

Article

Enhancing the Applicability of a Parametric Sixth-Order Convergent Method for Solving Nonlinear Equations

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Abstract

The applicability of a highly efficient sixth-order convergent method, originally proposed by Kansal et al., is extended in this study to a Banach space setting. The initial development of this method relied upon Taylor series expansions in $\mathbb{R}^n$ and the assumption that the nonlinear operator is sufficiently differentiable. This vague condition implies the existence of high-order derivatives that are not actually utilized by the algorithm. This study transcends these limitations by establishing convergence based solely on generalized continuity conditions of the first Fréchet derivative. By dispensing with these strong smoothness requirements, the domain of applicability is significantly widened. We derive computable radii for the ball of convergence and establish error bounds under local analysis. Furthermore, a rigorous semi-local convergence analysis is presented, a feature previously absent in the literature for this specific scheme, utilizing a majorizing sequence technique to guarantee the existence and uniqueness of the solution. The theoretical results are validated through numerical experiments, which demonstrate that the method converges even when the standard sufficiently differentiable conditions are violated.

Keywords: parametric family of two-step methods; Banach space; majorizing sequence; local/semi-local convergence; generalized continuity

MSC: 65G99; 65H10; 49M15; 47J25

1. Introduction

The task of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \quad (1)$$

is of great importance in computational mathematics and engineering. Here, $F : \Omega \subset X \rightarrow Y$ is a Fréchet differentiable operator [1,2] acting between Banach spaces X and Y , and Ω is a convex subset of X . Analytical solutions to such equations are rarely obtainable, necessitating the use of iterative algorithms.

Newton's method is the most widely used iterative solver due to its simplicity and quadratic convergence [1,2]. However, to achieve higher efficiency and faster convergence, numerous researchers have developed multistep iterative methods that increase convergence order. Jarratt's fourth-order convergent method [3] is a classic example of such a scheme. Following this direction, various variants of Newton's method have been proposed to improve computational efficiency [4–6].



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In recent years, the focus has shifted toward developing high-order convergent methods (fifth, sixth, eighth order, etc.) [7–14] that optimize the computational efficiency index [15].

Recently, Kansal et al. [16] constructed a new parametric family of iterative methods for solving systems of nonlinear equations. By designing a specialized three-step scheme, they achieved a sixth-order of convergence. The method is defined for an initial approximation $x_0 \in \Omega$ and each $n = 0, 1, 2, \dots$ by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - A_n^{-1}B_nF'(x_n)^{-1}F(x_n), \\ x_{n+1} &= z_n - \left[\left(a + \frac{5}{2} \right) I + F'(x_n)^{-1}F'(y_n) \left(\left(-\frac{3}{2} - 2a \right) I \right. \right. \\ &\quad \left. \left. + aF'(x_n)^{-1}F'(y_n) \right) \right] F'(x_n)^{-1}F(z_n), \end{aligned} \quad (2)$$

where

$$\begin{aligned} A_n &= 2 \left[q_5 I - 3q_6 F'(x_n)^{-1}F'(y_n) + 9q_7 F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n) \right. \\ &\quad \left. - 27q_8 F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n) \right], \\ B_n &= q_1 I + 3q_2 F'(x_n)^{-1}F'(y_n) + 9q_3 F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n) \\ &\quad - 27q_4 F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F'(y_n), \end{aligned}$$

and $q_1, \dots, q_8, p$ are real numbers,

$$\begin{aligned} q_1 &= 54p^3 - 135p^2 + 102p - 23, \\ q_2 &= -54p^3 + 135p^2 - 112p + 29, \\ q_3 &= 18p^3 - 45p^2 + 38p - 11, \\ q_4 &= (p-1)^2(2p-1), \\ q_5 &= 27p^3 - 54p^2 + 33p - 4, \\ q_6 &= 27p^3 - 54p^2 + 35p - 6, \\ q_7 &= 9p^3 - 18p^2 + 11p - 2, \end{aligned}$$

and

$$q_8 = p(p-1)^2.$$

Let $B(x, \rho)$ and $B[x, \rho]$ stand for the open and closed balls, respectively, in X with center $x \in X$ and of radius $\rho > 0$.

There are two popular types of convergence analysis for iterative methods: local and semi-local [1,2]. In the former, conditions are provided depending on the solution x^* , which provide a radius of convergence r for the method. Then, as long as the initial point $x_0 \in B(x^*, \rho)$, the convergence of the method to x^* is assured. The set $B(x^*, \rho)$ is called the convergence ball for the iterative method. Moreover, in the semi-local analysis, the conditions ensuring convergence depend on the initial point x_0 .

Method (2) is computationally efficient; however, the convergence analysis in [16] has a number of drawbacks limiting its applicability.

Motivation

(P₁) The local convergence analysis is derived using Taylor series expansions in the finite-dimensional space $\mathbb{R}^m$. This approach requires the operator F to be at least six times

differentiable. Notice that only the first derivative appears in method (2). However, such conditions are restrictive; in many practical applications, such as those involving integral equations or differential equations, high-order derivatives may not exist or may be computationally expensive to evaluate [17]. To illustrate this limitation, consider the scalar function φ defined on $\Omega = [-1.5, 1.5]$ by:

$$\varphi(t) = \begin{cases} \mu t^6 \ln t + \mu_1 t^7 + \mu_2 t^8, & t \neq 0, \\ 0, & t = 0, \end{cases} \quad (3)$$

where $\mu \neq 0$ and $\mu_1 + \mu_2 = 0$. It is straightforward to verify that $t^* = 1$ is a solution to $\varphi(t) = 0$. However, the sixth derivative of this function is unbounded as $t \rightarrow 0$. Therefore, the convergence analysis presented in [16] cannot guarantee convergence for this example, even though the method may converge. As an example, choose $p = \frac{1}{2}$ and $x_0 = 0.95$. Then, $\lim_{n \rightarrow \infty} x_n = 1$. To overcome these limitations and enhance the applicability of the method, this paper extends the study of scheme (2) to a general Banach space setting. Our analysis eliminates the requirement for high-order derivatives by establishing convergence based strictly on the first derivative and generalized continuity conditions. This approach allows the method to be applied to a much broader class of problems.

- (P₂) There is no advanced knowledge on the minimum number of iterations to be carried out in order to achieve a desired error tolerance.
- (P₃) No isolation of the solution results are given.
- (P₄) The local convergence analysis is restricted on the m -dimensional Euclidean space R^m .
- (P₅) The more important and challenging semi-local convergence analysis is not studied.

The applicability of method (2) is extended by addressing the following issues:

Novelty

- (P'₁) The local convergence is established using only conditions on the operators on method (2), i.e., F and F' .
- (P'₂) Computable a priori estimates are presented.
- (P'₃) The uniqueness of the solution results becomes available.
- (P'₄) The convergence for both types of analysis is given in the setting of a Banach space.
- (P'₅) The semi-local analysis is presented using majorizing sequences [1,2].

The technique of this paper can be used to extend the applicability of other methods using the Taylor series approach along the same lines [3–15,17–20].

The convergence analysis is heavily dependent on an extending generalized continuity for the operator F' . In particular, the local convergence of Newton-type methods [1,2] has used the generalized continuity of the operator F' given by

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq \bar{g}_1(\|y - x\|)$$

for all $x, y \in \Omega$, where the function $\bar{g}_1$ is continuous and nondecreasing on the interval $[0, +\infty)$. Notice that if $\bar{g}_1(t) = \ell_1 t$ or $\bar{g}_1(t) = \ell_1 t^\sigma$ ($\sigma \in [0, 1)$), we obtain the usual Lipschitz or Hölder conditions.

Here, we use

$$\|L^{-1}(F'(x) - L)\| \leq \bar{g}_0(\|x - x^*\|)$$

for each $x \in \Omega$ and

$$\|L^{-1}(F'(y) - F'(x))\| \leq \bar{g}(\|y - x\|)$$

for each $x, y \in \tilde{\Omega}_0 = B(x^*, \rho_0) \cap \Omega$ for some $\rho_0 > 0$, where L is an invertible operator (see (H_1)). In view of the estimate $\tilde{\Omega}_0 \subseteq \Omega$ and if $L = F'(x^*)$, we get for each $t \in [0, \rho_0]$:

$$\bar{g}_0(t) \leq \bar{g}_1(t)$$

and

$$\bar{g}(t) \leq \bar{g}_1(t).$$

Thus, the new error bounds using the functions $\bar{g}_0$ and $\bar{g}$ are sharper than the old ones using $\bar{g}_1$.

The remainder of this paper is organized as follows: Section 2 presents the local convergence analysis. Section 3 details the semi-local convergence analysis. Section 4 validates the theoretical results through numerical experiments and dynamical analysis. Finally, conclusions are drawn in Section 5.

2. Local Analysis of Convergence

The analysis in this section depends on some real functions g_0 and g , which are assumed to be continuous and nondecreasing on the interval $T = [0, +\infty)$ or a subset of it. Moreover, these functions are used to control the derivative F' of the operator F .

Suppose:

- (H_1) There exists a function $g_0 : T \rightarrow T$ such that the equation $g_0(t) - 1 = 0$ has at least one zero in the interval $T_0 - \{0\}$. We shall denote the smallest such zero by s_0 and define the interval $T_0 = [0, s_0)$.
- (H_2) There exists a function $g : T_0 \rightarrow T$ such that for $h_1 : T_0 \rightarrow T$ defined by

$$h_1(t) = \frac{\int_0^1 g((1-\theta)t) d\theta + \frac{1}{3} \left(1 + \int_0^1 g_0(\theta t) d\theta \right)}{1 - g_0(t)}$$

the equation $h_1(t) - 1 = 0$ has at least one zero in the interval $T_0 - \{0\}$. We shall denote the smallest such zero by s_1 .

It is convenient to define the constants

$$\begin{aligned} d_0 &= q_1 + 3q_2 + 9q_3 - 27q_4 - 1, \\ d_1 &= 3(q_2 + 6q_3 - 27q_4), \\ d_2 &= 9(q_3 - 9q_4), \\ d_3 &= -27q_4, \\ b_0 &= q_1 - 2q_5 + 3q_2 + 9q_6 + 9q_3 - 18q_7, \\ b_1 &= 3q_2 + 6q_6 + 18q_3 - 36q_7 + 81q_6 - 81q_4, \\ b_2 &= 9(q_3 - 2q_7 + 9q_8 - 9q_4), \\ b_3 &= 27(q_8 - q_4). \end{aligned}$$

and the functions, $\bar{w}$, u , e and c on the interval T_1 by

$$\begin{aligned}\bar{w}(t) &= \begin{cases} g((1+h_1(t))t) \\ \text{or} \\ g_0(t) + g_0(h_1(t)t), \end{cases} \\ u(t) &= \frac{\bar{w}(t)}{1-g_0(t)}, \\ e(t) &= |d_0| + |d_1|u(t) + |d_2|u^2(t) + |d_3|u^3(t), \\ \text{and} \\ c(t) &= |b_0| + |b_1|u(t) + |b_2|u^2(t) + |b_3|u^3(t).\end{aligned}$$

(H₃) The equation $e(t) - 1 = 0$ has at least one zero in the interval $T_0 - \{0\}$. Let us denote the smallest such zero by s_0 and let $T_1 = [0, s_0)$.

(H₄) For function $h_2 : T_1 \rightarrow T$ defined by

$$h_2(t) = \frac{\int_0^1 g((1-\theta)t)d\theta}{1-g_0(t)} + \frac{c(t)(1 + \int_0^1 g_0(\theta t)d\theta)}{(1-g_0(t))(1-e(t))},$$

the equation $h_2(t) - 1 = 0$ has at least one solution in the interval $T_1 - \{0\}$. Denote by s_2 the smallest such zero.

(H₅) For function $h_3 : T_1 \rightarrow T$ defined by

$$\begin{aligned}h_3(t) &= \left[\frac{\int_0^1 g((1-\theta)h_2(t)t)d\theta}{1-g_0(h_2(t)t)} + \frac{\bar{u}(t)(1 + \int_0^1 g_0(\theta h_2(t)t)d\theta)}{(1-g_0(t))(1-g_0(h_2(t)t))} \right. \\ &\quad \left. + u(t) \left(|a|u(t) + \frac{3}{2} \right) \frac{(1 + \int_0^1 g_0(\theta h_2(t)t)d\theta)}{1-g_0(t)} \right] h_2(t),\end{aligned}$$

the equation $h_3(t) - 1 = 0$ has at least one zero in the interval $T_2 - \{0\}$, where

$$\bar{u}(t) = \begin{cases} g((1+h_2(t))t) \\ \text{or} \\ g_0(t) + g_0(h_2(t)t). \end{cases}$$

Denote by s_3 the smallest such solution.

Define the parameter s and the interval T_2 by

$$s = \min\{s_j\}, j = 1, 2, 3 \quad (4)$$

and

$$T_2 = [0, s). \quad (5)$$

These definitions imply that for all $t \in T_2$

$$0 \leq g_0(t) < 1, \quad (6)$$

$$0 \leq g_0(h_2(t)t) < 1, \quad (7)$$

$$0 \leq e(t) < 1, \quad (8)$$

$$0 \leq c(t), \quad (9)$$

and

$$0 \leq h_j(t) < 1. \quad (10)$$

Next, we relate the functions g_0 and g to F' .

(H₆) There exists a solution $x^* \in \Omega$ and a linear operator $L \in \mathcal{L}(X, Y)$ that is invertible such that for all $v \in \Omega$

$$\|L^{-1}(F'(v) - L)\| \leq g_0(\|v - x^*\|).$$

Define the region $\Omega_0 = B(x^*, s) \cap \Omega$.

(H₇) $\|L^{-1}(F'(v_2) - F'(v_1))\| \leq g(\|v_2 - v_1\|)$

for all $v_2, v_1 \in \Omega_0$.

(H₈) $B[x^*, s] \subset \Omega$.

Remark 1. (i) The linear operator L can be selected as $L = I$ the identity operator on Ω or $L = F'(\bar{x})$ for an auxiliary point $\bar{x} \in \Omega$ with $\bar{x} \neq x^*$ or $L = F'(x^*)$. In the last case, x^* is a simple solution of the equation $F(x) = 0$. Notice however that such a condition is not assumed or implied by the conditions (H₆) and (H₇). Hence, we can also use the method to approximate solutions of multiplicity that are greater than one. The main local convergence result follows next. (ii) It is worth noting that the conditions (H₁)–(H₈) are divided into two types. The conditions (H₁)–(H₅) are needed to determine the radius of convergence provided that the functions g_0 and g are given. Moreover, the conditions (H₆) and (H₇) are needed to determine the functions g_0 and g . Such conditions and (H₈) are standard in the convergence analysis of iterative methods (see also [1,2]).

Set $\Omega_1 = B(x^*, s) - \{x^*\}$.

Theorem 1. Suppose that the conditions (H₁)–(H₈) hold and select the initial point $x_0 \in \Omega_1$. Then, the sequence $\{x_n\}$ generated by method (2) exists in the ball $B(x^*, s)$, remains in $B(x^*, s)$ for all $n = 0, 1, 2, \dots$, and converges to the solution x^* of the equation $F(x) = 0$. Moreover, the following assertions hold for all $n = 0, 1, 2, \dots$

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < s \quad (11)$$

$$\|z_n - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \quad (12)$$

and

$$\|x_{n+1} - x^*\| \leq h_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|. \quad (13)$$

Proof. The assertions (11)–(13) are shown by induction. Let $v \in \Omega_1$. Using the conditions (H₁), (H₆) and (4), we get

$$\|L^{-1}(F'(v) - L)\| \leq g_0(\|v - x^*\|) \leq g_0(s) < 1. \quad (14)$$

Estimate (14) and the Banach lemma on invertible linear operators [1,2] imply the existence of $F'(v)^{-1}$ and the estimate

$$\|F'(v)^{-1}L\| \leq \frac{1}{1 - g_0(\|v - x^*\|)}. \quad (15)$$

In particular, if $v = x_0$, then $F'(x_0)^{-1}$ exists, since $x_0 \in \Omega_1$. It follows by method (2) that iterate y_0 is well defined. Then, by the first substep of method (2) we can write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0) \\ &= [F'(x_0)^{-1}L] \left[\int_0^1 L^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))d\theta \right] (x_0 - x^*) \\ &\quad + \frac{1}{3}[F'(x_0)^{-1}L][L^{-1}F(x_0)]. \end{aligned} \quad (16)$$

In view of the conditions (H₇) and (H₈), and the definitions (4) and (15) (for $v = x_0$), identity (16) gives

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\left[\int_0^1 g((1-\theta)\|x_0 - x^*\|)d\theta + \frac{1}{3} \left(1 + \int_0^1 g_0(\theta\|x_0 - x^*\|)d\theta \right) \right] \|x_0 - x^*\|}{1 - g_0(\|x_0 - x^*\|)} \\ &\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < s. \end{aligned} \quad (17)$$

Thus, assertion (11) holds if $n = 0$ and the iterate $y_0 \in B(x^*, s)$, where we also used the estimates

$$F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))d\theta(x_0 - x^*)$$

and

$$\begin{aligned} \|L^{-1}F(x_0)\| &\leq \left\| L^{-1} \left(\int_0^1 (F'(x^* + \theta(x_0 - x^*)) - L + L)d\theta(x_0 - x^*) \right) \right\| \\ &\leq \left(1 + \int_0^1 g_0(\theta\|x_0 - x^*\|)d\theta \right) \|x_0 - x^*\|. \end{aligned} \quad (18)$$

It is convenient to introduce the notation $H_0 = F'(x_0)^{-1}(F'(y_0) - F'(x_0))$ and set $H = H_0$. Then, we have

$$\begin{aligned} A_n - I &= (q_1 - 1)I + 3q_2(H + I) + 9q_3(H + I)(H + I) \\ &\quad - 27q_4(H + I)(H + I)(H + I) \\ &= (q_1 - 1)I + 3q_2H + 3q_2I + 9q_3H^2 + 18q_3H + 9q_3I \\ &\quad - 27q_4H^3 - 81q_4H^2 - 81q_4H - 27q_4I \\ &= (q_1 + 3q_2 + 9q_3 - 27q_4 - 1)I + 3(q_2 + 6q_3 - 27q_4)H \\ &\quad + 9(q_3 - 9q_4)H^2 - 27q_4H^3, \end{aligned} \quad (19)$$

from which it follows

$$\|A_0 - I\| \leq |d_0| + |d_1|u_0 + |d_2|u_0^2 + |d_3|u_0^3 = e_0. \quad (20)$$

Similarly, we have in turn that

$$\begin{aligned}
 A_0 - B_0 &= (q_1 - 2q_5)I + (3q_2 + 6q_6)(H + I) \\
 &+ (9q_3 - 18q_7)(H + I)(H + I) + 27(q_8 - q_4)(H + I)(H + I)(H + I) \\
 &= (q_1 - 2q_5)I + (3q_2 + 6q_6)H + (3q_2 + 9q_6)I \\
 &+ (9q_3 - 18q_7)HH + 2(9q_3 - 18q_7)H + (9q_3 - 18q_7)I \\
 &+ 27(q_8 - q_4)HHH + 81(q_8 - q_4)HH + 81(q_8 - q_4)H \\
 &+ 27(q_8 - q_4)I \\
 &= (q_1 - 2q_5 + 3q_2 + 9q_6 + 9q_3 - 18q_7)I \\
 &+ (3q_2 + 9q_6 + 18q_3 - 36q_7 + 81q_6 - 81q_4)H \\
 &+ (9q_3 - 18q_7 + 81q_8 - 81q_4)HH + 27(q_8 - q_4)HHH,
 \end{aligned} \tag{21}$$

so

$$\|A_0 - B_0\| \leq |b_0| + |b_1|u_0 + |b_2|u_0^2 + |b_3|u_0^3 = c_0. \tag{22}$$

Notice that by (8) and the definition of e_0 , the linear operator A_0 is invertible, z_0 is well defined by the second substep of method (2) and

$$\|A_0^{-1}L\| \leq \frac{1}{1 - e_0}. \tag{23}$$

Next, we can write by the second substep of method (2) in turn that

$$\begin{aligned}
 z_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (I - A_0^{-1}B_0)F'(x_0)^{-1}F(x_0) \\
 &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + A_0^{-1}(A_0 - B_0)F'(x_0)^{-1}F(x_0).
 \end{aligned} \tag{24}$$

Using estimates (4), (8), (9), (10) (for $j = 2$), (17), (18), (22), and (23) we can get from (24)

$$\begin{aligned}
 \|z_0 - x^*\| &\leq \left[\frac{\int_0^1 g((1 - \theta)\|x_0 - x^*\|)d\theta}{1 - g_0(\|x_0 - x^*\|)} + \frac{c_0(1 + \int_0^1 g_0(\theta\|x_0 - x^*\|)d\theta)}{(1 - e_0)(1 - g_0(\|x_0 - x^*\|))} \right] \|x_0 - x^*\| \\
 &\leq h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.
 \end{aligned} \tag{25}$$

Thus, the assertion (12) holds if $n = 0$ and the iterate $z_0 \in B(x^*, s)$. It also follows that $F'(z_0)^{-1}$ is well defined and the estimate (15) holds if $v = z_0$. Notice that the iterate x_1 is well defined by the third substep of the method (2), and we can write

$$\begin{aligned}
 x_1 - x^* &= z_0 - x^* - F'(z_0)^{-1}F(z_0) + [F'(z_0)^{-1} - F'(x_0)^{-1}]F(z_0) \\
 &- \left[\left(a + \frac{3}{2} \right) I + F'(x_0)^{-1}F'(y_0) \left(-\frac{3}{2} - 2a \right) \right. \\
 &\quad \left. + a(F'(x_0)^{-1}F'(y_0))^2 \right] F'(x_0)^{-1}F(z_0) \\
 &= z_0 - x^* - F'(z_0)^{-1}F(z_0) \\
 &+ F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1}F(z_0) \\
 &+ F'(x_0)^{-1}(F'(y_0) - F'(x_0)) \\
 &\quad \left[aF'(x_0)^{-1}(F'(y_0) - F'(x_0)) - \frac{3}{2}I \right] F'(x_0)^{-1}F(z_0).
 \end{aligned} \tag{26}$$

Using (4), (10) (for $j = 3$), (15) (for $v = x_0, z_0$), (18) (for $z = x_0$), and (26), we obtain from (26)

$$\begin{aligned} \|x_1 - x^*\| &\leq \left[\frac{\int_0^1 g((1-\theta)\|z_0 - x^*\|)d\theta}{1 - g_0(\|z_0 - x^*\|)} \right. \\ &\quad + \frac{\bar{u}_0(1 + \int_0^1 g_0(\theta\|z_0 - x^*\|)d\theta)}{(1 - g_0(\|x_0 - x^*\|))(1 - g_0(\|z_0 - x^*\|))} \\ &\quad \left. + \frac{u_0(|a|u_0 + \frac{3}{2})(1 + \int_0^1 g_0(\theta\|z_0 - x^*\|)d\theta)}{1 - g_0(\|x_0 - x^*\|)} \right] \|z_0 - x^*\| \\ &\leq h_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \quad (27)$$

So, assertion (13) holds if $n = 0$ and the iterate $x_1 \in B(x^*, s)$. The induction for the assertions (11)–(13) is completed if we simply exchange x_0, y_0, z_0, x_1 by x_m, y_m, z_m, x_{m+1} (m a natural number) in the preceding calculations. Define the constant $\lambda = h_3(\|x_0 - x^*\|)$. Notice that by (10), it follows that $\lambda \in [0, 1)$. Then, from the estimate

$$\|x_{m+1} - x^*\| \leq \lambda \|x_m - x^*\| \leq \lambda^{m+1} \|x_0 - x^*\| < s, \quad (28)$$

we deduce that $\lim_{m \rightarrow \infty} x_m = x^*$ and all the iterates $x_m \in B(x^*, s)$. $\square$

Next, a region is identified containing only one solution of the equation $F(x) = 0$.

Proposition 1. Suppose that the condition (H_6) holds in the ball $B(x^*, \xi_0)$ for some $\xi_0 > 0$ and there exists $\xi_1 \geq \xi_0$ such that

$$\int_0^1 g_0(\theta \xi_1) d\theta < 1. \quad (29)$$

Define the region $\Omega_2 = B[x^*, s] \cap \Omega$.

Then, the point x^* is the only solution of the equation $F(x) = 0$ in the region Ω_2 .

Proof. Suppose that there exists a solution $w \in \Omega$ of the equation $F(x) = 0$ such that $w \neq x^*$. Define the linear operator $M = \int_0^1 F'(x^* + \theta(w - x^*))d\theta$. The application of the conditions (H_6) and (29) gives $\|L^{-1}(M - L)\| \leq \int_0^1 g_0(\theta\|w - x^*\|)d\theta \leq \int_0^1 g_0(\theta\xi_1)d\theta < 1$. Therefore, the linear operator M is invertible. Finally, from the identity $w - x^* = M^{-1}(F(w) - F(x^*)) = M^{-1}(0) = 0$, we deduce that $w = x^*$. $\square$

Remark 2. We can certainly take $\xi = s$ provided that all the conditions (H_1) – (H_8) hold in Proposition 1.

3. Semi-Local Analysis of Convergence

The calculations and formulas remain the same as in the local analysis. But items x^* , g_0 and g are replaced by x_0 , f_0 and f , respectively.

Suppose:

- (C₁) There exists a function $f_0 : T \rightarrow T$ such that the equation $f_0(t) - 1 = 0$ has at least one zero in the interval $T - \{0\}$. We shall denote the smallest such zero by l_0 and define the interval $Q = [0, l_0)$.

(C₂) There exists a function $f : Q \rightarrow T$. Define the sequence $\{\alpha_n\}$ for $\alpha_0 = 0$, some $\beta_0 \geq 0$ and each $n = 0, 1, 2, \dots$ by

$$\bar{v}_n = \begin{cases} f(\beta_n - \alpha_n) \\ \text{or} \\ f_0(\alpha_n) + f_0(\beta_n) \end{cases}, \quad u_n = \frac{\bar{v}_n}{1 - f_0(\alpha_n)},$$

$$\begin{aligned} \bar{c}_n &= |b_0| + |b_1|u_n + |b_2|u_n^2 + |b_3|u_n^3, \\ \bar{e}_n &= |d_0| + |d_1|u_n + |d_2|u_n^2 + |d_3|u_n^3, \\ \gamma_n &= \beta_n + \frac{1}{2} \left[1 + \frac{3\bar{c}_n}{1 - \bar{e}_n} \right] (\beta_n - \alpha_n), \\ \lambda_n &= \int_0^1 f((1 - \theta)(\gamma_n - \alpha_n)) d\theta (\gamma_n - \alpha_n) \\ &\quad + (1 + f_0(\alpha_n))(\gamma_n - \beta_n) + (1 + f_0(\alpha_n))(\beta_n - \alpha_n), \\ \alpha_{n+1} &= \gamma_n + \left[1 + u_n \left(|a|u_n + \frac{3}{2} \right) \right] \frac{\lambda_n}{1 - f_0(\alpha_n)}, \\ \mu_{n+1} &= \int_0^1 f((1 - \theta)(\alpha_{n+1} - \alpha_n)) d\theta (\alpha_{n+1} - \alpha_n) \\ &\quad + (1 + f_0(\alpha_n))(\alpha_{n+1} - \beta_n) + \frac{1}{2} (1 + f_0(\alpha_n))(\beta_n - \alpha_n) \end{aligned} \quad (30)$$

and

$$\beta_{n+1} = \alpha_{n+1} + \frac{2}{3} \frac{\mu_{n+1}}{1 - f_0(\alpha_{n+1})}.$$

The sequence $\{\alpha_n\}$ is shown to be majorizing for $\{x_n\}$ (see Theorem 2). But first we present a convergence condition for it.

(C₃) There exists $\ell \in [0, \ell_0)$ such that for all $n = 0, 1, 2, \dots$

$$f_0(\alpha_n) < 1, \quad \bar{e}_n < 1 \quad \text{and} \quad \alpha_n \leq \ell.$$

This condition together with (30) imply that there exists $\ell^* \in [0, \ell]$ such that $\lim_{n \rightarrow \infty} \alpha_n = \ell^*$ and $0 \leq \alpha_n \leq \beta_n \leq \gamma_n \leq \alpha_{n+1} \leq \ell$.

The functions f_0 and f can be connected to F' as in the local analysis.

(C₄) There exist $x_0 \in \Omega$ and an invertible operator L such that for all $v \in \Omega$

$$\|L^{-1}(F'(v) - L)\| \leq f_0(\|v - x_0\|).$$

Define the region $S = B(x_0, \ell_0) \cap \Omega$. Notice that $f_0(0) < 1$, so $F'(x_0)^{-1}$ exists. We shall take $\beta_0 \geq \frac{2}{3} \|F'(x_0)^{-1}F(x_0)\|$.

(C₅) $\|L^{-1}(F'(v_2) - F'(v_1))\| \leq f(\|v_2 - v_1\|)$ for all $v_2, v_1 \in S$

and

(C₆) $B[x_0, \ell^*] \subset \Omega$.

Remark 3. As in the local analysis, we can select $L = I$ or $L = F'(\bar{x})$ where $\bar{x} \in \Omega$ is an auxiliary point or some other choice as long as the conditions (C₁)–(C₆) hold.

The main semi-local convergence result for method (2) follows next.

Theorem 2. Suppose that the conditions (C₁)–(C₆) hold. Then, there exists $x^* \in B[x_0, \ell^*]$ solving the equation $F(x) = 0$ such that $\lim_{n \rightarrow \infty} x_n = x^*$. Moreover, the following assertions hold for all $n = 0, 1, 2, \dots$

$$\|y_n - x_n\| \leq \beta_n - \alpha_n, \quad (31)$$

$$\|z_n - y_n\| \leq \gamma_n - \beta_n, \quad (32)$$

$$\|x_{n+1} - z_n\| \leq \alpha_{n+1} - \gamma_n \quad (33)$$

and

$$\|x^* - x_n\| \leq \ell^* - \alpha_n. \quad (34)$$

Proof. Mathematical induction is used to first show assertions (31)–(33). The definition of β_0 and the first substep of method (2) give

$$\|y_0 - x_0\| = \frac{2}{3} \|F'(x_0)^{-1} F(x_0)\| \leq \beta_0 = \beta_0 - \alpha_0 < \ell^*. \quad (35)$$

Thus, the iterate $y_0 \in B(x_0, \ell^*)$ and assertion (31) holds if $n = 0$.

By subtracting the first from the second substep of method (2) for m denoting a natural number, we get

$$\begin{aligned} z_m - y_m &= \frac{2}{3} F'(x_m)^{-1} F(x_m) - A_m^{-1} B_m F'(x_m)^{-1} F(x_m) \\ &= -\frac{1}{3} (y_m - x_m) + (I - A_m^{-1} B_m) F'(x_m)^{-1} F(x_m) \\ &= -\frac{1}{3} (y_m - x_m) + A_m^{-1} (A_m - B_m) F'(x_m)^{-1} F(x_m), \end{aligned}$$

so

$$\begin{aligned} \|z_m - y_m\| &\leq \frac{1}{2} (\beta_m - \alpha_m) + \frac{c_m}{1 - \bar{e}_m} \frac{3}{2} (\beta_m - \alpha_m) \\ &= \gamma_m - \beta_m, \end{aligned} \quad (36)$$

$$\begin{aligned} \|z_m - x_0\| &\leq \|z_m - y_m\| + \|y_m - x_0\| \\ &\leq \gamma_m - \beta_m + \beta_m - \alpha_0 = \gamma_m < \ell^*, \end{aligned} \quad (37)$$

(see also (20), (23)).

Thus, the iterate $z_m \in B(x_0, \ell^*)$ and assertion (32) holds if $n = m$. By the third substep of method (2) we can write in turn

$$\begin{aligned} x_{m+1} - z_m &= -F'(x_m)^{-1} F(z_m) \\ &\quad + F'(x_m)^{-1} (F'(y_m) - F'(x_m)) \left(a F'(x_m)^{-1} (F'(y_m) - F'(x_m)) + \frac{3}{2} I \right) \\ &\quad F'(x_m)^{-1} F(z_m). \end{aligned} \quad (38)$$

We need the estimate

$$\begin{aligned} F(z_m) &= F(z_m) - F(x_m) - \frac{3}{2} F'(x_m)(y_m - x_m) \\ &= [F(z_m) - F(x_m) - F'(x_m)(z_m - x_m)] \\ &\quad + [F'(x_m)(z_m - y_m)] - \frac{1}{2} [F'(x_m)(y_m - x_m)] \end{aligned}$$

leading to

$$\begin{aligned}\|L^{-1}F(z_m)\| &\leq \int_0^1 f((1-\theta)(\gamma_m - \alpha_m))d\theta(\gamma_m - \alpha_m) \\ &\quad + (1 + f_0(\alpha_m))(\gamma_m - \beta_m) + \frac{1}{2}(1 + f_0(\alpha_m))(\beta_m - \alpha_m) = \lambda_m.\end{aligned}\quad (39)$$

Then, (38) gives

$$\begin{aligned}\|x_{m+1} - z_m\| &\leq \left[1 + u_m \left(|a|u_m + \frac{3}{2}\right)\right] \frac{\lambda_m}{1 - f_0(\alpha_m)} \\ &= \alpha_{m+1} - \gamma_m\end{aligned}\quad (40)$$

and

$$\begin{aligned}\|x_{m+1} - x_0\| &\leq \|x_{m+1} - z_m\| + \|z_m - x_0\| \\ &\leq \alpha_{m+1} - \gamma_m + \gamma_m - \alpha_0 = \alpha_{m+1} < \ell^*.\end{aligned}\quad (41)$$

Hence, the iterate $x_{m+1} \in B(x_0, \ell^*)$ and the assertion (33) holds if $m = n$. By switching z_m by x_{m+1} in (39) we get

$$\begin{aligned}\|L^{-1}F(x_{m+1})\| &\leq \int_0^1 f((1-\theta)(\alpha_{m+1} - \alpha_m))d\theta(\alpha_{m+1} - \alpha_m) \\ &\quad + (1 + f_0(\alpha_m))(\alpha_{m+1} - \beta_m) + \frac{1}{2}(1 + f_0(\alpha_m))(\beta_m - \alpha_m) \\ &= \mu_{m+1},\end{aligned}\quad (42)$$

so

$$\begin{aligned}\|y_{m+1} - x_{m+1}\| &\leq \|F'(x_{m+1})^{-1}L\| \|L^{-1}F(x_{m+1})\| \\ &\leq \frac{\mu_{m+1}}{1 - f_0(\|x_{m+1} - x_0\|)} \leq \frac{\mu_{m+1}}{1 - f_0(\alpha_{m+1})} = \beta_{m+1} - \alpha_{m+1}\end{aligned}\quad (43)$$

and

$$\begin{aligned}\|y_{m+1} - x_0\| &\leq \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\| \\ &\leq \beta_{m+1} - \alpha_{m+1} + \alpha_{m+1} - \alpha_0 = \beta_{m+1} < \ell^*.\end{aligned}\quad (44)$$

Thus, the iterate $y_{m+1} \in B(x_0, \ell^*)$ and the assertion (31) holds for $n = m + 1$. In view of (31)–(33) and the triangle inequality we get

$$\|x_{m+1} - x_m\| \leq \alpha_{m+1} - \alpha_m. \quad (45)$$

Therefore, the sequence $\{x_m\}$ is fundamental in a Banach space X , and as such it converges to some $x^* \in B[x_0, \ell^*]$ (since $B[x_0, \ell^*]$ is a closed set). Moreover, by the continuity of the operator F and (42), estimate (42) gives for $m \rightarrow \infty$ that $F(x^*) = 0$. Furthermore, for i , a natural number (45) and the triangle inequality imply

$$\|x_{m+i} - x_m\| \leq \alpha_{m+i} - \alpha_m. \quad (46)$$

Finally, by letting $i \rightarrow \infty$, we show the assertion (34). $\square$

A region containing only one solution is determined next.

Proposition 2. Suppose that there exists a solution $w_1 \in B(x_0, \ell_1)$ for some $\ell_1 > 0$, the condition (C_4) holds in the ball $B(x_0, \ell_1)$ and there exists $\ell_2 \geq \ell_1$ such that

$$\int_0^1 f_0((1-\theta)\ell_1 + \theta\ell_2)d\theta < 1. \quad (47)$$

Define the region $S_1 = B[x_0, \ell_2] \cap \Omega$. Then, the point w_1 is the only solution of the equation $F(x) = 0$ in the region S_1 .

Proof. Suppose that there exists a solution $w_2 \in S_1$ of the equation $F(x) = 0$ such that $w_2 \neq w_1$. Define the linear operator

$$M_2 = \int_0^1 F'(w_1 + \theta(w_2 - w_1))d\theta.$$

Using the condition (C_4) and (47) we get in turn

$$\begin{aligned} \|L^{-1}(M_2 - L)\| &\leq \int_0^1 f_0((1-\theta)\|w_1 - x_0\| + \theta\|w_2 - x_0\|)d\theta \\ &\leq \int_0^1 f_0((1-\theta)\ell_1 + \theta\ell_2)d\theta < 1. \end{aligned} \quad (48)$$

It follows that the operator M_2 is invertible. Finally, from the identity

$$w_2 - w_1 = M_2^{-1}(F(w_2) - F(w_1)) = M_2^{-1}(0) = 0,$$

we conclude that $w_2 = w_1$. $\square$

Remark 4. (i) The limit point ℓ^* can be replaced by ℓ_0 (given in (C_1)) in condition (C_6) . (ii) If all the conditions (C_1) – (C_6) hold then we can set $w_1 = x^*$ and $\ell_1 = \ell^*$ in Proposition 2.

4. Numerical Experiments

In this section, we evaluate the computational efficiency and convergence behavior of the sixth-order method (2) (denoted CM). To validate its performance, we compare it against two established high-order Newton-type methods from the literature: the sixth-order method by Lotfi et al. [10] (denoted as BM) and the sixth-order method by Wang and Li [11] (denoted as WM).

4.1. Experimental Setup

All numerical computations were performed using the Python programming language. To ensure high precision and minimize the impact of round-off errors, particularly for large-scale systems and high-order convergence verification, we utilized flint. The arithmetic precision was set to 3000 decimal digits. The numerical experiments were conducted on a computer equipped with an Intel(R) Core(TM) i7-10510U CPU @ 1.80 GHz and 32 GB of RAM, running Microsoft Windows 11 Pro (64-bit).

The iterative processes were terminated when the error norm satisfied the following tolerance:

$$\|x_{k+1} - x_k\| + \|F(x_k)\| < \epsilon, \quad (49)$$

where $\epsilon = 100$. The Computational Order of Convergence (COC) was approximated using the standard formula:

$$\text{COC} \approx \frac{\ln\left(\frac{\|x_{k+1} - x_k\|}{\|x_k - x_{k-1}\|}\right)}{\ln\left(\frac{\|x_k - x_{k-1}\|}{\|x_{k-1} - x_{k-2}\|}\right)}. \quad (50)$$

4.2. Academic Examples

To verify the theoretical results and assess the performance of the analyzed method (CM) (2), we compare it with two efficient sixth-order methods from the literature.

- Method BM [10]: The three-step method selected for comparison is the sixth-order scheme introduced by Lotfi et al., which we denote as BM. This iterative method utilizes two evaluations of the function and two evaluations of the Jacobian matrix. The iterative scheme is given by:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - 2(F'(x_k) + F'(y_k))^{-1}F(x_k), \\ x_{k+1} &= z_k - \left(\frac{7}{2}I - 4F'(x_k)^{-1}F'(y_k) + \frac{3}{2}(F'(x_k)^{-1}F'(y_k))^2 \right) F'(x_k)^{-1}F(z_k). \end{aligned} \quad (51)$$

- Method WM [11]: The second method, denoted as WM, is the sixth-order family proposed by Wang and Li:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= y_k - \left(2I - F'(x_k)^{-1}F'(y_k) \right) F'(x_k)^{-1}F(y_k), \\ x_{k+1} &= z_k - \left(2I - F'(x_k)^{-1}F'(y_k) \right) F'(x_k)^{-1}F(z_k), \end{aligned} \quad (52)$$

- Method CM: The analyzed method (2) with $p = 0.5$.

We compare the performance of the methods on three nonlinear systems with varying complexity and dimensions.

4.2.1. Example 1: Small-Scale Nonlinear System

The first test problem is a system of three nonlinear equations used to validate iterative schemes [16]:

$$F(x) = \begin{cases} x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ 2x_1^2 + x_2^2 - 4x_3 = 0, \\ 3x_1^2 - 4x_2^2 + x_3^2 = 0. \end{cases} \quad (53)$$

The solution of this system is $x^* \approx (0.6982886..., 0.6285243..., 0.3425642...)^\top$. We utilized the initial approximation $x_0 = (1, 1, 1)^\top$.

The quantitative results for Example 1 are summarized in Table 1, where columns contain the iteration count (k), the error estimates given by the step difference $\|x_k - x_{k-1}\|$ and the residual norm $\|F(x_k)\|$, as well as the Computational Order of Convergence (COC) and CPU timing. The analyzed method (CM) exhibits superior computational efficiency for this small-scale problem, satisfying the prescribed stopping criterion in only four iterations. In contrast, methods BM and WM require five iterations to reach the tolerance threshold.

Table 1. Comparison of the methods' performance for Example 1.

Method	k	$\ x_k - x_{k-1}\ $	$\ F(x_k)\ $	COC	CPU Time (s)
BM	5	9.24×10^{-458}	5.13×10^{-1000}	5.9996	2.10×10^{-3}
CM	4	2.51×10^{-120}	2.17×10^{-717}	5.9683	1.83×10^{-3}
WM	5	7.04×10^{-455}	1.13×10^{-995}	5.9971	2.46×10^{-3}

Regarding numerical precision, although BM and WM achieve a smaller step difference at the fifth iteration ($\approx 10^{-458}$), the analyzed method achieves a sufficiently small residual

norm of 2.17×10^{-717} one step earlier, highlighting its rapid convergence in the initial phase. The COC column further provides numerical validation of the theoretical derivation. Methods BM and WM display values of 5.9996 and 5.9971, respectively, while CM exhibits a slightly lower value of 5.9683. This minor deviation is attributable to the method reaching the required precision one iteration earlier, preventing the full stabilization of the asymptotic constant. Nevertheless, the COC metrics for all methods are in excellent agreement with the theoretical sixth order. As a consequence of the reduced iteration count, CM achieves the lowest CPU execution time (1.83×10^{-3} s), outperforming BM (2.10×10^{-3} s) and WM (2.46×10^{-3} s).

Figure 1 provides a graphical representation of the convergence behavior. The left side plots the logarithm of the consecutive step error, $\log_{10} \|x_k - x_{k-1}\|$, serving as a practical error estimate, while the right side displays the logarithm of the residual norm, $\log_{10} \|F(x_k)\|$. The vertical axes use a logarithmic scale extending to approximately 10^{-1000} . The trajectories of the analyzed method (CM) and the comparison methods (BM, WM) show a distinct acceleration in convergence after the third iteration, a characteristic feature of sixth-order schemes. It is also worth noting that the curves for BM and WM coincide in this example. While they exhibit a substantial reduction in error at the fifth iteration, the CM method achieves the required precision at the previous step.

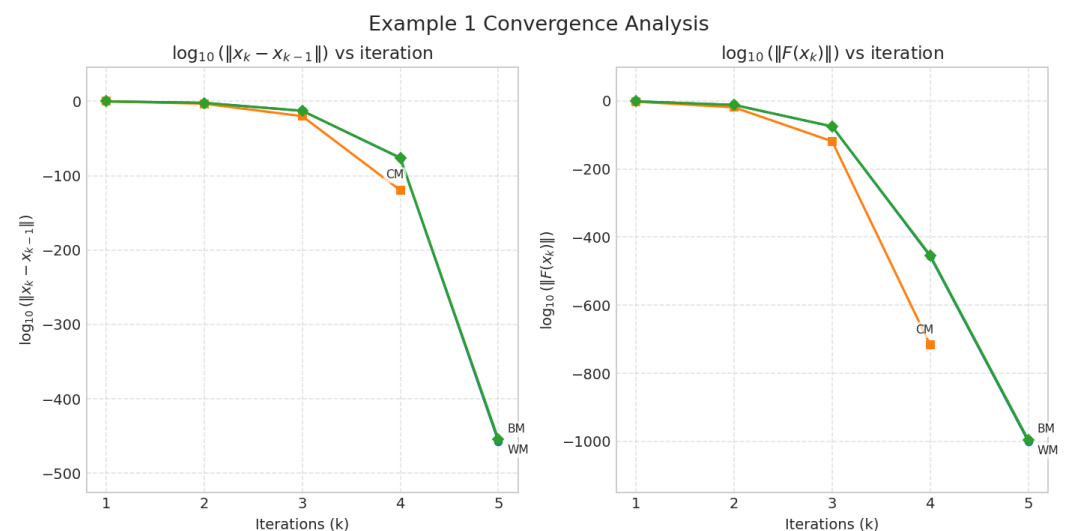


Figure 1. Convergence history for Example 1 showing the step difference (left) and residual norm (right).

4.2.2. Example 2: Trigonometric System

We consider a system of three transcendental equations involving trigonometric functions [20]:

$$F(x) = \begin{cases} 10x_1 + \sin(x_1 + x_2) - 1 = 0, \\ 8x_2 - \cos^2(x_3 - x_2) - 1 = 0, \\ 12x_3 + \sin(x_3) - 1 = 0. \end{cases} \quad (54)$$

The solution of this system is $x^* \approx (0.0690..., 0.2464..., 0.0769...)^\top$. The initial approximation was chosen as $x_0 = (0, 1, 0)^\top$. The results are summarized in Table 2.

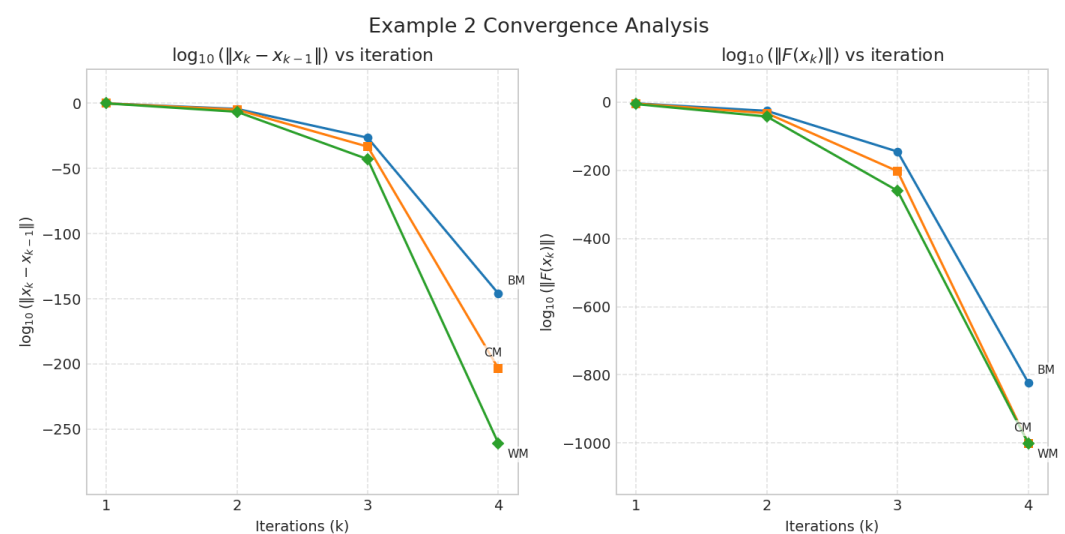
The numerical results for Example 2 are collected in Table 2. All three iterative schemes successfully converged to the solution in four iterations. In terms of approximation accuracy, method WM yielded the smallest step difference ($\|x_4 - x_3\| \approx 1.56 \times 10^{-261}$), followed by the proposed method (CM) with a comparable magnitude of $\approx 3.69 \times 10^{-204}$. Notably, for the residual norm, both CM and WM performed better than BM, reaching the extended precision limit of approximately 10^{-1001} .

Table 2. Comparison of the methods' performance for Example 2.

Method	k	$\ x_k - x_{k-1}\ $	$\ F(x_k)\ $	COC	CPU Time (s)
BM	4	9.86×10^{-147}	3.48×10^{-823}	5.4039	3.62×10^{-3}
CM	4	3.69×10^{-204}	2.66×10^{-1001}	6.0004	4.88×10^{-3}
WM	4	1.56×10^{-261}	2.66×10^{-1001}	5.9954	4.36×10^{-3}

Regarding computational cost, the analyzed method had slightly higher CPU time (4.88×10^{-3} s) than BM (3.62×10^{-3} s) and WM (4.36×10^{-3} s). This suggests a minor computational overhead attributable to the specific algebraic structure of CM for this system. However, the stability of the proposed scheme is robustly validated by COC. The CM method achieved a COC of 6.0004, perfectly aligning with the theoretical derivation of sixth-order convergence. Method WM also showed consistent behavior with a COC of 5.9954, whereas method BM exhibited a notable deviation from the theoretical value, recording a COC of 5.4039.

Figure 2 illustrates the convergence metrics for Example 2. The left panel reveals that method WM attains the lowest step difference at the final iteration, followed by the analyzed method (CM). The right panel demonstrates that both CM and WM outperform BM, reaching the arithmetic precision limit of approximately 10^{-1001} by the fourth iteration. While the steep convergence trajectories of all schemes confirm their high theoretical order, method WM exhibits superior terminal accuracy in minimizing the residual.

**Figure 2.** Convergence history for Example 2 showing the step difference (left) and residual norm (right).

4.2.3. Example 3: Large-Scale Exponential System

To evaluate the scalability of the method, we consider a large system of nonlinear equations ($n = 300$):

$$F_i(x) = \sum_{j=1, j \neq i}^n x_j - e^{-x_i} = 0, \quad i = 1, 2, \dots, n. \quad (55)$$

The solution of this system is $x^* = (1, 1, \dots, 1)^T$. The initial approximation is selected as $x_0 = (0.94, 0.94, \dots, 0.94)^T$. The results are shown in Table 3.

Table 3 summarizes the numerical results for the high-dimensional exponential system ($n = 300$). All considered iterative schemes exhibited robust convergence behavior, successfully satisfying the stopping criterion within four iterations. In terms of numerical

precision, the analyzed method (CM) and method WM demonstrated comparable performance, achieving error norms of 2.28×10^{-293} and 5.19×10^{-295} , respectively. Both methods slightly outperformed method BM, which attained an error norm of 1.41×10^{-270} . A similar trend is observed in the residual norms, where all methods effectively minimized the residual to the order of roughly 10^{-960} .

Table 3. Comparison of the methods' performance for Example 3.

Method	k	$\ x_k - x_{k-1}\ $	$\ F(x_k)\ $	COC	CPU Time (s)
BM	4	1.41×10^{-270}	3.59×10^{-971}	6.0000	4.68×10^{-2}
CM	4	2.28×10^{-293}	6.22×10^{-969}	6.0000	7.80×10^{-2}
WM	4	5.19×10^{-295}	2.21×10^{-959}	6.0000	4.39×10^{-2}

Regarding computational efficiency, a distinction arises in the CPU execution times. The analyzed method required 7.80×10^{-2} s, which is higher than the times recorded for BM (4.68×10^{-2} s) and WM (4.39×10^{-2} s). This increased computational cost is attributed to the algebraic complexity of the analyzed method, specifically the evaluation of the matrix coefficients A_n and B_n , which becomes more pronounced as the system dimension increases. However, the theoretical validity of the scheme is strictly confirmed by the Computational Order of Convergence (COC). All three methods achieved a COC of 6.0000, demonstrating that the analyzed method maintains its high-order convergence properties and stability even when applied to large-scale nonlinear systems.

Figure 3 depicts the convergence characteristics of the large-scale exponential system ($n = 300$). The left panel highlights that the analyzed method (CM) and method WM exhibit analogous trajectories, with both achieving a superior error norm (approximately 10^{-295}) at the fourth iteration compared to method BM. The right panel demonstrates that all three methods display robust convergence, effectively reducing the residual norm to the order of 10^{-970} .

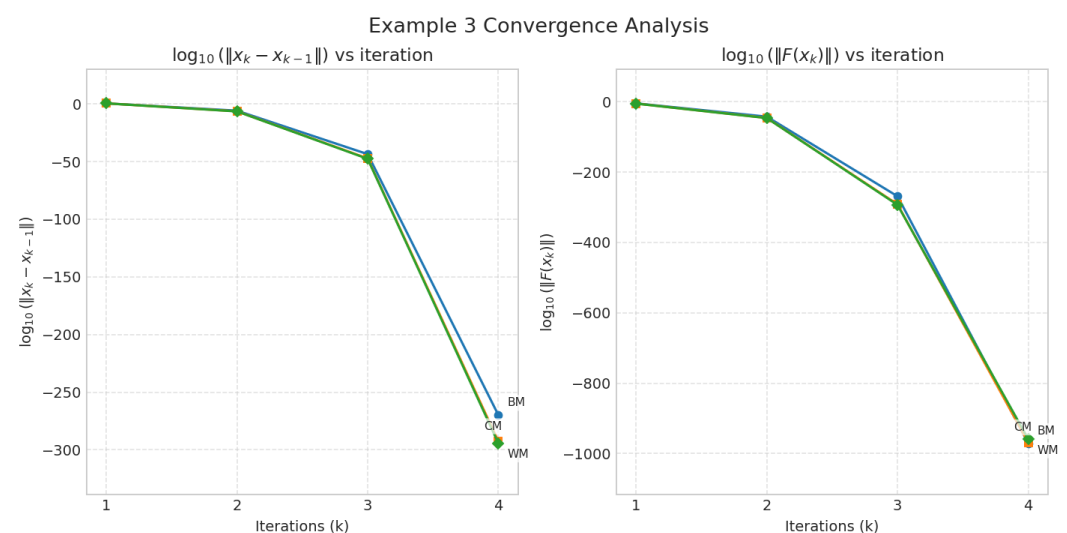


Figure 3. Convergence history for Example 3 showing the step difference (left) and residual norm (right).

4.2.4. Example 4: Nonlinear Partial Derivatives Equation

To evaluate the performance of the proposed method on a larger-scale system, we consider a modification of the nonlinear boundary-value problem commonly used in gas dynamics. In the formulation from the literature [17], the equation contains a non-differentiable term. However, to ensure the Fréchet differentiability of the operator and

strictly satisfy the conditions required for sixth-order convergence analysis, we modify the problem by replacing the non-smooth term with a linear term. The modified problem is defined on the unit square $\Omega = [0, 1] \times [0, 1]$ as follows:

$$w_{xx} + w_{yy} = w^3 + w, \quad (x, y) \in \Omega, \quad (56)$$

subject to the Dirichlet boundary conditions $w(x, 0) = 2x^2 - x + 1$ and $w(x, 1) = 2$ for $0 \leq x \leq 1$, and $w(0, y) = 2y^2 - y + 1$ and $w(1, y) = 2$ for $0 \leq y \leq 1$.

We create a mesh for discretizing the problem with step sizes $h = \frac{1}{n+1}$ and $\tau = \frac{1}{m+1}$, resulting in the mesh points (x_i, y_j) with $x_i = ih, i = 0, \dots, n+1$ and $y_j = j\tau, j = 0, \dots, m+1$. By approximating the partial derivatives using standard central divided differences, we obtain the following system of nonlinear equations:

$$\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\tau^2} = w_{i,j}^3 + w_{i,j}, \quad (57)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$, where $w_{i,j}$ denotes the approximation of $w(x_i, y_j)$. The discretized boundary conditions are given by:

$$\begin{aligned} w_{i,0} &= 2x_i^2 - x_i + 1, & w_{i,m+1} &= 2, & i &= 1, \dots, n, \\ w_{0,j} &= 2y_j^2 - y_j + 1, & w_{n+1,j} &= 2, & j &= 0, \dots, m+1. \end{aligned}$$

In our experiment, we set $n = m = 14$, which results in a sparse but large-scale system of nonlinear equations with $14 \times 14 = 196$ variables. The initial approximation was chosen as $w_0 = (1, 1, \dots, 1)^T$. The convergence results for methods BM, CM, and WM are summarized in Table 4.

Table 4. Convergence history of methods BM, CM, and WM for Example 4 (modified PDE with 196 variables).

Method	k	Error Norm $\ x_k - x_{k-1}\ $	Residual Norm $\ F(x_k)\ $	CPU Time (s)
BM	1	6.90	8.18×10^{-3}	27.35
	2	1.61×10^{-4}	7.08×10^{-26}	
	3	1.04×10^{-27}	1.95×10^{-141}	
	4	4.00×10^{-143}	7.77×10^{-719}	
CM	1	6.90	1.78×10^{-3}	57.72
	2	5.62×10^{-5}	7.56×10^{-33}	
	3	2.59×10^{-34}	7.30×10^{-209}	
	4	2.57×10^{-210}	3.01×10^{-989}	
WM	1	6.90	8.58×10^{-3}	14.38
	2	2.54×10^{-4}	1.99×10^{-28}	
	3	6.55×10^{-30}	6.70×10^{-182}	
	4	2.26×10^{-183}	1.70×10^{-983}	

Even for the partial derivatives problem, the analyzed method (CM) exhibits excellent stability and superior accuracy. After four iterations, CM reaches an error norm of 2.57×10^{-210} and a residual norm of 3.01×10^{-989} , which are substantially lower than the values achieved by BM (4.00×10^{-143} error, 7.77×10^{-719} residual) and WM (2.26×10^{-183} error, 1.70×10^{-983} residual).

A significant disparity in CPU execution time was observed: the analyzed method (CM) required 57.72 s, notably higher than BM (27.35 s) and WM (14.38 s). This performance gap is largely attributable to implementation specifics and the inherent limitations of the high-precision Python library utilized. The CM scheme requires constructing complex

coefficient matrices A_n and B_n , which involve multiple dense matrix operations. In the context of 3000-digit arithmetic, the Python wrapper for the underlying arithmetic library incurs substantial overhead for these specific matrix-algebra steps, disproportionately increasing the runtime of CM compared to the algebraically simpler WM method. Thus, while the analyzed method is computationally more expensive per iteration in this specific environment, it remains the optimal choice when the primary requirements are accuracy and stability in large-scale systems.

To empirically assess the sensitivity to initial approximations in high-dimensional spaces where basins of attraction cannot be visualized, an additional stability test was conducted for Example 4. We tested 10,000 random initial approximation vectors uniformly distributed within the hypercube $[-5, 5]^{196}$. The iterative schemes were permitted a maximum of 30 iterations per trial to achieve a residual tolerance of 10^{-8} . All tested methods (BM, WM, and the analyzed CM) successfully converged in all 10,000 trials with zero divergence points, demonstrating robust global convergence and minimal sensitivity to the initial guess for this large-scale application.

4.2.5. Example 5: Radius of Convergence Calculation

To practically illustrate the applicability of the local convergence analysis and to compute the theoretical radius of convergence, we evaluate a nonlinear system derived from a kinematic initial value problem.

Let us define a mapping $F : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the domain $\Omega = U[0, 1]$ for a state vector $v = (v_1, v_2, v_3)^T$. The components of the mapping are obtained by integrating specific rates of change, yielding the following system:

$$F(v) = \begin{pmatrix} e^{v_1} - 1 \\ v_2 \\ \frac{1}{2}(e - 1)v_3^2 + v_3 \end{pmatrix}. \quad (58)$$

The first-order Fréchet derivative of this operator is given by the diagonal matrix:

$$F'(v) = \begin{bmatrix} e^{v_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (e - 1)v_3 + 1 \end{bmatrix}. \quad (59)$$

It is straightforward to verify that the origin $v^* = (0, 0, 0)^T$ is a solution to the equation $F(v) = 0$. Furthermore, the derivative evaluated at this root is exactly the identity matrix, meaning $F(v^*) = I$.

By setting the arbitrary linear operator $L = F'(v^*)$, we can satisfy the theoretical local convergence conditions defined in Section 2 by choosing the following parameters and bounding functions:

$$s_0 = \frac{1}{e - 1}, \quad g_0(t) = (e - 1)t, \quad \text{and} \quad g(t) = e^{\frac{1}{e-1}t}. \quad (60)$$

Using these bounding functions, we sequentially solve the scalar equations $h_1(t) = 1$, $h_2(t) = 1$, and $h_3(t) = 1$ to determine the threshold parameters s_1 , s_2 , and s_3 , respectively. The overall guaranteed radius of the convergence ball is defined as $r = \min\{s_1, s_2, s_3\}$.

We computed these values for forms of the functions $\bar{w}(t)$ ($\bar{w}_1(t) = g((1 + h_1(t))t)$, $\bar{w}_2(t) = g_0(t) + g_0(h_1(t)t)$) and $\bar{u}(t)$ ($\bar{u}_1(t) = g((1 + h_2(t))t)$, $\bar{u}_2(t) = g_0(t) + g_0(h_2(t)t)$) defined in our theoretical analysis. The calculated radii are summarized in Table 5.

Table 5. Computed local convergence radii parameters for Example 5.

$\bar{w}(t), \bar{u}(t)$	s_1	s_2	s_3	$r = \min(s_j)$
$\bar{w}_1(t), \bar{u}_1(t)$	0.229929	0.049883	0.047600	0.047600
$\bar{w}_2(t), \bar{u}_2(t)$	0.229929	0.052634	0.048225	0.048225

The numerical evaluation confirms that the theoretical conditions successfully establish a computable radius of convergence for the scheme. Notably, the second form provides a slightly larger radius of convergence ($r \approx 0.048225$) compared to the first form ($r \approx 0.047600$) for this specific dynamical system.

4.3. Basins of Attraction Analysis

The dynamical behavior of iterative methods is investigated by generating basins of attraction. This visualization technique provides a global perspective on the stability and reliability of the methods, allowing us to assess their sensitivity to the choice of the initial guess. The basin of attraction for a root x^* is defined as the set of all initial points x_0 in a domain Ω that converge to x^* under the iterative mapping. Points that fail to converge within a specified maximum number of iterations or diverge to infinity are classified as divergent.

To generate the basins, we selected a rectangular domain $\Omega = [-2, 2] \times [-2, 2] \subset \mathbb{R}^2$. A uniform mesh of 400×400 points was generated over Ω , resulting in a total of 160,000 initial starting points. Each method was applied to every point in the mesh, with a maximum of 50 iterations. The stopping criterion used was a tolerance of 10^{-3} for the residual or step norm.

The visual representation uses a color-coding scheme in which each distinct root of the system is assigned a specific base color. To visualize the computational cost, color brightness is scaled by the number of iterations required to reach convergence; brighter colors indicate fewer iterations. Black regions represent points that failed to converge.

We analyze the basins of attraction for three nonlinear systems defined as follows:

Case 1:

$$F_1(x, y) = \begin{cases} x^2 + y^2 - 1 = 0, \\ x^3 - y = 0. \end{cases} \quad (61)$$

Case 2:

$$F_2(x, y) = \begin{cases} x^3 - y + 0.2 = 0, \\ y^3 - x - 0.2 = 0. \end{cases} \quad (62)$$

Case 3:

$$F_3(x, y) = \begin{cases} x^2 + 4y^2 - 4 = 0, \\ 4x^2 + y^2 - 4 = 0. \end{cases} \quad (63)$$

To provide a rigorous quantitative assessment of the boundary complexity and the severity of non-convergence, we calculated the fractal dimensions of the basins of attraction. Specifically, we employed the box-counting dimension method utilizing coordinate hashing, defined as:

$$D_a = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}, \quad (64)$$

where $N(\epsilon)$ is the number of boxes of side length ϵ required to cover the target set. We compute two distinct metrics:

- **Boundary Dimension (D_b):** The fractal dimension of the boundaries separating different basins of attraction. A value approaching 2.0 implies a highly fractal, space-filling

boundary characteristic of chaotic behavior, whereas a lower value indicates smoother, more predictable transitions between roots.

- Divergence Dimension (D_d): The fractal dimension of the strictly divergent set (points failing to converge). A stable method should yield $D_d = 0.0$ (indicating isolated or no divergence). A high D_d approaching 2.0 quantifies catastrophic, space-filling failure.

The numerical stability metrics, including the divergent points, stability percentages, average iterations, and the computed fractal dimensions, are summarized in Table 6. The visual results are presented in Figures 4–6.

Table 6. Stability metrics and fractal dimensions for the basins of attraction on a grid of 40,000 points. D_b represents the boundary dimension, and D_d represents the divergence dimension.

System	Method	Divergent Points	Stability (%)	Avg Iter.	D_b	D_d
Case 1	CM	1089	99.28%	10.84	1.6689	1.0570
	WM	14,318	64.20%	10.33	1.6667	1.7163
	BM	2830	92.93%	12.00	1.7447	1.3621
Case 2	CM	62	99.85%	8.01	1.6443	0.5036
	WM	9356	76.61%	11.49	1.5458	1.5862
	BM	210	99.47%	9.63	1.4559	0.6839
Case 3	CM	4	99.99%	4.07	1.6844	0.0000
	WM	1621	95.95%	5.40	1.4845	1.3106
	BM	4	99.99%	3.82	1.2698	0.0000

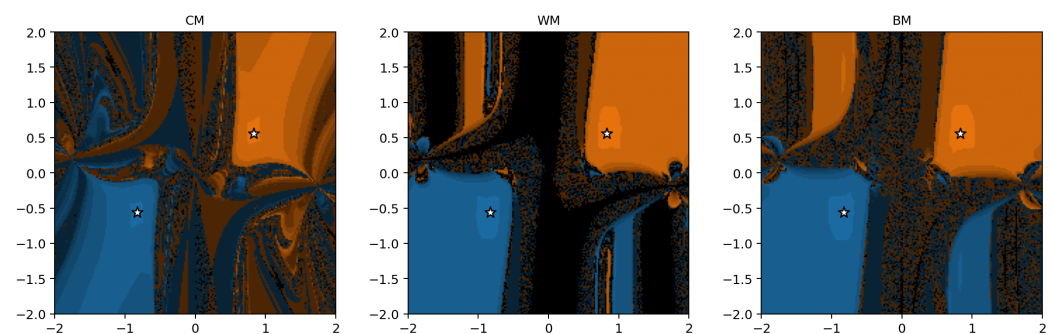


Figure 4. Basins of attraction for Case 1. The star symbols indicate the locations of the exact roots, distinct colors represent the basins of attraction corresponding to each root, and the black regions denote initial points that failed to converge.

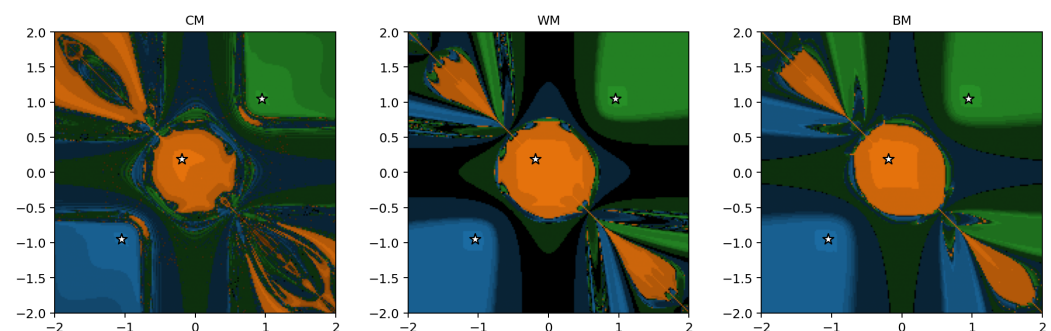


Figure 5. Basins of attraction for Case 2. The star symbols indicate the locations of the exact roots, distinct colors represent the basins of attraction corresponding to each root, and the black regions denote initial points that failed to converge.

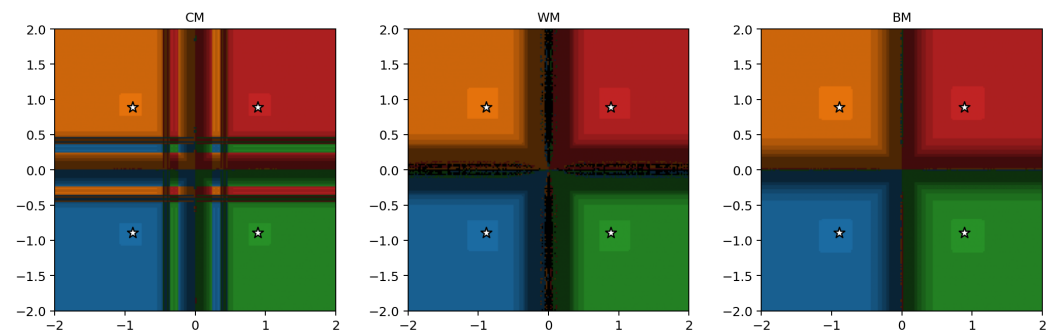


Figure 6. Basins of attraction for Case 3. The star symbols indicate the locations of the exact roots, distinct colors represent the basins of attraction corresponding to each root, and the black regions denote initial points that failed to converge.

The analysis reveals a clear distinction between the methods in terms of the trade-off between stability and efficiency:

1. Case 1: This asymmetric system highlights the superior robustness of the proposed method. CM achieves a stability rate of 99.28%, significantly outperforming BM (92.93%). In stark contrast, method WM, while computationally fast in previous examples, proves to be the least stable here, failing to converge for over 35% of the initial points. Visual inspection of Figure 4 corroborates these statistical findings. The basins of attraction for the analyzed method (CM) appear well-defined, with broad, continuous regions of convergence (shown in orange and blue) corresponding to the two roots. The fractal boundaries, where chaos typically ensues, are thin and constrained. In contrast, the plot for method WM exhibits extensive black regions indicating divergence, particularly along the separatrices between basins. Method BM shows improved stability over WM, yet the chaotic noise along the boundaries is visibly more pronounced than in the CM plot, where the transition between basins is sharper and cleaner. These visual observations are rigorously supported by the fractional dimension metrics. Method CM yields the lowest divergence dimension ($D_d = 1.0570$), confirming its failure zones are highly constrained. Conversely, WM's massive failure regions are quantified by a severe divergence dimension of $D_d = 1.7163$, proving its instability is space-filling. Furthermore, while the boundary dimensions (D_b) for CM (1.6689) and WM (1.6667) are mathematically similar, BM exhibits the most highly fractal boundaries ($D_b = 1.7447$), corroborating the visible chaotic noise.
2. Case 2: The difference in stability is most dramatic in this coupled system. CM achieves near-perfect global convergence (99.85% stability), with only 62 divergent points out of 160,000. While method WM is known for its speed, it is highly unstable in this topology, diverging for nearly a quarter of the domain (23.39%). This demonstrates that for complex coupled systems, the speed of WM comes at the cost of reliability. Figure 5 offers the most striking visual evidence of the stability differences among the methods. The dynamic plane for method WM is characterized by expansive black regions, indicating a significant failure of convergence in approximately a quarter of the domain. These divergence zones are not merely confined to boundaries but occupy significant portions of the phase space between roots. Conversely, the analyzed methods (CM and BM) exhibit robust global convergence, with basins covering nearly the entire rectangular domain. Between these two methods, the analyzed method (CM) performs better, as evidenced by the virtual absence of black pixels ($<0.05\%$) and sharper, less fractalized boundaries compared to BM, confirming its suitability for complex coupled nonlinear systems. The fractional metrics further illuminate this behavior. CM maintains an exceptionally low divergence dimension

sion ($D_d = 0.5036$). Method WM exhibits a seemingly "better" boundary dimension ($D_b = 1.5458$) than CM (1.6443), but this is a mathematical artifact of monolithic failure: WM replaces intricate, valid fractal boundaries with massive, solid blocks of divergence. This catastrophic failure mode is strictly quantified by WM's excessively high divergence dimension ($D_d = 1.5862$). Thus, CM preserves dynamical fidelity without succumbing to global instability.

3. Case 3: In this symmetric case, both CM and BM achieve practically perfect stability (99.99%). Even here, however, WM exhibits fractal instability along the basin boundaries, resulting in 4.05% divergence. Figure 6 depicts the basins of attraction for the symmetric system of ellipses. In this highly structured topology, all three methods demonstrate excellent global stability, with the basins clearly divided into four symmetric quadrants corresponding to the four roots. The analyzed methods (CM and BM) exhibit virtually indistinguishable performance, characterized by smooth boundaries and an almost total absence of divergent points. While method WM also successfully resolves the four roots, a close inspection of the axes (the boundaries between quadrants) reveals a thin band of fractal noise and divergent points (black pixels), confirming the slight instability noted in Table 6. The fractional dimensions perfectly align with this assessment. Both CM and BM yield a divergence dimension of $D_d \approx 0$, confirming an absolute absence of space-filling divergent behavior. In contrast, WM's fractal noise along the boundaries translates to a chaotic failure pattern strictly quantified by $D_d = 1.3106$.

The dynamical analysis leads to a crucial conclusion regarding the performance characteristics of the tested schemes. While the proposed method (CM) is not the most computationally efficient in terms of per-iteration execution time (as observed in the PDE example), it is unequivocally the most stable algorithm. It consistently minimizes chaotic behavior and maximizes the probability of convergence, even for difficult initial guesses. Conversely, method WM, despite being very fast, is the least stable method, exhibiting large regions of divergence in asymmetric and coupled systems. Therefore, CM is the preferred choice for applications where reliability and robustness are prioritized over raw speed.

In conclusion, the analyzed method (CM) is consistently the most stable algorithm across all tested nonlinear systems. It successfully minimizes chaotic behavior and fractal boundaries, maximizing the probability of convergence even for difficult initial guesses. While BM often rivals CM in simpler symmetric cases, CM outperforms it in more complex, asymmetric topologies.

5. Conclusions

In this work, the theoretical domain of applicability for the sixth-order iterative solver proposed by Kansal et al [16] was significantly expanded. The primary contribution of this study is establishing a rigorous convergence framework that overcomes the limitations of standard analyses. Previous studies relied on Taylor series expansions in Euclidean spaces, requiring derivatives up to the seventh order, which is often restrictive or unverifiable in complex nonlinear problems. By shifting the analysis to a Banach space setting and using generalized continuity conditions on the first Fréchet derivative alone, we have demonstrated that the method converges under much weaker hypotheses.

We derived precise local convergence criteria, establishing computable radii for the convergence ball and error bounds that do not depend on higher-order derivatives. Furthermore, a rigorous semi-local convergence analysis was presented using the majorizing sequence technique. This approach guarantees the existence and uniqueness of the solution based strictly on the initial data, providing a theoretical guarantee that was previously absent for this specific scheme. These analytical results allow the method to be used in

a broader class of problems, including those involving non-smooth operators for which standard sufficient convergence conditions are violated.

The theoretical propositions were validated through numerical experiments on systems ranging from small-scale transcendental equations to large-scale nonlinear partial differential equations. The results confirmed that the method maintains sixth-order convergence, as verified by the Computational Order of Convergence (COC) metrics. Additionally, the dynamical analysis via basins of attraction provided a global perspective on stability, revealing that the scheme minimizes chaotic behavior and offers robust convergence in complex topologies.

In future investigations, we intend to apply similar analytical techniques to other families of high-order iterative methods [3–15,17–20]. By doing so, we aim to systematically surpass the theoretical limitations of existing solvers, thereby enhancing their reliability and applicability in solving challenging nonlinear operator equations.

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