K))^{\alpha, \beta}$. We have: $p \in (\Gamma^{-1}(K))^{\alpha, \beta} \iff \Gamma^{-1}(\nu_K)(p) \geq \alpha$ and $\Gamma^{-1}(\lambda_K)(p) \leq \beta \iff \bigvee_{a \in \Gamma(p)} (\nu_K)(a) \geq \alpha$ and $\bigwedge_{a \in \Gamma(p)} (\lambda_K)(a) \leq \beta \iff \forall a \in \Gamma(p), (\nu_K)(a) \geq \alpha$ and $(\lambda_K)(a) \leq \beta \iff \Gamma(p) \cap K^{\alpha, \beta} \neq \emptyset \iff p \in \Gamma^{-1}(K^{\alpha, \beta})$

□

5. Applications of Γ -Rough Intuitionistic Fuzzy Filters in Artificial Intelligence and Medical Diagnosis

The applications discussed in this section are intended to be theoretical and illustrative. Their purpose is to demonstrate how the proposed framework of rough intuitionistic fuzzy filters and Γ -rough intuitionistic fuzzy filters may be conceptually integrated into artificial intelligence and medical decision-support settings. No empirical experiments, clinical validation, or performance comparisons are claimed in this work. Rather, the focus is on highlighting the potential modeling advantages offered by the algebraic structure of rough intuitionistic fuzzy filters when dealing with uncertainty, vagueness, and incomplete information.

In a classical fuzzy filter framework, uncertainty is represented solely by a membership function $\nu : \Omega \rightarrow [0, 1]$, which quantifies the degree to which an element belongs to a filter. While effective in modeling gradual transitions, this approach does not explicitly distinguish between rejection and hesitation, nor does it account for indiscernibility or granularity in the underlying information structure.

In contrast, rough intuitionistic fuzzy filters extend fuzzy filters in two essential ways. First, intuitionistic fuzzy sets introduce both membership and non-membership functions, allowing acceptance, rejection, and hesitation to be modeled simultaneously. Second, rough approximations provide lower and upper bounds induced by congruence relations or set-valued homomorphisms, thereby capturing uncertainty arising from incomplete knowledge or indistinguishable elements.

As a result, rough intuitionistic fuzzy filters enable a three-level interpretation of decision states: necessarily accepted elements (lower approximation), possibly accepted elements (upper approximation), and rejected elements. This layered representation offers a more expressive and structured treatment of uncertainty than standard fuzzy filters, while

preserving algebraic consistency within BE-algebras. Consequently, the proposed framework is particularly well suited for theoretical modeling of uncertainty-aware reasoning in implication-based systems.

Γ -rough intuitionistic fuzzy filters (RIFFs) constitute a mathematically rigorous framework for modeling uncertainty and imprecision in logical reasoning, intelligent decision-making, medical diagnosis, and artificial intelligence (AI) [1–3,6,9]. By extending classical filter theory within BE-algebras, Γ -RIFFs combine rough set approximations with intuitionistic fuzzy membership structures, thereby enabling the systematic treatment of ambiguity, incompleteness, and hesitation in complex systems.

From an algebraic perspective, filters in BE-algebras represent consistent and stable decision states under implication-based reasoning. The introduction of rough and intuitionistic fuzzy approximations enriches this structure by allowing decisions to be evaluated through lower and upper bounds, rather than crisp membership alone. Consequently, Γ -RIFFs provide a refined decision framework in which certainty, possibility, and rejection coexist within a unified mathematical model.

A key advantage of Γ -RIFFs lies in their ability to represent multi-level uncertainty in environments where traditional fuzzy or probabilistic models are insufficient. Rough approximations capture indiscernibility and data granularity, while intuitionistic fuzzy logic explicitly models acceptance, rejection, and hesitation [7,8]. This combination is particularly effective in domains such as medical diagnosis and artificial intelligence, where data may be incomplete, contradictory, or noisy.

Moreover, Γ -RIFFs enhance AI-driven decision-support systems by enforcing algebraic consistency while improving interpretability and robustness [10,11]. Their integration into intelligent systems supports reliable reasoning in applications such as risk assessment, anomaly detection, autonomous decision-making, and predictive analytics. Recent advances in artificial intelligence highlight the importance of spatio-temporal and interaction-aware models, particularly in trajectory prediction and complex decision-making systems [26]. Deep learning techniques, particularly convolutional neural networks, have demonstrated strong performance in complex vision-based tasks such as stereo image segmentation and structured scene understanding [27].

This section examines the applicability of Γ -RIFFs in real-world contexts, with particular emphasis on:

- Improving AI-based decision-making under uncertainty;
- Supporting medical diagnosis through structured uncertainty modeling;
- Strengthening AI-driven risk analysis in finance, cybersecurity, and autonomous systems.

5.1. Illustrative AI Decision-Support Pipeline Based on Γ -RIFFs

To clarify how Γ -rough intuitionistic fuzzy filters (RIFFs) may be employed in artificial intelligence and medical decision-support systems, we present a brief, self-contained illustrative pipeline. This example is intended to demonstrate the *integration mechanism* of RIFFs within a standard AI workflow rather than to propose a complete diagnostic or clinical deployment model.

Input. Let P denote a patient case characterized by a collection of heterogeneous observations

$$X_P = \{\text{vital signs, laboratory tests, symptoms, medical history}\}.$$

Let $(\Omega, *, 1)$ be a BE-algebra whose elements represent diagnostic hypotheses or decision rules.

Step 1: Intuitionistic fuzzy encoding. Each observation $x \in X_P$ is mapped to an intuitionistic fuzzy set

$$K_x = (\nu_x, \lambda_x)$$

on Ω , where $\nu_x(h)$ and $\lambda_x(h)$ represent the degrees of support and rejection of hypothesis $h \in \Omega$, respectively.

Step 2: Rough approximation via a set-valued homomorphism. Let $\Gamma : \Omega_1 \rightarrow \mathcal{P}^*(\Omega_2)$ be a set-valued homomorphism capturing uncertainty induced by incomplete or noisy data. The lower and upper approximations of each K_x are computed as

$$\Gamma^+(K_x), \quad \Gamma^-(K_x),$$

yielding a Γ -rough intuitionistic fuzzy representation.

Step 3: Aggregation and filter verification. Aggregated acceptance and rejection degrees are obtained using BE-algebraic operations:

$$K_P = \bigcap_{x \in X_P} \Gamma^+(K_x).$$

If both $\Gamma^+(K_P)$ and $\Gamma^-(K_P)$ satisfy the intuitionistic fuzzy filter conditions, then K_P constitutes a Γ -rough intuitionistic fuzzy filter.

Step 4: Decision-support output. Hypotheses belonging to the lower approximation $\Gamma^+(K_P)$ are interpreted as *necessarily supported* decisions, while those in $\Gamma^-(K_P) \setminus \Gamma^+(K_P)$ are considered *possibly supported*. These outputs can be provided to clinicians or AI systems as ranked recommendations with explicit uncertainty bounds.

Remark 1. This pipeline illustrates how Γ -RIFFs function as an algebraic uncertainty-aware representation layer within AI decision-support systems. Empirical validation, calibration, and clinical assessment are required before any real-world deployment.

5.2. Application of Γ -RIFFs in Medical Diagnosis

Medical diagnosis inherently involves uncertainty due to incomplete patient data, overlapping symptoms, and imprecise clinical observations [5,10]. Γ -rough intuitionistic fuzzy filters provide a formal mechanism for managing such uncertainty by defining lower and upper approximations of potential diagnoses within an algebraic framework. These approximations distinguish between diagnoses that are certainly supported, possibly supported, or rejected, thereby improving clinical decision-making.

5.2.1. Problem Context and Diagnostic Uncertainty

In many clinical scenarios, different diseases share common symptoms, making precise diagnosis difficult. Consider a patient presenting with respiratory symptoms, where possible diagnoses include:

1. Asthma;
2. Chronic Obstructive Pulmonary Disease (COPD);
3. Pneumonia;
4. Bronchitis.

Common symptoms such as wheezing, coughing, shortness of breath, chest tightness, and fever often overlap among these conditions. This overlap introduces diagnostic ambiguity that cannot be adequately addressed using crisp classification methods alone. To resolve this issue, Γ -RIFFs are employed to model diagnostic uncertainty algebraically [1–3,28].

5.2.2. Intuitionistic Fuzzy Representation and Rough Approximation

Each symptom–disease relationship is represented using intuitionistic fuzzy membership and non-membership degrees [11]. The corresponding membership and non-membership values for selected respiratory diseases are summarized in Table 2. For instance, the association between wheezing and asthma may be modeled as

$$\nu_{S_1}(D_3) = 0.8, \quad \lambda_{S_1}(D_3) = 0.1.$$

indicating strong diagnostic support with limited contradiction and an explicit hesitation margin.

Table 2. Membership and non-membership values for respiratory diseases. [21]

Disease	Membership (ν)	Non-Membership (λ)
D_1 (Pneumonia)	0.7	0.2
D_2 (Bronchitis)	0.6	0.3
D_3 (Asthma)	0.8	0.1

Using Γ -approximations, the diagnostic bounds are computed as

$$\Gamma^+(D_j) = \min\{\mu(D_j)\}, \quad \Gamma^-(D_j) = \max\{\lambda(D_j)\}.$$

As observed from Table 2, asthma (D_3) attains the highest membership degree and the lowest non-membership degree, yielding the most favorable lower and upper approximation values. Consequently, asthma emerges as the most plausible diagnosis, supported by both conservative and optimistic evaluations.

5.2.3. Integration with AI-Based Diagnostic Systems

When embedded into AI-driven diagnostic frameworks, Γ -RIFFs can enhance classification reliability by providing an algebraic mechanism for modeling uncertainty and imprecision [6,9]. Machine learning models enhanced with rough intuitionistic fuzzy reasoning refine their predictions by incorporating approximation-based uncertainty bounds, leading to improved performance in ambiguous or borderline cases.

Let D_j denote a possible diagnosis and $S_1, \dots, S_n$ the observed symptoms. The AI system evaluates

$$A(D_j) = \min\{\Gamma^+(S_i)(D_j) : i = 1, \dots, n\}, \quad B(D_j) = \max\{\Gamma^-(S_i)(D_j) : i = 1, \dots, n\},$$

where $A(D_j)$ and $B(D_j)$ represent the guaranteed and possible diagnostic confidence levels, respectively.

5.3. Application of Γ -RIFFs in Artificial Intelligence

Artificial intelligence (AI) systems frequently operate in environments characterized by vagueness, incomplete information, and conflicting evidence [6,9,16]. Classical deterministic or probabilistic approaches often impose crisp decision boundaries, which may fail to capture the intrinsic uncertainty of real-world data. Γ -rough intuitionistic fuzzy filters (Γ -RIFFs) address these limitations by embedding uncertainty directly into algebraic decision rules defined on BE-algebras, thereby enhancing adaptability, robustness, and logical consistency [3,7]. Recent advances in deep generative models have enabled reference-based image generation and reconstruction from sparse or symbolic inputs, such as line art drawings, demonstrating the effectiveness of deep neural architectures in uncertainty-aware visual reasoning [29].

From a mathematical standpoint, AI decision-making processes can be modeled within a BE-algebra $(\Omega, *, 1)$, where elements of Ω represent hypotheses, actions, or classification rules, and the operation $*$ encodes implication-based reasoning between them. Filters in Ω correspond to coherent decision states that are stable under inference. Introducing intuitionistic fuzzy membership functions allows each decision to be evaluated with degrees of acceptance and rejection, while Γ -based rough approximations model uncertainty arising from indiscernibility, noise, or incomplete observations.

5.3.1. Decision-Making and Uncertainty Modeling

Let $K = (\nu_K, \lambda_K)$ be an intuitionistic fuzzy set on Ω , where $\nu_K(x)$ and $\lambda_K(x)$ denote the degrees of acceptance and rejection of a hypothesis $x \in \Omega$, respectively. In the presence of uncertainty induced by multiple interpretations of input data, a set-valued homomorphism Γ defines lower and upper approximations of K :

$$\Gamma^+(K) = (\Gamma^+(\nu_K), \Gamma^+(\lambda_K)), \quad \Gamma^-(K) = (\Gamma^-(\nu_K), \Gamma^-(\lambda_K)).$$

If both $\Gamma^+(K)$ and $\Gamma^-(K)$ form intuitionistic fuzzy filters of Ω , then K is a Γ -rough intuitionistic fuzzy filter. Algebraically, this guarantees that accepted AI decisions satisfy closure and consistency properties under implication, even when uncertainty is present.

In AI-based expert systems, decisions can thus be classified into three algebraically meaningful regions:

- Decisions belonging to the lower approximation $\Gamma^-(K)$, representing necessarily accepted outcomes;
- Decisions belonging to the upper approximation $\Gamma^+(K)$ but not to $\Gamma^-(K)$, representing possibly accepted outcomes;
- Decisions outside the upper approximation, representing rejected outcomes.

This trichotomy provides a mathematically precise interpretation of uncertainty-aware decision-making, ensuring logical coherence while distinguishing certainty from plausibility [8].

Applications of this framework include expert systems, robotics, reinforcement learning, healthcare AI, and cybersecurity [9–11]. In cybersecurity, for example, network behaviors are mapped into Ω , and Γ -RIFFs enable the separation of definite threats (lower approximation) from suspicious but uncertain anomalies (upper approximation), thereby reducing false-positive rates.

5.3.2. Risk Assessment and Intelligent Control

In financial risk assessment, loan approval decisions depend on multiple uncertain indicators such as credit score, income stability, and debt-to-income ratio. Let $(\Omega, *, 1)$ be a BE-algebra whose elements represent risk-related decision rules, where the operation $*$ encodes implication-based reasoning between financial conditions and risk outcomes. An intuitionistic fuzzy set $K = (\nu_K, \lambda_K)$ on Ω assigns to each rule $x \in \Omega$ a degree of acceptance $\nu_K(x)$ and a degree of rejection $\lambda_K(x)$ derived from observed financial data.

Uncertainty arising from incomplete records, noisy measurements, or heterogeneous data sources is modeled via a set-valued homomorphism Γ , which induces rough approximations of K . The corresponding lower and upper risk bounds are defined by

$$R^+ = \inf\{\nu_K(x) \mid x \in \Gamma(y)\}, \quad R^- = \sup\{\lambda_K(x) \mid x \in \Gamma(y)\},$$

for a given risk assessment rule $y \in \Omega$. If both approximations form intuitionistic fuzzy filters, then K is a Γ -rough intuitionistic fuzzy filter, guaranteeing algebraic closure and consistency under implication.

From a decision-theoretic perspective, a loan decision is considered *reliable* if it belongs to the lower approximation R^+ and *admissible* if it belongs to the upper approximation R^{-1} . This algebraic distinction replaces crisp binary decisions with a structured hierarchy of certainty, enabling AI systems to recommend conditional approvals, adaptive interest rates, or additional verification steps. As a result, decision-making becomes both fairer and more robust under uncertainty.

A similar formulation applies to intelligent control in autonomous systems and robotics, where sensor measurements are frequently noisy or partially observable. Let Ω denote a BE-algebra of control rules derived from sensor inputs and system constraints. Applying Γ -rough intuitionistic fuzzy filters ensures that real-time control decisions remain stable under uncertainty: actions corresponding to the lower approximation are guaranteed to satisfy safety constraints, while actions in the upper approximation are treated as potentially admissible but require caution or further validation. This algebraic separation significantly enhances system safety and performance in dynamic environments.

The mathematical structure of Γ -rough intuitionistic fuzzy filters further makes them well suited to emerging paradigms in artificial intelligence, including explainable AI, federated learning, healthcare analytics, quantum AI, autonomous systems, and advanced cybersecurity frameworks [9,17]. In explainable AI, the explicit representation of lower and upper approximations provides transparent justifications for model decisions by formally distinguishing certainty, possibility, and hesitation. This feature is essential for constructing trustworthy AI systems that require both rigorous uncertainty quantification and logical consistency.

Overall, Γ -rough intuitionistic fuzzy filters provide a mathematically grounded framework for uncertainty-aware reasoning in artificial intelligence and medical diagnosis. By integrating rough approximations, intuitionistic fuzzy logic, and BE-algebraic filter theory, they enable consistent, interpretable, and robust decision-making. Their successful application across AI, healthcare, finance, and cybersecurity highlights both their theoretical significance and practical relevance, paving the way for future advancements in intelligent systems. Recent advances in artificial intelligence further highlight the relevance of uncertainty-aware and structure-preserving models such as Γ -rough intuitionistic fuzzy filters. In particular, heterogeneous and multi-view learning frameworks have been shown to improve learner modeling and decision reliability in complex online environments, where incomplete and diverse data sources must be integrated coherently [30]. Such settings naturally motivate algebraic uncertainty models capable of handling indiscernibility and partial information.

Similarly, deep learning approaches for fault diagnosis and pattern recognition, including convolutional neural networks combined with advanced signal transforms, demonstrate that real-world diagnostic tasks often involve noisy, nonstationary, and ambiguous data [31]. Rough Intuitionistic Fuzzy frameworks provide a complementary mathematical perspective by offering interpretable lower and upper decision bounds rather than purely data-driven predictions.

Moreover, recent progress in generative models and transformer-based architectures for image reconstruction and completion further underscores the importance of robust reasoning under incomplete observations [32]. In such contexts, Γ -rough Intuitionistic Fuzzy Filters can be viewed as a theoretical decision-support layer that formalizes uncertainty handling and logical consistency, thereby complementing modern AI architectures rather than replacing them.

6. Conclusions

In this work, we have developed a unified algebraic framework for rough intuitionistic fuzzy filters in BE-algebras by integrating rough set approximations with intuitionistic fuzzy structures. Starting from fundamental definitions and preliminaries, we introduced lower and upper approximation operators induced by congruence relations and investigated their role in characterizing rough intuitionistic fuzzy filters. To extend the expressive power of the framework, we defined set-valued homomorphisms and established the notion of Γ -rough intuitionistic fuzzy filters, along with their main structural properties.

The results presented in this paper extend existing studies on fuzzy and rough structures in algebraic systems by focusing on BE-algebras and by employing generalized approximation mechanisms. Our findings clarify how intuitionistic fuzziness and rough approximations interact with filter theory in implication-based algebras, contributing to a deeper theoretical understanding of uncertainty modeling in algebraic logic.

The applications discussed in the paper are intended to be illustrative and conceptual, highlighting how the proposed framework may serve as a mathematical foundation for uncertainty-aware reasoning in areas such as artificial intelligence and medical decision-support systems. Future research may focus on developing computational models or algorithms based on Γ -rough intuitionistic fuzzy filters, as well as exploring empirical validation in specific application domains. Further investigations may also examine connections with other algebraic structures, order-theoretic frameworks, and logical systems, thereby broadening the scope and applicability of the proposed theory.

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List of Notations

Ω	Universe (domain of a BE-algebra)
$(\Omega, *, 1)$	BE-algebra with operation $*$ and identity 1
p, q, r	Elements of Ω
$\nu_K(p)$	Membership degree of p in an IFS K
$\lambda_K(p)$	Non-membership degree of p in K
$K = (\nu_K, \lambda_K)$	Intuitionistic fuzzy set
$\mathcal{N}$	Congruence or covering relation on Ω
$[p]_{\mathcal{N}}$	Equivalence class of p under $\mathcal{N}$
$\mathcal{N}^+(K)$	Lower rough approximation of K
$\mathcal{N}^-(K)$	Upper rough approximation of K
Γ	Set-valued homomorphism
$\Gamma^+(K)$	Γ -rough lower approximation of K
$\Gamma^-(K)$	Γ -rough upper approximation of K
$K_{\alpha, \beta}$	(α, β) -cut of an IFS K
$K_{\alpha, \beta}^s$	Strong (α, β) -cut
RIFF	Rough intuitionistic fuzzy filter
Γ -RIFF	Γ -rough intuitionistic fuzzy filter

Appendix A. Detailed Verification of Example 3

Step 1: A finite BE-algebra. Let $\Omega = \{1, a, b\}$ and define the operation $*$ by the Cayley table

*	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then $(\Omega, *, 1)$ is a BE-algebra.

Step 2: An intuitionistic fuzzy set on Ω .

Define an IF set $K = (v_K, \lambda_K)$ on Ω by

$$v_K(1) = 0.8, \quad v_K(a) = 0.7, \quad v_K(b) = 0.4,$$

$$\lambda_K(1) = 0.1, \quad \lambda_K(a) = 0.2, \quad \lambda_K(b) = 0.5.$$

Note that $0 \leq v_K(x) + \lambda_K(x) \leq 1$ for all $x \in \Omega$, so K is well-defined.

Step 3: A congruence relation and its classes. Define a relation $\mathcal{N}$ on Ω by two equivalence classes

$$[1]_{\mathcal{N}} = \{1, a\}, \quad [b]_{\mathcal{N}} = \{b\}.$$

Equivalently,

$$\mathcal{N} = \{(1, 1), (a, a), (b, b), (1, a), (a, 1)\}.$$

It is straightforward to verify $\mathcal{N}$ is a congruence relation: if $x \mathcal{N} y$ then $x * z \mathcal{N} y * z$ for all $z \in \Omega$ (check cases using the table; the only nontrivial class is $\{1, a\}$, and $1 * z$ and $a * z$ always belong to the same class).

Step 4: Lower and upper approximations of K with respect to $\mathcal{N}$. For $p \in \Omega$, define the bottom (lower) and exceeding (upper) approximations by

$$\mathcal{N}_{\bullet}(v_K)(p) = \inf_{x \in [p]_{\mathcal{N}}} v_K(x), \quad \mathcal{N}_{\bullet}(\lambda_K)(p) = \sup_{x \in [p]_{\mathcal{N}}} \lambda_K(x),$$

$$\mathcal{N}^{\bullet}(v_K)(p) = \sup_{x \in [p]_{\mathcal{N}}} v_K(x), \quad \mathcal{N}^{\bullet}(\lambda_K)(p) = \inf_{x \in [p]_{\mathcal{N}}} \lambda_K(x).$$

Hence,

$$\mathcal{N}_{\bullet}(K) = (\mathcal{N}_{\bullet}(v_K), \mathcal{N}_{\bullet}(\lambda_K)), \quad \mathcal{N}^{\bullet}(K) = (\mathcal{N}^{\bullet}(v_K), \mathcal{N}^{\bullet}(\lambda_K)).$$

Since $[1]_{\mathcal{N}} = \{1, a\}$ and $[b]_{\mathcal{N}} = \{b\}$, we obtain the explicit values:

p	1	a	b
$\mathcal{N}_{\bullet}(v_K)(p)$	$\min\{0.8, 0.7\} = 0.7$	$\min\{0.8, 0.7\} = 0.7$	0.4
$\mathcal{N}_{\bullet}(\lambda_K)(p)$	$\max\{0.1, 0.2\} = 0.2$	$\max\{0.1, 0.2\} = 0.2$	0.5
$\mathcal{N}^{\bullet}(v_K)(p)$	$\max\{0.8, 0.7\} = 0.8$	$\max\{0.8, 0.7\} = 0.8$	0.4
$\mathcal{N}^{\bullet}(\lambda_K)(p)$	$\min\{0.1, 0.2\} = 0.1$	$\min\{0.1, 0.2\} = 0.1$	0.5

Step 5: Verifying that $\mathcal{N}_{\bullet}(K)$ is an IF filter. Recall that an IF set $F = (v_F, \lambda_F)$ is an IF filter if for all $p, q \in \Omega$:

1. $v_F(1) \geq v_F(p)$.
2. $v_F(q) \geq \min\{v_F(p * q), v_F(p)\}$
3. $\lambda_F(1) \leq \lambda_F(p)$
4. $\lambda_F(q) \leq \max\{\lambda_F(p * q), \lambda_F(p)\}$

Let $F = \mathcal{N}_{\bullet}(K)$. From the table above,

$$v_F(1) = 0.7 \geq v_F(a) = 0.7 \geq v_F(b) = 0.4, \quad \lambda_F(1) = 0.2 \leq \lambda_F(a) = 0.2 \leq \lambda_F(b) = 0.5,$$

so (1) and (3) hold.

To check (2) and (4), it suffices to consider the possible outputs of $p * q$ from the Cayley table:

Case A: $p = a$ or $p = b$. Then $p * q = 1$ for every $q \in \Omega$. Hence

$$\min\{\nu_F(p * q), \nu_F(p)\} = \min\{\nu_F(1), \nu_F(p)\} = \nu_F(p) \leq \nu_F(q)$$

because $\nu_F(q) \in \{0.7, 0.4\}$ and $\nu_F(p) \in \{0.7, 0.4\}$, and the only strict inequality needed occurs when $\nu_F(p) = 0.4$, which always satisfies $0.4 \leq \nu_F(q)$. Thus (2) holds. Similarly,

$$\max\{\lambda_F(p * q), \lambda_F(p)\} = \max\{\lambda_F(1), \lambda_F(p)\} = \lambda_F(p) \geq \lambda_F(q),$$

since $\lambda_F(p) \in \{0.2, 0.5\}$ and $\lambda_F(q) \in \{0.2, 0.5\}$; if $\lambda_F(p) = 0.2$ then $\lambda_F(q) \leq 0.5$ and the maximum becomes 0.2 only when $\lambda_F(q) = 0.2$, so the inequality holds. Hence (4) holds.

Case B: $p = 1$. Then, $1 * q = q$. Therefore,

$$\min\{\nu_F(1 * q), \nu_F(1)\} = \min\{\nu_F(q), 0.7\} = \nu_F(q),$$

so (2) becomes $\nu_F(q) \geq \nu_F(q)$, which is true. Likewise,

$$\max\{\lambda_F(1 * q), \lambda_F(1)\} = \max\{\lambda_F(q), 0.2\} = \lambda_F(q),$$

so (4) becomes $\lambda_F(q) \leq \lambda_F(q)$, true.

Thus, $\mathcal{N}_\bullet(K)$ is an IF filter of Ω .

Step 6: Verifying that $\mathcal{N}^\bullet(K)$ is an IF filter. Let $G = \mathcal{N}^\bullet(K)$. From the table,

$$\nu_G(1) = 0.8 \geq \nu_G(a) = 0.8 \geq \nu_G(b) = 0.4, \quad \lambda_G(1) = 0.1 \leq \lambda_G(a) = 0.1 \leq \lambda_G(b) = 0.5,$$

so (1) and (3) hold.

For (2) and (4), the same case split works:

Case A: $p = a$ or $p = b$. Then $p * q = 1$ and

$$\min\{\nu_G(p * q), \nu_G(p)\} = \min\{\nu_G(1), \nu_G(p)\} = \nu_G(p) \leq \nu_G(q),$$

since $\nu_G(p) \in \{0.8, 0.4\}$ and $\nu_G(q) \in \{0.8, 0.4\}$. Also,

$$\max\{\lambda_G(p * q), \lambda_G(p)\} = \max\{\lambda_G(1), \lambda_G(p)\} = \lambda_G(p) \geq \lambda_G(q),$$

because $\lambda_G(p) \in \{0.1, 0.5\}$.

Case B: $p = 1$. Then $1 * q = q$ and the inequalities reduce to equalities as before.

Hence, $\mathcal{N}^\bullet(K)$ is an IF filter of Ω .

Both $\mathcal{N}_\bullet(K)$ and $\mathcal{N}^\bullet(K)$ are intuitionistic fuzzy filters of Ω . Therefore, K is a rough intuitionistic fuzzy filter with respect to the congruence relation $\mathcal{N}$.

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