

Article

Delay-Induced Stability Transitions and Hopf Bifurcation in a Model of Cumulative Hot Pollutant Concentration

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Abstract

The present work provides a detailed discussion of the dynamical behavior of the delay-induced model of cumulative concentration of hot pollutants, including the contribution of the time-delay parameter to the system's stability. Analytical results indicate that time delay is a bifurcation mechanism that leads to a critical threshold, at which a steady state loses asymptotic stability and a Hopf bifurcation occurs. The directional analysis is carried out to further explain the behavior of the system in the neighborhood of this transition, and this offers some understanding of the nature and stability of the resulting periodic solutions, as well as the qualitative evolution. Numerical simulations are done on representative parameter values to support the theoretical results. Comprehensively, the findings reveal the strong dependence of the accumulation processes of pollutants on the effects of time delays and the significance of considering the temporal lags in environmental modeling. The study provides a viable analytical and numerical system of interpreting transitions caused by delays in pollutant concentration systems.

Keywords: pollutant concentration dynamics; directional analysis; Hopf bifurcation; stability; delay

1. Introduction

Mathematical modeling is essential for atmospheric processes involving water vapor, cloud droplets, and precipitation. Initial theoretical investigations indicated that evaporation, condensation, drop formation, and rainfall are time-dependent processes resulting from nonlinear interactions among atmospheric constituents [1–3]. Such processes are commonly modeled by systems of differential equations that describe how the vapor density, cloud droplets, and raindrops vary over time.

Cloud microphysics shows that both natural processes of growth and depletion, as well as interactions among system variables, control how water vapor is converted into cloud droplets and then into rainfall [4,5]. Nonlinear models are primarily used to understand feedbacks, threshold dynamics, and steady-state dynamics in real atmospheric systems [6,7].

Atmospheric pollutants significantly affect cloud and precipitation processes. Experimental and modeling research has found that pollutants may reduce the efficacy of condensation, decrease droplet size, and delay the onset of rainfall [8,9]. The high levels of pollutants prevent the formation of cloud droplets, thereby increasing cloud duration and

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decreasing precipitation efficiency [10]. Mathematical and numerical models have extensively investigated the role of pollutant concentration as a dynamical quantity in cloud microphysics [11,12]. As it has been shown by numerous theoretical and empirical works, the faster the pollutant is removed, the greater the level of water droplet densities [13]. According to these results, coupled models of precipitation and pollution with washout processes are routinely considered [14].

Some of the proposed practical interventions aimed at minimizing atmospheric pollutant levels include fogging systems, mist sprays, and artificial rain technologies [15,16]. The studies also show that the introduced water droplets undergo natural depletion through evaporation and contact with atmospheric pollutants [17].

One of the fundamental properties of atmospheric systems is time delay, because most physical and chemical processes are not instantaneous. Transport times, chemical reaction latencies, or slow interactions between pollutants and water droplets may lead to delays [18,19]. Delays in responses of environmental and ecological systems are usually modeled by delay differential equations [20]. Recent studies use fractional-order models and refined numerical techniques to study alcohol dependence, nonlinear delay differential equations, and singly perturbed integro-differential equations, and the accuracy is verified [21,22].

The dynamics in the atmosphere are complicated with nonlinear interactions between water vapor, cloud droplets, raindrops, and pollutants, which need mathematical modeling to be an important tool in the study of the dynamics. But time delay is one basic characteristic of actual environmental processes that have frequently been simplified or ignored. Pollutant transport, mixing, thermal conditioning, and washout by water droplets do not happen in reality instantaneously. It will always be finite until pollutants are effectively able to interact with mitigation measures. Such delays can be overlooked and result in incomplete forecasts of accumulation and removal processes of pollutants. Based on this observation, the current paper is an attempt to study the effects of adding a time-delay parameter in the pollutant washout term on the overall stability and qualitative behavior of a coupled vapor-cloud-pollution system.

The primary value of the research is the development and thorough investigation of a delay differential equation of cumulative concentration of hot pollutants with a realistic delay of washout. It defines time delay as a key bifurcation parameter that destabilize the steady state and cause Hopf bifurcation-based oscillations. The work offers clear analytical and computational understanding on delay-induced transitions in the pollutant dynamics through directional analysis and numerical simulations of supporting numerical simulations.

2. Mathematical Model

According to this model, the densities of water vapor, cloud droplets, raindrops, and water droplets, as well as the concentrations of pollutants, are treated as time-varying variables. The interaction between these elements occurs through natural development and decay, as pollutants affect various aspects of the system. The time delay in the pollutant term is used to model the slow response of the pollutant washout due to delays in reacting with the water droplets. The time delay is introduced only in the pollutant washout term $b_1 C_{HP}(t - \tau) W_D(t)$ to represent the finite time required for hot pollutants to undergo transport, mixing, and thermal conditioning before being effectively removed by water droplets. The water droplet density W_D is treated as an immediately controllable variable with no intrinsic memory, so no delay is imposed on it.

$$\frac{dD_v}{dt} = V_F - r_0 D_v - r_1 D_v C_{HP} + \lambda_1 C_d C_{HP} \quad (1)$$

$$\frac{dC_d}{dt} = aD_v - a_0C_d - a_1C_dC_{HP} \quad (2)$$

$$\frac{dC_r}{dt} = RC_d - R_0C_r \quad (3)$$

$$\frac{dC_{HP}}{dt} = b_0C_{HP} - b_1C_{HP}(t - \tau)W_D \quad (4)$$

$$\frac{dW_D}{dt} = cC_{HP} - c_0W_D - c_1CW_D \quad (5)$$

where the parameters and variables are as follows: D_v represents the density of vapors, C_d shows the cloud droplets, C_r shows raindrops, C_{HP} represent the cumulative concentration of hot pollutants in the surrounding, W_D shows the density of water drops introduced to reduce the cumulative concentration of hot pollutants in the surrounding. V_F shows the constant rate of vapor formation, r_0 is the natural rate of depletion of water vapors, r_1 is the rate of decrease in water vapors due to interaction with hot pollutants, a is the rate of formation of cloud droplets, a_0 is the natural rate of depletion of droplets, a_1 is the rate of decrease in water cloud droplets due to interaction with hot pollutants, R is the rate of formation of raindrops, R_0 is the natural rate of depletion of raindrop, b_0 is the natural rate of depletion of hot pollutants, and b_1 is the rate of depletion of hot pollutants due to washout by water drops. c is the rate of inflow of water drops in the atmosphere, c_0 is the natural rate of depletion of water drops and c_1 is the rate of depletion of water drops due to hot pollutants. All state variables are in terms of concentrations L^{-1} , time is taken in hours (h), the first-order rate parameters are in the units of h^{-1} , parameters of interaction are measured in Lh^{-1} , and the rate of formation of the vapor V_F is measured in units of $L^{-1} h^{-1}$.

Positivity of the dynamical model: We define the state vector

$$X(t) = (D_v(t), C_d(t), C_r(t), C_{HP}(t), W_D(t)) \in \mathbb{R}_{\geq 0}^5. \quad (6)$$

Parameters are all non-negative and the right-hand side is locally Lipschitz, which implies local existence and uniqueness.

Theorem 1 (Positivity Invariance). *Under initial conditions:*

$$D_v(0), C_d(0), C_r(0), C_{HP}(0), W_D(0) \geq 0,$$

then the solution is non-negative to all $t > 0$.

$$X(t) \in \mathbb{R}_{\geq 0}^5 \forall t \geq 0.$$

Proof. We verify that all the variables do not enter the negative orthant. This is based on the classical Nagumo Invariance Condition [23]:

Assuming that the formula of a variable $x(t)$ takes the form

$$f(x) |_{x=0} \geq 0,$$

then the set $x \geq 0$ is positively invariant. $\square$

We examine equation-by-equation.

1. Vapor density D_v :

At the boundary $D_v = 0$,

$$\frac{dD_v}{dt} |_{D_v=0} = V_F + \lambda_1 C_d C_{HP} \geq 0.$$

Thus, D_v cannot be negative.

2. Cloud droplets C_d :

At $C_d = 0$,

$$\frac{dC_d}{dt} \big|_{C_d=0} = aD_v \geq 0.$$

Hence, $C_d \geq 0$ is invariant.

3. Raindrops C_r :

At $C_r = 0$,

$$\frac{dC_r}{dt} \big|_{C_r=0} = RC_d \geq 0.$$

Thus, C_r will not therefore be negative.

4. Hot pollutant concentration C_{HP} :

At $C_{HP} = 0$, the equation becomes

$$\frac{dC_{HP}}{dt} \big|_{C_{HP}=0} = 0.$$

Accordingly, the boundary is implicit and cannot pass over the negative values.

5. Water drops W_D :

At $W_D = 0$,

$$\frac{dW_D}{dt} \big|_{W_D=0} = cC_{HP} \geq 0.$$

Thus, $W_D \geq 0$ is invariant. For each component x_i , we have

$$\dot{x}_i \big|_{x_i=0} \geq 0,$$

And the non-negative orthant is, therefore, positively invariant.

Boundedness of dynamical system:

Theorem 2. Assume that $X(t) = (D_v, C_d, C_r, C_{HP}, W_D)$ is any of the solutions of (1) whose initial data are nonnegative. Then, there is a constant $M > 0$, which only depends on the parameters and initial data and such that

$$\sup_{t \geq 0} \|X(t)\| \leq M,$$

and in fact

$$\limsup_{t \rightarrow \infty} \|X(t)\| \leq M^*,$$

where $M^* > 0$ is an explicit computable constant.

Proof. Using standard techniques of dissipative population models and delay differential equations of Hale and Lunel [24], Smith [25], and Brauer and Chavez [26]. $\square$

We construct a Lyapunov-like linear functional

$$S(t) = \alpha_1 D_v(t) + \alpha_2 C_d(t) + \alpha_3 C_r(t) + \alpha_4 C_{HP}(t) + \alpha_5 W_D(t), \quad (7)$$

with constants $\alpha_i > 0$ to be chosen. Differentiate (2) w.r.t (1):

$$\begin{aligned} \dot{S}(t) = & \alpha_1 (V_F - r_0 D_v - r_1 D_v C_{HP} + \lambda_1 C_d C_{HP}) \\ & + \alpha_2 (a D_v - a_0 C_d - a_1 C_d C_{HP}) \\ & + \alpha_3 (R C_d - R_0 C_r) \\ & + \alpha_4 (b_0 C_{HP} - b_1 C_{HP}(t - \tau) W_D) \\ & + \alpha_5 (c C_{HP} - c_0 W_D - c_1 C_{HP} W_D). \end{aligned} \quad (8)$$

Choose

$$\alpha_1 = 1, \alpha_2 = \frac{a_0}{2a}, \alpha_3 = \frac{R_0}{2R}, \alpha_4 = 1, \alpha_5 = \frac{c_0}{2c}. \quad (9)$$

This ensures

$$\alpha_2 a D_v \leq \frac{r_0}{2} D_v + C_1, \alpha_3 R C_d \leq \frac{a_0}{2} C_d + C_2, \quad (10)$$

by Young's inequality

$$xy \leq \frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} y^2, \quad (11)$$

and since $x^2 \leq kx$ for all $x \geq 0$ once k is chosen large.

In this way, the negative linear dissipation will always predominate on the positive linear production terms. Every bilinear term, such as

$$D_v C_{HP}, C_d C_{HP}, C_{HP} W_D \quad (12)$$

is enclosed in the Young's inequality. For any $\varepsilon > 0$,

$$xy \leq \varepsilon x + \frac{1}{4\varepsilon} y^2. \quad (13)$$

according to Lakshmikantham and Leela [27]. This being a positive system, we may continue to use

$$y^2 \leq Ky + K^2, \quad (14)$$

and assuming that there is a (distinctly) small change in y , the sum of all the bilinear positive terms is dominated by

$$xy \leq \varepsilon x + Cy + C', \quad (15)$$

for explicit constants $C, C' > 0$.

Replacing (15) by (8) and provided that $\varepsilon > 0$ is small enough, all contributions of bilinear positive terms are dominated by

$$\delta_1 D_v + \delta_2 C_d + \delta_3 C_r + \delta_4 C_{HP} + \delta_5 W_D + K_0, \quad (16)$$

in which $\delta_i > 0$ may be arbitrarily small.

The linear part of (13) contains

$$-r_0 D_v - a_0 C_d - R_0 C_r - c_0 W_D. \quad (17)$$

On subtraction of the little contributions δ_i coming from Step 3, we have

$$\dot{S}(t) \leq \alpha_1 V_F + K_0 - \mu_1 D_v - \mu_2 C_d - \mu_3 C_r - \mu_4 C_{HP} - \mu_5 W_D, \quad (18)$$

where

$$\mu_i = \text{original dissipation} - \delta_i > 0. \quad (19)$$

Finally, as both the variables are positive coefficients of $S(t)$, we can have

$$D_v, C_d, C_r, C_{HP}, W_D \geq cS(t) \text{ for } c = \min_i \alpha_i. \quad (20)$$

Thus

$$\dot{S}(t) \leq K - \mu S(t), \quad (21)$$

where

$$K = \alpha_1 V_F + K_0, \mu = c \min \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\} > 0. \quad (22)$$

the differential inequality

$$\dot{S}(t) \leq K - \mu S(t).$$

The standard comparison lemma of ODEs is as follows.

$$S(t) \leq S(0)e^{-\mu t} + \frac{K}{\mu}(1 - e^{-\mu t}). \quad (23)$$

Thus,

$$\limsup_{t \rightarrow \infty} S(t) \leq \frac{K}{\mu}. \quad (24)$$

As each of the variables is restricted by $S(t)/c$, then the variables are all ultimately bound in a uniform manner. Therefore, the solution is always contained in a small subset of $\mathbb{R}_{\geq 0}^5$, and the blow-up cannot be suffered.

Equilibrium point:

Using linear equation C_r and C_d we find (the denominators below assume $\neq 0$)

$$C_r^* = \frac{R}{R_0} C_d^*, C_d^* = \frac{a D_v^*}{a_0 + a_1 C_{HP}^*}, (a_0 + a_1 C_{HP}^* \neq 0). \quad (25)$$

Replacement of C_d^* into the D_v equation at equilibrium to eliminate C_d^* . Define the auxiliary scalar

$$G(z) = r_0 + r_1 z - \frac{\lambda_1 a z}{a_0 + a_1 z}, z \geq 0. \quad (26)$$

If $G(C_{HP}^*) \neq 0$ then

$$D_v^* = \frac{V_F}{G(C_{HP}^*)} \quad (27)$$

Equivalently (G), with $z = C_{HP}^*$,

$$D_v^* = \frac{V_F (a_0 + a_1 z)}{(r_0 + r_1 z)(a_0 + a_1 z) - \lambda_1 a z} \quad (28)$$

Then, C_d^* and C_r^* become

$$C_d^* = \frac{a V_F}{(a_0 + a_1 z) G(z)}, C_r^* = \frac{R a V_F}{R_0 (a_0 + a_1 z) G(z)} \quad (29)$$

Thus, all balances are fixed at a value $z = C_{HP}^* \geq 0$, which is a solution to the pollutant–water algebraic subsystem that follows the past two equations.

Pollutant–water subsystem (two branches):

The C_{HP} equation gives us the equilibrium, and, as a result, the following equation:

$$0 = C_{HP}^* (b_0 - b_1 W_D^*).$$

Hence, either $C_{HP}^* = 0$, or, if $C_{HP}^* \neq 0$, then

$$W_D^* = \frac{b_0}{b_1} \text{ (requires } b_1 \neq 0 \text{)}. \quad (30)$$

Pollutant-free equilibrium ($C_{HP}^* = 0$)

Set $z = 0$. At equilibrium W_D simplifies $-c_0 W_D^* = 0$, so (assuming $c_0 > 0$) $W_D^* = 0$. Applying (8) and (9) with $z = 0$, we find that there is a unique pollutant-free equilibrium:

$$E_0 = (D_v^*, C_d^*, C_r^*, C_{HP}^*, W_D^*) = \left(\frac{V_F}{r_0}, \frac{a V_F}{a_0 r_0}, \frac{R a V_F}{R_0 a_0 r_0}, 0, 0 \right) \quad (31)$$

(If $c_0 = 0$ degeneracy occurs: W_D^* is undetermined by the last eqn when $C_{HP}^* = 0$; treat separately.)

Endemic (pollutant-present) equilibrium ($C_{HP}^* > 0$):

Assume $C_{HP}^* = z > 0$. Then $W_D^* = b_0/b_1$ (require $b_1 \neq 0$). Replace in W_D -equation (steady state):

$$0 = cz - c_0 \frac{b_0}{b_1} - c_1 z \frac{b_0}{b_1}.$$

Solve for z (provided $b_1 c - b_0 c_1 \neq 0$):

$$C_{HP}^* = z = \frac{b_0 c_0}{b_1 c - b_0 c_1} \quad (32)$$

Existence/positivity condition for this branch:

$$b_1 \neq 0, b_1 c - b_0 c_1 \neq 0, \text{ and } C_{HP}^* > 0, \text{ i.e., } b_1 c - b_0 c_1 > 0 \text{ if } b_0, c_0 > 0. \quad (33)$$

then (32) gives a positive z and $G(z) \neq 0$ and then the rest of the coordinates are obtained uniquely by (31) and (32):

$$\begin{aligned} W_D^* &= \frac{b_0}{b_1}, D_v^* = \frac{V_F(a_0 + a_1 z)}{(r_0 + r_1 z)(a_0 + a_1 z) - \lambda_1 a z} \\ C_d^* &= \frac{a V_F}{(a_0 + a_1 z) G(z)}, C_r^* = \frac{R a V_F}{R_0(a_0 + a_1 z) G(z)} \end{aligned} \quad (34)$$

Ensure the denominator D_v^* to be positive, i.e.,

$$(r_0 + r_1 z)(a_0 + a_1 z) - \lambda_1 a z > 0 \Leftrightarrow G(z) > 0. \quad (35)$$

Assuming (33), (34) are true, the pollutant-present equilibrium E^* is unique (the pollutant subsystem takes on only one algebraic value of z and the other ones are algebraic).

If (33) and (34) hold, the pollutant-present equilibrium E^* exists and is unique (the pollutant subsystem yields a single algebraic value of z and the rest follow algebraically).

Degenerate cases:

- $b_1 = 0$: then Equation (4) gives $b_0 C_{HP}^* = 0$. When $b_0 \neq 0$, then $C_{HP}^* = 0$ (only clean, pollutant-free branch). When $b_0 = b_1 = 0$, there is further degeneracy and Equation (4) gives no constraint; solve Equation (5) jointly possible continua of equilibria.
- $b_1 c - b_0 c_1 = 0$: Formula (6) is singular. Then, Equation (4–5) is dependent, and either there is no endemic solution of the interior or a continuum might be seen, solving Equation (4), Equation (5) at the same time (they become linearly dependent).
- $a_0 + a_1 z = 0$ or $G(z) = 0$: denominators disappear in (41) and (42); treat separately (non-generic for standard positive parameters $a_0, a_1 \geq 0$).

Pollutant-free equilibrium (always present under standard dissipation):

$$E_0 = \left(\frac{V_F}{r_0}, \frac{a V_F}{a_0 r_0}, \frac{R a V_F}{R_0 a_0 r_0}, 0, 0 \right)$$

Endemic (pollutant-present) equilibrium (exists) iff (12) and (13) hold:

let $z = \frac{b_0 c_0}{b_1 c - b_0 c_1}$ (with $b_1 \neq 0$ and denominator $\neq 0$).

If $z > 0$ and $(r_0 + r_1 z)(a_0 + a_1 z) - \lambda_1 a z > 0$, then

$$\begin{aligned} C_{HP}^* &= z, W_D^* = \frac{b_0}{b_1}, D_v^* = \frac{V_F(a_0 + a_1 z)}{(r_0 + r_1 z)(a_0 + a_1 z) - \lambda_1 a z}, \\ C_d^* &= \frac{a V_F}{(a_0 + a_1 z) G(z)}, C_r^* = \frac{R a V_F}{R_0(a_0 + a_1 z) G(z)} \end{aligned}$$

Stability of dynamical model: linearization around E^* , and define perturbations:

$$x_1 = D_v - D_v^*, x_2 = C_d - C_d^*, x_3 = C_r - C_r^*, x_4 = C_{HP} - C_{HP}^*, x_5 = W_D - W_D^*.$$

Then, the linearized system will be

$$\dot{X}(t) = AX(t) + BX(t - \tau), \quad (36)$$

where $X(t) = (x_1, x_2, x_3, x_4, x_5)^T$.

The Jacobian matrix without delay:

$$A = \begin{pmatrix} -r_0 - r_1 C_{HP}^* & \lambda_1 C_{HP}^* & 0 & -r_1 D_v^* + \lambda_1 C_d^* & 0 \\ a & -a_0 - a_1 C_{HP}^* & 0 & -a_1 C_d^* & 0 \\ 0 & R & -R_0 & 0 & 0 \\ 0 & 0 & 0 & b_0 & -b_1 C_{HP}^* \\ 0 & 0 & 0 & c - c_1 W_D^* & -c_0 \end{pmatrix} \quad (37)$$

The delay contribution matrix:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_1 W_D^* & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (38)$$

Characteristic Equation: the characteristic equation is

$$\det(\lambda I - A - B e^{-\lambda \tau}) = 0, \quad (39)$$

This is extended into a fifth-degree quasi-polynomial of the form

$$\lambda^5 + p_1 \lambda^4 + p_2 \lambda^3 + p_3 \lambda^2 + p_4 \lambda + p_5 + (q_1 \lambda + q_0) e^{-\lambda \tau} = 0. \quad (40)$$

Parameters and equilibrium values explicitly affect p_i, q_i .

Stability Analysis for $\tau = 0$

When $\tau = 0$, the system reduces to an ODE:

$$\dot{X} = (A + B)X.$$

The characteristic equation becomes a fifth-degree polynomial:

$$\lambda^5 + a_1 \lambda^4 + a_2 \lambda^3 + a_3 \lambda^2 + a_4 \lambda + a_5 = 0 \quad (41)$$

4.1 Routh–Hurwitz Stability $\tau = 0$

An asymptotically stable equilibrium of the model exists provided that the following are true:

Routh–Hurwitz Conditions:

$$\begin{aligned} a_1 &> 0, a_5 > 0, \\ a_1 a_2 - a_3 &> 0, \\ (a_1 a_2 - a_3) a_3 - a_1^2 a_4 &> 0, \\ a_4 (a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) &> a_5 (a_1 a_2 - a_3)^2. \end{aligned}$$

Because all natural depletion rates are positive:

$$r_0, a_0, R_0, c_0 > 0, \text{ and similarly } r_1, a_1, c_1, b_1 > 0,$$

we have

$$\text{tr}(A + B) < 0$$

$$\det(A + B) > 0$$

All major minors positive when self-decay is most significant compared to self-feed-back:

$$b_0 < b_1 W_D^*, c_0 > c_1 C_{HP}^*.$$

This gives the following condition in the asymptotic stability case when $\tau = 0$:

$$b_1 W_D^* > b_0, c_0 > c_1 C_{HP}^*$$

Stability for $\tau > 0$ and Hopf bifurcation:

Delay has no effect except in the equation of pollution:

$$\dot{x}_4 = b_0 x_4 - b_1 C_{HP}^* x_5 - b_1 W_D^* x_4(t - \tau).$$

5.1 Characteristic equation relevant to delay:

A reduced 2D delayed subsystem:

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \Rightarrow \det \begin{pmatrix} \lambda - b_0 + b_1 W_D^* e^{-\lambda \tau} & b_1 C_{HP}^* \\ -(c - c_1 W_D^*) & \lambda + c_0 \end{pmatrix} = 0.$$

This yields

$$(\lambda - b_0)(\lambda + c_0) + b_1(c - c_1 W_D^*) C_{HP}^* + (\lambda + c_0) b_1 W_D^* e^{-\lambda \tau} = 0. \quad (42)$$

Hopf Bifurcation: a Hopf bifurcation is a bifurcation that happens when two eigenvalues cross the imaginary axis:

$$\lambda = i\omega, \omega > 0.$$

Substitute

$$(i\omega - b_0)(i\omega + c_0) + K + (i\omega + c_0) b_1 W_D^* e^{-i\omega \tau} = 0$$

where

$$K = b_1(c - c_1 W_D^*) C_{HP}^*.$$

Distinguish real and imaginary parts:

Real part:

$$-(\omega^2 + c_0 b_0) + K + b_1 W_D^* (c_0 \cos(\omega \tau) + \omega \sin(\omega \tau)) = 0. \quad (43)$$

Imaginary part:

$$\omega(c_0 - b_0) + b_1 W_D^* (\omega \cos(\omega \tau) - c_0 \sin(\omega \tau)) = 0. \quad (44)$$

Treat $\cos(\omega \tau) = C$ and $\sin(\omega \tau) = S$. These two equations are linear in C, S . They can be rearranged as follows:

$$\begin{cases} b_1 W_D^* \omega C - b_1 W_D^* c_0 S = -\omega(c_0 - b_0), \\ b_1 W_D^* c_0 C + b_1 W_D^* \omega S = \omega^2 + c_0 b_0 - K. \end{cases}$$

Divide by $b_1 W_D^*$ and put in matrix form $M \begin{pmatrix} C \\ S \end{pmatrix} = r$, where

$$M = \begin{pmatrix} \omega & -c_0 \\ c_0 & \omega \end{pmatrix}, r = \frac{1}{b_1 W_D^*} \begin{pmatrix} -\omega(c_0 - b_0) \\ \omega^2 + c_0 b_0 - K \end{pmatrix}.$$

The determinant $\det M = \omega^2 + c_0^2 = D > 0$ for $\omega > 0$. The inversion expressed in closed-form is as follows:

$$\cos(\omega \tau) = \frac{b_0}{b_1 W_D^*} - \frac{c_0 K}{b_1 W_D^* (\omega^2 + c_0^2)}, \quad (45)$$

$$\sin(\omega \tau) = \frac{\omega(\omega^2 + c_0^2 - K)}{b_1 W_D^* (\omega^2 + c_0^2)} \quad (46)$$

And $\cos^2(\omega \tau) + \sin^2(\omega \tau) = 1$ gives only the equation in ω . Change (A) and non-denominators:

Let $D = \omega^2 + c_0^2$. Multiply $\cos^2(\omega \tau) + \sin^2(\omega \tau) = 1$, through by $b_1^2 W_D^{*2} D^2$. Upon simplification we get the scalar equation

$$(b_0 D - c_0 K)^2 + \omega^2 (D - K)^2 = b_1^2 W_D^{*2}. \quad (47)$$

This is the master equation for ω . Take it to a polynomial form: it is algebraic of degree 6 in omega ($D = \omega^2 + c_0^2$), so it should have several positive roots $\omega > 0$ a priori.

Expand it to a polynomial form: it is a sixth-degree algebraic Equation (6) in ω ($D = \omega^2 + c_0^2$), so you will have finitely many positive solutions $\omega > 0$.

Given an individual root $\omega_j > 0$, calculate $C = \cos(\omega\tau)$, $S = \sin(\omega\tau)$.

The family of critical delays where $i\omega_j$ is a root is then

$$\tau_{j,k} = \frac{1}{\omega_j} (\text{atan } 2(S, C) + 2\pi k), k \in \mathbb{Z} \quad (48)$$

The correct quadrant will be obtained by use of the two-argument arctangent $\text{atan } 2(y, x)$. The first critical delay of that ω_j is the minimum value of $\tau_{j,k}$ (k chosen suitably).

From the real/imaginary separation, we eliminate trigonometric terms to obtain the frequency equation $F(\omega) = 0$. This reduces to a sixth-degree algebraic equation in ω .

Using Equation (48): $\tau_{j,k} = \frac{1}{\omega_j} (\text{atan } 2(S, C) + 2\pi k)$, $k \in \mathbb{Z}$, we compute $\tau^* = 1.29$.

The corresponding Hopf frequency is $\omega^* = 1.26$.

We substitute (τ^*, ω^*) back into the original characteristic equation and show numerically that

$$\text{Re}(\lambda) = 0, \quad \text{Im}(\lambda) = \omega^*$$

Equation (47) is necessary and sufficient for the existence of a pair of purely imaginary roots $\pm i\omega$. In practice you can check simpler necessary conditions before solving the full degree-6 equation:

1. $b_1^2 W_D^{*2} D^2$ is positive value on the right; therefore, a solution is expected with the left-hand side not surpassing this value at a given ω ; check the continuity of function

$$F(\omega) = (b_0 D - c_0 K)^2 + \omega^2 (D - K)^2 - b_1^2 W_D^{*2} D^2, \quad (49)$$

and look for sign changes for $\omega > 0$.

2. One of the simplest necessary conditions that at least one of the real ω satisfies is the opposite signs of the value at $\omega \rightarrow 0^+$ and $\omega \rightarrow \infty$:

$$\lim_{\omega \rightarrow 0} F(\omega) = (b_0 c_0^2 - c_0 K)^2 - b_1^2 W_D^{*2} c_0^4, \quad (50)$$

$$\lim_{\omega \rightarrow \infty} F(\omega) \sim \omega^6 (b_0^2 + 1 - b_1^2 W_D^{*2}) \quad (51)$$

So, if the $b_1^2 W_D^{*2} > b_0^2 + 1$ limit at infinity is negative, add a sign at zero to find the roots. (This is only heuristic—solve numerically for reliability.)

6.1 Critical Delay (τ^*)

Solve imaginary part for τ :

$$\tan(\omega\tau) = \frac{\omega(c_0 - b_0) - b_1 W_D^* \omega}{b_1 W_D^* c_0}. \quad (52)$$

Therefore, the critical delay which induces a Hopf bifurcation is

$$\tau^*(\omega) = \frac{1}{\omega} \arctan \left(\frac{\omega(c_0 - b_0) - b_1 W_D^* \omega}{b_1 W_D^* c_0} \right) \quad (53)$$

Put the value of $c_0 = 0.001$, $b_0 = 0.0009$, $b_1 = 0.13$ in Equation (53):

$$\tau^*(\omega) = \frac{1}{\omega} \arctan \left(\frac{\omega(0.001 - 0.0009) - 0.13 W_D^* \omega}{0.13 \times 0.001 \times W_D^*} \right) \approx 1.29.$$

The ω that corresponds will have to meet the real-part equation.

7. The condition of the Hopf bifurcation theorem:

A Hopf bifurcation occurs iff:

(H1) Pure imaginary roots exist

$$\lambda = \pm i\omega \text{ solves the characteristic equation.}$$

(H2) Crossing condition (transversality)

$$\frac{d\operatorname{Re}(\lambda)}{d\tau} \Big|_{\tau=\tau^*} \neq 0.$$

Compute by implicit differentiation of $\Delta(\lambda, \tau) = 0$. This means that in the present case, where the scalar characteristic function of interest is, the scalar characteristic equations are 2×2 delayed block equations:

$$\Delta(\lambda, \tau) = (\lambda - b_0)(\lambda + c_0) + K + (\lambda + c_0)Ae^{-\lambda\tau}. \quad (54)$$

Differentiate $\Delta(\lambda(\tau), \tau) \equiv 0$ w.r.t. τ :

$$\frac{d\lambda}{d\tau} = -\frac{\partial_\tau \Delta}{\partial_\lambda \Delta} = -\frac{-(\lambda + c_0)A(-\lambda)e^{-\lambda\tau}}{2\lambda + (c_0 - b_0) + Ae^{-\lambda\tau} - \tau A(\lambda + c_0)e^{-\lambda\tau}}. \quad (55)$$

Evaluate at $\lambda = i\omega_j$ and $\tau = \tau_{j,k}$. The transversality condition is lowered to

$$\Re \left\{ \frac{(\lambda + c_0)\lambda Ae^{-\lambda\tau}}{\partial_\lambda \Delta(\lambda, \tau)} \right\} \Big|_{\lambda=i\omega_j} \neq 0. \quad (56)$$

Since $\lambda = i\omega_j$ and $A \neq 0$, the numerator is generically nonzero; we just have to make sure that practically we calculate numerically the complex quotient and verify that its real part $\neq 0$. On increasing τ the pair shifts to the right (loss of stability) and vice versa. the denominator $\partial_\lambda \Delta(i\omega_j, \tau_{j,k})$ is nonzero (i.e., the root is simple), and then we evaluate the real part.

We prove the transversality condition for the characteristic Equation (42):

$$(\lambda - b_0)(\lambda + c_0) + b_1(c - c_1 W_D^*) C_{HP}^* + (\lambda + c_0)b_1 W_D^* e^{-\lambda\tau} = 0$$

For simplicity, $A = b_0$, $B = c_0$, $K = b_1(c - c_1 W_D^*) C_{HP}^*$, $L = b_1 W_D^*$

Then (42) becomes

$$F(\lambda, \tau) = (\lambda - A)(\lambda + B) + K + (\lambda + B)Le^{-\lambda\tau} = 0 \quad (57)$$

Assume at $\tau = \tau^*$ that the characteristic equation admits $\lambda = i\omega^*$, $\omega^* > 0$

Substitute into (57). First, expand the polynomial part:

$$(\lambda - A)(\lambda + B) = \lambda^2 + (B - A)\lambda - AB$$

Thus,

$$F(\lambda, \tau) = \lambda^2 + (B - A)\lambda - AB + K + (\lambda + B)Le^{-\lambda\tau} \quad (58)$$

Substitute $\lambda = i\omega$:

$$-\omega^2 + i(B - A)\omega - AB + K + (i\omega + B)L(\cos\omega\tau - i\sin\omega\tau) = 0$$

Real part:

$$-\omega^2 - AB + K + BL\cos(\omega\tau) + \omega L\sin(\omega\tau) = 0$$

Imaginary part:

$$(B - A)\omega + \omega L\cos(\omega\tau) - BL\sin(\omega\tau) = 0$$

These determine ω^* and τ^* . We differentiate (57) implicitly. Since

$$F(\lambda, \tau) = 0, \quad \frac{d\lambda}{d\tau} = -\frac{F_\tau}{F_\lambda}.$$

We compute both derivatives. In terms of the derivative w.r.t. τ , only the exponential depends on τ :

$$F_\tau = -(\lambda + B)L\lambda e^{-\lambda\tau}$$

For the derivative w.r.t. λ , differentiate (58):

$$F_\lambda = 2\lambda + (B - A) + Le^{-\lambda\tau} - \tau(\lambda + B)Le^{-\lambda\tau} \quad (59)$$

Numerator:

$$F_\tau = -(i\omega)(i\omega + B)Le^{-i\omega\tau}$$

Compute product

$$(i\omega)(i\omega + B) = i\omega B - \omega^2$$

Thus,

$$F_\tau = -(i\omega B - \omega^2)Le^{-i\omega\tau} = (\omega^2 - i\omega B)Le^{-i\omega\tau}$$

Denominator:

$$F_\lambda = 2i\omega + (B - A) + Le^{-i\omega\tau} - \tau(i\omega + B)Le^{-i\omega\tau}$$

$$\frac{d\lambda}{d\tau} = \frac{(\omega^2 - i\omega B)Le^{-i\omega\tau}}{F_\lambda} \quad (60)$$

The denominator never vanishes because the eigenvalue is simple. To obtain the real part, multiply the numerator and denominator by the complex conjugate of F_λ .

After standard algebra (using (R) and (I) to eliminate trigonometric terms), one obtains

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{\omega L[2\omega^2 + (B - A)^2]}{|F_\lambda|^2} \quad (61)$$

Since

$$|F_\lambda|^2 > 0$$

the sign is determined by the numerator. Recall

$$L = b_1 W_D^*, \quad B - A = c_0 - b_0.$$

Thus,

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{\omega^* b_1 W_D^* [2(\omega^*)^2 + (c_0 - b_0)^2]}{|F_\lambda|^2} \quad (62)$$

All factors: $\omega^* > 0$, $b_1 > 0$ (washout rate), $W_D^* > 0$, $2(\omega^*)^2 + (c_0 - b_0)^2 > 0$.

Therefore,

$$\Re\left(\frac{d\lambda}{d\tau}\right) > 0$$

Hence the eigenvalues cross the imaginary axis with non-zero speed. Therefore, the transversality condition holds, and a Hopf bifurcation occurs at $\tau = \tau^*$.

Practically, one calculates numerically the complex quotient and verifies that its real part $\neq 0$. On increasing τ the pair shifts to the right (loss of stability) and vice versa.

Directional Analysis of Dynamical Model: State vector $x(t) = (D_v, C_d, C_r, C_{HP}, W_D)^T$. Equilibrium $E^* = (D_v^*, C_d^*, C_r^*, C_{HP}^*, W_D^*)$. Linearization yields

$$\dot{y}(t) = \mathcal{L}_0 y_t + \mathcal{L}_1 y(t - \tau), \quad (63)$$

or in matrices as earlier

$$\dot{y}(t) = A_0 y(t) + A_1 y(t - \tau), \quad (64)$$

with A_0, A_1 given explicitly in Jacobian.

Assume that at $\tau = \tau^*$, the characteristic equation is the sum of a simple pair of purely complex roots $\pm i\omega_0$ and no others on the imaginary axis.

Compute l_1 (Hassard method):

Eigenvectors $q(\theta)$ and adjoint $p(\theta)$: In the case of DDEs, eigenvectors are functions on $[-\tau, 0]$.

1. Solve the adjoint eigenproblem for the vector $v \in \mathbb{C}^5$:

$$(i\omega_0 I - A_0 - A_1 e^{-i\omega_0 \tau^*}) v = 0. \quad (65)$$

Take $q(\theta) = e^{i\omega_0 \theta} v$ (for $\theta \in [-\tau^*, 0]$).

2. Solve the adjoint eigenproblem for $w \in \mathbb{C}^5$ (row vector):

$$(-i\omega_0 I - A_0^T - A_1^T e^{i\omega_0 \tau^*}) w^T = 0. \quad (66)$$

Assume the adjoint eigenfunction $p(\theta) = e^{-i\omega_0 \theta} w$ for $\theta \in [0, \tau^*]$ in Hassard's inner product conventions).

3. Normalization. Enforce the bilinear inner-product normalization:

$$\langle p, q \rangle = 1,$$

in which case the bilinear form $\langle \cdot, \cdot \rangle$ is (Hassard convention) in the case of DDEs.

$$\langle p, q \rangle = \bar{w}^T v - \int_{-\tau^*}^0 \bar{w}^T A_1 q(\theta) d\theta \quad (67)$$

(or) the corresponding finite form which results from the exponential form of q ; Hassard et al. [28]) In practice, with $q(\theta) = e^{i\omega_0 \theta} v$ and $p(\theta) = e^{-i\omega_0 \theta} w$, the normalization reduces to an explicit algebraic formula in v, w, A_1, ω . Solve a scalar normalization constant such that $\langle p, q \rangle = 1$.

Taylor expansion of the right-hand side functional F (the vector field of the RFDE, with delayed arguments) around the equilibrium defines bilinear map $B: \mathbb{C}^5 \times \mathbb{C}^5 \rightarrow \mathbb{C}^5$ and trilinear map $C: \mathbb{C}^5 \times \mathbb{C}^5 \times \mathbb{C}^5 \rightarrow \mathbb{C}^5$.

$$F(\phi) = A_0 \phi(0) + A_1 \phi(-\tau) + \frac{1}{2} B(\phi(0), \phi(0)) + \frac{1}{6} C(\phi(0), \phi(0), \phi(0)) + (\text{higher}). \quad (68)$$

Since the dynamical model is a polynomial (only bilinear terms), the trilinear map C will be zero except due to the composition of lower-order nonlinearities (but in the model, most cubic partial derivatives are zero). Precisely, calculate the second derivatives of the original right-hand side (x, x_-) , about up-to-date and deferred arguments and equations that have nonzero second derivatives (at equilibrium) of the following:

From $\dot{D}_v = -r_1 D_v C_{HP} + \lambda_1 C_d C_{HP} + \dots$:

Nonzero second partials:

$$\frac{\partial^2 f_1}{\partial D_v \partial C_{HP}} = -r_1, \quad \frac{\partial^2 f_1}{\partial C_d \partial C_{HP}} = \lambda_1.$$

From $\dot{C}_d = -a_1 C_d C_{HP} + a D_v + \dots$:

$$\frac{\partial^2 f_2}{\partial C_d \partial C_{HP}} = -a_1.$$

From $\dot{C}_{HP} = -b_1 C_{HP}(t - \tau) W_D + b_0 C_{HP} + \dots$:

$$\frac{\partial^2 f_4}{\partial(C_{HP}(t-\tau)) \partial W_D(t)} = -b_1,$$

and are symmetrical in differentiation (in your representation). Note: The delayed factor causes incorporated delayed dependence; the formulas of Hassard deal with this by considering the contribution of the delayed argument as a contribution to the multilinear maps by evaluating it at $\theta = -\tau$. From $\dot{W}_D = -c_1 C_{HP} W_D + c C_{HP} + \dots$:

$$\frac{\partial^2 f_5}{\partial C_{HP} \partial W_D} = -c_1.$$

All other second derivatives are zero. Construct $B(u, v)$: for two vectors $u, v \in \mathbb{C}^5$ (treated as values of perturbations at time 0); the values of $B(u, v)$ are

$$[B(u, v)]_1 = -r_1(u_1 v_4 + v_1 u_4) + \lambda_1(u_2 v_4 + v_2 u_4),$$

$$[B(u, v)]_2 = -a_1(u_2 v_4 + v_2 u_4),$$

$$[B(u, v)]_3 = 0,$$

$$[B(u, v)]_4 = -b_1(u_4(-\tau) v_5(0) + v_4(-\tau) u_5(0))$$

$$[B(u, v)]_5 = -c_1(u_4 v_5 + v_4 u_5).$$

In this case $u_4(-\tau)$ is the value of the history at time $-\tau$; in a reduction in Hassard with an ansatz $q(\theta) = e^{i\omega\theta} v$, you will evaluate such terms by the exponential factor $e^{-i\omega\tau}$.

Trilinear map C : in a pure quadratic model of polynomials, the pure third partial derivatives are all zero; but in the Hassard normal-form formula there is an additional term which contains such compositions as $B(q, (i\omega I - \mathcal{L})^{-1} B(q, q))$, which is not zero.

Calculate the scalar coefficients $g_{20}, g_{11}, g_{02}, g_{21}$

Using $q(\theta) = e^{i\omega\theta} v$ and $p(\theta) = e^{-i\omega\theta} w$ you form

$$g_{20} = \frac{1}{2} \langle p, B(q, q) \rangle, g_{11} = \frac{1}{2} \langle p, B(q, \bar{q}) \rangle, g_{02} = \frac{1}{2} \langle p, B(\bar{q}, \bar{q}) \rangle,$$

in which $\langle p, \cdot \rangle$ is the inner product of the Hassard (in the case of DDEs this is reduced to an algebraic pairing since B is computed at times 0 and τ).

The auxiliary function $H(\theta)$ solves the inhomogeneous equation on the range orthogonal to the center subspace:

$$(i2\omega_0 I - \mathcal{L}_0 - \mathcal{L}_1 e^{-i2\omega_0 \tau^*}) U(\theta) = -B(q, q)(\theta),$$

Then, solve for U (resolvent/inverse evaluation). This can be represented in terms of matrices by a linear algebraic system of equations to get the Fourier coefficient of U , since it is all exponential in θ .

$$(i2\omega_0 I - A_0 - A_1 e^{-i2\omega_0 \tau^*}) u = -B(v, v)_{\text{effective}},$$

for $u \in \mathbb{C}^5$. The same exponential evaluations are used on the right-hand side to take into consideration delay. Then, define

$$g_{21} = \langle p, B(q, \bar{U}) + 2B(\bar{q}, U) + \frac{1}{2} C(q, q, \bar{q}) \rangle,$$

in which U is the already calculated solution and C is zero for purely quadratic fields of vectors: the C term need not be present, although the resolvent term is essential.

From the sum of coefficients into $c_1(0)$ and then l_1 :

The complex constant $c_1(0)$ (often simply referred to as c_1 in other texts) is obtained by the formula of Hassard; and the first Lyapunov coefficient is then

$$l_1 = \frac{\Re(c_1(0))}{\omega_0}$$

One convenient expression (Hassard et al. [28], Thm. 3.4.2) is

$$c_1(0) = \frac{1}{2} g_{21} - \frac{g_{20}g_{11}}{2i\omega_0} + \frac{g_{02}g_{11}}{6i\omega_0},$$

The choice of combination varies depending upon the normalization conventions of g_{ij} ; use the version in Hassard et al. (1981) [28] consistently. After computing $c_1(0)$, we evaluate $l_1 = \frac{\Re(c_1(0))}{\omega_0}$. If $l_1 < 0$ supercritical, if $l_1 > 0$ subcritical.

Numerical Example: To confirm the analytical results, a numerical example is given where the system is simulated by values of representative parameters. The calculated trajectories demonstrate the qualitative characteristics of all state variables and validate the expected change in stable and oscillatory states. These findings also underline the importance of the delay parameter in determining the dynamics of a long-term system. The parametric values are: $V_F = 22, r_0 = 0.1, r_1 = 0.001, \lambda_1 = 0.001, a = 0.20, a_0 = 0.9, a_1 = 0.00990, R = 0.22, R_0 = 0.2, b_0 = 0.0009, b_1 = 0.13, c = 0.1, c_0 = 0.001, c_1 = 0.01$.

3. Result and Discussion

The simulated curves represent the qualitative dynamics of all the state variables and confirm the predicted changes in the dynamical regimes (stable, asymptotically stable, and oscillatory). Figure 1 depicts the dynamical system and how its different parameters interact. These numerical data also underscore the critical role of the delay parameter (τ) in the long-term behavior of the system and in regulating the onset of oscillatory behavior. Figure 2 shows that, in the absence of delay ($\tau = 0$), the cumulative concentration of hot pollutants is constant. The curves approach the equilibrium in a monotonous manner, which is the entire proof that the system itself maintains the desired kind of stability in the absence of a time lag. Figure 3 shows the delay values below the critical value ($\tau < 1.29999999$); the cumulative concentration exhibits a simple instability loss.

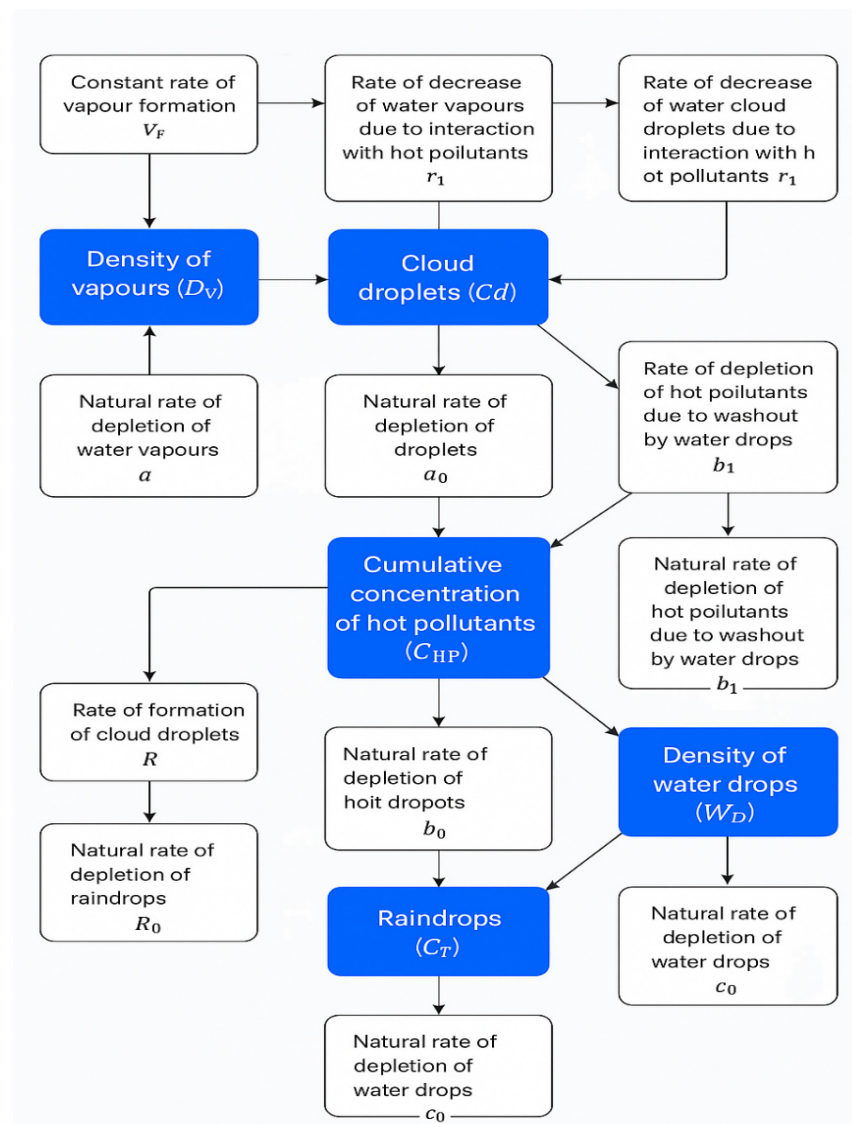


Figure 1. Flow chart of dynamical system.

Nevertheless, the cumulative concentration moves toward an asymptotically stable equilibrium. Though there may be small transient oscillations that fade away over time, this indicates that the system is globally asymptotically stable at this range of delay. Figure 4 shows that the delay is as significant as, or larger than, the critical value ($\tau \geq 1.29999999$); there is a qualitative change in the dynamical structure of the system. The equilibrium loses its asymptotic stability, and the curves develop into continuous oscillations, indicating the presence of a Hopf bifurcation. Such a transition marks the onset of sustained periodic behavior, consistent with the theoretical definition of a delay-induced bifurcation. The quantitative results indicate how time delay is an important factor in defining the behavior of the system in the long run. Although the system remains stable under small delays, a more pronounced delay may ruin the balance to create sustained oscillations. This kind of behavior is realistic in the long-term systems where cyclical behavior can be caused by delayed responsiveness.

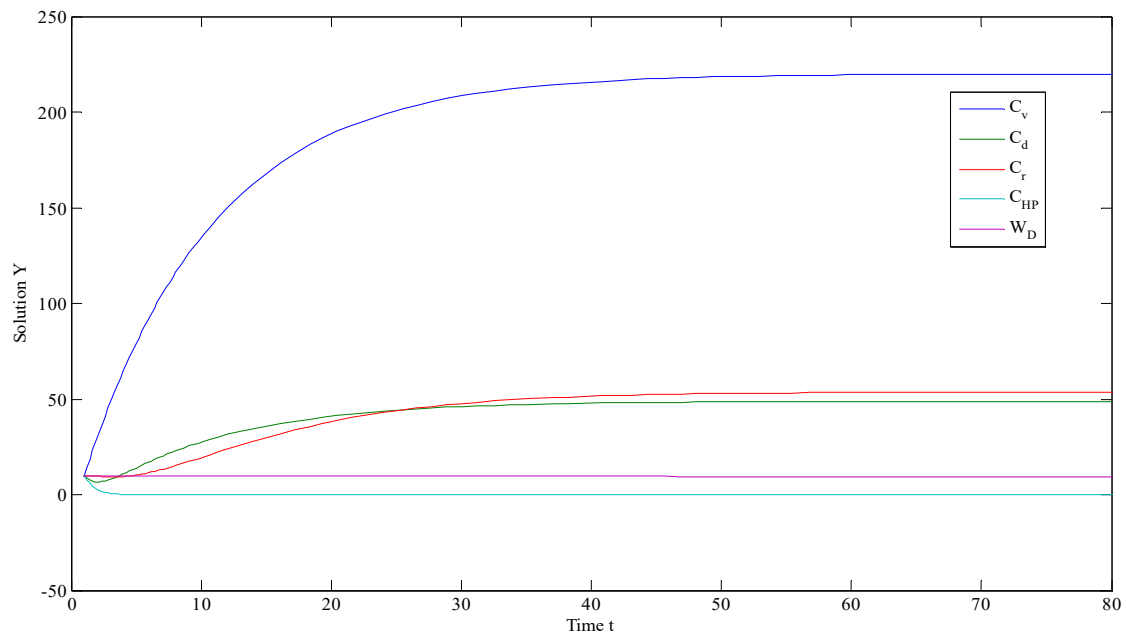


Figure 2. The cumulative concentration of hot pollutants is stable, when $\tau = 0$.

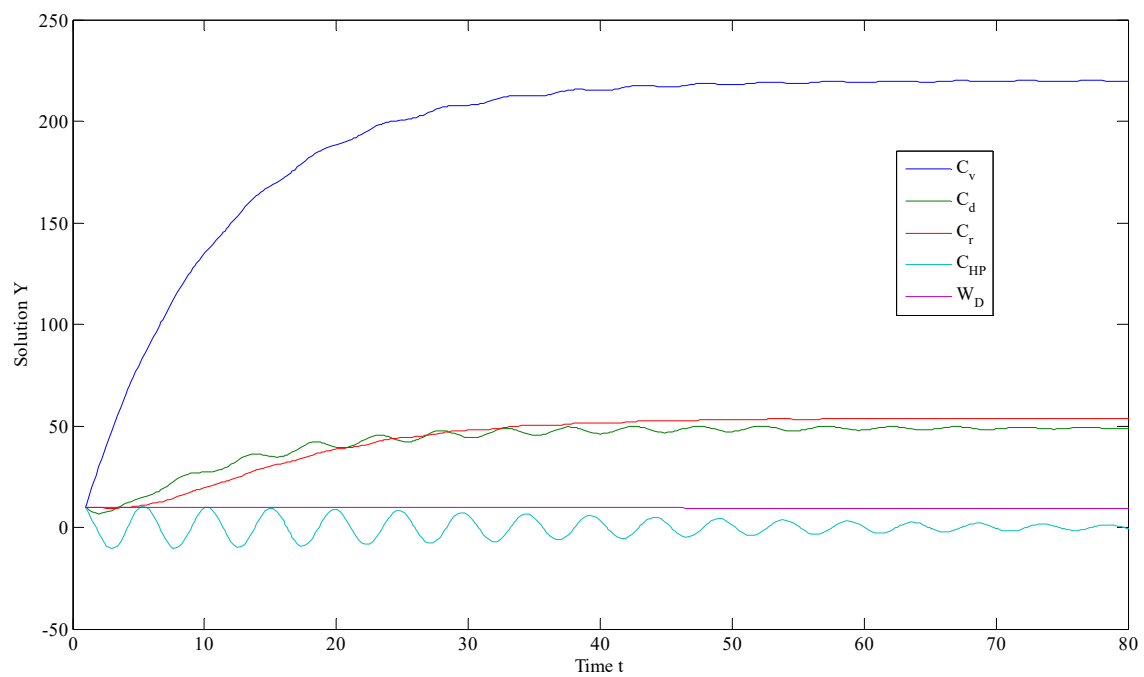


Figure 3. The cumulative concentration loses the stability and move towards asymptotic stability when $\tau < 1.29999999$.

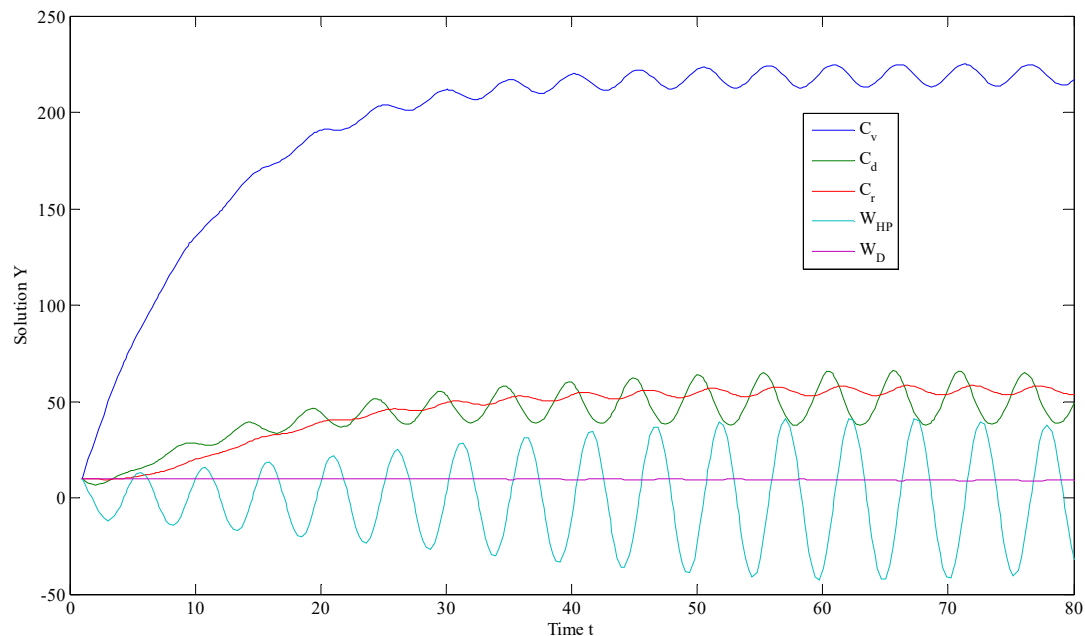


Figure 4. The cumulative concentration loses the asymptotic stability and moves towards Hopf bifurcation when $\tau \geq 1.29999999$.

4. Conclusions

This paper investigates a dynamical model of the cumulative concentration of hot pollutants to explore its dynamic behavior with a delay. The time delay (τ) is determined analytically as an important bifurcation parameter that can change the qualitative nature of the system. The critical delay value is calculated at the point at which the equilibrium state becomes asymptotically unstable, and a Hopf bifurcation takes place, which indicates the beginning of sustained oscillatory behavior. Moreover, the character and permanence of the bifurcating periodic solutions are also found by an elaborate directional analysis of the model. The analysis defines the direction of the Hopf bifurcation. It explains the qualitative development of oscillatory behavior near the critical threshold, thereby providing a more in-depth understanding of the system's structural changes. The results of the analysis are confirmed by numerical simulations performed with representative parameter sets. The findings indicate that the system is stable even without delay, switches to an asymptotically stable state when delay is present $\tau < 1.3$, and finally exhibits sustained oscillations. The results highlight the significant role of the delay parameter on the long-term dynamics of pollutants $\tau \geq 1.3$.

All the analytical, directional, and numerical investigations provide a standard for assessing the model's behavior. The findings of the work underpin the critical role of time delays in regulating stability and oscillatory behavior, with practical implications for developing future models, predictions, and management strategies in systems that incorporate time lags.

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