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Bonferroni Mean Operators on Weighted Hesitant Fuzzy Set and Its Application in Multi-Criteria Symmetric Group Decision Making

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Abstract

The Bonferroni mean operator, as a powerful aggregation operator, is widely applied as a solution to various problems, owing to its strong ability to capture relevance between different variables. Compared with other extensions, the weighted hesitant fuzzy set (WHFS) can depict the fuzziness of relationships between things better. Considering the advantage of weighted hesitant fuzzy sets (describing fuzziness more objectively in real problems), the Bonferroni mean is introduced into weighted hesitant fuzzy set theory. In this paper, by merging and transforming weighted hesitant fuzzy weighted average/geometric (WHFWA/WHFWG) operators, a novel weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) operator is developed using weighted hesitant fuzzy theory, so as to fuse information and provide idea support for practical tasks under various experts' common judgment. To show the effectiveness of the novel operator intuitively, the comparison results of symmetric numeric examples are displayed.

Keywords: Bonferroni mean operator; multi-criteria decision making; information fusion; weighted hesitant fuzzy set; geometric aggregation operator

1. Introduction

Fuzzy set employs a membership degree to describe fuzziness between things, which was proposed by Zadeh [1] in 1965, attracted many researchers' interest, and is widely applied in many information-fusion-related fields, such as intelligence control, etc. To enlarge the application range and meet actual needs, more and more extensions are proposed by scholars so that they can describe uncertainty in the real world better. Intuitionistic fuzzy sets [2] introduce a positive membership degree and negative membership degree to depict the relationship between belonging and not belonging. Based on intuitionistic fuzzy sets, Yager [3] proposed a Pythagorean fuzzy set by releasing the restriction, which can depict fuzzy information in a wider range. In some real scenarios, it is more realistic to adopt interval values to describe the membership degree; the interval-valued fuzzy set [4–6] plays a more adaptable role in fuzzy information fusion. The type-2 fuzzy set [7] introduces a lower membership function and upper membership function to further fuzzify the membership degree and describe fuzziness comprehensively. To simulate multiple experts' judgment, the hesitant fuzzy set (HFS) [8,9] adopts several values in the range [0, 1] to describe grades from different experts. The data structure of a hesitant fuzzy element (HFE) is n elements tuple, where the value n depends on the number of experts. Compared



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with classical fuzzy sets and other extensions, HFS depicts hesitancy more objectively, owing to the common decision-making process. With the efforts of scholars, HFS-related topics and meaningful conclusions are proposed. Weighted hesitant fuzzy set theory can further describe fuzziness by adding weights for every value in HFE, which helps decision makers give more objective results from a broad perspective.

Based on these extensions of fuzzy theory, researchers proposed many operators aiming at information fusion under different scenarios. Wang et al. [10] proposed power operators based on Aczel–Alsina T-norm/conorm for logistic service provider selection. Ulucay [11] proposed prioritized operators for the evaluation of renewable energy sources. Kavitha et al. [12] employed Einstein and Dombi aggregation operators under a hesitant fuzzy scenario for ensemble feature selection. Shit and Ghorai [13] developed a novel kind of Aczel–Alsina operator to obtain a solution for the selection of the best brand of educational institution. Sun et al. [14] developed novel aggregation operators under the hybrid Pythagorean hesitant fuzzy scenario combining the Aczel–Alsina norm for decision making. Özlü [15] developed novel Dombi operators for Bipolar-valued complex aggregation. Dong et al. [16] proposed operators under a probabilistic linguistic hesitant fuzzy scenario for a rural revitalization project selection in China. Muhammad [17] developed Muirhead mean aggregation operators for regenerative energy source selection under interval-valued probabilistic dual hesitant fuzzy scenarios. Gupta et al. [18] proposed a Bonferroni mean pre-aggregation operator-assisted dynamic fuzzy histogram equalization for retina vascular segmentation. Maneckshaw et al. [19] proposed hesitant fuzzy aggregation with a non-dominated sorting genetic algorithm for multi-object allocation. Mahmood et al. [20] constructed complex fuzzy frank aggregation operators for choosing an optimal encryption algorithm candidate. Jiang et al. [21] proposed a rough integrated asymmetric cloud model under multi-granularity linguistic environments for MCDM. Hu et al. [22] proposed a three-parameter generalized weighted Heronian mean for MADM problems. For MCDM tasks in different scenarios, operators were combined with different extensions of fuzzy theory display availability and flexibility.

The Bonferroni mean operator [23], as a popular aggregation operator, has attracted researchers' interests since being proposed by Bonferroni. It can complete information fusion and yield value for obtaining rankings. It consists of common arithmetic mean and geometric mean operators. Owing to its prominent ability to integrate and compute averages of different elements, many scholars have investigated it and given it more extension operators. Wei et al. [24] introduced an uncertain linguistic Bonferroni mean and investigated related properties. Xu and Chen [25] developed an interval-valued Bonferroni mean. Xia et al. [26] provided a generalization and extension of the Bonferroni mean operator based on an intuitionistic fuzzy environment. Beliakov et al. [27] provided generalization by composing aggregation functions. Recently, more scholars gave more extensions in novel scenarios. Liu et al. [28] introduced a novel definition and concept of the Bonferroni mean operator based on picture fuzzy set theory. Debnath [29] introduced the Aczel–Alsina generalized weighted Bonferroni mean operator into MABAC methods. Gowtham [30] employed a Bonferroni mean and fuzzy pattern tree in epileptic EEG classification. Vommi [31] constructed a novel filter for feature selection with the help of a fuzzy kNN based on the Bonferroni mean. In this paper, HFWA/HFWG is introduced to propose a definition of the novel operator, hesitant fuzzy geometric Bonferroni mean (HFGBM); further, the Bonferroni mean WHFS is integrated to propose a weighted hesitant fuzzy geometric mean (WHFGBM) operator. The novel operator will aggregate information in a symmetrical way. Finally, comparisons with other operators are given to show effectiveness.

The organization of this paper is as follows: In Section 2, basic concepts, including HFS and their related functions, are reviewed. Classical and geometric Bonferroni operators

are given, and their properties are investigated. In Section 3, WHFWA and WHFWG are introduced. The Bonferroni mean operator is integrated to propose WHFGBM based on WHFE, and related properties are provided. In Section 4, the proposed operator is applied in practical tasks to display validation and effectiveness. Conclusions are given in the last section.

2. Preliminaries

In this paper, the symbol $X = \{x_1, x_2, \dots, x_n\}$ is utilized to represent the set. HFS represents the set of multiple HFEs. By adding weights, multiple WHFEs will be integrated into WHFS. For convenience, the symbols h and h^w are utilized as HFE and WHFE, respectively.

2.1. Hesitant Fuzzy Set

Definition 1 ([8]). For a fixed set X , an HFS on X is in terms of a function that, when applied to X , returns a subset of $[0, 1]$. Xia and Xu [8] proposed a mathematical representation of HFS as follows:

$$E = \{ \langle x, h_E(x) \rangle \} \quad (1)$$

where $h_E(x)$ contains multiple values in $[0, 1]$, which denotes the membership degree of the element x to the set X . For convenience, the HFE is called $h_E(x)$ and denoted by h .

Definition 2 ([8]). Considering three HFEs h, h_1, h_2 , h^- denotes the minimum of h , h^+ denotes the maximum of h , and h^c , complement of h , denotes $\bigcup_{\gamma \in h} \{1 - \gamma\}$. The union and intersection operations of two HFEs are defined as $h_1 \cup h_2 = \{\gamma \in h_1 \cup h_2 \mid \gamma \geq \max(h_1^-, h_2^-)\}$ and $h_1 \cap h_2 = \{\gamma \in h_1 \cap h_2 \mid \gamma \leq \min(h_1^+, h_2^+)\}$. Further, Xia and Xu [8] proposed new representation: $h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$ and $h_1 \cap h_2 = \bigcap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Furthermore, for given HFEs h, h_1, h_2 , Xia [32] and Liao [33] introduced the operations below. These operators are used to construct the whole computing system based on HFEs.

1. $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}, \lambda > 0;$
2. $\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}, \lambda > 0;$
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\};$
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\};$
5. $\bigoplus_{i=1}^n h_i = \bigcup_{\gamma_i \in h_i} \{1 - \prod_{i=1}^n (1 - \gamma_i)\};$
6. $\bigotimes_{i=1}^n h_i = \bigcup_{\gamma_i \in h_i} \{\prod_{i=1}^n \gamma_i\}.$

To compare two HFEs, a score function [32] is introduced as the “average” concept to achieve comparison. For a given HFE h , the score function can be defined as

$$s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma \quad (2)$$

where $l(h)$ is the length of the HFE, which generally means the number of decision makers in decision-making tasks.

To compare two HFEs, rankings can be obtained through this rule: If $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$. Scholars further give more rules in their research [34,35].

Definition 3 ([33]). Even though the score function can be used to compare two HFEs, to obtain final rankings when they have the same score values, the scholars [33] gave a variance function, these two indices will be used in combination. For a given HFE h , the variance function can be written as follows:

$$v(h) = \frac{1}{l(h)} \sqrt{\sum_{\gamma_i, \gamma_j \in h} (\gamma_i - \gamma_j)^2} \quad (3)$$

Then, the score function and variance function are used in combination to produce the rankings of HFEs:

- (a) When $s(h_1) > s(h_2)$, then $h_1 > h_2$.
- (b) When $s(h_1) = s(h_2)$, then
 - (1) When $v(h_1) = v(h_2)$, then $h_1 = h_2$.
 - (2) When $v(h_1) > v(h_2)$, then $h_1 < h_2$.

2.2. Bonferroni Mean

The Bonferroni mean was proposed by Bonferroni. To expand the application scope, Yager gave a generalization of the Bonferroni mean:

Definition 4. $p, q \geq 0$, and $a_i (i = 1, \dots, n)$ is a non-negative, real number.

$$B^{p,q}(a_1, \dots, a_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n a_i^p a_j^q \right)^{\frac{1}{p+q}} \quad (4)$$

p, q are two important parameters for Bonferroni. If $q = 0$, the Bonferroni mean will degenerate into a generalized operator:

$$B^{p,0}(a_1, \dots, a_n) = \left(\frac{1}{n} \sum_{i=1}^n a_i^p \left(\frac{1}{n-1} \sum_{j=1, j \neq i}^n a_j^0 \right) \right)^{\frac{1}{p+0}} = \left(\frac{1}{n} \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \quad (5)$$

Further, if $p = 1$ and $q = 0$, then the Bonferroni mean operator will degenerate into basic arithmetic mean:

$$B^{1,0}(a_1, \dots, a_n) = \frac{1}{n} \sum_{i=1}^n a_i \quad (6)$$

Generally, mean operators contains two common types, arithmetic and geometric ($A(a_1, \dots, a_n) = \frac{1}{n} \sum_{i=1}^n a_i$ and $G(a_1, \dots, a_n) = (\prod_{i=1}^n a_i)^{\frac{1}{n}}$).

The Bonferroni mean was proposed by Bonferroni in 1950. Many researchers have given different generalizations and investigated some properties. Xia et al. [26] proposed a geometric Bonferroni mean (GB), and its definition is given as follows:

$$GB^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^n (pa_i + qa_j)^{\frac{1}{n(n-1)}} = \frac{1}{p+q} \prod_{i=1}^n \left(\prod_{j \neq i}^n (pa_i + qa_j)^{\frac{1}{n-1}} \right)^{\frac{1}{n}} \quad (7)$$

Li et al. [36] introduced geometric Bonferroni mean (GBM) by comparing arithmetic and geometric means and investigating the relationship inspired by Xia et al. [26], and its definition is given as follows:

$$GBM^{p,q}(a_1, \dots, a_n) = \frac{1}{p+q} \prod_{i=1}^n \left(pa_i + \prod_{j \neq i}^n (qa_j)^{\frac{1}{n-1}} \right)^{\frac{1}{n}} \quad (8)$$

Example 1. Let $a_1 = 0.0185, a_2 = 0.8214, a_3 = 0.4447$ and $p = 1, q = 2$ is assigned, then it can be known:

$$GB^{p,q}(0.0185, 0.8214, 0.4447) = 0.3809 \neq GBM^{1,2}(0.0185, 0.8214, 0.4447) = 0.3159$$

Some properties of the GBM are listed as follows:

1. **Idempotency:** $GBM^{p,q}(a, a, \dots, a) = a$ if $a_i = a$ for all i .
2. **Monotonicity:** $GBM^{p,q}(a_1, a_2, \dots, a_n) \geq GBM^{p,q}(b_1, b_2, \dots, b_n)$, if $a_i \geq b_i$ for all i .
3. **Boundedness:** $\min_i \{a_i\} \leq GBM^{p,q}(a_1, a_2, \dots, a_n) \leq \max_i \{a_i\}$

3. Weighted Hesitant Fuzzy Geometric Bonferroni Mean

For a given set $X = \{x_1, \dots, x_n\}$, Zhang and Wu [37] introduced the concept of WHFS E^w defined on X , which can be represented by the following mathematical symbol:

$$E^w = \{ \langle x, h_E^w(x) \rangle \mid x \in X \}$$

where $h_E^w(x) = \{ \langle \gamma_1, w_1 \rangle, \langle \gamma_2, w_2 \rangle, \dots, \langle \gamma_m, w_m \rangle \}$; $\gamma_j, j = 1, 2, \dots, m$, forming a non-descending collection containing m membership degrees in range $[0, 1]$; and weight $w_j \in [0, 1], j = 1, 2, \dots, m, \sum_{j=1}^m w_j = 1$, which represents the importance of the membership degree γ_j . On the other hand, weights display the preference of the decision maker in real decision-making processes. Therefore, $h^w = h_E^w(x)$ is called a WHFE.

Furthermore, Zhang and Wu [37] introduced some operations of WHFSs below. These operators are used to construct the whole computing system based on WHFEs.

Given three WHFEs $h^w = \bigcup_{\gamma \in h^w} \{ \langle \gamma, w_\gamma \rangle \}$, $h_1^w = \bigcup_{\gamma_1 \in h_1^w} \{ \langle \gamma_1, w_{\gamma_1} \rangle \}$ and $h_2^w = \bigcup_{\gamma_2 \in h_2^w} \{ \langle \gamma_2, w_{\gamma_2} \rangle \}$, for $\lambda > 0$, then

- (1) $(h^w)^c = \bigcup_{\gamma \in h^w} \{ \langle 1 - \gamma, w_\gamma \rangle \}$;
- (2) $h_1^w \cup h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{ \langle \gamma_1 \vee \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2} \rangle \}$;
- (3) $h_1^w \cap h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{ \langle \gamma_1 \wedge \gamma_2, w_{\gamma_1} \cdot w_{\gamma_2} \rangle \}$;
- (4) $(h^w)^\lambda = \bigcup_{\gamma \in h^w} \{ \langle \gamma^\lambda, w_\gamma \rangle \}$;
- (5) $\lambda h^w = \bigcup_{\gamma \in h^w} \{ \langle 1 - (1 - \gamma)^\lambda, w_\gamma \rangle \}$;
- (6) $h_1^w \oplus h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{ \langle \gamma_1 + \gamma_2 - \gamma_1 \gamma_2, w_{\gamma_1} w_{\gamma_2} \rangle \}$;
- (7) $h_1^w \otimes h_2^w = \bigcup_{\gamma_1 \in h_1^w, \gamma_2 \in h_2^w} \{ \langle \gamma_1 \gamma_2, w_{\gamma_1} w_{\gamma_2} \rangle \}$.

After that, Zeng et al. [38] pointed out the shortcoming of “ $\cup$ ” and “ $\cap$ ” operations on WHFSs, redefined their novel operations and investigated their properties.

Here, the operations, such as “ $\oplus$ ” and “ $\otimes$ ”, on WHFEs are investigated in the following:

Example 2. For a given WHFE $h_1^w = \{ \langle 0.6, 0.3 \rangle, \langle 0.7, 0.25 \rangle, \langle 0.8, 0.45 \rangle \}$, there is

$$h_1^w \oplus h_1^w = \{ \langle 0.84, 0.09 \rangle, \langle 0.88, 0.15 \rangle, \langle 0.91, 0.065 \rangle, \\ \langle 0.92, 0.27 \rangle, \langle 0.94, 0.225 \rangle, \langle 0.96, 0.2025 \rangle \}$$

$$h_1^w \otimes h_1^w = \{ \langle 0.36, 0.09 \rangle, \langle 0.42, 0.15 \rangle, \langle 0.48, 0.27 \rangle, \\ \langle 0.49, 0.0625 \rangle, \langle 0.56, 0.225 \rangle, \langle 0.64, 0.2025 \rangle \}$$

Based on the above operations, the following results are obtained:

$$2h_1^w = \bigcup_{\gamma \in h_1^w} \{ \langle 1 - (1 - \gamma)^2, w_\gamma \rangle \} = \{ \langle 0.84, 0.3 \rangle, \langle 0.91, 0.25 \rangle, \langle 0.96, 0.45 \rangle \}$$

$$(h_1^w)^2 = \bigcup_{\gamma \in h_1^w} \{ \langle \gamma^2, w_\gamma \rangle \} = \{ \langle 0.36, 0.3 \rangle, \langle 0.49, 0.25 \rangle, \langle 0.64, 0.45 \rangle \}$$

Hence, $h_1^w \oplus h_1^w \neq 2h_1^w, h_1^w \otimes h_1^w \neq (h_1^w)^2$. Obviously, these results are in contradiction with our tuition. Hence, providing novel definitions of “ $\oplus$ ” and “ $\otimes$ ” not only will make results more objective and accurate but also accord with human perception.

Given WHFEs $h_1^w = \bigcup_{i=1}^{l_1} \{ \langle \gamma_1^{(i)}, w_{\gamma_1^{(i)}} \rangle \}$, $h_2^w = \bigcup_{i=1}^{l_2} \{ \langle \gamma_2^{(i)}, w_{\gamma_2^{(i)}} \rangle \}$, where l_1 and l_2 are the lengths of WHFEs h_1^w and h_2^w , respectively.

Owing to the difference in the number of experts, the number of corresponding elements in WHFEs is also different. In most cases, h_1^w and h_2^w have different lengths ($l_1 \neq l_2$). To make WHFEs comparable, every WHFE with a less-than-optimal length will be complemented by adding a maximal, minimal, or any value. Optimists and pessimists will decide which value is used for complementation to make all the WHFEs have the same length. In this paper, the minimal value will be utilized for complementation. For convenience, $l = \max\{l_1, l_2\}$ is adopted.

Definition 5. For given WHFEs $h_1^w = \bigcup_{i=1}^{l_1} \{ \langle \gamma_1^{(i)}, w_{\gamma_1^{(i)}} \rangle \}$, $h_2^w = \bigcup_{i=1}^{l_2} \{ \langle \gamma_2^{(i)}, w_{\gamma_2^{(i)}} \rangle \}$, then

1. $h_1^w \oplus h_2^w = \bigcup_{i=1}^l \{ \langle \gamma_i, w_i \rangle \}$, where $\gamma_i = \gamma_1^{(i)} + \gamma_2^{(i)} - \gamma_1^{(i)} \gamma_2^{(i)}$, $w_i = \frac{w_{\gamma_1^{(i)}} \vee w_{\gamma_2^{(i)}}}{\sum_{k=1}^l (w_{\gamma_1^{(k)}} \vee w_{\gamma_2^{(k)}})}$;
2. $h_1^w \otimes h_2^w = \bigcup_{i=1}^l \{ \langle \gamma_i, w_i \rangle \}$, where $\gamma_i = \gamma_1^{(i)} \gamma_2^{(i)}$, $w_i = \frac{w_{\gamma_1^{(i)}} \vee w_{\gamma_2^{(i)}}}{\sum_{k=1}^l (w_{\gamma_1^{(k)}} \vee w_{\gamma_2^{(k)}})}$.

For $h_1^w = \{ \langle 0.6, 0.3 \rangle, \langle 0.7, 0.25 \rangle, \langle 0.8, 0.45 \rangle \}$, $h_2^w = \{ \langle 0.65, 0.4 \rangle, \langle 0.7, 0.35 \rangle, \langle 0.9, 0.25 \rangle \}$, then:

$$h_1^w \oplus h_2^w = \{ \langle 0.86, 0.333 \rangle, \langle 0.91, 0.292 \rangle, \langle 0.98, 0.375 \rangle \};$$

$$h_1^w \otimes h_2^w = \{ \langle 0.39, 0.333 \rangle, \langle 0.49, 0.292 \rangle, \langle 0.72, 0.375 \rangle \}.$$

Here, the Example 2 is recalculated.

For a given WHFE, $h_1^w = \{ \langle 0.6, 0.3 \rangle, \langle 0.7, 0.25 \rangle, \langle 0.8, 0.45 \rangle \}$; then

$$h_1^w \oplus h_1^w = \{ \langle 0.84, 0.3 \rangle, \langle 0.91, 0.25 \rangle, \langle 0.96, 0.45 \rangle \} = 2h_1^w$$

$$h_1^w \otimes h_1^w = \{ \langle 0.36, 0.3 \rangle, \langle 0.49, 0.25 \rangle, \langle 0.64, 0.45 \rangle \} = (h_1^w)^2$$

Furthermore, the following property is obtained:

Property: For a given WHFE h^w , then:

- (1) $\underbrace{h^w \oplus h^w \oplus \dots \oplus h^w}_n = nh^w;$
- (2) $\underbrace{h^w \otimes h^w \otimes \dots \otimes h^w}_n = (h^w)^n.$

Proof. (1) Firstly, for $h^w = \bigcup_{i=1}^l \{ \langle \gamma^{(i)}, w_{\gamma^{(i)}} \rangle \}$, applying the mathematical induction, then:

Case 1: Basic step: For $n = 2$, then

$$\begin{aligned} h^w \oplus h^w &= \bigcup_{i=1}^l \{ \langle \gamma^{(i)} + \gamma^{(i)} - \gamma^{(i)} \gamma^{(i)}, w_{\gamma^{(i)}} \vee w_{\gamma^{(i)}} \rangle \} \\ &= \bigcup_{i=1}^l \{ \langle 2\gamma^{(i)} - (\gamma^{(i)})^2, w_{\gamma^{(i)}} \rangle \} \\ &= \bigcup_{i=1}^l \{ \langle 1 - (1 - \gamma^{(i)})^2, w_{\gamma^{(i)}} \rangle \} \\ &= 2h^w \end{aligned}$$

Case 2: Induction step: Assume that the property (1) holds for some fixed $n = k \geq 2$. Namely, $\underbrace{h^w \oplus h^w \oplus \dots \oplus h^w}_k = kh^w$.

Now, for $n = k + 1$, then $\underbrace{h^w \oplus h^w \oplus \dots \oplus h^w}_n = nh^w$ holds.

$$\begin{aligned}
\underbrace{h^w \oplus h^w \oplus \dots \oplus h^w}_{k+1} &= \underbrace{h^w \oplus h^w \oplus \dots \oplus h^w}_k \oplus h^w \\
&= \bigcup_{i=1}^l \{ < 1 - (1 - \gamma^{(i)})^k, w_{\gamma^{(i)}} > \} \oplus h^w \\
&= \bigcup_{i=1}^l \{ < 1 - (1 - \gamma^{(i)})^k + \gamma^{(i)} - \gamma^{(i)}[1 - (1 - \gamma^{(i)})^k], w_{\gamma^{(i)}} \vee w_{\gamma^{(i)}} > \} \\
&= \bigcup_{i=1}^l \{ < 1 - (1 - \gamma^{(i)})^k + \gamma^{(i)}(1 - \gamma^{(i)})^k, w_{\gamma^{(i)}} > \} \\
&= \bigcup_{i=1}^l \{ < 1 - (1 - \gamma^{(i)})^{k+1}, w_{\gamma^{(i)}} > \} \\
&= (k+1)h^w
\end{aligned}$$

(2) For $h^w = \bigcup_{i=1}^l \{ \gamma^{(i)}, w_{\gamma^{(i)}} \}$, applying the mathematical induction:

Case 3: Basic step: For $n = 2$, then

$$\begin{aligned}
h^w \otimes h^w &= \bigcup_{i=1}^l \{ < \gamma^{(i)} \gamma^{(i)}, w_{\gamma^{(i)}} \vee w_{\gamma^{(i)}} > \} \\
&= \bigcup_{i=1}^l \{ (\gamma^{(i)})^2, w_{\gamma^{(i)}} \} \\
&= (h^w)^2
\end{aligned}$$

Case 4: Induction step: Assume that the property (2) holds for some fixed $n = k \geq 2$. Namely, $\underbrace{h^w \otimes h^w \otimes \dots \otimes h^w}_k = (h^w)^k$.

$$\begin{aligned}
\underbrace{h^w \otimes h^w \otimes \dots \otimes h^w}_{k+1} &= \underbrace{h^w \otimes h^w \otimes \dots \otimes h^w}_k \otimes h^w \\
&= \bigcup_{i=1}^l \{ < (\gamma^{(i)})^k, w_{\gamma^{(i)}} > \} \otimes \{ \gamma^{(i)}, w_{\gamma^{(i)}} \} \\
&= \bigcup_{i=1}^l \{ (\gamma^{(i)})^k \gamma^{(i)}, w_{\gamma^{(i)}} \vee w_{\gamma^{(i)}} \} \\
&= \bigcup_{i=1}^l \{ (\gamma^{(i)})^{(k+1)}, w_{\gamma^{(i)}} \} \\
&= (h^w)^{(k+1)}
\end{aligned}$$

Hence, the proof of Property 1 is completed. $\square$

Furthermore, Zhang and Wu [37] introduced the score function and variance function of WHFE h^w , respectively, and Zeng et al. [38] proposed the ranking rule to compare any two WHFEs, h_1^w and h_2^w .

Considering that WHFS is a useful tool for handling uncertainty and fuzzy information and is widely applied in the decision-making field and evaluation system, motivated by the idea of the geometric Bonferroni mean proposed by Li et al. [36], in this paper, the geometric Bonferroni mean operator and WHFS theory are integrated.

Let $a_1, a_2, \dots, a_n$ be a collection of HFEs, $p, q \geq 0$, then the geometric Bonferroni mean of HFE is proposed as follows:

$$HFGBM^{p,q}(a_1, \dots, a_n) = \frac{1}{p+q} \left(\bigotimes_{i=1}^n \left(pa_i \oplus \bigotimes_{j \neq i}^n (qa_j)^{\frac{1}{n-1}} \right)^{\frac{1}{n}} \right) \quad (9)$$

For the HFGBM operator, three properties are discussed as follows:

1. **Idempotency** $HFGBM^{p,q}(a, a, \dots, a) = a$ if $a_i = a$ for all i .
2. **Monotonicity** $HFGBM^{p,q}(a_1, a_2, \dots, a_n) \geq HFGBM^{p,q}(b_1, b_2, \dots, b_n)$, if $a_i \geq b_i$ for all i .
3. **Boundedness** $\min_i \{a_i\} \leq HFGBM^{p,q}(a_1, a_2, \dots, a_n) \leq \max_i \{a_i\}$

Let $a_1, a_2, \dots, a_n$ be a collection of WHFEs, $p, q \geq 0$, then the geometric Bonferroni mean of WHFE is proposed in the following:

$$WHFGBM^{p,q}(a_1, \dots, a_n) = \frac{1}{p+q} \left(\bigotimes_{i=1}^n \left(pa_i \oplus \bigotimes_{j \neq i}^n (qa_j)^{\frac{1}{n-1}} \right)^{w_i} \right)$$

where the $\otimes$ operator can be implemented through:

$$\otimes_{i=1}^n a_i = \bigcup_{\gamma_1 \in a_1, \dots, \gamma_n \in a_n} \{ \langle \prod_{i=1}^n \gamma_i, w_{\gamma_1} w_{\gamma_2} \dots w_{\gamma_n} \rangle \}; \quad (10)$$

the $\oplus$ operator can be implemented through:

$$a_1 \oplus a_2 = \bigcup_{\gamma_1 \in a_1, \gamma_2 \in a_2} \{ \langle \gamma_1 + \gamma_2 - \gamma_1 \gamma_2, w_{\gamma_1} w_{\gamma_2} \rangle \}; \quad (11)$$

and the multiplication and power operator can be implemented through:

$$\lambda a = \bigcup_{\gamma \in a} \{ \langle 1 - (1 - \gamma)^\lambda, w_\gamma \rangle \} \quad (12)$$

$$(a)^\lambda = \bigcup_{\gamma \in a} \{ \langle \gamma^\lambda, w_\gamma \rangle \} \quad (13)$$

For the WHFGBM operator, three properties are discussed as follows:

1. Idempotency: For the WHFGBM operator, when $a_i = a$ for all i , the WHFGBM operator can be written as $WHFGBM^{p,q}(a, a, \dots, a)$.

$$\begin{aligned} WHFGBM^{p,q}(a, a, \dots, a) &= \frac{1}{p+q} \left(\otimes_{i=1}^n \left(pa \oplus \otimes_{j \neq i}^n (qa)^{\frac{1}{n-1}} \right)^{w_i} \right) \\ &= \frac{1}{p+q} (\otimes_{i=1}^n (pa \oplus qa)^{w_i}) \\ &= \frac{1}{p+q} (p+q)a = a \end{aligned} \quad (14)$$

Therefore, idempotency holds.

2. Monotonicity: For two groups of WHFEs, $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$, which meet the condition that $a_i \geq b_i$ for all i . In the WHFGBM operator, basic operators are multiplication and power operators. These two operators are based on element-by-element multiplication and power, which meets monotonicity. Owing to weights $w_i \geq 0$, if $a_i > b_i$, then $WHFGBM^{p,q}(a_1, a_2, \dots, a_n) \succ WHFGBM^{p,q}(b_1, b_2, \dots, b_n)$. Therefore, monotonicity holds.

3. Boundedness: Let $\min_i \{a_i\} = \min\{a_1, a_2, \dots, a_n\}$, $\max_i \{a_i\} = \max\{a_1, a_2, \dots, a_n\}$, owing to $a_i > \min_i \{a_i\}$, then $WHFGBM^{p,q}(a_i) \succ WHFGBM^{p,q}(\min_i \{a_i\})$ holds according to monotonicity. On the other hand, the $WHFGBM^{p,q}(\max_i \{a_i\}) \succ WHFGBM^{p,q}(a_i)$ holds, owing to $a_i < \max_i \{a_i\}$. Therefore, owing to the existence of upper and lower boundaries, the boundedness of the WHFGBM operator holds.

The WHFGBM operator is specially developed for handling weighted hesitant fuzzy data. Since WHFS allows every membership degree to have weights, a special extension of Bonferroni mean operator is necessary. Existing Bonferroni mean operators extensions are based on different environments, such as hesitant fuzzy scenarios and other fuzzy environments. The main differences among these operators lie in the types of data they handle. Traditional hesitant fuzzy data lack weights, interval valued fuzzy data discard single value to describe things, intuitionistic fuzzy data adopt positive and negative membership degrees to depict fuzzy relation.

WHFS includes multiple membership degrees with weights, which improves the depiction of relationships between things. However, multiple membership degrees with weights will cause computation burden in practical operators running process. When the problem to be solved become more and more complex, the attributes will also increase. In programming implementation, multiple loops are adopted to construct the proposed operator. Hence, if decision-making problems become complex enough, the computation burden will lead to difficulty in providing solutions within limited time.

The flowchart and algorithm of the proposed operator, WHFGBM, are displayed in Figure 1 and Algorithm 1, respectively.

Algorithm 1 Weighted hesitant fuzzy geometric Bonferroni mean

Input: The collection of WHFEs $a_1, \dots, a_n$, the constant p, q .

Output: Aggregation result f .

- 1: Initialize relative parameters and WHFEs.
 - 2: Construct empty set $All = \{\}$.
 - 3: **for** $i \leftarrow 1$ to n **do**
 - 4: Construct empty set $Temp = \{\}$.
 - 5: **for** $j \leftarrow 1$ to n **do**
 - 6: **if** j is not equal to i **then**
 - 7: Compute $(qa_j)^{\frac{1}{n-1}}$ through lh^w and $(h^w)^\lambda$ operators and put into set $Temp$.
 - 8: **end if**
 - 9: **end for**
 - 10: Adopt $\otimes$ to aggregate WHFEs from $Temp$ to produce a_T .
 - 11: Adopt $\oplus$ to aggregate pa_i and a_T , obtain result and put in All .
 - 12: **end for**
 - 13: Adopt $\oplus$ operator to aggregate all WHFEs from All with the constant p, q .
 - 14: Obtain final result f of fuzzy information fusion.
-

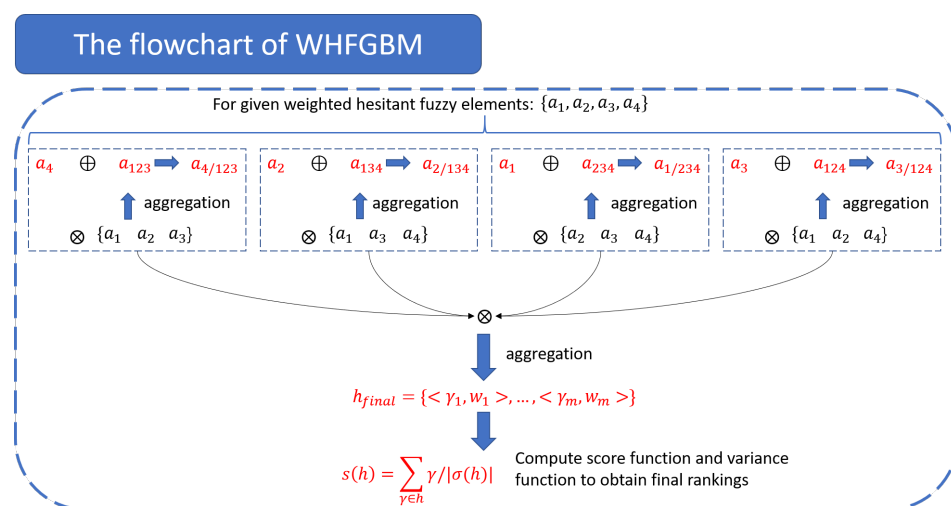


Figure 1. The flowchart of the proposed operator WHFGBM.

4. Numerical Example

In this section, a series of numerical examples is provided to compare the proposed method with existing methods.

Example 3. To enhance the working efficiency, the university is considering purchasing a new information system. Assuming there are some software packages available, the managers from the university need to obtain the best plan after comprehensive consideration. After the initial selection, A_1, A_2, A_3, A_4 are the four candidates. Then, three experts, $E_k (k = 1, 2, 3)$, will give their final scores from different views to decide which information system will be the best choice. In the whole decision-making process, three experts give their own preferences, respectively. The following four attributes will be the main factors taken into consideration:

- Capitalized costs for information system construction (C_1).
- Performance improvement for the whole organization (C_2).
- Migration costs and sunk costs (C_3).
- The reliability of software developer (C_4).

For different attributes ($C_j (j = 1, 2, 3, 4)$), these attributes have different weights $w = (0.3, 0.25, 0.25, 0.2)^T$. $A^{(k)} = (r_{ij}^{(k)}) (k = 1, 2, 3)$ (seen in Table 1) is constructed, and $r_{ij}^{(k)} \in [0, 1]$ represents the evaluation degree of every expert $E_k (k = 1, 2, 3)$ to every alternative A_i under attribute C_j . Through our proposed method, we provide solutions to this problem.

In comparison, two aggregation operators, WHFHA/WHFHG, are used to aggregate hesitant fuzzy information. To illustrate results, a comparison with WHFWA and WHFWG operators [37] is made. The definition of these operators is as follows:

$$\text{WHFWA}(h_1^w, \dots, h_n^w) = \oplus_{j=1}^n (w_j h_j^w) = \bigcup_{\gamma_1 \in h_1^w, \dots, \gamma_n \in h_n^w} \left\{ \left\langle 1 - \prod_{j=1}^n (1 - \gamma_j)^{w_j}, w_{\gamma_1} \dots w_{\gamma_n} \right\rangle \right\}$$

$$\text{WHFWG}(h_1^w, \dots, h_n^w) = \otimes_{j=1}^n (h_j^w)^{w_j} = \bigcup_{\gamma_1 \in h_1^w, \dots, \gamma_n \in h_n^w} \left\{ \left\langle \prod_{j=1}^n \gamma_j^{w_j}, w_{\gamma_1} w_{\gamma_2} \dots w_{\gamma_n} \right\rangle \right\}$$

Table 1. The preference matrix provided by E_1, E_2 , and E_3 .

	E_1				E_2				E_3			
	C_1	C_2	C_3	C_4	C_1	C_2	C_3	C_4	C_1	C_2	C_3	C_4
A_1	0.6	0.8	0.5	0.6	0.6	0.85	0.65	0.6	0.8	0.85	0.65	0.7
A_2	0.5	0.9	0.8	0.7	0.7	0.8	0.8	0.75	0.7	0.8	0.8	0.75
A_3	0.7	0.6	0.7	0.6	0.8	0.6	0.75	0.6	0.7	0.8	0.75	0.9
A_4	0.8	0.7	0.8	0.9	0.8	0.7	0.8	0.85	0.9	0.75	0.85	0.9

Case 5 In this case, assume that the weight of every expert is unknown, and evaluation results from decision makers are anonymous.

Step 1 Firstly, WHFSs $A = \{ \langle r_{ij}, w_{r_{ij}} \rangle \}$ are obtained as follows:

For A_1

$$h_{A_1}^w(C_1) = \{ \langle 0.6, \frac{2}{3} \rangle, \langle 0.8, \frac{1}{3} \rangle \}, h_{A_1}^w(C_2) = \{ \langle 0.8, \frac{1}{3} \rangle, \langle 0.85, \frac{2}{3} \rangle \}$$

$$h_{A_1}^w(C_3) = \{ \langle 0.5, \frac{1}{3} \rangle, \langle 0.65, \frac{2}{3} \rangle \}, h_{A_1}^w(C_4) = \{ \langle 0.6, \frac{2}{3} \rangle, \langle 0.7, \frac{1}{3} \rangle \}$$

For A_2

$$h_{A_2}^w(C_1) = \{ \langle 0.5, \frac{1}{3} \rangle, \langle 0.7, \frac{2}{3} \rangle \}, h_{A_2}^w(C_2) = \{ \langle 0.8, \frac{2}{3} \rangle, \langle 0.9, \frac{1}{3} \rangle \}$$

$$h_{A_2}^w(C_3) = \{ \langle 0.8, 1 \rangle \}, h_{A_2}^w(C_4) = \{ \langle 0.7, \frac{1}{3} \rangle, \langle 0.75, \frac{2}{3} \rangle \}$$

For A_3

$$h_{A_3}^w(C_1) = \{ \langle 0.7, \frac{2}{3} \rangle, \langle 0.8, \frac{1}{3} \rangle \}, h_{A_3}^w(C_2) = \{ \langle 0.6, \frac{2}{3} \rangle, \langle 0.8, \frac{1}{3} \rangle \}$$

$$h_{A_3}^w(C_3) = \{ \langle 0.7, \frac{1}{3} \rangle, \langle 0.75, \frac{2}{3} \rangle \}, h_{A_3}^w(C_4) = \{ \langle 0.6, \frac{2}{3} \rangle, \langle 0.9, \frac{1}{3} \rangle \}$$

For A_4

$$h_{A_4}^w(C_1) = \{ \langle 0.8, \frac{2}{3} \rangle, \langle 0.9, \frac{1}{3} \rangle \}, h_{A_4}^w(C_2) = \{ \langle 0.7, \frac{2}{3} \rangle, \langle 0.75, \frac{1}{3} \rangle \}$$

$$h_{A_4}^w(C_3) = \{ \langle 0.8, \frac{2}{3} \rangle, \langle 0.85, \frac{1}{3} \rangle \}, h_{A_4}^w(C_4) = \{ \langle 0.85, \frac{1}{3} \rangle, \langle 0.9, \frac{2}{3} \rangle \}$$

Step 2 The WHFHA operator is applied to aggregate WHFEs of every sample. The results are displayed in Table 2.

Step 3 Through the equation above, the score and variance of $h^w(A_i) (i = 1, 2, 3, 4)$ are obtained:

$$s(h^w(A_1)) = 0.9256, v(h^w(A_1)) = 0.0345, s(h^w(A_2)) = 0.9790, v(h^w(A_2)) = 0.0090, \\ s(h^w(A_3)) = 0.9307, v(h^w(A_3)) = 0.0213, s(h^w(A_4)) = 0.9747, v(h^w(A_4)) = 0.0148.$$

Step 4 To produce the final ranking results, these samples are ranked according to $s(h^w(A_i))$. Therefore, the rankings of every sample: $A_2 \succ A_4 \succ A_3 \succ A_1$ are obtained, and the best sample is A_2 .

Table 2. The results of the WHFWA operator.

WHFS	The Results of the WHFWA Operator
$h_{A_1}^w$	$\{< 0.8183, 0.0123 >, \{< 0.8313, 0.0247 >, \{< 0.8526, 0.0247 >, \{< 0.8632, 0.0494 >, \{< 0.8863, 0.0247 >, \{< 0.8944, 0.0494 >, \{< 0.9078, 0.0494 >, \{0.9123, 0.0247\}, \{< 0.9144, 0.0988 >, \{< 0.9186, 0.0494 >, \{< 0.9289, 0.0494 >, \{< 0.934, 0.0988 >, \{< 0.9451, 0.0494 >, \{< 0.949, 0.0988 >, \{< 0.9555, 0.0988 >, \{< 0.9587, 0.1975 >\}$
$h_{A_2}^w$	$\{< 0.9507, 0.037 >, \{< 0.9637, 0.0741 >, \{< 0.9708, 0.0741 >, \{< 0.9722, 0.0741 >, \{< 0.9785, 0.1481 >, \{< 0.9795, 0.1481 >, \{< 0.9835, 0.1481 >, \{< 0.9878, 0.2963 >\}$
$h_{A_3}^w$	$\{< 0.8757, 0.0247 >, \{< 0.8846, 0.0494 >, \{< 0.8937, 0.0123 >, \{< 0.9014, 0.0247 >, \{< 0.9048, 0.0494 >, \{< 0.9116, 0.0988 >, \{< 0.9186, 0.0247 >, \{< 0.9244, 0.0494 >, \{< 0.9263, 0.0494 >, \{< 0.9316, 0.0988 >, \{< 0.937, 0.0247 >, \{< 0.9415, 0.0494 >, \{< 0.9435, 0.0988 >, \{< 0.9476, 0.1975 >, \{< 0.9517, 0.0494 >, \{< 0.9552, 0.0988 >\}$
$h_{A_4}^w$	$\{< 0.9175, 0.0123 >, \{< 0.9392, 0.0247 >, \{< 0.9413, 0.0247 >, \{< 0.9469, 0.0247 >, \{< 0.9567, 0.0494 >, \{< 0.9609, 0.0494 >, \{< 0.9622, 0.0494 >, \{< 0.9665, 0.0247 >, \{< 0.9722, 0.0988 >, \{< 0.9753, 0.0494 >, \{< 0.9762, 0.0494 >, \{< 0.9785, 0.0494 >, \{< 0.9825, 0.0988 >, \{< 0.9841, 0.0988 >, \{< 0.9847, 0.0988 >, \{< 0.9887, 0.1975 >\}$

Step 2: The WHFWG operator is applied to aggregate WHFEs of every sample. The results of the WHFWG operator are displayed in Table 3.

Table 3. The results of the WHFWG operator.

WHFS	The Results of the WHFWG Operator
$h_{A_1}^w$	$\{< 0.3407, 0.1975 >, \{< 0.3525, 0.0988 >, \{< 0.3604, 0.0988 >, \{< 0.3729, 0.0494 >, \{< 0.4253, 0.0988 >, \{< 0.44, 0.0494 >, \{< 0.4447, 0.0988 >, \{< 0.4499, 0.0494 >, \{< 0.46, 0.0494 >, \{< 0.4654, 0.0247 >, \{< 0.4703, 0.0494 >, \{< 0.4866, 0.0247 >, \{< 0.555, 0.0494 >, \{< 0.5742, 0.0247 >, \{< 0.587, 0.0247 >, \{< 0.6073, 0.0123 >, \}$
$h_{A_2}^w$	$\{< 0.4487, 0.2963 >, \{< 0.4517, 0.1481 >, \{< 0.4826, 0.1481 >, \{< 0.4859, 0.0741 >, \{< 0.5027, 0.1481 >, \{< 0.5061, 0.0741 >, \{< 0.5407, 0.0741 >, \{< 0.5443, 0.037 >\}$
$h_{A_3}^w$	$\{< 0.3293, 0.1975 >, \{< 0.3542, 0.0988 >, \{< 0.3878, 0.0988 >, \{< 0.4171, 0.0494 >, \{< 0.4298, 0.0988 >, \{< 0.447, 0.0988 >, \{< 0.4623, 0.0494 >, \{< 0.4808, 0.0494 >, \{< 0.5061, 0.0494 >, \{< 0.5263, 0.0494 >, \{< 0.5443, 0.0247 >, \{< 0.5661, 0.0247 >, \{< 0.5833, 0.0494 >, \{< 0.6274, 0.0247 >, \{< 0.6868, 0.0247 >, \{< 0.7388, 0.0123 >, \}$
$h_{A_4}^w$	$\{< 0.5458, 0.1975 >, \{< 0.5546, 0.0988 >, \{< 0.5999, 0.0988 >, \{< 0.6096, 0.0494 >, \{< 0.6115, 0.0988 >, \{< 0.6214, 0.0494 >, \{< 0.629, 0.0988 >, \{< 0.6392, 0.0494 >, \{< 0.6721, 0.0494 >, \{< 0.683, 0.0247 >, \{< 0.6914, 0.0494 >, \{< 0.7026, 0.0247 >, \{< 0.7047, 0.0494 >, \{< 0.7161, 0.0247 >, \{< 0.7746, 0.0247 >, \{< 0.7871, 0.0123 >\}$

Step 3: Through the equation above, the score and variance of the $h^w(A_i)$ of every sample are obtained. $s(h^w(A_1)) = 0.4190$, $v(h^w(A_1)) = 0.0724$, $s(h^w(A_2)) = 0.4975$, $v(h^w(A_2)) = 0.0312$, $s(h^w(A_3)) = 0.4409$, $v(h^w(A_3)) = 0.0964$, $s(h^w(A_4)) = 0.6196$, $v(h^w(A_4)) = 0.0613$.

Step 4: To produce the final ranking results, these samples are ranked according to $s(h^w(A_i))$ and $v(h^w(A_i))$. Therefore, the rankings of every sample ($A_4 \succ A_2 \succ A_3 \succ A_1$) are obtained, and the best sample is A_4 .

For this example, the proposed method—WHFGMB operator—is applied to aggregate these WHFEs. After computation, the novel rankings of these examples is obtained ($A_4 \succ A_2 \succ A_3 \succ A_1$), and the best sample is A_4 . It is easy to know that WHFGMB has the same ranking as WHFWG; these two operators provide the same best choice. However, WHFWA provides a divergent result, owing to the difference between A_2 and A_4 .

Example 4 ([37]). Site choice is a common multi-criteria decision-making problem a factory considers when adding new construction for production capacity expansion. Three attributes G_i ($i = 1, 2, 3$) are factors influencing the final decision, including

- G_1 : Price;
- G_2 : Location;

- G_3 : Environment.

After the initial selection and experts' evaluation, the decision-making matrix, including weighted hesitant fuzzy data, is obtained. WHFE can be represented by $h_{ij}^w = \bigcup_{\gamma_{ij} \in h_{ij}^w} \{ < \gamma_{ij}, w_{\gamma_{ij}} > \}$, where $w_{\gamma_{ij}}$ and γ_{ij} are weights and membership degree values, respectively, which describe the candidate Y_i to the attribute G_j . These data, described by WHFEs, are displayed in Table 4.

Table 4. Weighted hesitant fuzzy decision matrix.

	G_1	G_2	G_3
Y_1	{(0.6,0.3),(0.5,0.3),(0.4,0.4)}	{(0.6,0.8),(0.4,0.2)}	{(0.5,0.3),(0.3,0.7)}
Y_2	{(0.4,0.6),(0.3,0.4)}	{(0.8,1)}	{(0.4,0.2),(0.3,0.3),(0.2,0.5)}
Y_3	{(0.8,1)}	{(0.7,0.1),(0.6,0.3),(0.5,0.6)}	{(0.2,0.5),(0.1,0.5)}

Remark 1. The score and variance of $h^w(Y_i)$ of every sample are obtained. The results of WHFWA: $s(h^w(Y_1)) = 0.6445$, $v(h^w(Y_1)) = 0.1072$, $s(h^w(Y_2)) = 0.8585$, $v(h^w(Y_2)) = 0.0115$, $s(h^w(Y_3)) = 0.8686$, $v(h^w(Y_3)) = 0.0140$. The results of WHFWG: $s(h^w(Y_1)) = 0.2962$, $v(h^w(Y_1)) = 0.1037$, $s(h^w(Y_2)) = 0.2844$, $v(h^w(Y_2)) = 0.0762$, $s(h^w(Y_3)) = 0.2289$, $v(h^w(Y_3)) = 0.0528$. The rankings of WHFWA and WHFWG are: $Y_3 \succ Y_2 \succ Y_1$ and $Y_1 \succ Y_2 \succ Y_3$, respectively. The proposed operator WHFGBM is applied to obtain the final rankings: $Y_1 \succ Y_3 \succ Y_2$. WHFGBM gives the same best choice as WHFWG, but both of them have differences regarding Y_2 and Y_3 . Even though this case has only three samples, WHFWA yields contradiction rankings with WHFWG.

Example 5. To further validate the proposed operator, the hesitant fuzzy data from the paper [32] are utilized. Weights to transfer HFEs data to WHFEs data are added by fusing equal weights. All the data are shown in Table 5.

Table 5. Weighted hesitant fuzzy decision matrix.

	G_1	G_2
Y_1	{(0.2,1/3),(0.4,1/3),(0.7,1/3)}	{(0.2,1/3),(0.6,1/3),(0.8,1/3)}
Y_2	{(0.2,1/4),(0.4,1/4),(0.7,1/4),(0.9,1/4)}	{(0.1,1/4),(0.2,1/4),(0.4,1/4),(0.5,1/4)}
Y_3	{(0.3,1/4),(0.5,1/4),(0.6,1/4),(0.7,1/4)}	{(0.2,1/4),(0.4,1/4),(0.5,1/4),(0.6,1/4)}
Y_4	{(0.3,1/3),(0.5,1/3),(0.6,1/3)}	{(0.2,1/2),(0.4,1/2)}
	G_3	
Y_1	{(0.2,1/5),(0.3,1/5),(0.6,1/5),(0.7,1/5),(0.9,1/5)}	
Y_2	{(0.3,1/4),(0.4,1/4),(0.6,1/4),(0.9,1/4)}	
Y_3	{(0.3,1/4),(0.5,1/4),(0.7,1/4),(0.8,1/4)}	
Y_4	{(0.5,1/3),(0.6,1/3),(0.7,1/3)}	

Remark 2. The score and variance of $h^w(Y_i)$ for each sample are obtained. The results of WHFWA: $s(h^w(Y_1)) = 0.4950$, $v(h^w(Y_1)) = 0.1375$, $s(h^w(Y_2)) = 0.4390$, $v(h^w(Y_2)) = 0.1488$, $s(h^w(Y_3)) = 0.4367$, $v(h^w(Y_3)) = 0.0880$, $s(h^w(Y_4)) = 0.5067$, $v(h^w(Y_4)) = 0.0624$. The results of WHFWG: $s(h^w(Y_1)) = 0.5003$, $v(h^w(Y_1)) = 0.1384$, $s(h^w(Y_2)) = 0.5091$, $v(h^w(Y_2)) = 0.1189$, $s(h^w(Y_3)) = 0.5782$, $v(h^w(Y_3)) = 0.0907$, $s(h^w(Y_4)) = 0.3493$, $v(h^w(Y_4)) = 0.0707$. The rankings of WHFWA and WHFWG are $Y_4 \succ Y_1 \succ Y_3 \succ Y_2$ and $Y_3 \succ Y_2 \succ Y_1 \succ Y_4$, respectively. The proposed operator WHFGBM is applied to obtain the final rankings of samples: $Y_3 \succ Y_1 \succ Y_2 \succ Y_4$. WHFGBM gives the same best choice as WHFWG. These two operators have a slight difference between Y_1 and Y_2 samples. However, WHFWA provides an almost contradiction best choice.

Example 6. To compare different methods, data from the paper [39] are adopted. Assume the ideal solution is $A^* = \{1\}$, and aggregation operator and score function are used for obtaining results. Related decision-making data, as shown in Table 6, are collected to form weighted hesitant fuzzy sets.

Table 6. Weighted hesitant fuzzy decision matrix.

	P_1	P_2
A_1	$\{(0.5,1/3),(0.4,1/3),(0.3,1/3)\}$	$\{(0.9,1/4),(0.8,1/4),(0.7,1/4),(0.1,1/4)\}$
A_2	$\{(0.5,1/2),(0.3,1/2)\}$	$\{(0.9,1/4),(0.7,1/4),(0.6,1/4),(0.2,1/4)\}$
A_3	$\{(0.7,1/2),(0.6,1/2)\}$	$\{(0.9,1/2),(0.6,1/2)\}$
A_4	$\{(0.8,1/4),(0.7,1/4),(0.4,1/4),(0.3,1/4)\}$	$\{(0.7,1/3),(0.4,1/3),(0.2,1/3)\}$
A_5	$\{(0.9,1/5),(0.7,1/5),(0.6,1/5),(0.3,1/5),(0.1,1/5)\}$	$\{(0.8,1/4),(0.7,1/4),(0.6,1/4),(0.4,1/4)\}$
	P_3	
A_1	$\{(0.5,1/3),(0.4,1/3),(0.2,1/3)\}$	
A_2	$\{(0.8,1/4),(0.6,1/4),(0.5,1/4),(0.1,1/4)\}$	
A_3	$\{(0.7,1/3),(0.5,1/3),(0.3,1/3)\}$	
A_4	$\{(0.8,1/2),(0.1,1/2)\}$	
A_5	$\{(0.9,1/3),(0.8,1/3),(0.7,1/3)\}$	

Remark 3. Score and variance of $h^w(A_i)$ for each sample are obtained. The results of WHFWA: $s(h^w(A_1)) = 0.4700$, $v(h^w(A_1)) = 0.1170$, $s(h^w(A_2)) = 0.4841$, $v(h^w(A_2)) = 0.1067$, $s(h^w(A_3)) = 0.7806$, $v(h^w(A_3)) = 0.0795$, $s(h^w(A_4)) = 0.5451$, $v(h^w(A_4)) = 0.1859$, $s(h^w(A_5)) = 0.6326$, $v(h^w(A_5)) = 0.0854$. The results of WHFWG: $s(h^w(A_1)) = 0.4391$, $v(h^w(A_1)) = 0.1075$, $s(h^w(A_2)) = 0.4466$, $v(h^w(A_2)) = 0.1090$, $s(h^w(A_3)) = 0.5440$, $v(h^w(A_3)) = 0.0858$, $s(h^w(A_4)) = 0.3785$, $v(h^w(A_4)) = 0.1986$, $s(h^w(A_5)) = 0.6934$, $v(h^w(A_5)) = 0.1142$. The rankings of WHFWA and WHFWG are $A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$ and $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$, respectively. Generalized hybrid hesitant weighted distance is proposed by [39]; when $\lambda = 1$, the ranking is $A_3 \succ A_5 \succ A_4 \succ A_1 \succ A_2$. The proposed operator WHFGBM is applied to obtain the final rankings of samples: $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$. This case includes the most samples compared with other cases. Experts give different judgments based on attributes P_1, P_2, P_3 . Experts give a concentrated judgment for A_3 , which can be seen from WHFEs with short length. WHFWA and generalized hybrid hesitant weighted distance give an almost consistent ranking, only having a slight difference in inferior samples. WHFWG and WHFGBM provide consistent rankings.

Parameter selection: For the WHFGBM operator, there exist two parameters, p and q . When these two parameters are assigned, the WHFGBM operator will finalize. When the decision-making process begins, the WHFEs, as input data, will be sent to the WHFGBM to complete aggregation. Typically, the parameters p and q are small values, which makes the values of the final results to be in a reasonable range for comparison. For above numeric examples, the two parameters values, $p = 1$ and $q = 2$, are assigned, respectively. If the values of parameters become bigger, the power operator and $\otimes$ operator will make the values become very small, which is not conducive to comparison. Hence, these two parameters are usually 1 or 2.

Parameter sensitivity: Due to the influence of parameter values, the sensitivity of the proposed operator should be considered. From the above analysis, the values of two parameters should be greater than 0 but not too large. Values of $p = 1$ and $q = 2$ are assigned for the above numeric examples. Combinations of $p = 1, q = 1$, $p = 2, q = 1$, and $p = 2, q = 2$ are assigned for further analysis. For example 3, the same ranking results are obtained: $A_4 \succ A_2 \succ A_1 \succ A_3$. For example 4, when $p = 1, q = 1$, $p = 2, q = 2$, and $p = 2, q = 1$, the results are $Y_3 \succ Y_1 \succ Y_2$, which are slightly different from those

when $p = 1, q = 2$. For example 5, when $p = 1, q = 1$, the same results are obtained: $Y_3 \succ Y_2 \succ Y_1 \succ Y_4$. When $p = 2, q = 2$ and $p = 2, q = 1$, the results are slightly different: $Y_3 \succ Y_1 \succ Y_2 \succ Y_4$. For example 6, the same results are obtained: $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$. From the above results, in example 3 and example 6, any p, q combination can provide the same results. In example 5, one combination provides slightly different results from the other three combinations. In example 4, results from different combinations are slightly different. Hence, the proposed operator can provide stable results, and it has robustness for decision-making tasks.

5. Conclusions

Weighted hesitant fuzzy theory allows every membership degree to have positive value weights to further depict fuzzy information. In this paper, some operations (“ $\oplus$ ” and “ $\otimes$ ” operations of WHFEs) are redefined intuitively and logically, and properties and corresponding proofs are given. Further, the WHFWA and WHFWG aggregation operators are introduced to develop a novel WHFGBM operator. The novel operator will integrate fuzzy information better in multi-criteria decision-making problems. Finally, the results of numeric examples are displayed to validate the effectiveness and feasibility of the proposed operator. Although our operator can provide an aggregation method based on WHFS, when attributes increase, the computation burden will become heavier, which is the main limitation for subsequent large-scale expansion.

In the future, we will be dedicated to developing novel aggregation operators based on WHFSs and applying them to multi-criteria group decision-making problems, enhancing the information fusion ability under different scenarios. We will propose more types of mean operators, such as extending the operator to interval-valued WHFS. Then, we will consider introducing these operators to broader fuzzy decision-making fields, practical and complex scenarios. Further, we hope that our research will inspire more efficient tools for MCDM. On the other hand, introducing the advantages of existing operators is considered to improve the robustness of the operator. In addition, the weights for WHFS are important. However, the most common way to obtain weights is by experience, which increases the uncertainty. Hence, obtaining attributes objectively is definitely one of the directions for future research.

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