

Article

Statistical Inference for the Inverted Kumaraswamy Accelerated Model Under Type-I Generalized Hybrid Censoring with Applications

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Abstract

This study investigates the methodologies for robust parameter estimation within the context of the parameters of the inverted Kumaraswamy model using data derived from step-stress partially accelerated life testing with Type-I generalized hybrid censoring. We formulate estimation procedures within both frequentist (maximum likelihood) and Bayesian frameworks, including the construction of asymptotic and credible intervals. Subsequently, we provide a formal derivation of the associated asymptotic and bootstrap confidence intervals. To address the analytical intractability of the Bayesian estimation, we employ Markov Chain Monte Carlo techniques. The proposed methods are illustrated through an illustrative example, an application to real-world precipitation data, and a simulation study.

Keywords: inverted Kumaraswamy distribution; step-stress partially accelerated life tests; bootstrap confidence interval; maximum likelihood estimation; Bayesian estimation; generalized hybrid censored scheme

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1. Introduction

Modern technological advancements have resulted in highly reliable products with exceptionally long lifespans. Consequently, traditional life testing under normal operating conditions often yields insufficient failure data within practical timeframes. Accelerated life tests (ALTs) and partially accelerated life tests (PALTs) are employed to rapidly obtain reliability data. ALTs exclusively employ high-stress environments, whereas PALTs integrate the normal use conditions and the accelerated stress condition to facili-

tate more realistic reliability assessments. The data from these tests are then used to predict product failure behavior under standard use. Common methodologies include constant-stress and step-stress PALTs. In step-stress PALTs, items initially run at normal conditions are switched to accelerated conditions if they survive a specified duration, and the test continues until failure or test termination. Constant-stress PALTs, however, constant high-magnitude stress state must be sustained until an item fails or the test concludes.

Much of the existing research on step-stress PALTs (SS-PALTs) focuses on Type-I and Type-II censoring frameworks and significant research has been conducted in this area, most notably in Refs. [1–3]. Further advancements were provided by Refs. [4–7]. Furthermore, research into PALTs under progressive censoring has been advanced by studies conducted by Ismail and Sarhan [8], Srivastava and Mittal [9], and Mohie EL-Din et al. [10]. Ideally, life-testing experiments yield complete data where the failure time of every unit is observed. However, practical limitations such as time and cost often necessitate the use of censored data. Traditional approaches include Type-I censoring (fixed time and random failures) and Type-II censoring (fixed failures and random time). Despite their popularity, these conventional schemes suffer from a major drawback: they do not allow for the withdrawal of surviving units during the experiment. To overcome this inflexibility, progressive censoring was introduced. Specifically, the progressive Type-II censoring scheme allows researchers to systematically remove active experimental units before the test concludes. Specifically, it permits the withdrawal of surviving units at various intermediate stages of the experiment. The mechanics of this model are described as follows (Balakrishnan and Aggarwala [11]). By combining the properties of Type-I and Type-II censoring, the HCS procedure is defined as follows: A set of identical units is subjected to testing, and the experiment is designed to conclude upon the occurrence of a predetermined quantity of failures (r) or upon reaching a fixed duration (T) elapses. Epstein [12] is credited with introducing this scheme, which has proven highly effective for reliability acceptance strategies. Many authors, such as Draper and Guttman [13], have applied this method in their work. Because the termination depends on the specified parameters r and T , the experimental results fall into two categories based on which limit is reached first, under Type-I HCS, with n units under test, the experimental procedure concludes at by $\tau = \min(x_{r:n}, T)$, where r and T are fixed constants. A significant drawback of this method is the potential for very few (or zero) observed failures, which negatively impacts estimation efficiency. In the Type-II HCS, the test termination time is given by $\tau = \max(x_{r:n}, T)$ (see Gupta and Kundu [14], Dube et al. [15]). While this guarantees at least r failures, it can lead to prolonged testing times. Since both standard schemes suffer from either insufficient data or unconstrained testing duration, the generalized HCS was proposed by Chandrasekar et al. [16] to balance these elements. This scheme is defined as follows:

Type-I generalized HCS:

Consider a life-testing scenario involving n units employing two failure count thresholds, (r_1, r_2) (where, $1 < r_1 < r_2 < n$, and a fixed time limit T). Failure times are recorded as they occur. The Type-I generalized HCS ensures that the test does not end before r_1 failures have occurred. The actual termination time τ^* and the failures number κ are determined by the following three scenarios:

- At $(T < x_{r_1})$: If the time T elapses before r_1 failures occur, the test continues and stops exactly when the r_1 -th failure is detected ($\kappa = r_1, \tau^* = x_{r_1}$).
- At $(T < x_{r_2})$: If the r_2 -th failure occurs prior to time T , the experimental procedure is terminated instantaneously at that failure ($\kappa = r_2, \tau^* = x_{r_2}$).

- At $(x_{r_1} < T < x_{r_2})$: If r_1 failures have occurred but the r_2 -th failure has not yet happened by time T , the test stops at T (K is the count of failures by time T) ($\tau^* = T$).

Type-II generalized HCS: Assume an experiment is conducted with n units. This setup utilizes the Type-II generalized HCS (GHCS), which relies on a target number of failures r , and two specific time limits, τ_1 and τ_2 (where $\tau_1 < \tau_2$). The failure times are recorded continuously. The stopping rule is designed to ensure the test runs at least until τ_1 but no longer than τ_2 . Consequently, the observed data is given by

$x = (x_1, x_2, \dots, x_K)$, where K is the number of failed units and τ^* is the termination time. The definitions are as follows:

- At r -th failure happens before τ_1 ($x_r < \tau_1$), the test extends to τ_1 . Thus, ($K \geq r$, $\tau^* = \tau_1$).
- At r -th failure happens between the two time limits ($\tau_1 < x_r < \tau_2$), the test stops at that exact failure time. Thus, ($K = r$, $\tau^* = x_r$).
- At r -th failure has not occurred by τ_2 ($x_r > \tau_2$), the test stops. Thus ($K < r$, $\tau^* = \tau_2$).

In contrast to generalized progressive censoring systems, which allow for the removal of survivors during the test, in both Type-I and Type-II GHCSs, active experimental units cannot be removed before the test concludes. For a comprehensive overview of such progressive methods, comprehensive reviews of this subject can be found in the works of Balakrishnan [17], Balakrishnan and Cramer [18], and Elsherpieny et al. [19]. This research adopts the Type-I GHCS, which is adopted here due to its ability to balance observation requirements with temporal efficiency. Unlike standard censoring frameworks, it allows for the staged disassembly of units before reaching the intended sample size. It incorporates a pre-specified unit failures threshold to prevent excessively long experimental durations.

The inverted Kumaraswamy (IKum) distribution parameterized by $\gamma, \eta > 0$ and denoted IKum (γ, η) , originates from the Kumaraswamy distribution via the transformation $X = 1/Y - 1$. This distribution is notable for its direct relationships to other established distributions. Specifically, the IKum model simplifies to several well-known distributions based on its parameters:

- Lomax distribution when $\eta = 1$.
- Inverted beta Type-II distribution when $\gamma = 1$.
- Log-logistic (Fisk) distribution when $\gamma = \eta = 1$.

Let X is a random variable distributed according to the IKum distribution; the corresponding cumulative distribution function (CDF), survival function, probability density function (PDF), and hazard rate function are expressed as follows:

$$F(x) = (1 - (1+x)^{-\gamma})^\eta, \quad x > 0, \gamma, \eta > 0, \quad (1)$$

$$S(x) = 1 - (1 - (1+x)^{-\gamma})^\eta, \quad x > 0, \gamma, \eta > 0, \quad (2)$$

$$f(x) = \gamma\eta(1+x)^{-\gamma-1}(1 - (1+x)^{-\gamma})^{\eta-1}, \quad x > 0, \gamma, \eta > 0, \quad (3)$$

$$h(x) = \frac{\gamma\eta(1+x)^{-\gamma-1}(1 - (1+x)^{-\gamma})^{\eta-1}}{(1 - (1 - (1+x)^{-\gamma})^\eta)}, \quad x > 0, \gamma, \eta > 0. \quad (4)$$

As illustrated in Figure 1, the IKum distribution exhibits a notably longer right tail compared to other common distributions, as observed in its PDF curve. This characteristic implies that the IKum distribution may yield more optimistic long-term reliability predictions for rare events occurring in the right tail. The IKum distribution, introduced by [20], has demonstrated excellent fit across various datasets in the literature. Also, Ref. [20] investigated its structural properties and addressed parameter estimation under Type-II censoring. Subsequent studies include [21], who evaluated unknown parameters using general progressive censored samples, and [22], who derived moment relations for dual generalized order statistics using the IKum distribution. Additional information on this distribution is available in [23,24].

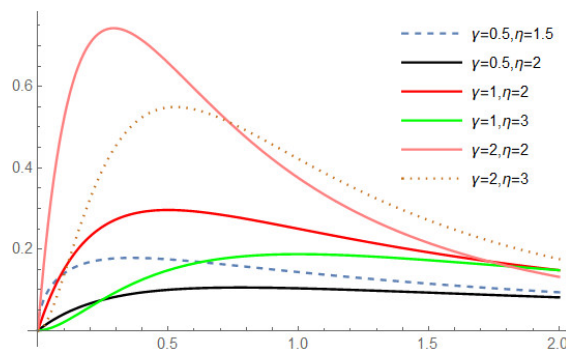


Figure 1. PDF of IKum distribution for different values of parameters.

This study provides a comprehensive statistical analysis of SS-PALTs under Type-I GHCSs, utilizing the IKum distribution. Frequentist point estimates are derived using the ML method. The construction of confidence intervals is achieved via the observed Fisher information matrix and bootstrap resampling procedures. This study focuses on the Bayesian estimation of model parameters. Without a closed-form expression for the posterior distribution, we rely on Markov Chain Monte Carlo (MCMC) methods and we adopt independent gamma prior for the shape and scale parameters. To obtain point estimates and credible intervals under the squared error (SE) loss function, the Metropolis–Hastings (MH) algorithm is employed to generate the necessary posterior samples. We use Monte Carlo simulations to investigate the behavior of the Bayesian estimates in comparison to the MLEs. We also apply the proposed methodology to a real data set for illustration.

In Section 2, we present the mathematical formulation of the model alongside the SS-PALT setup. Section 3 presents the ML method for point and interval estimation. Section 4 details the construction of bootstrap confidence intervals. Bayesian point and interval estimation are detailed in Section 5. The performance of these methods is subsequently evaluated in Section 6 through the analysis of a real dataset and simulated dataset and a Monte Carlo simulation study. Section 7 provides a comprehensive analysis of the results and their discussion, while Section 8 offers concluding remarks.

2. SS-PALT Modeling via IKum Distribution

Supposed units are placed under an SS-PALT using a Type-I GHCS framework. The test duration is governed by a predetermined time T and integers κ and r are the lower and upper failure limits, with the stress-change time Y set such that $Y < T$. Units are observed under normal use conditions until Y , after which they are subjected to accelerated stress. Test termination is determined by the κ -th failure time: if $x_\kappa < T$, the test stops at $\min(T, x_r)$, otherwise it stops at x_κ . The resulting Type-I GHCS data is $\mathbf{x} = \{x_1 < x_2 < x_3, \dots, x_J < Y < x_{J+1} < \dots < x_m\}$, where $\kappa < m < r$. In this setup, J failures are

recorded during the normal usage stage, while the remaining failures $m - J$ occur after switching to accelerated conditions. This increase in stress is applied specifically to accelerate the failure process.

Denoting the total accumulated test time by T , the formulation for the SS-PALT model is given by $\mathbf{W} = \{w_1 < w_2 < w_3, \dots, w_J < Y < w_{J+1} < \dots < w_m\}$. The acceleration factor is θ . As a result, the total observed lifetime $\mathbf{W}$ reflects the total time spent under both normal and accelerated conditions, the lifetime variable for a unit within the SS-PALT framework is formulated as follows:

$$\mathbf{W} = \begin{cases} X & \text{if } X \leq Y, \\ Y + \frac{X-Y}{\theta} & \text{if } X > Y. \end{cases} \quad (5)$$

Here, X is the lifetime under normal conditions, where θ is the acceleration factor (mean life ratio, use-to-accelerated, with $\theta > 1$, and Y indicates the switching time between stress levels). We assume the test units' lifetime follows an IKum (γ, η) distribution.

The CDF and PDF associated with the lifetime model described in Equation (5) are presented below:

$$F(\mathbf{w}) = \begin{cases} F_1(\mathbf{w}) = (1 - (1 + w)^{-\gamma})^\eta, & \text{if } w \leq Y, \\ F_2(\mathbf{w}) = (1 - (1 + (Y + \theta(w - Y)))^{-\gamma})^\eta, & \text{if } w > Y, \end{cases}$$

$$f(\mathbf{w}) = \begin{cases} f_1(\mathbf{w}) = \gamma\eta(1 + w)^{-\gamma-1}(1 - (1 + w)^{-\gamma})^{\eta-1}, & \text{if } w \leq Y, \\ f_2(\mathbf{w}) = \gamma\eta(1 + (Y + \theta(w - Y)))^{-\gamma-1}(1 - (1 + (Y + \theta(w - Y)))^{-\gamma})^{\eta-1}, & \text{if } w > Y. \end{cases}$$

The total test time is expressed as, the likelihood function of a Type-I (GHCS) SS-PALT $\mathbf{W} = \{w_1 < w_2 < w_3, \dots, w_J < Y < w_{J+1} < \dots < w_m\}$, which is given by

$$f(\mathbf{w} | \gamma, \eta, \theta) \propto (S_2(w_r))^{(n-r)} \prod_{i=1}^J (f_1(w_i)) \prod_{i=J+1}^m (f_2(w_i)). \quad (6)$$

3. Frequentist Inferences Under SS-PALT

This section focuses on obtaining MLEs for the acceleration factor and the model's parameters. MLE is chosen for its consistency and efficiency, offering statistically precise estimates and quantifiable uncertainty through confidence limits. From Equations (2), (3), (5) and (6) and the likelihood function derived from the observed accelerated Type-I GHCS data $\mathbf{W} = (w_1, w_2, w_3, \dots, w_J, Y, w_{J+1}, \dots, w_m)$, we can obtain

$$L(\gamma, \eta, \theta | \mathbf{w}) \propto (1 - (1 - (1 + (Y + \theta(w_r - Y)))^{-\gamma})^\eta)^{(n-m)} \\ \times \prod_{i=1}^J (\gamma\eta(1 + w_i)^{-\gamma-1}(1 - (1 + w_i)^{-\gamma})^{\eta-1}) \\ \times \prod_{i=J+1}^m (\gamma\eta(1 + (Y + \theta(w_i - Y)))^{-\gamma-1}(1 - (1 + (Y + \theta(w_i - Y)))^{-\gamma})^{\eta-1}). \quad (7)$$

The natural logarithms of the likelihood function are written by

$$\begin{aligned}
\ell(\gamma, \eta, \theta | \mathbf{w}) &= m \log(\gamma\eta) + (n-m) \log(1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^\eta) \\
&- (\gamma + 1) \sum_{i=1}^J \log(1 + w_i) + (\eta - 1) \sum_{i=1}^J \log(1 - (1 + w_i)^{-\gamma}) \\
&- (\gamma + 1) \sum_{i=J+1}^m \log(1 + (\Upsilon + \theta(w_i - \Upsilon))) + (\eta - 1) \sum_{i=J+1}^m \log(1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma}).
\end{aligned} \tag{8}$$

The following system is produced by setting the first partial derivatives of Equation (8) with respect to parameters to zero.

$$\begin{aligned}
\frac{\ell(\gamma, \eta, \theta | \mathbf{w})}{\partial \gamma} &= \frac{m}{\gamma} - \sum_{i=1}^J \log(1 + w_i) + (\eta + 1) \sum_{i=1}^J \frac{(1 + w_i)^{-\gamma} \log(1 + w_i)}{(1 - (1 + w_i)^{-\gamma})} \\
&\frac{\eta(n-m)(1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma} (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta-1} \log(1 + (\Upsilon + \theta(w_m - \Upsilon)))}{(1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^\eta)} - \\
&\sum_{i=J+1}^m \log(1 + (\Upsilon + \theta(w_i - \Upsilon))) + (\eta + 1) \sum_{i=1}^J \frac{(1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma} \log(1 + (\Upsilon + \theta(w_i - \Upsilon)))}{(1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\frac{\ell(\gamma, \eta, \theta | \mathbf{w})}{\partial \eta} &= \frac{m}{\eta} + \sum_{i=1}^J \log(1 - (1 + w_i)^{-\gamma}) + \sum_{i=J+1}^m \log(1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma}) \\
&+ \frac{(n-m)(1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^\eta \log(1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})}{(1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^\eta)} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\ell(\gamma, \eta, \theta | \mathbf{w})}{\partial \theta} &= -\frac{\gamma\eta(n-m)(w_m - \Upsilon)(1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma-1} (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta-1}}{(1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^\eta)} \\
&- (\gamma + 1) \sum_{i=J+1}^r \frac{(w_i - \Upsilon)}{(1 + (\Upsilon + \theta(w_i - \Upsilon)))} + (\eta + 1) \sum_{i=J+1}^r \frac{\gamma(w_i - \Upsilon)(1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma-1}}{(1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})} = 0.
\end{aligned} \tag{11}$$

The MLEs $\hat{\gamma}$, $\hat{\eta}$ and $\hat{\theta}$ of the model parameters γ , η and θ are obtained by solving the system of non-linear equations presented in (9)–(11). Such cases necessitate the application of numerical iterative schemes, exemplified by the Newton–Raphson algorithm or fixed-point iteration.

Approximate Interval Estimation for Frequentist Inferences

Although the Fisher information matrix is theoretically defined as the negative anticipated value of the log-likelihood function's second derivatives, the approximation ranges are constructed using the analytical derivation of this expectation can be difficult. Consequently, the observed Fisher information matrix is widely used as a suitable approximation for facilitating interval estimation. The specific second derivatives of the log-likelihood function regarding the parameter vector $\chi = (\gamma, \eta, \theta)$ are derived as follows:

$$I(\chi) = \left[-\frac{\partial^2 \ell(\mathbf{w}; \gamma, \eta, \theta)}{\partial \gamma \partial \eta \partial \theta} \right]. \tag{12}$$

Let $I(\hat{\chi})$ denote the observed information matrix evaluated at the maximum likelihood estimate (MLE). Invoking the asymptotic properties of MLEs, the estimator. Based on the asymptotic properties of MLEs, $\hat{\chi}$ is approximately distributed as a multivariate normal vector with mean χ and variance–covariance matrix $I^{-1}(\hat{\chi})$, defined as

$$\hat{\chi} \rightarrow N(\chi, I^{-1}(\hat{\chi})).$$

The approximate $100(1 - \vartheta)\%$ interval estimates for the components of the parameter vector χ are

$$\begin{aligned} \hat{\gamma} \pm Z_{\vartheta/2} \sqrt{\text{Var}(\hat{\gamma})}, \\ \hat{\eta} \pm Z_{\vartheta/2} \sqrt{\text{Var}(\hat{\eta})}, \\ \hat{\theta} \pm Z_{\vartheta/2} \sqrt{\text{Var}(\hat{\theta})}. \end{aligned} \quad (13)$$

4. Bootstrap Confidence Interval

The bootstrap method described in Hinkley and Davison [25], parametric bootstrap methodology, serves as a standard technique for interval estimation, is used here to compute the confidence intervals. It can also be computed to estimate the variance and bias of an estimator. For further information, read Efron and Tibshirani [26], where the following procedure is described, as for Algorithm 1.

Algorithm 1: Parametric Bootstrap Procedure

- Step 1: Based on an empirical dataset of size n , the stress-change time Y and pre-specified integers κ and r obtain the original failure data vector $\mathbf{w} = (w_1, w_2, w_3, \dots, w_J, Y, w_{J+1}, \dots, w_m)$.
- Step 2: Compute the initial MLEs, denoted as $\hat{\gamma}, \hat{\eta}$ and $\hat{\theta}$, from the original dataset $\mathbf{w}$.
- Step 3: Initialize the bootstrap iteration counter at $i = 1$.
- Step 4: Generate a Type-I GHC random sample from the specified distribution using the estimated parameters ($\hat{\gamma}, \hat{\eta}$, and $\hat{\theta}$) and the original censoring constraints (n, κ, r , and T).
- Step 5: Apply the ALT transformation (Equation (4)) to all simulated observations exceeding the stress-change time Y to generate the accelerated bootstrap sample $\mathbf{w}^* = (w_1^*, w_2^*, \dots, w_J^*, Y, w_{J+1}^*, \dots, w_m^*)$.
- Step 6: Obtain the bootstrap MLEs, $\hat{\gamma}^*, \hat{\eta}^*$, and $\hat{\theta}^*$ using the bootstrap sample $\mathbf{w}^*$.
- Step 7: Increment the counter $i = i + 1$.
- Step 8: Repeat Steps 4 through 7 for N iterations to obtain a sequence of bootstrap estimates:

$$\hat{\chi}^* = \frac{1}{N} \sum_{i=1}^N \hat{\chi}^{*(i)}, \chi = (\gamma, \eta, \theta), \quad (14)$$

- Step 9: Bootstrap-based approximate $100(1 - \vartheta)\%$ confidence intervals are constructed for the model parameters $\hat{\gamma}, \hat{\eta}, \hat{\theta}$, and $l = 1, \dots, N$ are given, respectively, by

$$\begin{aligned} &(\hat{\gamma}^{*[N(\vartheta/2)]}, \hat{\gamma}^{*[N(1-\vartheta/2)]}), \\ &(\hat{\eta}^{*[N(\vartheta/2)]}, \hat{\eta}^{*[N(1-\vartheta/2)]}), \\ &(\hat{\theta}^{*[N(\vartheta/2)]}, \hat{\theta}^{*[N(1-\vartheta/2)]}). \end{aligned}$$

5. Bayesian Inferences Under SS-PALT

Here, we compute the Bayes estimators for γ, η and θ . The Squared Error (SE) loss function is adopted, which is expressed as follows.

We assign independent gamma priors to γ , and η . The two parameters in this Bayesian model are assumed to be random variables following Gamma distributions.

This choice is appropriate because the parameters are strictly positive and the Gamma distribution is defined on the positive real line. Additionally, Gamma priors offer significant shape flexibility and allow for the seamless integration of prior information, ensuring that the posterior inference remains manageable. The independent gamma priors given by

$$\gamma \propto \gamma^{a-1} e^{-b\gamma}, \quad \eta \propto \eta^{c-1} e^{-d\eta}, \quad (15)$$

where (a, b, c, d) , are known hyper parameters. The acceleration factor θ has noninformative prior information:

$$\theta \propto 1/\theta. \quad (16)$$

When a non-informative prior is adopted, the joint posterior density coincides with the likelihood function up to a normalizing constant, implying a non-informative joint prior for the parameter vector $\chi = (\gamma, \eta, \theta)$ is expressed as

$$P \propto \frac{\gamma^{a-1} \eta^{c-1}}{\theta} e^{-b\gamma - d\eta}. \quad (17)$$

Using Equation (17) to combine the likelihood and the previous density, We formulate the joint posterior density of the parameters γ, η and θ , which are derived as

$$\Pi(\mathbf{w}; \gamma, \eta, \theta) \propto \frac{\ell(\mathbf{w}; \gamma, \eta, \theta) \times \frac{\gamma^{a-1} \eta^{c-1}}{\theta} e^{-b\gamma - d\eta}}{\int_{\chi} \ell(\mathbf{w}; \gamma, \eta, \theta) \times \frac{\gamma^{a-1} \eta^{c-1}}{\theta} e^{-b\gamma - d\eta} d\gamma d\eta d\theta}. \quad (18)$$

Then, the Bayes estimator for the parameters $\chi = (\gamma, \eta, \text{and } \theta)$ under the SE loss function is described as

$$\hat{\chi}_{SE} = \int_{\chi} \Pi(\mathbf{w}; \gamma, \eta, \theta) d\gamma d\eta d\theta. \quad (19)$$

The integrals in Equations (17) and (18) are generally difficult to evaluate and lack closed-form solutions, especially in high-dimensional settings. While approximation techniques like Lindley's method exist, MCMC methods are often more effective. These methods generate an empirical posterior distribution through simulation using algorithms such as Gibbs sampling, Metropolis–Hastings (MH) within Gibbs, or importance sampling.

The vast body of recent statistical literature highlights the importance of MCMC methodology. Its utility lies in its ability to provide not only point estimates but also interval estimates via the final simulated empirical distribution. MCMC functions by iterative sampling from the model's posterior distributions. Among the most important techniques in this category is Gibbs sampling and the MH-within-Gibbs sampler. Readers are referred to Robert and Casella [27] and Ahmed [28] for further details.

We can formulate the joint posterior density of γ, η and θ , given the data, as

$$\begin{aligned} \Pi(\gamma, \eta, \theta | \mathbf{w}) &\propto \theta^{-1} \gamma^{m+a-1} \eta^{m+c-1} e^{-b\gamma - d\eta} \times (1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta})^{(n-m)} \\ &\times \prod_{i=1}^J ((1 + w_i)^{-\gamma-1} (1 - (1 + w_i)^{-\gamma})^{\eta-1}) \\ &\times \prod_{i=J+1}^m ((1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma-1} (1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})^{\eta-1}). \end{aligned} \quad (20)$$

Therefore, the conditional posterior densities are defined as

$$\begin{aligned}\Pi_{\gamma}(\gamma | \eta, \theta, \mathbf{w}) &\propto \gamma^{m+a-1} e^{-b\gamma} \times (1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta})^{(n-m)} \\ &\times \prod_{i=1}^J ((1 + w_i)^{-\gamma-1} (1 - (1 + w_i)^{-\gamma})^{\eta-1}) \\ &\times \prod_{i=J+1}^m ((1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma-1} (1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})^{\eta-1}),\end{aligned}\quad (21)$$

$$\begin{aligned}\Pi_{\eta}(\eta | \gamma, \theta, \mathbf{w}) &\propto \eta^{m+c-1} e^{-d\eta} \times (1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta})^{(n-m)} \\ &\times \prod_{i=1}^J ((1 - (1 + w_i)^{-\gamma})^{\eta-1}) \times \prod_{i=J+1}^m ((1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})^{\eta-1}),\end{aligned}\quad (22)$$

$$\begin{aligned}\Pi_{\theta}(\theta | \gamma, \eta, \mathbf{w}) &\propto \theta^{-1} \times (1 - (1 - (1 + (\Upsilon + \theta(w_m - \Upsilon)))^{-\gamma})^{\eta})^{(n-m)} \\ &\times \prod_{i=J+1}^m ((1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma-1} (1 - (1 + (\Upsilon + \theta(w_i - \Upsilon)))^{-\gamma})^{\eta-1}).\end{aligned}\quad (23)$$

Since the distributions presented in Equations (21)–(23) are not amenable to direct sampling via conventional techniques, an MH step with a normal proposal distribution is employed. This choice is motivated by the fact that the conditional posteriors of the parameters exhibit are approximately normally distributed. This procedure facilitates the estimation of marginal posterior distributions and posterior means, both of which are critical elements of a comprehensive Bayesian inference. The Algorithm 2 describes the MCMC (MH) procedure used to generate posterior samples for the parameters γ , η and θ :

Algorithm 2: MCMC procedure

- Step 1: Select an initial parameter vector $(\gamma^{(0)}, \eta^{(0)}, \theta^{(0)})$. A practical choice is to use the MLEs, $(\hat{\gamma}, \hat{\eta}, \hat{\theta})$.
- Step 2: Set $i = 1$.
- Step 3: Generate a candidate value γ^* from the proposal distribution $N(\gamma^{(i-1)}, \text{var}(\hat{\gamma}))$
 1. Calculate the acceptance probability:
$$q(\gamma) = \min \left[1, \frac{\Pi_{\gamma}(\gamma^* | \eta^{(i-1)}, \theta^{(i-1)}, w)}{\Pi_{\gamma}(\gamma^{(i-1)} | \eta^{(i-1)}, \theta^{(i-1)}, w)} \right].$$
 2. Generate $U \sim \text{Uniform}(0, 1)$, if $U \leq q(\gamma)$, set $\gamma^{(i)} = \gamma^*$; otherwise, set $\gamma^{(i)} = \gamma^{(i-1)}$.
- Step 4: Generate from the proposal distribution $N(\eta^{(i-1)}, \text{var}(\hat{\eta}))$ a candidate value η^* .
- Step 5: Generate a candidate value θ^* from the proposal distribution $N(\theta^{(i-1)}, \text{var}(\hat{\theta}))$.
- Step 6: Increment i and repeat steps 3 – 5 N times until the desired number of a sequence of Bayes (MCMC) estimates, then the mean of estimates is given by

$$\hat{\chi} = \frac{1}{N-M} \sum_{i=1}^{N-M} \chi^{(i)}, \chi = (\gamma, \eta, \theta), M \text{ is burn-in}, \quad (24)$$

- Step 7: Order $\chi^{(M+1)}, \chi^{(M+2)}, \dots, \chi^{(N)}$ as $\chi_{(1)} < \chi_{(2)} < \dots < \chi_{(N-M)}$,
 - The approximate $100(1 - \vartheta)\%$ credible intervals for parameters $\hat{\gamma}$, $\hat{\eta}$, and $\hat{\theta}$, are given, respectively, by
-

$$(\hat{\chi}^{[(N-M)(\theta/2)]}, \hat{\chi}^{[(N-M)(1-\theta/2)]}).$$

6. Applications

The main goal of this part is to show how you can use these methods in real-world and simulated data.

6.1. Data Analysis

Application 1: (real data)

To illustrate our discussed methodology, a real dataset referenced in [29] is analyzed in this section. The data consist of thirty sequential records of March precipitation (in inches) for Minneapolis/St. Paul. The specific observations are provided below as

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Accelerated Type-I GHCS is shown below as

0.32, 0.47, 0.52, 0.59, 0.77, 0.81, 0.81, 0.9, 0.96, 1.18, 1.2, 1.2, 1.31, 1.35, 1.43, 1.51, 1.62, 1.74, 1.87, 1.89, 1.95, 2.05, 2.1, 2.2, 2.48.

Standard goodness-of-fit criteria were used to the provided data set in order to assess the goodness-of-fit of the IKum distribution, as shown in Figure 2. The Kolmogorov–Smirnov (K–S) statistics, defined as the maximum absolute difference between the empirical and fitted cumulative distribution functions, was found to be $K-S = 0.1143$ and p -value = 0.828. Using the data for $n = 30$, we generated an accelerated Type-I GHCS sample. The experiment parameters were defined as $k = 12$, $r = 25$, and $T = 35.0$, with the stress level changing at time $\gamma = 3$. The resulting dataset yielded $J = 13$ failures before the stress change and a total of $m = 22$ failures. In light of this result using Type-I GHCS, we computed an estimate of parameters, MLE, and Bayes estimate using MCMC methods, employing 33,000 MCMC samples, with the first 3000 values being discarded as “burn-in”. Non-informative gamma priors are assumed for the unknown parameters in order to calculate the Bayes estimates without prior knowledge, with hyperparameters set to $a = b = c = d = 0.0001$. The results are provided in Tables 1 and 2 for Application 1. The marginal posterior density estimates and their corresponding histograms for the parameters γ, η and θ are presented in Figures 3–5. These visualizations confirm that the MCMC algorithm has successfully converged to the distribution, reflecting the high quality of the posterior samples obtained through the Bayesian approach.

Table 1. The point estimate of MLE, bootstrap, and MCMC for real data.

Par.	MLE	MCMC	Bootstrap
γ	1.1609	1.2185	1.1953
η	3.3669	3.3782	3.2430
Θ	8.1219	8.5870	8.4520

Table 2. The 95% confidence interval of MLE, bootstrap, and MCMC for real data.

Par.	95% ACI	MCMC 95% CI	95% Bootstrap
γ	(1.1281, 5.6057)	(1.9280, 5.2049)	(1.8531, 5.018)
η	(0.3466, 1.9751)	(0.7512, 1.5210)	(0.7631, 1.6007)
θ	(0.9161, 18.1601)	(4.2450, 18.2140)	(4.200, 18.9930)

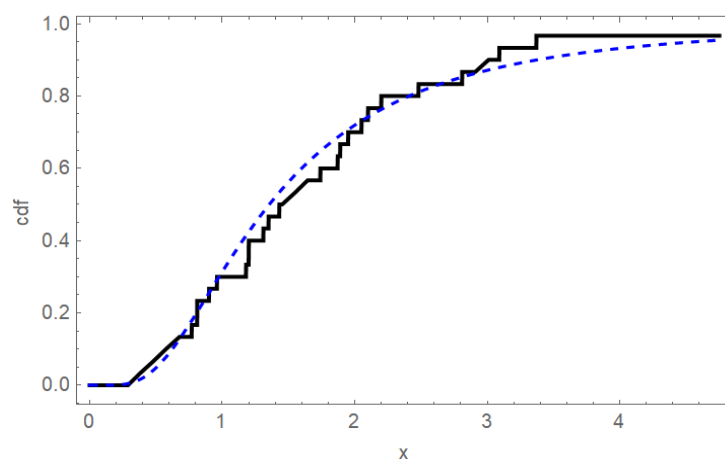


Figure 2. Comparison of the empirical function (black line) and fitted CDF (blue dashed line) for real data.

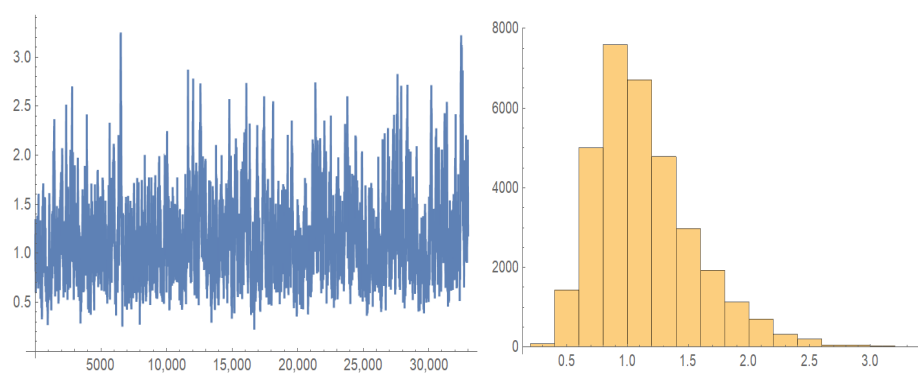


Figure 3. Trace plots and histograms of γ for MCMC method (real data).

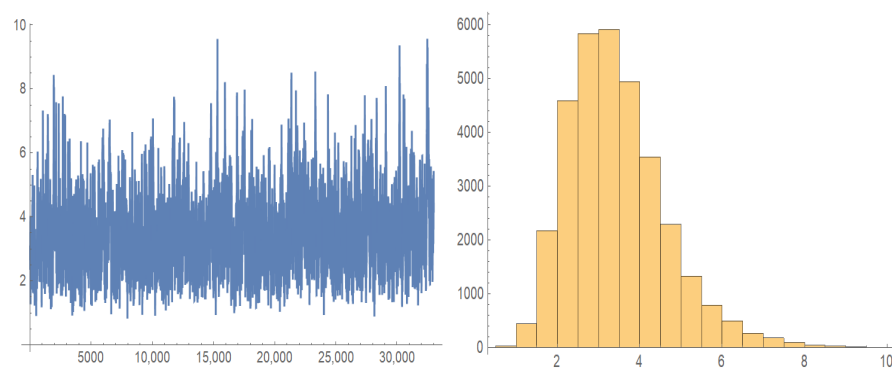


Figure 4. Trace plots and histograms of η for MCMC method (real data).

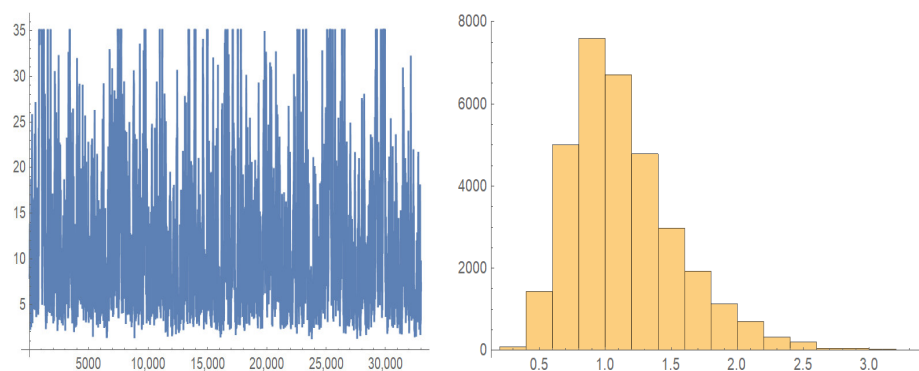


Figure 5. Trace plots and histograms of θ for MCMC method (real data).**Application 2: (Simulated data)**

To study and compare the proposed Bayes estimate with the MLE, we performed a Monte Carlo simulation study employing the IKum distribution, with the parameter values set to $(\gamma, \eta, \theta) = (0.5, 1.5, 6)$. The significance threshold that we employed was $\vartheta = 0.05$. We created Type-I GHCS where the parameters' true values for the IKum distribution were arbitrarily assumed to be $(\gamma, \eta, \theta) = (0.5, 1.5, 6)$. The PDF in Figure 1 were used to determine the values of γ and η under different combinations of n, r, κ and informative priors ($a = 1, b = 2, c = 3, d = 2$). In computing the estimate, we generated 10000 samples from the IKum distribution. We generated data for $n = 40$. Suppose $r = 35$, $\kappa = 30$, $\theta = 6.0$ and the stress change time $Y = 0.6$, then the accelerated Type-I GHCS can be obtained from the generated data. The integer numbers were $m = 35$ and $J = 0$. The generated observations are provided below as

0.0413474, 0.259038, 0.36493, 0.49007, 0.547116, 0.571806, 0.651004, 0.688497
 0.771678, 0.789858, 0.81758, 0.905022, 0.910081, 0.955493, 1.21984, 1.22893, 1.59267,
 1.62449, 1.70297, 1.76225, 1.83755, 1.95158, 2.64267, 2.68324, 2.75216, 2.86852, 3.09434, 4.07312, 4.19091, 4.26755.

We obtained the accelerated Type-I GHC data below as

{0.0413474, 0.221085, 0.258904, 0.303597, 0.32397, 0.332788, 0.361073, 0.374463, 0.404171, 0.410663, 0.420564, 0.451794, 0.4536, 0.469819, 0.564229, 0.567474, 0.697382, 0.708747, 0.736775, 0.757947, 0.784839, 0.825563, 1.07238, 1.08687, 1.11149, 1.15304, 1.23369, 1.58326, 1.62532, 1.6527}. The results are provided in Tables 3 and 4 for Example 2. Figures 6–8 show the marginal posterior density estimates and the marginal posterior density estimates histograms for the parameters γ , η , and θ .

These visualizations confirm that the MCMC algorithm has successfully converged to the stationary distribution, reflecting the high quality of the posterior samples obtained through the Bayesian approach. We used the Maximize command in Mathematica 12.

Table 3. The point estimate of MLE, bootstrap, and MCMC for simulated data.

Par.	MLE	MCMC	Bootstrap
γ	0.6022	0.6100	0.6204
η	1.6347	1.7103	1.6648
θ	6.4857	6.5240	6.4280

Table 4. The 95% different confidence interval of MLE, bootstrap, and MCMC for simulated data.

Par.	95% ACI	95% MCMC CI	95% Bootstrap
γ	(0.27, 0.9343)	(0.4138, 0.7850)	(0.4208, 0.7889)
η	(0.7616, 2.5077)	(0.7512, 1.5210)	(0.7650, 1.5327)
θ	(0.9213, 13.8784)	(2.1200, 14.5247)	(2.0150, 15.0193)

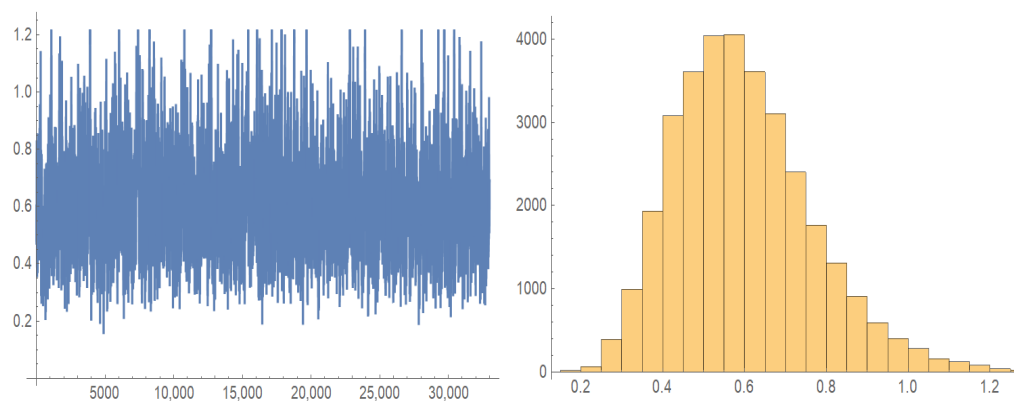


Figure 6. Trace plots and histograms for γ using the MCMC (simulated data).

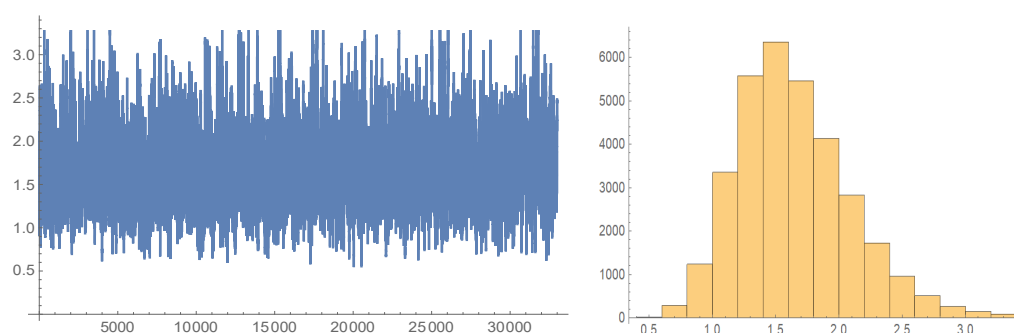


Figure 7. Trace plots and histograms for η using the MCMC (simulated data).

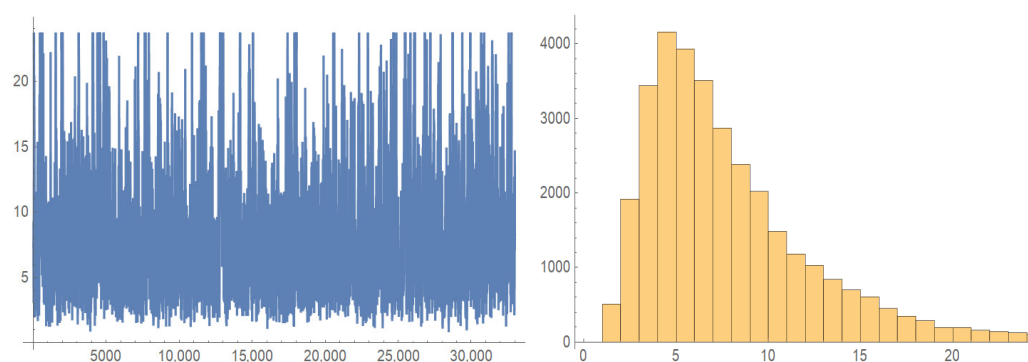


Figure 8. Trace plots and histograms for θ using the MCMC (simulated data).

6.2. Simulation Study

To study and compare the proposed Bayes estimate with the MLE, We used the IKum distribution in a Monte Carlo simulation research with the parameter values set to $(\gamma, \eta, \theta) = (0.5, 1.5, 6)$. The significance threshold that we employed was $\vartheta = 0.05$. We created Type-I GHCS, where the parameters' true values for the IKum distribution were arbitrarily assumed to be $(\gamma, \eta, \theta) = (0.5, 1.5, 6)$. Figure 1 was used to determine the values of γ and η .

Two distinct sets of priors are assumed for the four parameters, which are denoted as follows:

- **Non-Informative Prior (prior 1):** The Hyperparameters are $a = b = c = d = 0.0001$. The selected priors are diffusing (uninformative), intended to exert minimal influence on the inference. This allows the likelihood of the observed data to shape the posterior distribution, which is a strategy often chosen to avoid bias in the absence of prior knowledge.

- **Informative Prior (prior 2):** The Hyperparameters are $a = 1$, $b = 2$, $c = 3$, and $d = 2$. The informative priors are specified such that the mean aligns with findings from $a/b \approx E(\gamma)$ and $\frac{c}{d} \approx E(\eta)$; conversely, the informative priors are chosen such that their expectations align with the true values of parameter, ensuring the prior means match the true means for the subsequent SE loss function analyses.

In computing the estimate as in Algorithm 3, we generated 1000 samples from the IKum distribution. The Bias and MSEs of the estimate of parameters are presented in Table 5. The coverage percentages (CPs) and average credible interval lengths (ALs) are presented in Table 6.

A sample of size $n = 40$, was generated. Assuming $r = 35$, $k = 27$, and $\theta = 6.0$, and there is a stress change time of 0.6, the accelerated Type-I GHCS data was constructed.

Algorithm 3: Simulation for MSEs and CPs

- Step 1: Generate a Type-I GHCS from the IKum distribution based on pre-specified values of (θ, n, r, k) .
 - Step 2: Estimate the parameters using the ML and Bayesian approach under the SE loss function we estimate the parameters.
 - Step 3: Construct interval estimates, including asymptotic (MLE), bootstrap, and Bayesian credible intervals.
 - Step 4: Repeat the process N times to generate a distribution of estimates.
 - Step 5: Aggregate the results by computing the MSEs and CPs.
 - Step 6: Report the summarized findings as shown in Tables 5 and 6.
-

The integers corresponding to the failures are based on the observed Type-I GHCS data. We compute the parameter estimates using both ML and Bayesian methods. The Bayes estimates are obtained via MCMC simulations consisting of 33,000 samples, with the first 3000 discarded as ‘burn-in’. We assume non-informative priors to reflect the lack of prior information. The results are presented in Tables 5 and 6. We use Mathematica 12 for calculation.

Table 5. Average bias and MSEs of different estimates of parameters at γ , η and $\theta = 2, 3$ and 4.

$(\Upsilon, \tau, n, m, k)$		MLE			Bayes (Prior 1)			Bayes (Prior 2)		
		γ	η	θ	γ	η	θ	γ	η	θ
(0.5, 2.5, 60, 40, 30)	Bias	0.2478	0.5248	0.6201	0.2314	0.3200	0.7520	0.1182	0.3110	0.6607
	MSE	0.0987	0.2014	0.4820	0.0542	0.0932	0.248	0.0157	0.0745	0.2017
(0.8, 2.5, 60, 40, 30)	Bias	0.2150	0.5201	0.6541	0.1985	0.3304	0.7106	0.1173	0.3013	0.6270
	MSE	0.0850	0.1890	0.4106	0.0214	0.0885	0.2260	0.0155	0.0755	0.1998
(0.5, 3.5, 60, 40, 30)	Bias	0.2045	0.5017	0.6207	0.1856	0.3145	0.690	0.1085	0.3218	0.6006
	MSE	0.0632	0.2163	0.4005	0.0220	0.0907	0.2107	0.0114	0.0854	0.1935
(0.8, 3.5, 60, 40, 30)	Bias	0.2107	0.5104	0.6048	0.1624	0.3018	0.6110	0.1155	0.3007	0.6041
	MSE	0.0624	0.1932	0.4016	0.0200	0.0930	0.2458	0.0110	0.0854	0.1882
(0.5, 2.5, 80, 50, 40)	Bias	0.2013	0.5183	0.6149	0.1852	0.2904	0.6078	0.1852	0.2770	0.6032
	MSE	0.0621	0.2006	0.3990	0.0190	0.0877	0.2804	0.0111	0.0770	0.1850
(0.8, 2.5, 80, 50, 40)	Bias	0.2204	0.5109	0.6043	0.1824	0.2880	0.6008	0.1485	0.2907	0.6016
	MSE	0.0842	0.2015	0.3991	0.0166	0.0860	0.2017	0.0101	0.0771	0.1750
(0.5, 3.5, 80, 50, 40)	Bias	0.2105	0.5016	0.6004	0.2046	0.4993	0.5990	0.2040	0.4806	0.5705
	MSE	0.0698	0.1993	0.3880	0.0572	0.2043	0.2450	0.0576	0.1750	0.2730
(0.8, 3.5, 80, 50, 40)	Bias	0.2421	0.5007	0.6054	0.2134	0.4990	0.7120	0.1980	0.5100	0.5740
	MSE	0.0587	0.1998	0.3870	0.0480	0.1999	0.3008	0.0470	0.1740	0.3006

Table 6. The ALs and CPs of parameters at γ , η and $\theta = 2, 3$ and 4.

(Y, τ , n, m, k)		MLE			Bayes(Prior 1)			Bayes (Prior 2)		
		γ	η	θ	γ	η	θ	γ	η	θ
(0.5, 2.5, 60, 40, 30)	AL	0.7527	1.4520	11.6227	0.7664	1.540	10.5782	0.7107	1.2490	10.1520
	CP	0.9540	0.9440	0.9640	0.9520	0.9560	0.9620	0.9600	0.9520	0.9720
(0.8, 2.5, 60, 40, 30)	AL	0.7450	1.3480	11.5781	0.7770	1.4480	10.6570	0.7127	1.2180	10.5271
	CP	0.9600	0.9460	0.9640	0.9640	0.9500	0.9600	0.9620	0.9520	0.9620
(0.5, 3.5, 60, 40, 30)	AL	0.7358	1.5015	10.2537	0.7456	1.6480	10.3297	0.7180	1.420	10.1480
	CP	0.9640	0.9500	0.9540	0.9660	0.9600	0.9620	0.9740	0.9620	0.9620
(0.8, 3.5, 60, 40, 30)	AL	0.7245	1.3244	10.5276	0.7105	1.5120	10.4580	0.7046	1.337	10.1230
	CP	0.9600	0.9640	0.9540	0.9540	0.9720	0.9640	0.9640	0.9700	0.9600
(0.5, 2.5, 80, 50, 40)	AL	0.7210	1.3348	10.4556	0.7453	1.2280	10.4110	0.7018	1.2280	10.2590
	CP	0.9740	0.9700	0.9740	0.9660	0.9650	0.9640	0.9800	0.9640	0.9820
(0.8, 2.5, 80, 50, 40)	AL	0.7120	1.2475	10.6043	0.7340	1.486	10.4480	0.7000	1.2278	10.2189
	CP	0.9760	0.9660	0.9700	0.9740	0.9620	0.9740	0.9640	0.9680	0.9800
(0.5, 3.5, 80, 50, 40)	AL	0.7166	1.5011	10.6052	0.7014	1.2485	10.2482	0.7012	1.1183	10.2008
	CP	0.9760	0.9640	0.9760	0.9780	0.9680	0.9800	0.9680	0.9740	0.9780
(0.8, 3.5, 80, 50, 40)	AL	0.7005	1.5022	9.2586	0.7185	1.3297	10.2289	0.7013	1.3270	10.0548
	CP	0.9700	0.9740	0.9640	0.9720	0.9740	0.9700	0.9740	0.9800	0.9800

7. Results and Discussion

The following observations stem from the numerical analysis and Monte Carlo simulations:

- In both examples, the ML and Bayes estimates are similar because a non-informative previous was used. Point and interval estimates work best when using larger sample sizes and longer test times.
- The Bayesian credible interval outperforms both the ML and boot-p confidence intervals.
- The MSEs exhibit a downward trend as the sample size and effective sample size grow.
- Results indicate that incorporating prior information yields better point estimates; thus, the informative Bayes method surpasses both the ML and non-informative approaches.
- For interval estimation, the bootstrap and informative Bayes approaches outperform the ML method, especially when τ is large.

8. Concluding Remarks

Due to cost and time limitations, life-testing and reliability experiments often require analyzing incomplete information regarding failure times. This paper focuses on the IKum distribution, which is widely applied in biomedical sciences and engineering. We utilize a Type-I GHCS under an SS-PALT. This conventional censoring method is chosen specifically to save time and reduce experimental costs. The proposed method is validated using real data analysis, simulated data, and a Monte Carlo simulation study. Making inferences in a frequentist setting is often hampered by the difficulty of mathematically deriving theoretical sampling distributions. This is especially true in field reliability studies, where non-Gaussian and non-negative data complicate interval estimation. To overcome these limitations, we propose a Bayesian approach. This approach yields a combined posterior distribution that is proportionate to the product of the informative priors and the likelihood function. The suggested Bayes estimates perform exceptionally well, according to the results reported in Tables 5 and 6. Moreover, Tables 5

and 6 demonstrate that Bayesian methods are superior to classical methods when prior information is utilized, showing the expected decrease in MSEs as the effective sample size m grows. Lastly, this framework offers a solid foundation for the model's future advancement.

Future Work

Future research can build upon this study in several key directions. Methodologically, the framework could be extended to include alternative estimation techniques such as E-Bayesian, modified moments, and least squares, as well as one-sample prediction problems. Regarding experimental design, there is a clear need to develop optimal stress-loading strategies specifically for the IKum distribution. Furthermore, the applicability of this research can be broadened by incorporating diverse censoring schemes (Type-I, joint, or hybrid) and exploring non-constant stress patterns like cyclic or progressive loading.

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